Chapter 2
Dialogues with Play-Objects

The dialogical approach to logic is not a specific logical system but rather a rule-based semantic framework in which different logics can be developed, combined and compared. An important point is that there are different kinds of rules fixing meaning. This feature of the underlying semantics of the dialogical approach has often caused it to be called a *pragmatist* semantics.[1]

More clearly, in a dialogue two parties argue about a thesis while respecting certain fixed rules. The player stating the thesis is called Proponent (**P**) and his rival, the one contesting the thesis, is called Opponent (**O**). In their original form, dialogues were designed in such a way that each play ends after a finite number of moves with one player winning, and the other losing. Actions or moves in a dialogue are often understood as speech-acts involving declarative utterances (i.e. posits) and interrogative utterances (i.e. requests). The point is that the rules of the dialogue do not operate on expressions or sentences isolated from the act of uttering them. The rules are divided into particle rules or rules for logical constants (*Partikelregeln*) and

[1]The main original papers are collected in Lorenzen and Lorenz (1978). For a historical overview see Lorenz (2001). For a presentation about the initial role of the framework as a foundation for intuitionistic logic, see Felscher (1985). Other papers have been collected more recently in Lorenz (2008, 2010a, b). A detailed account of recent developments, since Rahman (1993), can be found in Rahman and Keiff (2005), Keiff (2009) and Rahman (2012). For the underlying metalogic see Clerbout (2014a, c). For textbook presentations: Kamlah and Lorenzen (1972, 1984), Lorenzen and Schwemmer (1975), Redmond and Fontaine (2011) and Rückert (2011a). For the key role of dialogic in linking dialectics and logic again, see Rahman and Keiff (2010). Keiff (2004a, b, 2007) and Rahman (2009) study the dialogical approach to Modal Logic. Fiutek et al. (2010) study the dialogical approach to belief revision. Clerbout et al. (2011) study Jain Logic in the dialogical framework. Popek (2012) developed a dialogical reconstruction of medieval obligationes. Rahman and Tulenheimo (2009) study the links between GTS and Dialogical games. For other books see Redmond (2010)— on fiction and dialogic—Fontaine (2013)—on intentionality, fiction and dialogues—and Magnier (2013)—on dynamic epistemic logic (van Ditmarsch et al. 2007) and legal reasoning in a dialogical framework.

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structural rules (*Rahmenregeln*). The structural rules establish the general course of a dialogue game, whereas the particle rules regulate the moves (or utterances) that are either requests (against the rival's moves) or answers (to the requests).

The following points are crucial to the dialogical approach[2]:

1. There is a distinction between the local meaning (rules for logical constants) and the global meaning (included in the structural rules that determine how to play);
2. The local meaning is player-independent;
3. There is a distinction between the play level (local winning or winning of a play) and the strategic level (existence of a winning strategy);
4. The notion of validity amounts to the existence of a winning strategy independently of any model instead of the existence of a winning strategy for every model;
5. Non formal and formal plays are differentiated. Formal plays concern plays in which positing elementary sentences does not require a meta-language level providing their truth.

In the framework of Constructive Type Theory propositions are sets whose elements are called proof-objects. When such a set is not empty, then the proposition has a proof and it is true. In his 1988 paper, Ranta proposed a way to use this approach in relation to game-theoretical approaches. Ranta took Hintikka's Game Theoretical Semantics as a case study, but his point does not depend on that particular framework. Ranta's idea was that in the context of game-based approaches, a proposition is a set of winning strategies for the player positing the proposition.[3] In game-based approaches, the notion of truth is found at the level of such winning strategies. Ranta's idea should therefore let us apply safely and directly methods taken from Constructive Type Theory to cases of game-based approaches.

But from the perspective of game theoretical approaches, reducing a game to a set of winning strategies is quite unsatisfactory, especially when it comes to a theory of meaning. This is particularly clear in the dialogical approach in which different levels of meaning are carefully distinguished. There is thus the level of strategies which is a level of meaning analysis, but there is also a level prior to it which is usually called the level of plays. The role of the latter level for developing an analysis is crucial according to the dialogical approach, as pointed out by Kuno Lorenz in his 2001 paper, p. 258:

> [. . .] for an entity [A] to be a proposition there must exist a dialogue game associated with this entity [. . .] such that an individual play of the game where A occupies the initial position [. . .] reaches a final position with either win or loss after a finite number of moves [. . .]

For this reason we would rather have propositions interpreted as sets of what we shall call play-objects and read an expression

$$p : \varphi$$

as "p is a play-object for φ".

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[2] Cf. Rahman (2012).

[3] That player can be called Player 1, Myself, or Proponent.

Thus, Ranta's work on proof-objects and strategies constitutes the end, not the beginning, of the dialogical project.

In order to present the dialogical framework which we will link with CTT, we will proceed in two steps. In the first section of this chapter, we present quite briefly the standard dialogical framework. The purpose is to introduce or recall the basics of the dialogical approach before delving into the more sophisticated system which we are interested in. This progressive introduction should be particularly helpful to readers not familiar with the dialogical approach. Covering the basics in the first section allows us to focus in the rest of the chapter on the framework modifications triggered by adding play-objects to the language.

2.1 Standard Dialogical Games

Let $\mathscr{L}$ be a first-order language built as usual upon the propositional connectives, the quantifiers, a denumerable set of individual variables, a denumerable set of individual constants and a denumerable set of predicate symbols (each with a fixed arity).

We extend the language $\mathscr{L}$ with two labels **O** and **P**, standing for the players of the game, and the two symbols '!' and '?'. When the identity of the player does not matter, we use variables **X** or **Y** (with $\mathbf{X} \neq \mathbf{Y}$). A move is an expression of the form '**X**-e', where e is either of the form $!\varphi$ for some sentence φ of $\mathscr{L}$ or of the form '$?[\varphi_1, \ldots, \varphi_n]$'.

The particle (or local) rules for standard dialogical games are given in Table 2.1.

Table 2.1 Particle rules in standard dialogical games

Posit	Challenge	Defence
$\mathbf{X}\ !\varphi \wedge \psi$	$\mathbf{Y}\ ?[\varphi]$ or $\mathbf{Y}\ ?[\psi]$	$\mathbf{X}\ !\varphi$ $\mathbf{X}\ !\psi$
$\mathbf{X}\ !\varphi \vee \psi$	$\mathbf{Y}\ ?[\varphi, \psi]$	$\mathbf{X}\ !\varphi$ or $\mathbf{X}\ !\psi$
$\mathbf{X}\ !\varphi \rightarrow \psi$	$\mathbf{Y}\ !\varphi$	$\mathbf{X}\ !\psi$
$\mathbf{X}\ !\neg\varphi$	$\mathbf{Y}\ !\varphi$	–
$\mathbf{X}\ !\forall x\varphi$	$\mathbf{Y}\ ?[\varphi(x/a_i)]$	$\mathbf{X}\ !\varphi(x/a_i)$
$\mathbf{X}\ !\exists x\varphi$	$\mathbf{Y}\ ?[\varphi(x/a_1), \ldots, \varphi(x/a_n)]$	$\mathbf{X}\ !\varphi(x/a_i)$ with $1 \leq i \leq n$

In this table, the a_is are individual constants and $\varphi(x/a_i)$ denotes the formula obtained by replacing every free occurrence of x in φ by a_i. When a move consists in a question of the form '$?[\varphi_1, \ldots, \varphi_n]$', the other player chooses one formula among $\varphi_1, \ldots, \varphi_n$ and plays it. We thus distinguish conjunction from disjunction and universal quantification from existential quantification in terms of which player has a choice. With conjunction and universal quantification, the challenger chooses which formula he asks for. With disjunction and existential quantification, it is the defender who can choose between various formulas. Notice that there is no defence in the particle rule for negation.

Particle rules provide an abstract description of how the game can proceed locally: they specify the way a formula can be challenged and defended according to its main logical constant. In this way the particle rules govern the local level of meaning. Strictly speaking, the expressions occurring in the table above are not actual moves because they feature formula schemata and the players are not specified. Moreover, these rules are indifferent to any particular situation that might occur during the game. For these reasons we say that the description provided by the particle rules is abstract.

Since the players' identities are not specified in these rules, particle rules are symmetric: the rules are the same for the two players. The local meaning being symmetric (in this sense) is one of the greatest strengths of the dialogical approach to meaning. It is in particular the reason why the dialogical approach is immune to a wide range of trivializing connectives such as Prior's *tonk*.[4]

The expressions occurring in particle rules are all move schematas. The words "challenge" and "defence" are convenient to name certain moves according to their relation with other moves which can be defined in the following way. Let σ be a sequence of moves. The function p_σ assigns a position to each move in σ, starting with 0. The function F_σ assigns a pair $[m, Z]$ to certain moves N in σ, where m denotes a position smaller than $p_\sigma(N)$ and Z is either C or D, standing respectively for "challenge" and "defence". That is, the function F_σ keeps track of the relations of challenge and defence as they are given by the particle rules. Consider for example the following sequence σ:

$$\mathbf{P}\,!\varphi \wedge \psi, \mathbf{P}\,!\chi \wedge \psi, \mathbf{O}\,?[\varphi], \mathbf{P}\,!\varphi$$

In this sequence we have for example $p_\sigma(\mathbf{P}\,!\chi \wedge \psi) = 1$.

A *play* is a legal sequence of moves, i.e., a sequence of moves which observes the game rules. Particle rules are not the only rules which must be observed in this respect. In fact, it can be said that the second kind of rules named *structural rules* are the ones giving the precise conditions under which a given sequence is a play. The dialogical game for φ, written $\mathscr{D}(\varphi)$, is the set of all plays with φ being the *thesis* (see the Starting Rule below). The structural rules are the following:

SR0 (Starting Rule). Let φ be a complex sentence of $\mathscr{L}$ and i, j be positive integers. For every $\varsigma \in \mathscr{D}(\varphi)$ we have:

[4] See Rahman et al. (2009) and Rahman (2012).

- $p_\varsigma(\mathbf{P}\,!\varphi) = 0$,
- $p_\varsigma(\mathbf{O}\,\mathtt{n} := \mathtt{i}) = 1$,
- $p_\varsigma(\mathbf{P}\,\mathtt{m} := \mathtt{j}) = 2$.

In other words, any play ς in $\mathscr{D}(\varphi)$ starts with **P** positing φ. We call φ the thesis of both the play and the dialogical game. After that, the Opponent and the Proponent successively choose a positive integer called repetition rank. The role of these integers is to ensure that every play ends after finitely many moves in the way specified by the next structural rule.

SR1 (Classical game-playing Rule).

- Let $\varsigma \in \mathscr{D}(\varphi)$. For every M in ς with $p_\varsigma(M) > 2$ we have $F_\varsigma(M) = [m', Z]$ with $m' < p_\varsigma(M)$ and $Z \in \{C, D\}$.
- Let $\mathtt{r}$ be the repetition rank of player **X** and $\varsigma \in \mathscr{D}(\varphi)$ such that
  - the last member of ς is a **Y**-move,
  - M_0 is a **Y**-move of position m_0 in ς,
  - $M_1, \ldots, M_n$ are **X**-moves in ς such that $F_\varsigma(M_1) = \cdots = F_\varsigma(M_n) = [m_0, Z]$.

  Consider the sequence[5] $\varsigma' = \varsigma \star N$ where N is an **X**-move such that $F_{\varsigma'}(N) = [m_0, Z]$. We have $\varsigma' \in \mathscr{D}(\varphi)$ only if $n < \mathtt{r}$.

The first part of the rule states that, after repetition ranks have been chosen, every move is either a challenge or a defence. The second part ensures finiteness of plays by setting the player's repetition rank as the maximum number of times he can challenge or defend against a given move by the other player.

SR2 (Formal Rule). Let ψ be an elementary sentence, N be the move $\mathbf{P}\,!\psi$ and M be the move $\mathbf{O}\,!\psi$. A sequence ς of moves is a play only if we have: if $N \in \varsigma$ then $M \in \varsigma$ and $p_\varsigma(M) < p_\varsigma(N)$.

That is, the Proponent can play an elementary sentence only if the Opponent has played it previously. The Formal Rule is one of the characteristic features of the dialogical approach: other game-based approaches do not have it.

Helge Rückert pointed out that the Formal Rule triggers a novel notion of validity: *Geltung* (Legitimacy).[6] Indeed with this rule the dialogical framework comes with an internal account for elementary sentences: an account in terms of interaction only, without depending on metalogical meaning explanations for the non-logical vocabulary. More prominently this means that the dialogical account does not rely—contrary to Hintikka's GTS games—on the model-theoretical approach to meaning for atomic formulas.

From there Rückert claims, and on this point we disagree with him, that *Geltung* is the idea that interaction emerges without knowing (or without needing to know) what the meaning of elementary sentences are. We disagree because the question of the meaning of elementary sentences (and more generally, of non-logical vocabulary)

[5] We use $\varsigma \star N$ to denote the sequence obtained by adding move N to the play ς.

[6] See Rückert (2011b).

cannot be disregarded if the dialogical framework is meant to provide a general theory of meaning.

In our view, Rückert's interpretation of *Geltung* unfortunately dissolves the question of the meaning of elementary sentences in the Formal Rule. This is mainly due to the fact that the standard version of the framework does not have the means to express a semantic at the object language level in terms of asking and giving reasons for elementary sentences. As a consequence, the standard formulation simply relies on the Formal Rule which amounts to entitle **P** to copy-cat the elementary sentences brought forward by **O**. According to us, the introduction of play-objects provides a solution to this without giving up the internal aspect linked with *Geltung*. We will develop this idea when giving the particle rules in Sect. 2.3 and after having introduced a "Modified Formal Rule" in Sect. 2.4.

Here is some terminology for the last structural rule in standard dialogical games. A play is called *terminal* when it cannot be extended by further moves in compliance with the rules. We say it is **X**-terminal when the last move in the play is an **X**-move.

SR3 (Winning Rule). Player **X** wins the play ς only if it is **X**-terminal.

Consider for example the following sequences of moves:

$$\mathbf{P}\,!Q(a) \wedge Q(b), \mathbf{O}\,\mathtt{n} := 1, \mathbf{P}\,\mathtt{m} := 6, \mathbf{O}\,?[Q(a)], \mathbf{P}\,!Q(a)$$
$$\mathbf{P}\,!Q(a) \rightarrow Q(a), \mathbf{O}\,\mathtt{n} := 1, \mathbf{P}\,\mathtt{m} := 12, \mathbf{O}\,!Q(a), \mathbf{P}\,!Q(a)$$

The first one is not a play because it breaks the Formal Rule: with his last move, the Proponent plays an elementary sentence which the Opponent has not played beforehand. By contrast, the second sequence is a play in $\mathscr{D}(Q(a) \rightarrow Q(a))$.

We often use a convenient table notation for plays. For example, we can write this play as follows:

	O			**P**	
				$!\,Q(a) \rightarrow Q(a)$	0
1	$\mathtt{n} := 1$			$\mathtt{m} := 12$	2
3	$!\,Q(a)$	(0)		$!\,Q(a)$	4

The numbers in the external columns are the positions of the moves in the play. When a move is a challenge, the position of the challenged move is indicated in the internal columns, as with move 3 in this example. Notice that such tables carry the information given by the functions p and F in addition to represent the play.

However, when we want to consider several plays together—for example when building a strategy—such tables are not that perspicuous. So we do not use them to deal with dialogical games for which we prefer another perspective. The *extensive form* of the dialogical game $\mathscr{D}(\varphi)$ is simply the tree representation of it, also often called the game-tree. More precisely, the extensive form $\mathfrak{E}_\varphi$ of $\mathscr{D}(\varphi)$ is the tree (T, ℓ, S) such that:

(i) Every node t in T is labelled with a move occurring in $\mathscr{D}(\varphi)$.
(ii) $\ell : T \mapsto \mathbb{N}$.
(iii) $S \subseteq T^2$ with the following:

– There is a unique t_0 (the root) in T such that $\ell(t_0) = 0$, and t_0 is labelled with the thesis of the game.
– For every $t \neq t_0$ there is a unique t' such that $t'St$.
– For every t and t' in T, if tSt' then $\ell(t') = \ell(t) + 1$.
– Let $\varsigma \in \mathscr{D}(\varphi)$ such that $p_\varsigma(M') = p_\varsigma(M) + 1$. If t and t' are respectively labelled with M and M', then tSt'.

Many metalogical results about dialogical games are obtained by leaving the level of rules and plays to move to the level of strategies. Significant among these results are the ones concerning the existence of winning strategies for a player. We will now define these notions and give examples of such results.

A *strategy* for player **X** in $\mathscr{D}(\varphi)$ is a function which assigns an **X**-move M to every non terminal play ς having a **Y**-move as last member such that extending ς with M results in a play. An **X**-strategy is winning if playing according to it leads to **X**'s victory no matter how **Y** plays.

Strategies can be considered from the perspective of extensive forms: the extensive form of an **X**-strategy s in $\mathscr{D}(\varphi)$ is the tree-fragment $\mathfrak{S}_\varphi = (T_s, \ell_s, S_s)$ of $\mathfrak{E}_\varphi$ such that:

(i) The root of $\mathfrak{S}_\varphi$ is the root of $\mathfrak{E}_\varphi$,
(ii) Given a node t in $\mathfrak{E}_\varphi$ labelled with an **X**-move, we have $t' \in T_s$ and tS_st' whenever tSt'.
(iii) Given a node t in $\mathfrak{E}_\varphi$ labelled with a **Y**-move and with at least one t' such that tSt', there is a unique t_s in T_s with tS_st_s and t_s is labelled with the **X**-move prescribed by s.

Here are some results pertaining to the level of strategies.[7]

- Winning **P**-Strategies and Leaves. *Let w be a winning* **P**-*strategy in* $\mathscr{D}(\varphi)$. *Then every leaf in the extensive form* $\mathfrak{W}_\varphi$ *of w is labelled with a* **P** *elementary sentence.*
- Determinacy. *There is a winning* **X**-*strategy in* $\mathscr{D}(\varphi)$ *if and only if there is no winning* **Y**-*strategy in* $\mathscr{D}(\varphi)$.
- Soundness and Completeness of Tableaux. *Consider first-order tableaux and first-order dialogical games. There is a tableau proof for φ if and only if there is a winning* **P**-*strategy in* $\mathscr{D}(\varphi)$.
  The fact that the existence of a winning **P**-strategy coincides with validity (there is a winning **P**-strategy in $\mathscr{D}(\varphi)$ if and only if φ is valid) follows from the soundness and completeness of the tableau method with respect to model-theoretical semantics.

Regarding several results, extensive forms of strategies have key parts: one of the parts of a winning strategy, called the *core* of the strategy, is actually that on which one works when considering translation algorithms such as the procedures given in Chaps. 3 and 5. We will give the details in Chap. 3, but the basic idea behind the notion of core is to get rid of redundant information (for example, different orders of

[7]These results are proven, together with others, in Clerbout (2014c).

moves) which we find in extensive forms of strategies. Now that we have recalled the standard dialogical approach, we will focus on the enriched dialogical framework we are interested in for the equivalence result with CTT.

2.2 The Formation of Propositions

Before delving into the details about play-objects, let us first discuss the issue of forming expressions and especially propositions in the dialogical approach.

It is presupposed in standard dialogical systems that the players use well-formed formulas (wff). The well formation can be checked at will, but only with the usual meta reasoning by which the formula is checked to indeed observe the definition of a wff. The first enrichment we want to make is to allow players to question the status of expressions, and in particular to ask if a certain expression is a proposition. We thus start with rules explaining dialogically the *formation* of propositions. These rules are local rules which are added to the particle rules giving the local meaning of logical constants (see next section).

A remark before displaying the formation rules: because the dialogical theory of meaning is based on argumentative interaction, dialogues feature expressions which are not posits of sentences. That is they also feature requests, used for challenges, as the formation rules below and the particle rules in the next section illustrate. Because of the *no entity without type* principle, it seems at first glance that we should specify the type of these actions during a dialogue: the type "*formation-request*". It turns out we should not: an expression such as "$?_F$: *formation-request*" is a judgement that some action $?_F$ is a formation-request, which should not be confused with the actual act of requesting. We also consider that the force symbol $?_F$ makes the type explicit. Hence the way requests are written in rules and dialogues in this work.

The formation rules are given in Table 2.2, p. 17. Notice that a posit '$\perp : prop$' cannot be challenged: this is the dialogical account of the fact that the falsum $\perp$ is by definition a proposition.

Table 2.2 Formation rules

Posit	Challenge (when different challenges are possible, the challenger chooses)	Defence
$\mathbf{X}\,!\,\Gamma : set$	$\mathbf{Y}\,?_{can}\,\Gamma$ or $\mathbf{Y}\,?_{gen}\,\Gamma$ or $\mathbf{Y}\,?_{eq}\,\Gamma$	$\mathbf{X}\,!\,a_1 : \Gamma\,, \mathbf{X}\,!\,a_2 : \Gamma\,, \ldots$ **X** gives the canonical elements of Γ $\mathbf{X}\,!\,a_i : \Gamma \Rightarrow a_j : \Gamma$ **X** provides a generation method for Γ **X** gives the equality rule for Γ (see Sect. 2.3)
$\mathbf{X}\,!\,\varphi \vee \psi : prop$	$\mathbf{Y}\,?_{F\vee 1}$ or $\mathbf{Y}\,?_{F\vee 2}$	$\mathbf{X}\,!\,\varphi : prop$ respectively $\mathbf{X}\,!\,\psi : prop$
$\mathbf{X}\,!\,\varphi \wedge \psi : prop$	$\mathbf{Y}\,?_{F\wedge 1}$ or $\mathbf{Y}\,?_{F\wedge 2}$	$\mathbf{X}\,!\,\varphi : prop$ respectively $\mathbf{X}\,!\,\psi : prop$
$\mathbf{X}\,!\,\varphi \rightarrow \psi : prop$	$\mathbf{Y}\,?_{F\rightarrow 1}$ or $\mathbf{Y}\,?_{F\rightarrow 2}$	$\mathbf{X}\,!\,\varphi : prop$ respectively $\mathbf{X}\,!\,\psi : prop$
$\mathbf{X}\,!\,(\forall x : A)\varphi(x) : prop$	$\mathbf{Y}\,?_{F\forall 1}$ or $\mathbf{Y}\,?_{F\forall 2}$	$\mathbf{X}\,!\,A : set$ respectively $\mathbf{X}\,!\,\varphi(x) : prop\ (x : A)$
$\mathbf{X}\,!\,(\exists x : A)\varphi(x) : prop$	$\mathbf{Y}\,?_{F\exists 1}$ or $\mathbf{Y}\,?_{F\exists 2}$	$\mathbf{X}\,!\,A : set$ respectively $\mathbf{X}\,!\,\varphi(x) : prop\ (x : A)$
$\mathbf{X}\,!\,B(k) : prop$ for atomic B	$\mathbf{Y}\,?_{F}$	$\mathbf{X}\,sic(n)$ **X** indicates that **Y** posited it at move n
$\mathbf{X}\,!\,\bot : prop$	–	–

The next rule is not a formation rule per se but rather a substitution rule.[8] When φ is an elementary sentence, the substitution rule helps explaining the formation of such sentences.

Posit-substitution

When a list of variables occurs in a posit with proviso, the challenger **Y** can ask **X** to replace those variables: he does so by positing an instantiation of the proviso, in which he (**Y**) is the one who chooses the instantiations for the variables.[9]

[8] It is an application of the original rule from CTT given in Ranta (1994, p. 30).

[9] More precisely: in the case where the defender did not commit himself to the proviso. The dialogical approach allows a distinction here which we discuss in the next section.

Posit	Challenge	Defence
$\mathbf{X} ! \pi(x_1, \ldots, x_n)\ (x_i : A_i)$	$\mathbf{Y} ! \tau_1 : A_1, \ldots, \tau_n : A_n$	$\mathbf{X} ! \pi(\tau_1, \ldots, \tau_n)$

A particular case of posit substitution is when the challenger simply posits the whole assumption as it is without introducing new instanciation terms. This is particularly useful in the case of formation plays: see an application in the example given in Table 2.4, p. 20.

Posit	Challenge	Defence
$\mathbf{X} ! \pi(\tau_1, \ldots, \tau_n)\ (\tau_i : A_i)$	$\mathbf{Y} ! \tau_1 : A_1, \ldots, \tau_n : A_n$	$\mathbf{X} ! \pi(\tau_1, \ldots, \tau_n)$

Remarks on the formation dialogues:

(a) Conditional formation posits: A crucial feature of formation rules is that they enable the displaying of the syntactic and semantic presuppositions of a given thesis which can thus be examined by the Opponent before running the actual dialogue on the thesis. For instance if the thesis amounts to positing φ, then the Opponent can ask for its formation before launching an attack. Defending on the formation of φ might bring the Proponent to posit that φ is a proposition, provided for instance that A being a set is conceded. In this situation the Opponent might concede A is a set, but only after the Proponent displayed the constitution of A.

(b) Elementary sentences, definitional consistency and material-analytic dialogues: Following the idea of formation rules through and through, the defence $sic(n)$ for elementary sentences is somehow unsatisfactory as it does not really explore the formation of the expression. A defence which applies fitting predicator rules previously conceded, if such a concession has been made, would be a possibility. See Rahman and Clerbout (2014). What would then happen is that the challenge of elementary sentences would be based on the definitional consistency of the use of the conceded predicator rules. This is what we think material dialogues are about: definitional consistency dialogues. This leads to the following material analytic rule for formation dialogues:

> **O**'s elementary sentences cannot be challenged, however **O** can challenge an elementary sentence (posited by **P**) iff she herself (the Opponent) did not posit it before.

Remark Once **P** forced **O** to concede the elementary sentence in the formation dialogue, the dialogue proceeds using the copy-cat strategy. The version of the rule we work with, in which the defence is $sic(n)$, is related to that.

By way of illustration, Table 2.3 gives an example where the Proponent posits the thesis $(\forall x : A)(B(x) \rightarrow C(x)) : prop$ given that $A : set$, $B(x) : prop\ (x : A)$ and $C(x) : prop\ (x : A)$. The three provisos appear as initial concessions by the Opponent.[10] Normally we should give all the rules of the game before giving an example, but we make an exception here because the standard structural rules of Sect. 2.1 are enough to understand the following plays. We can thus focus on illustrating the way formation rules can be used.

[10]The example comes from Ranta (1994, p. 31).

Table 2.3 Example of a formation dialogue 1

	O			P	
I	$!\, A : set$				
II	$!\, B(x) : prop\ (x : A)$				
III	$!\, C(x) : prop\ (x : A)$				
				$!\, (\forall x : A)(B(x) \to C(x)) : prop$	0
1	$\mathtt{n} := 1$			$\mathtt{m} := 2$	2
3	$?_{F\forall 1}$	(0)		$!\, A : set$	4

Explanations:

- I to III: **O** concedes that A is a set and that $B(x)$ and $C(x)$ are propositions provided x is an element of A.
- Move 0: **P** posits that the main sentence, universally quantified, is a proposition (under the concessions made by **O**).
- Moves 1 and 2: the players choose their repetition ranks.[11]
- Move 3: **O** challenges the thesis by asking the left-hand part as specified by the formation rule for universal quantification.
- Move 4: **P** responds by positing that A is a set. This has already been granted with the premiss I so even if **O** were to challenge this posit, the Proponent could refer to this initial concession. Later, we will introduce the structural rule $SR3$ to deal with this phenomenon (see Sect. 2.4). Thus **O** has no further possible move, the dialogue ends here and is won by **P**.[12]

Obviously, this dialogue does not cover all the aspects related to the formation of $(\forall x : A)(B(x) \to C(x))$. Notice however that the formation rules allow an alternative move for the Opponent's move 3.[13] Hence another possible course of action for **O** arises (Table 2.4).

[11]The device of repetition ranks is introduced in the structural rules which we present in Sect. 2.4. See also Clerbout (2014a, b, c) for detailed explanations on this notion.

[12]See Sect. 2.4.

[13]As a matter of fact, increasing her repetition rank would allow her to play the two alternatives for move 3 within a single play. But increasing the Opponent's rank usually yields redundancies (Clerbout 2014b, c) making things harder to understand for readers not familiar with the dialogical approach. Hence our choice to divide the example into different simple plays.

Table 2.4 Example of a formation dialogue 2

	O			P	
I	$!\,A : set$				
II	$!\,B(x) : prop\ (x : A)$				
III	$!\,C(x) : prop\ (x : A)$				
				$!\,(\forall x : A)(B(x) \to C(x)) : prop$	0
1	$\mathtt{n} := 1$			$\mathtt{m} := 2$	2
3	$?_{F_{\forall 2}}$	(0)		$!\,B(x) \to C(x) : prop\ (x : A)$	4
5	$!\,x : A$	(4)		$!\,B(x) \to C(x) : prop$	6
7	$?_{F_{\to 1}}$	(6)		$!\,B(x) : prop$	10
9	$!\,B(x) : prop$		(II)	$!\,x : A$	8

Explanations:

The second dialogue starts like the first one until move 2. Then:

- Move 3: This time **O** challenges the thesis by asking for the right-hand part.
- Move 4: **P** responds, positing that $B(x) \to C(x)$ is a proposition provided $x : A$.
- Move 5: **O** uses the substitution rule to challenge move 4 by granting the proviso.
- Move 6: **P** responds by positing that $B(x) \to C(x)$ is a proposition.
- Move 7: **O** then challenges move 6 by asking the left-hand part as specified by the formation rule for material implication.

To defend this **P** needs to make an elementary move. But since **O** has not played it yet, **P** cannot defend it at this point. Thus:

- Move 8: **P** launches a counterattack against assumption II by applying the substitution rule.
- Move 9: **O** answers to move 8 and posits that $B(x)$ is a proposition.
- Move 10: **P** can now defend in reaction to move 7 and win this dialogue.

Then again, there is another possible path for the Opponent because she has another possible choice for her move 7, namely asking the right-hand part. This yields a dialogue similar to the one above except that the last moves are about $C(x)$ instead of $B(x)$.

By displaying these various possibilities for the Opponent, we have entered the *strategical* level. This is the level at which the question of the good formation of the thesis gets a definitive answer, depending on whether the Proponent can always win, that is, whether he has a winning strategy. We have introduced the basic notions related to this level in the previous section. See also the end of the next section, as well as Chaps. 3 and 5 for more explanations.

Now that the dialogical account of formation rules has been clarified, we may further develop our analysis of plays by introducing play-objects.

2.3 Play-Objects

The idea now is to design dialogical games in which the players' posits are of the form "$p : \varphi$" and get their meaning by the way they are used in the game: how they are challenged and defended. This requires analysing the form of a given play-object p, which depends on φ, and how a play-object can be obtained from other, simpler, play-objects. The standard dialogical semantics (Sect. 2.1) for logical constants gives us the information we need. The main logical constant of the expression at stake provides the basic information as to what a play-object for that expression consists of:

A play for $\mathbf{X} ! \varphi \vee \psi$ is obtained from two plays p_1 and p_2, in which p_1 is a play for $\mathbf{X} ! \varphi$ and p_2 is a play for $\mathbf{X} ! \psi$. According to the standard dialogical approach to disjunction, the player $\mathbf{X}$ is the one who can switch from p_1 to p_2 and conversely.

A play for $\mathbf{X} ! \varphi \wedge \psi$ is obtained similarly, except that the player $\mathbf{Y}$ is the one who can switch from p_1 to p_2.

A play for $\mathbf{X} ! \varphi \rightarrow \psi$ is obtained from two plays p_1 and p_2, in which p_1 is a play for $\mathbf{Y} ! \varphi$ and p_2 is a play for $\mathbf{X} ! \psi$. The player $\mathbf{X}$ is the one who can switch from p_1 to p_2.

The standard dialogical particle rule for negation rests on the interpretation of $\neg\varphi$ as an abbreviation for $\varphi \rightarrow \bot$, although it is usually left implicit. From this follows that one obtains plays for $\mathbf{X} ! \neg\varphi$ in a way similar to plays for a material implication, that is from two plays p_1 and p_2 in which p_1 is a play for $\mathbf{Y} ! \varphi$, p_2 is a play for $\mathbf{X} ! \bot$, and $\mathbf{X}$ can switch from p_1 to p_2. Notice that this approach covers the standard game-theoretical interpretation of negation as role-switch: p_1 is a play for a $\mathbf{Y}$-move.

As for quantifiers, a detailed discussion will be given after the particle rules. We would like to point out for now that, just like what is done in CTT, we are dealing with quantifiers for which the type of the bound variable is always specified. We thus consider expressions of the form $(Qx : A)\varphi$, where Q is a quantifier symbol.

Table 2.5, on p. 22, presents the particle rules.

Let us point out that we have added a challenge of the form $\mathbf{Y}\ ?_{prop}$ by which the challenger questions the fact that the expression at the right-hand side of the semi-colon is a proposition. This connects back with the formation rules of Sect. 2.2 via **X**'s defence. Further details will be given in the discussion following the structural rules.

It may happen that the form of a play-object is not explicit at first. In such cases we deal with expressions of the form (for example) "$p : \varphi \wedge \psi$". In the relevant challenges and defences, we then use expressions such as $L^{\wedge}(p)$ and $R^{\wedge}(p)$ used in our example. We call these expressions *instructions*. Their respective interpretations are "take the left part of p" and "take the right part of p". In instructions we indicate the logical constant at stake.[14] This keeps the formulations explicit enough, in particular in the case of embedded instructions. We must keep in mind the important differences between play-objects depending on the logical constant. Consider for example the cases of conjunction and disjunction:

[14]If needed, we use subscripts to prevent scope ambiguities in the case of embedded occurrences of the same quantifier.

Table 2.5 Particle rules

Posit	Challenge	Defence
$\mathbf{X}\,!\,\varphi$ (where no play-object has been specified for φ)	$\mathbf{Y}\,?\,play\text{-}object$	$\mathbf{X}\,!\,p : \varphi$
$\mathbf{X}\,!\,p : \varphi \vee \psi$	$\mathbf{Y}\,?_{prop}$	$\mathbf{X}\,!\,\varphi \vee \psi : prop$
	$\mathbf{Y}\,?\,[\varphi/\psi]$	$\mathbf{X}\,!\,L^{\vee}(p) : \varphi$ or $\mathbf{X}\,!\,R^{\vee}(p) : \psi$ **[the defender has the choice]**
$\mathbf{X}\,!\,p : \varphi \wedge \psi$	$\mathbf{Y}\,?_{prop}$	$\mathbf{X}\,!\,\varphi \wedge \psi : prop$
	$\mathbf{Y}\,?\,L$ or $\mathbf{Y}\,?\,R$ **[the challenger has the choice]**	$\mathbf{X}\,!\,L^{\wedge}(p) : \varphi$ respectively $\mathbf{X}\,!\,R^{\wedge}(p) : \psi$
$\mathbf{X}\,!\,p : \varphi \rightarrow \psi$	$\mathbf{Y}\,?_{prop}$	$\mathbf{X}\,!\,\varphi \rightarrow \psi : prop$
	$\mathbf{Y}\,!\,L^{\rightarrow}(p) : \varphi$	$\mathbf{X}\,!\,R^{\rightarrow}(p) : \psi$
$\mathbf{X}\,!\,p : \neg\varphi$	$\mathbf{Y}\,?_{prop}$	$\mathbf{X}\,!\,\neg\varphi : prop$
	$\mathbf{Y}\,!\,L^{\perp}(p) : \varphi$	$\mathbf{X}\,!\,R^{\perp}(p) : \perp$
$\mathbf{X}\,!\,p : (\exists x : A)\varphi$	$\mathbf{Y}\,?_{prop}$	$\mathbf{X}\,!\,(\exists x : A)\varphi : prop$
	$\mathbf{Y}\,?\,L$ or $\mathbf{Y}\,?\,R$ **[the challenger has the choice]**	$\mathbf{X}\,!\,L^{\exists}(p) : A$ respectively $\mathbf{X}\,!\,R^{\exists}(p) : \varphi(L^{\exists}(p))$
$\mathbf{X}\,!\,p : \{x : A \mid \varphi\}$	$\mathbf{Y}\,?\,L$ or $\mathbf{Y}\,?\,R$ **[the challenger has the choice]**	$\mathbf{X}\,!\,L^{\{\ldots\}}(p) : A$ respectively $\mathbf{X}\,!\,R^{\{\ldots\}}(p) : \varphi(L^{\{\ldots\}}(p))$
$\mathbf{X}\,!\,p : (\forall x : A)\varphi$	$\mathbf{Y}\,?_{prop}$	$\mathbf{X}\,!\,(\forall x : A)\varphi : prop$
	$\mathbf{Y}\,!\,L^{\forall}(p) : A$	$\mathbf{X}\,!\,R^{\forall}(p) : \varphi(L^{\forall}(p))$
$\mathbf{X}\,!\,p : B(k)$ (for atomic B)	$\mathbf{Y}\,?_{prop}$	$\mathbf{X}\,!\,B(k) : prop$
	$\mathbf{Y}\; p : B(k)?$	$\mathbf{X}\; sic(n)$ (**X** indicates that **Y** posited it at move n)

- A play-object p for a disjunction is composed of two play-objects, but each of them constitutes a sufficient play-object for the disjunction. Moreover it is the defender who makes the choice between $L^{\vee}(p)$ and $R^{\vee}(p)$.
- A play-object p for a conjunction is also composed of two play-objects, but this time the two of them are necessary to constitute the play-object for the conjunction.

> It is then the challenger's privilege to ask for either or both (provided the other rules allow him to do so).[15]

Accordingly, $L^{\wedge}(p)$ and $L^{\vee}(p)$, say, are actually different things and the notation takes that into account.

Let us now focus on the quantifier rules. Dialogical semantics highlights the fact that there are two distinct moments when considering the meaning of quantifiers: choosing a suitable substitution term for the bound variable, and instantiating the formula after replacing the bound variable with the chosen substitution term. However, the standard dialogical approach presupposes a unique, global, collection of objects over which the quantifiers range. Things are different with the explicit language borrowed from CTT. Quantification is always relative to a set, and there are sets of many different kinds of objects (sets of individuals, sets of pairs, sets of functions, etc.). Owing to the instructions we can give a general form for the particle rules, and the object is specified in a third and later moment when instructions are "resolved" by means of the structural rule $SR4.1$ presented in the next section.

Constructive Type Theory clearly shows the basic similarity there is, as soon as propositions are thought of as sets, between conjunction and existential quantifier on the one hand and material implication and universal quantifier on the other hand. Briefly, the point is that they are formed in similar ways and their elements are generated by the same kind of operations.[16] In our approach, this similarity manifests itself in the fact that a play-object for an existentially quantified expression is of the same form as a play-object for a conjunction. Similarly, a play-object for a universally quantified expression is of the same form as one for a material implication.[17]

The particle rule just before the one for universal quantification is a novelty in the dialogical approach. It involves expressions commonly used in Constructive Type Theory to deal with separated subsets. The idea is to understand those elements of A *such that* φ as expressing that at least one element $L^{\{\cdots\}}(p)$ of A witnesses $\varphi(L^{\{\cdots\}}(p))$. The same correspondence that linked conjunction and existential quantification now appears.[18] This is not surprising since such posits actually have an

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[15] See in particular in the next section the repetition ranks in the structural rule $SR1i$.

[16] More precisely, conjunction and existential quantifiers are two particular cases of the Σ operator (disjoint union of sets), whereas material implication and universal quantifiers are two particular cases of the Π operator (indexed product on sets). See for example Ranta (1994, Chap. 2).

[17] Still, if we are playing with classical structural rules, there is a slight difference between material implication and universal quantification which we take from Ranta (1994, Table 2.3)., namely that in the second case p_2 always depends on p_1.

[18] As pointed out in Martin-Löf (1984), subset separation is another case of the Σ operator. See in particular p. 53:

> Let A be a set and $B(x)$ a proposition for $x \in A$. We want to define the set of all $a \in A$ such that $B(a)$ holds (which is usually written $\{x \in A : B(x)\}$). To have an element $a \in A$ such that $B(a)$ holds means to have an element $a \in A$ together with a proof of $B(a)$, namely an element $b \in B(a)$. So the elements of the set of all elements of A satisfying $B(x)$ are pairs (a, b) with $b \in B(a)$, i.e. elements of $(\Sigma x \in A)B(x)$. Then the Σ-rules play the role of the Comprehension Axiom (or the separation principle in ZF).

existential aspect: in $\{x : A \mid \varphi\}$ the left part "$x : A$" signals the existence of a play-object. Let us point out that since the expression stands for a set, it is not presupposed to be a proposition in **X**'s posit. This is why it cannot be challenged with the request "$?_{prop}$".

As we previously said, in the dialogical approach to CTT every object is known as instantiating a type and this constitutes the most elementary form of assertion $a : A$. Furthermore, instructions are in fact substitution commitments in a sense very close to the one mentioned by Brandom.[19] A thorough study is yet to be done on the substitutional approach to subsentential expressions and the role of instructions, though in our view it would be necessary for the exploration of both the formal consequences of Brandom's insights and the philosophical tenets underlying the notion of instruction.

Let us now consider the rule for the elementary case so as to close on the particle rules and complete our remarks of Sect. 2.1 on Rückert's point about legitimacy (*Geltung*). In this rule, but also in the associated formation rule of Sect. 2.2, the defence "$sic(n)$" recalls that the adversary has previously made the same posit. The rule works in a similar fashion as the Formal Rule of the standard formulation of Sect. 2.1, except that it is applicable to both players: it is not limited to the Proponent. We say similar in the sense that the rule allows players to perform a kind of copy-cat. Once that aspect of the Formal Rule is provided, we can work with a modified version of the rule which we will introduce with more explanations in the next section.

Despite the similarity we have just mentioned, there is a crucial difference with standard dialogical games. Elementary sentences are associated with play-objects, and one such sentence can be associated with many different play-objects in actual courses of the game. Therefore, and this is a most important point, the defence "$sic(n)$" does not express a copy-cat on the elementary sentence alone, but on the whole posit. We thus have a game rule such that, for a given elementary sentence, there are as many ways to give reasons for it (to defend it) as there are play-objects for it. Formulating the rule with the defence "$sic(n)$" is very different from merely integrating the standard Formal Rule at the local level: "$sic(n)$" is an abbreviation useful to provide an abstract rule, but because play-objects are introduced, it actually embodies a fully fledged semantics in terms of asking for and giving reasons.

So far, apart from the rule for subset-separation and the rule for elementary sentences, we have mostly adapted the rules of standard dialogical games to the explicit language we are working with. Now because of the explicit nature of this language,

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[19] See Brandom (1994, pp. 425–426):

> So for an expression to be used as a singular term, there must be **some** substantive substitutional commitment undertaken by the one who uses it. It is not necessary that either the one who undertakes that commitment or the one who attributes it—by attributing a doxastic commitment that would be avowed by the assertion of a sentence containing the singular term—be able to specify just what the content of that commitment is. [. . .]
>
> Purported reference to objects must be understood in terms of substitutional commitments linking diverse expressions.

there are more rules related to the meaning explanations of play-objects and types. The next rules involve what is known in CTT as definitional equality. These rules introduce a different kind of provisional clause, namely a clause in which the defender is the player committed to the expression within the clause and thus he, rather than the challenger, will eventually posit it. In standard CTT there is no need for such a distinction since there are no players. However, in dialogical games the distinction can and must be made depending on who posits the proviso. Accordingly we use the notation $< \cdots >$ to signal that it is the player making the posit who is committed to the expression in the proviso clause and $(\cdots)$ when it is the challenger.

We have already considered the latter case in this chapter. As for the first case, let π be a posit and $< \cdots >$ a proviso which the utterer is committed to. The general form of the rule is then the following:

Posit	Challenge	Defence
$\mathbf{X}\,!\pi < \cdots >$	$\mathbf{Y}\ ?_{[\pi]}$ or $\mathbf{Y}\ ?_{[<>]}$	$\mathbf{X}\,![\pi]$ $\mathbf{X}\,![<>]$
	where $?_{[\pi]}$ and $![\pi]$ stand respectively for the relevant challenge or defence against π, and similarly for $?_{[<>]}$ and $![<>]$	

In the initial posit, **X** commits himself to both π and the proviso. Hence **Y** is entitled to question either one, and he is the one to choose which to ask for. The rule states that the challenger can question either part of the initial posit, and that in each case he does so depending on the form of the expression. An illustration is helpful here. Assume the initial posit is $p : (\forall x : A)B(x) < c : C >$ which reads "given $c : C$ we have $B(x)$ for all $x : A$; and the player making the posit commits himself to the proviso". Then the rule is applied in the following way:

Posit	Challenge	Defence
$\mathbf{X}\,!p : (\forall x : A)B(x) < c : C >$	$\mathbf{Y}\,!L^{\forall}(p) : A$ or $\mathbf{Y}\,c : C?$	$\mathbf{X}\,!R^{\forall}(p) : B(L^{\forall}(p))$ $\mathbf{X}\,sic(n)$

In this case, the challenger can attack either part of the initial posit. To challenge the first part, he applies the particle rule for universal quantification. The second part is challenged by applying the particle rule for elementary posits.

A typical case in which provisos of the form $< \cdots >$ occur is functional substitution. Assume some function f has been introduced, for example with $f(x) : B(x : A)$. When a player uses $f(a)$ in a posit, for some $a : A$, the antagonist is entitled to ask him what the output of f is, given a as input. Now $f(a)$ can be used either at the left or at the right of the colon. Accordingly we have two rules:

(Function-substitution)

Posit	Challenge	Defence
$\mathbf{X}\,!f(a):\varphi$	$\mathbf{Y}\ f(a)/?_{<=>}$	$\mathbf{X}\,!f(a)/k_i:\varphi < f(a)=k_i:B >$
$\mathbf{X}\,!\alpha:\varphi[f(a)]$	$\mathbf{Y}\ f(a)/?_{<=>}$	$\mathbf{X}\,!\alpha:\varphi[f(a)/k_i] < \varphi[f(a)]=\varphi[f(a)/k_i]:set >$

The subscript '$<=>$' in the challenges indicates that the substitution is related to some equality, and the defender endorses an equality in the proviso of the defence. The second rule—in which α can be a play-object or an instruction—is applied in the dialogical take on the Axiom of Choice. See the "second play" in Sect. 4.1.1.

Important remark: These two rules express a double commitment for the defender who is committed to the proviso in the defence. One might therefore argue that the rules could also be formulated as involving two challenges (and two defences). There are however two problems with such an approach. For illustration purposes, let us consider such a formulation of the second rule involving two steps:

Posit	Challenge	Defence
$\mathbf{X}\,!\alpha:\varphi[f(a)]$	$\mathbf{Y}\ L(f(a))/?$	$\mathbf{X}\,!\,p:\varphi[f(a)/k_i]$
	$\mathbf{Y}\ R(f(a))/?$	$\mathbf{X}\,!\,\varphi[f(a)]=\varphi[f(a)/k_i]:set$

The first problem is that the second challenge works as if the proviso $\varphi[f(a)] = \varphi[f(a)/k_i]:set$ was implicit in the initial posit and had to be made explicit. However this is a slightly misguided approach since the proviso does not concern the initial posit: the proviso must be established only after **X** has chosen k_i for the substitution. The second problem is related to the first: in such a formulation the challenger is the one who can choose between asking **X** to perform the substitution and asking him to posit the proviso. It thus allows the challenger to perform just the second challenge without asking for the substitution, which brings us back to the first problem. Moreover, introducing a choice for one of the players results, when the rule can be applied, in multiplying the number of alternative plays (in particular when the repetition rank of the challenger is 1). For all these reasons, such an alternative formulation is less satisfactory than the one we gave above.

Functional substitution is closely related to the Π-Equality rule, which we now introduce together with Σ.

(Π-Equality) We use the CTT notation Π which covers the cases of universal quantification and material implication.

Posit	Challenge	Defence
$\mathbf{X}\,!p:(\Pi x:A)\varphi$ $\mathbf{Y}\,!L^{\Pi}(p)/a:A$ $\mathbf{X}\,!R^{\Pi}(p):\varphi(a/x)$	$\mathbf{Y}\ ?_{\Pi\text{-}Eq}$	$\mathbf{X}\,!p(a)=R^{\Pi}(p):\varphi(a/x)$

(Σ-Equality) The rule is similar for existential quantification, subset separation, and conjunction. Thus we use the notation from CTT with the Σ operator. In the following rule I^{Σ} can be either L^{Σ} or R^{Σ}, and i can be either 1 or 2: it is 1 when I is L and 2 when I is R.

Posit	Challenge	Defence
$\mathbf{X}\,!p : (\Sigma x : \varphi_1)\varphi_2$ $\mathbf{Y}\,I^{\Sigma}(p)/?$ $\mathbf{X}\,!p_i/I^{\Sigma}(p) : \varphi_i$	$\mathbf{Y}\,?_{\Sigma\text{-}Eq}$	$\mathbf{X}\,!I^{\Sigma}(p) = p_i : \varphi_i$

Notice that these rules have several preconditions: there is no lone initial posit triggering the application of the rule. From a dialogical perspective, these rules intend to allow the challenger to take advantage of information from the history of the current play—including resolutions of instructions—to make **X** posit some equality. For an application, see the second play in Sect. 4.1.1 in which the Π-Equality rules play a prominent role.

These rules strongly suggest a close connection between the CTT equality rules for logical constants and the dialogical instructions through what we will call in the next section their *resolution*. It is thus important to remind the significant differences between them, and especially that the particle rules define operations on propositions that are very different from the set-theoretical operations in CTT.

Let us discuss this topic before giving the remaining rules. The main point involves the *player-independence* of the rules, which we have only mentioned at the beginning of this chapter. By this we refer to the fact that the rules presented in this section are the same for the two players, which is why they are formulated with the variables **X** and **Y**. Various publications[20] have already linked the notion of player-independence to the immunity of the dialogical framework to different trivializing connectives such as Prior's *tonk*[21] and thus more generally to Dummett's requirement of *harmony* between Introduction and Elimination rules.[22]

In CTT, the harmonious relationship between Introduction and Elimination rules is made explicit by associating each logical constant with a suitable Equality rule (Sect. 1.2.2).[23] More precisely, the possibility to have such equality rules ensures harmony between Introduction and Elimination rules.

But at the same time the triplets Introduction-Elimination-Equality rules in the usual presentation of the CTT framework reinforces a certain form of asymmetry between Introduction and Elimination rules. The very idea of requiring rules to be harmonious advocates for the possibility of an approach with rules of only one kind (either Introduction or Elimination), and to design harmonious corresponding rules of the other kind—see Dummett (1993). By giving priority to Introduction rules, Gentzen (1934–1935) already observed one direction in this possible alternative,

[20] See Rahman and Keiff (2005), Rahman et al. (2009), Rahman (2012).

[21] Cf. Prior (1961).

[22] See Dummett (1973).

[23] They are, after all, linearized versions of Prawitz's reduction steps in Natural Deduction. See Sundholm (1997, Sect. 3.6).

and in this respect the standard presentation of CTT follows him: Equality rules establish the harmony of Elimination rules with respect to previously given Introduction rules. That is to say, we can start with the Introduction rules, then derive from them corresponding Elimination rules and check that they are harmonious.[24]

Therefore the Equality rules in the standard presentation cannot be used to achieve the other alternative, namely to start with Elimination rules then look for possible corresponding Introduction rules. To our knowledge this program has not been addressed in full details, but some hints and suggestions have been made: a promising candidate to replace the standard Equality rules for this purpose is the η-conversion (see in particular Primiero (2008) on this). But then again, the result would be the converse of the standard presentation, still featuring one kind of rule as conceptually prior to the other.

In our view, the dialogical approach is a promising one because the rules, being player-independent, do not dichotomise Introduction and Elimination. A key point is that this also holds in the case of the Π and Σ-Equality rules which we have given above: the dialogical rules do not have this "one-sided" aspect of the CTT Equality rules or their candidate alternative, the η-conversion. Still, as we will see in Chaps. 3 and 5, the connection between the dialogical and the CTT approaches becomes salient when we consider *applications* (by the players) *of the dialogical rules at the level of strategies*. The **P**-application versus **O**-application thus not only gives us Introduction versus Elimination rules in the sense of CTT, but also two versions of the Π and Σ-Equality. Moreover, it seems like these two versions yield on the one hand the standard Equality rules of CTT, and on the other the alternative η-conversion: this looks promising in respect to Dummett's 1993 remark mentioned above. It is also the topic of ongoing studies on the notion of harmony using the dialogical approach to CTT.

Applying Π or Σ-Equality rules is particularly useful for the Proponent when he is the challenger (**Y**): they provide a way for the challenger to make **X** perform a particular posit. The rule is obviously interesting for **P** when it comes to elementary posits: by compelling **O** to make an elementary posit, **P** ensures he can defend his own version of the elementary posit with "$sic(n)$", should he need to do so. In other words, the Proponent can use Π and Σ-Equality to be able to resort to copy-cat so as to defend his elementary posits.

This suggests a close connection between the Equality rules of CTT and copy-cat in dialogical games. In fact, we could probably replace the two rules above with rules conveying the idea that when the three preconditions are met within a play, one of the players is entitled to make the posits appearing as defences in the two rules above. That is to say, he is allowed to make the posits $!p(a) = R^{\Pi}(p) : \varphi(a/x)$ and $!I^{\Sigma}(p) = p_i : \varphi_i$, respectively. In the form of a local rule with attack and defence. This yields,

[24] This is the path taken for example in Thompson (1991, Sect. 8.4).

Posit	Challenge	Defence
$\mathbf{X}\,!p(a) = R^{\Pi}(p) : \varphi(a/x)$	$\mathbf{Y}\,?$	$\mathbf{X}\,\Pi - Eq - sic(j,k,l)$ where j, k, l are the moves at which the three preconditions have been played

Likewise for Σ-Eq. Such a formulation insists on the link with copy-cat by using the abbreviation "$sic(j,k,l)$", comparable to "$sic(n)$" in the case of elementary posits.

Formulating the rule in such a way comes quite naturally to the game perspective in which copy-cat is a well-established notion. It nevertheless gets more difficult when going in depth. We have left the challenge underspecified in this rule: it may not be obvious that the initial posit by **X** is related to a previous Π-expression (i.e. involves a universal or a material implication). In fact it is precisely the point of the defence to make it salient. But then what is there in the posit entitling a challenge that can be answered with the abbreviation "sic"? And what would such a challenge look like? To sum up, it is not easy to produce a convincing and explicit rule.

The first versions we gave, the ones in which three preconditions entitle **Y** to ask **X** to perform a certain posit in accordance to Π and Σ-Equality, look more promising because it can be of great importance to have rules as explicit as possible. For example, it is closely related to the way extension and intension are distinguished in CTT—see an application with the discussion on the differences between the intensional and extensional versions of the Axiom of Choice in Chap. 4.

These few remarks are surely not enough to deal with the different but related topics we have just mentioned. If anything, they suggest that there are various directions in which the link between CTT and dialogical games should be further explored, well beyond the single technical result at stake in this study.

Let us stop the digression here and come back to the other rules involving equality.

(Reflexivity within set)

Posit	Challenge	Defence
$\mathbf{X}\,!\,A : set$	$\mathbf{Y}\,?_{set\text{-refl}}$	$\mathbf{X}\,!\,A = A : set$

(Symmetry within set)

Posit	Challenge	Defence
$\mathbf{X}\,!\,A = B : set$	$\mathbf{Y}\,?_{set\text{-symm}}$	$\mathbf{X}\,!\,B = A : set$

(Transitivity within set)

Posit	Challenge	Defence
$\mathbf{X}\,!\,A = B : set$		
$\vdots$		
$\mathbf{X}\,!\,B = C : set$	$\mathbf{Y}\,?_{set\text{-trans}}$	$\mathbf{X}\,!\,A = C : set$

(Reflexivity within A)

Posit	Challenge	Defence
$\mathbf{X}\,!\,a : A$	$\mathbf{Y}\,?_{A\text{-refl}}$	$\mathbf{X}\,!\,a = a : A$

(Symmetry within A)

Posit	Challenge	Defence
$\mathbf{X}\,!\,a = b : A$	$\mathbf{Y}\,?_{A\text{-symm}}$	$\mathbf{X}\,!\,b = a : A$

(Transitivity within A)

Posit	Challenge	Defence
$\mathbf{X}\,!\,a = b : A$ $\vdots$ $\mathbf{X}\,!\,b = c : A$	$\mathbf{Y}\,?_{A\text{-trans}}$	$\mathbf{X}\,!\,a = c : A$

(Set-equality/Extensionality)

Posit	Challenge	Defence
$\mathbf{X}\,!\,A = B : set$	$\mathbf{Y}\,!\,a : A$ $\mathbf{Y}\,!\,a = b : A$	$\mathbf{X}\,!\,a : B$ $\mathbf{X}\,!\,a = b : B$

(Set-substitution)

Posit	Challenge	Defence
$\mathbf{X}\,!\,B(x) : set(x : A)$	$\mathbf{Y}\,!\,x = a : A$	$\mathbf{X}\,!\,B(x/a) : set$
$\mathbf{X}\,!\,B(x) : set(x : A)$	$\mathbf{Y}\,!\,a = c : A$	$\mathbf{X}\,!\,B(a) = B(c) : set$
$\mathbf{X}\,!\,b(x) : B(x)(x : A)$	$\mathbf{Y}\,!\,a : A$	$\mathbf{X}\,!\,b(a) : B(a)$
$\mathbf{X}\,!\,b(x) : B(x)(x : A)$	$\mathbf{Y}\,!\,a = c : A$	$\mathbf{X}\,!\,b(a) = b(c) : B(a)$

In these last rules, we have considered the simpler case in which there is only one assumption in the proviso or context. The rules can obviously be generalized for provisos featuring multiple assumptions.

This ends the presentation of the dialogical notion of play-object and of the rules which give an abstract description of the local proceedings of dialogical games. Next we consider the global conditions taking part in the development of dialogical plays.

2.4 The Development of a Play

We will deal in this section with the other kind of dialogical rules called structural rules. These rules govern the way plays globally proceed and are therefore an important aspect of dialogical semantics. We will work with the following structural rules:

SR0 (Starting Rule). Any dialogue starts with the Opponent positing initial concessions, if any, and the Proponent positing the thesis. After that the players each choose a positive integer called repetition rank.

SR1i (Intuitionisitic Development Rule). Players move alternately. After the repetition ranks have been chosen, each move is a challenge or a defence in reaction to a previous move, in accordance with the particle rules. The repetition rank of a player bounds the number of challenges he can play in reaction to a same move. Players can answer only against the *last non-answered* challenge by the adversary.[25]

SR2 ("Priority to Formation" Rule). **O** starts by challenging the thesis with the request '$?_{prop}$'. The game then proceeds by applying the formation rules first so as to check that the thesis is indeed a proposition. After that the Opponent is free to use the other local rules insofar as the other structural rules allow it.

SR3 (Modified Formal Rule). **O**'s elementary sentences cannot be challenged. However, **O** can challenge a **P** elementary move provided she did not herself play it before.

Since we have particle rules for elementary sentences involving the defence "$sic(n)$" we have no need for a Formal Rule which entitles a player to copy-cat some moves of the adversary.[26] We must however also ensure that the strictly internal aspect related to the idea of *Geltung* in the dialogical approach to meaning is not lost, and that the asymmetry between the player **P** who brings forward the thesis and his adversary **O** is accounted for. This is why the standard Formal Rule is replaced by this modified version.

SR4.1 (Resolution of Instructions). Whenever a player posits a move in which instructions $I_1, \ldots, I_n$ occur, the other player can ask him to replace these instructions (or some of them) by suitable play-objects.

If the instruction (or list of instructions) occurs at the right of the colon and the posit is the tail of an universally quantified sentence or of an implication (so that these instructions occur at the left of the colon in the posit of the head of the implication), then it is the challenger who can choose the play-object. In these cases the player who challenges the instruction is also the challenger of the universal quantifier and/or of the implication.

[25]This last clause is known as the *Last Duty First* condition, and is the clause making dialogical games suitable for Intuitionistic Logic, hence the name of this rule.

[26]But let us insist once more on the important point we raised in Sect. 2.3: contrary to standard dialogical games, copy-cat does not apply only to elementary sentences but also to posits in which such sentences are associated with play-objects.

Otherwise it is the defender of the instructions who chooses the suitable play-object:

Posit	Challenge	Defence
$\mathbf{X}\,\pi(I_1, \ldots, I_n)$	$\mathbf{Y}\, I_1, \ldots, I_m/?\ (m \leq n)$	$\mathbf{X}\,\pi(b_1, \ldots, b_m)$ If the instruction occurring at the right of the colon is the tail of either a universal or an implication (such that $I_1, \ldots, I_n$ also occurs at the left of the colon in the posit of the head), then $b_1, \ldots, b_m$ was previously chosen by the challenger Otherwise the defender chooses

Important remark. In the case of embedded instructions $I_1(\ldots(I_k)\ldots)$, the substitutions are thought of as being carried out from I_k to I_1: first substitute I_k with some play-object b_k, then $I_{k-1}(b_k)$ with b_{k-1} etc. until $I_1(b_2)$. If such a progressive substitution has already been carried out once, a player can then replace $I_1(\ldots(I_k)\ldots)$ directly.

SR4.2 (Substitution of Instructions). When, during the play, the play-object b has been chosen by any of the two players for an instruction I, and player **X** makes any posit $\pi(I)$, then the other player can ask for I to be substituted by b in this posit:

Posit	Challenge	Defence
$\mathbf{X}\,\pi(I)$ (where I/b has been previously established)	$\mathbf{Y}\,?\,I/b$	$\mathbf{X}\,\pi(b)$

The idea is that the resolution of an instruction yields a certain play-object for some substitution term, and therefore the same play-object can be assumed to result from any other occurrence of the same substitution term: instructions, after all, are functions and must yield as such the same play-object for the same substitution term.

SR5 (Winning Rule for plays). For any p, a player who posits "$p : \bot$" looses the current play. Otherwise the player who makes the last move in a dialogue wins it.

In comparison to the rules of standard dialogical games, some additions in the rules we just gave have been made, namely $SR2$ and $SR4.1$-2. Also, the so-called Formal Rule (here $SR3$) and the Winning Rule are a bit different. Since we made explicit the use of $\bot$ in our games, we need to add a rule for it: the point is that positing falsum leads to immediate loss. We could say that it amounts to a withdrawal.[27] Hence the formulation of the Winning Rule for plays above.

We need the rules $SR4.1$ and $SR4.2$ because of some features of CTT's explicit language. In CTT it is possible to account for questions of dependency, scope, etc.

[27] See Keiff (2007).

directly at the language level. In this way various puzzles, such as anaphora, get a convincing and successful treatment. The typical example, considered below, is the so-called donkey sentence “Every man who owns a donkey beats it”. The two rules account for the way play-objects can be ascribed to what we have called instructions. See the dialogue in Sect. 2.5 for an application.

The rule $SR2$ is consistent with the common practice in CTT to start demonstrations by checking or establishing the formation of propositions before proving their truth. Notice that this step also covers the formation of sets—membership, generation of elements, etc.—occurring in hypothetical posits and in quantifiers. In the current study, however, we can overlook this rule: we can take it for granted that expressions are well formed because we have restricted this study to the valid fragment of CTT. That is to say, we will only consider cases for which it is not necessary to carry out the formation steps since even if they were carried out, the players would always be able to justify that their expressions are well formed. We will, for this, always take examples in which good formation is guaranteed by hypotheses introduced as initial concessions by the Opponent at the beginning of the play.

What is more, it seems like we could liberalise the rule $SR2$. But because of the number of rules we have introduced, verifying this carefully is a delicate task that we will not carry out in this study. Let us for now simply mention that it seems sensible enough in dialogues to combine more freely the process linked to formation rules with the other rules at stake in the development of a play. Questioning the status of expressions as they are introduced in the course of the game does in fact seem perfectly consistent with actual practices. Suppose for example that player **P** has posited ‘$p : \varphi \vee \psi$’. As soon as he has posited that the disjunction is a proposition—i.e. as soon as he has posited ‘$\varphi \vee \psi : prop$’—the other player knows how to challenge the disjunction and should be free to either keep on exploring the formation of the expression or to challenge the first posit. The point is that in a way it makes more sense to check whether φ is a proposition or not once (or if) **X** posits it in order to defend the disjunction. Doing so in a ‘monological’ framework such as CTT would probably bring various confusions, but the dialogical approach to meaning should quite naturally allow this additional dynamic aspect. Nonetheless, in order to generalise the equivalence result beyond the valid fragment of CTT (the reason why we have introduced rule $SR2$), it seems sensible in our view to clearly distinguish in a fashion close to CTT the steps linked to the formation from the other aspects of meaning.

The definitions of plays, games and strategies are the same as those given in Sect. 2.1. Let us now recall them. A play for φ is a sequence of moves in which φ is the thesis posited by the Proponent and which complies with the game rules. The dialogical game for φ is the set of all possible plays for φ and its extensive form is nothing but its tree representation. Thus, every path in this tree which starts with the root is the linear representation of a play in the dialogical game at stake.

We say that a play for φ is terminal when the last move is made by player **X** and there is no further move for **Y**. A strategy for player **X** in a given dialogical game is a function which assigns a legal **X**-move to each non-terminal play where it is **X**'s turn to move. When the strategy is a winning one for **X**, the assignation results in terminal plays won by **X**. It is common practice to consider in an equivalent way an **X**-strategy s as the set of terminal plays resulting when **X** plays according to s. The extensive form of s is then the tree representation of that set—which is by the way a fragment of the extensive form of the dialogical game. For more explanations on these notions, see Clerbout (2014c). The equivalence result between dialogical games and CTT is established by procedures of translation between a certain part of an extensive form of winning **P**-strategies—and more precisely what we call their core—and CTT demonstrations. We will give more details on how to isolate cores of strategies in Chap. 3.

2.5 Example

We end this presentation of dialogical games with an illustration of the approach. The example comes from Rahman et al. (2014) and consists in a dialogue in which the famous donkey sentence "Every man who owns a donkey beats it" is involved.

In his 1986 paper, G. Sundholm thoroughly discussed this famous puzzle in the context of Constructive Type Theory. As is well-known, the problem is to give a way to capture the back-reference of the pronoun "it". The point in Sundholm (1986) is that the explicit language of CTT makes it possible to express and account for such dependencies as soon as one pays attention to the fact that "a man who owns a donkey" is a member of the set

$$\{x : M | (\exists y : D) O(x, y)\}$$

For the detailed explanation of the CTT approach, see Sundholm's paper. The point we are interested in here is that the back-reference of the pronoun is dealt with in a similar way by using dialogical instructions. That is, we write the donkey sentence as

$$\big(\forall z : \{x : M | (\exists y : D) O(x, y)\}\big) B(L^{\{\dots\}}(z), L^{\exists}(R^{\{\dots\}}(z)))$$

where M is the set of men, D is the set of donkeys, $O(x, y)$ stands for "x owns y" and $B(x, y)$ stands for "x beats y".

Table 2.6 presents a dialogue in which this sentence occurs as one of the initial concessions by the Opponent.

Table 2.6 A dialogue involving the donkey sentence

	O			P	
I	$M : set$				
II	$D : set$				
III	$O(x,y) : set(x : M, y : D)$				
IV	$B(x,y) : set(x : M, y : D)$				
V	$!p : (\forall z : \{x : M \mid (\exists y : D)O(x,y)\})B(L^{\{\cdots\}}(z), L^{\exists}(R^{\{\cdots\}}(z)))$				
VI	$!m : M$				
VII	$!d : D$				
VIII	$!p' : Omd$				
				$!B(m,d)$	0
1	$\mathtt{n} := \ldots$			$\mathtt{m} := \ldots$	2
3	$?\,play\text{-}object$	(0)		$!q : B(m,d)$	30
25	$!R^{\forall}(p) : B(L^{\{\cdots\}}(z), L^{\exists}(R^{\{\cdots\}}(z)))$		(V)	$!L^{\forall}(p) : \{x : M \mid (\exists y : D)O(x,y)\}$	4
5	$L^{\forall}(p)/?$	(4)		$!z : \{x : M \mid (\exists y : D)O(x,y)\}$	6
7	$?_L$	(6)		$!L^{\{\cdots\}}(z) : M$	8
9	$L^{\{\cdots\}}(z)/?$	(8)		$!m : M$	10
11	$?_R$	(6)		$!R^{\{\cdots\}}(z) : (\exists y : D)Omy$	12
13	$R^{\{\cdots\}}(z)/?$	(12)		$(L^{\exists}(R^{\{\cdots\}}(z)), R^{\exists}(R^{\{\cdots\}}(z))) : (\exists y : D)Omy$	14
15	$L^{\exists}(R^{\{\cdots\}}(z))/?, R^{\exists}(R^{\{\cdots\}}(z))/?$	(14)		$!(d,p') : (\exists y : D)Omy$	16
17	$?_L$	(16)		$!L^{\exists}(d,p') : D$	18
19	$L^{\exists}(R^{\{\cdots\}}(z))/?$	(18)		$!d : D$	20
21	$?_R$	(16)		$!R^{\exists}(d,p') : Omd$	22
23	$R^{\exists}(R^{\{\cdots\}}(z))/?$	(22)		$!p' : Omd$	24
27	$!R^{\forall}(p) : B(m,d)$		(25)	$L^{\{\cdots\}}(z)/m, L^{\exists}(R^{\{\cdots\}}(z))/d$	26
29	$!q : B(m,d)$		(27)	$R^{\forall}(p)/?$	28

Explanations:

- Moves I–VIII. These moves are **O**'s initial concessions. Moves I to IV deal with the formation of expressions. After that the Opponent concedes the donkey sentence and atomic expressions related to the sets M, D and $O(x, y)$.
- Moves 0-3. The Proponent posits the thesis. The players choose their repetition ranks in moves 1 and 2. The actual value they choose does not really matter for the point we illustrate here: we simply assume they are sufficient for this play and leave them unspecified. When **P** posited the thesis he did not specify a play-object so **O** asks for it in move 3.
- Move 4. The Proponent chooses to launch a counter-attack by challenging the donkey sentence which **O** conceded at V. The rules do allow him to directly answer to the challenge, but then he would not be able to win.[28]
- Moves 5-24. The dialogue then proceeds in a rather straightforward way by applications of the rules introduced in Sects. 2.3 and 2.4. More precisely, this dialogue displays the case where **O** chooses to challenge **P**'s posits as much as she can before answering to **P**'s challenge 4.
  Notice that the Opponent cannot challenge the Proponent's atomic expressions posited at moves 10, 20 and 24. Since **O** made the same posits in her initial concessions VI to $VIII$, the Modified Formal Rule $SR3$ forbids her to challenge them.
- Move 25. When there is nothing left for her to challenge, **O** comes back to the last unanswered challenge by **P** which was move 4 and makes the relevant defence according to the particle rule for universal quantification.
- Moves 26-27. The resolution for instructions $L^{\{\cdots\}}(z)$ and $L^{\exists}(R^{\{\cdots\}}(z))$ have been carried out during the dialogue with moves 9-10 and moves 23-24. The Proponent can thus use

[28]Indeed, after he answers challenge 3 he has to defend the atomic posit played in defence. To successfully do so, he must make **O** perform the same posit. But the reader can check that in this case nothing would compel **O** to choose the same play-object. Hence, the way to victory for **P** is to let **O** choose a play-object for $B(m, d)$ first as in this dialogue.

the established substitutions to challenge move 25 according to the structural rule *SR*4.2. The Opponent defends by performing the requested substitutions.

– Moves 28-30. The Proponent then asks the play-object for which the instruction $R^{\forall}(z)$ stands. When she answers, the Opponent posits exactly what **P** needs to defend against **O**'s challenge 3. Notice that at this point this is the last unanswered challenge by **O**, therefore **P** is allowed to answer it in accordance to the structural rule *SR*1*i*. He does so with his move 30.
  Since **O** made the same posit, the rule *SR*3 forbids her to challenge it. She then has no further possible move, and the Proponent wins this dialogue.

The example illustrates the applications of many rules of the dialogical approach. Let us insist on the way the device of dialogical instructions enriches the language in order to account for dependency relations, such as anaphora, through their resolution. This is the reason why we chose an example involving the donkey sentence.

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