Chapter 2
Linear Models for Portfolio Optimization

2.1 Introduction

Nowadays, Quadratic Programming (QP) models, like Markowitz model, are not hard to solve, thanks to technological and algorithmic progress. Nevertheless, Linear Programming (LP) models remain much more attractive from a computational point of view for several reasons. The design and development of commercial software for the solution of LP models is more advanced than for QP models. As a consequence, several commercial LP solvers are available and, in general, LP solvers tend to be more reliable than QP solvers. On average, LP solvers can solve in small time (the order of seconds) instances of much larger size than QP solvers.

Is it possible to have linear models for portfolio optimization? How can we measure the risk or safety in order to have a linear model? A first observation is that, in order to guarantee that a portfolio takes advantage of diversification, no risk or safety measure can be a linear function of the shares of the assets in the portfolio, that is of the variables x_j, $j = 1, \ldots, n$. Linear models, however, can be obtained through discretization of the return random variables or, equivalently, through the concept of scenarios.

2.2 Scenarios and LP Computability

We have indicated by R_j the random variable representing the rate of return of asset $j, j = 1, \ldots, n$, at the target time.

Now we change the way we look at the uncertainty of the rates of return of the assets at the target time and introduce the concept of *scenario*. A scenario is, informally, a possible situation that can happen at the target time, in our case a possible realization of the rates of return of the assets at the target time. Depending

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Table 2.1 Scenarios: An example

Asset	Scenario 1 (%)	Scenario 2 (%)	Scenario 3 (%)	Mean return rate (%)
1	$r_{11} = 3.1$	$r_{12} = -2.7$	$r_{13} = 1.60$	$\mu_1 = 0.67$
2	$r_{21} = 2.3$	$r_{22} = -2.3$	$r_{23} = 1.30$	$\mu_2 = 0.43$
3	$r_{31} = 4.2$	$r_{32} = -3.1$	$r_{33} = -0.2$	$\mu_3 = 0.43$
4	$r_{41} = 1.5$	$r_{42} = -2.0$	$r_{43} = -0.1$	$\mu_4 = -0.2$

on what will happen between the investment time and the target time, any of several different scenarios may become true. The scenarios may also be less or more likely to happen. More formally, a scenario is a realization of the multivariate random variable representing the rates of return of all the assets.

We now suppose that, on the basis of a careful preliminary analysis, T different scenarios have been identified as possible at the target time. The probability that scenario t, $t = 1, \ldots, T$, will happen is indicated by p_t, with $\sum_{t=1}^{T} p_t = 1$. We assume that for each random variable R_j, $j = 1, \ldots, n$, its realization r_{jt} under scenario t is known. The set of the returns of all the assets $\{r_{jt}, j = 1, \ldots, n\}$ defines the scenario t. The expected return of asset j, $j = 1, \ldots, n$, is calculated as $\mu_j = \sum_{t=1}^{T} p_t r_{jt}$. The concept of scenario captures the correlation among the rates of return of the assets.

In Table 2.1, we show an example of $n = 4$ assets and $T = 3$ scenarios. The table shows the rates of return of the assets in the different scenarios. Scenario 1 is an optimistic scenario: all rates of return are positive. Scenario 2 is negative, whereas scenario 3 is positive for assets 1 and 2 and negative for assets 3 and 4. The averages are computed under the assumptions that the scenarios are equally probable ($p_t = 1/3, t = 1, 2, 3$).

Identifying the scenarios, their probabilities and estimating the values of the rate of return r_{jt} of each asset j under each scenario t is crucial. To be statistically significant, the number of scenarios has to be sufficiently large.

Each portfolio $\mathbf{x}$ defines a corresponding random variable $R_{\mathbf{x}} = \sum_{j=1}^{n} R_j x_j$ that represents the portfolio rate of return. The step-wise cumulative distribution function (cdf) of $\{R_{\mathbf{x}}\}$ is defined as

$$F_{\mathbf{x}}(\xi) = P(R_{\mathbf{x}} \leq \xi). \tag{2.1}$$

The return y_t of a portfolio $\mathbf{x}$ in scenario t can be computed as

$$y_t = \sum_{j=1}^{n} r_{jt} x_j, \tag{2.2}$$

and the expected return of the portfolio $\mu(\mathbf{x})$ can be computed as a linear function of $\mathbf{x}$

$$\mu(\mathbf{x}) = \mathbb{E}\{R_{\mathbf{x}}\} = \sum_{t=1}^{T} p_t y_t = \sum_{t=1}^{T} p_t (\sum_{j=1}^{n} r_{jt} x_j) = \sum_{j=1}^{n} x_j \sum_{t=1}^{T} p_t r_{jt} = \sum_{j=1}^{n} \mu_j x_j. \tag{2.3}$$

We have defined a scenario as a realization of the multivariate random variable representing the rates of return of the assets. We may look at the set of scenarios as a discretization of the multivariate random variable.

We will say that the returns are *discretized* when they are defined by their realizations under the specified scenarios, that is by the set of values $\{r_{jt} : j = 1, \ldots, n, t = 1, \ldots, T\}$. We will say that a risk or a safety measure is *LP computable* if the portfolio optimization model takes a linear form in the case of discretized returns.

2.3 Basic LP Computable Risk Measures

The variance is the classical statistical quantity used to measure the dispersion of a random variable around its mean. There are, however, other ways to measure the dispersion of a random variable. The random variable, we are interested in, is the portfolio return $R_{\mathbf{x}}$.

The Mean Absolute Deviation (MAD) is a dispersion measure that is defined as

$$\delta(\mathbf{x}) = \mathbb{E}\{|R_{\mathbf{x}} - \mathbb{E}\{R_{\mathbf{x}}\}|\} = \mathbb{E}\{|\sum_{j=1}^{n} R_j x_j - \mathbb{E}\{\sum_{j=1}^{n} R_j x_j\}|\}. \tag{2.4}$$

The MAD measures the average of the absolute value of the difference between the random variable and its expected value. With respect to the variance, the MAD considers absolute values instead of squared values. We show in the following that, when the returns are discretized, the MAD is LP computable. Recalling that the expected return of the portfolio can be calculated as (2.3), the MAD can be written as

$$\delta(\mathbf{x}) = \sum_{t=1}^{T} p_t (|\sum_{j=1}^{n} r_{jt} x_j - \sum_{j=1}^{n} \mu_j x_j|). \tag{2.5}$$

The portfolio optimization problem then becomes

$$\min \delta(\mathbf{x}) = \sum_{t=1}^{T} p_t(|\sum_{j=1}^{n} r_{jt}x_j - \mu|) \tag{2.6a}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.6b}$$

$$\mu \geq \mu_0 \tag{2.6c}$$

$$\mathbf{x} \in Q, \tag{2.6d}$$

where μ_0 is the lower bound on the portfolio expected return required by the investor, and Q denotes the system of constraints defining the set of feasible portfolios as described in Chap. 1.

This form is not linear in the variables x_j but can be transformed into a linear form. Using (2.2) for the return of the portfolio in scenario t, y_t, $\delta(\mathbf{x})$ can also be written as

$$\delta(\mathbf{x}) = \sum_{t=1}^{T} p_t(|y_t - \sum_{j=1}^{n} \mu_j x_j|).$$

We now define the deviation in scenario t as d_t, that is $d_t = |y_t - \sum_{j=1}^{n} \mu_j x_j|$. Then, the portfolio optimization problem is

$$\min \sum_{t=1}^{T} p_t d_t \tag{2.7a}$$

$$d_t = |y_t - \mu| \qquad t = 1, \ldots, T \tag{2.7b}$$

$$y_t = \sum_{j=1}^{n} r_{jt}x_j \qquad t = 1, \ldots, T \tag{2.7c}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.7d}$$

$$\mu \geq \mu_0 \tag{2.7e}$$

$$\mathbf{x} \in Q. \tag{2.7f}$$

Since $|y_t - \mu| = \max\{(y_t - \mu); -(y_t - \mu)\}$, the problem can be written in the following equivalent linear form

$$\min \sum_{t=1}^{T} p_t d_t \tag{2.8a}$$

$$d_t \geq y_t - \mu \qquad t = 1, \dots, T \tag{2.8b}$$

$$d_t \geq -(y_t - \mu) \qquad t = 1, \dots, T \tag{2.8c}$$

$$y_t = \sum_{j=1}^{n} r_{jt} x_j \qquad t = 1, \dots, T \tag{2.8d}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.8e}$$

$$\mu \geq \mu_0 \tag{2.8f}$$

$$d_t \geq 0 \qquad t = 1, \dots, T \tag{2.8g}$$

$$\mathbf{x} \in Q. \tag{2.8h}$$

The equivalence comes from observing that if $y_t - \mu \geq 0$ constraints (2.8c) are redundant. In this case constraints (2.8b), combined with the minimization of $\sum_{t=1}^{T} p_t d_t$ in (2.8a) that pushes the value of each d_t to the minimum value allowed by the constraints, impose that $d_t = y_t - \mu = |y_t - \mu|$. If, on the contrary $y_t - \sum_{j=1}^{n} \mu_j x_j \leq 0$, constraints (2.8b) are redundant. In this case, constraints (2.8c), combined with the objective function, impose that $d_t = -(y_t - \sum_{j=1}^{n} \mu_j x_j) = |y_t - \sum_{j=1}^{n} \mu_j x_j|$. Thus, in conclusion, the optimization model (2.8) is a linear programming model for the optimization of a portfolio where the risk is measured through the MAD of the return of the portfolio.

In Fig. 2.1, we represent the calculation of the MAD measure. In other words, we assume that the values of the shares x_j are given. We represent over the horizontal axis the scenarios $t = 1, \dots, T$ and over the vertical axis the values y_t of the return

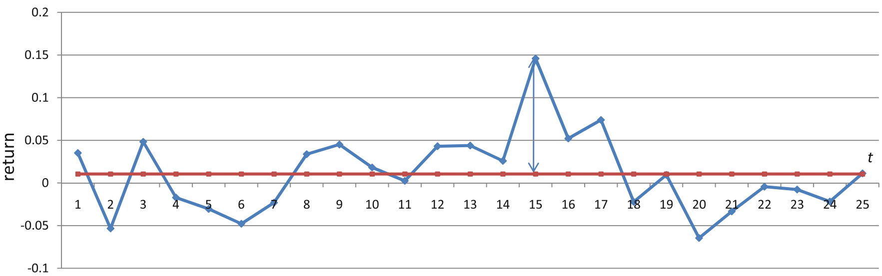

Fig. 2.1 The MAD measure

of the portfolio under the various scenarios t. The thick horizontal line identifies the expected return of the portfolio $\mu = \sum_{j=1}^{n} \mu_j x_j = \sum_{t=1}^{T} p_t y_t$. The length of a vertical segment is the absolute value of the deviation d_t (in Fig. 2.1, the deviation d_{15} corresponding to the scenario $t = 15$ is drawn as example). The MAD model aims at minimizing the average absolute deviation.

In the case the rates of return are a multivariate normally distributed random variable, the rate of return of the portfolio is normally distributed. Then, the proportionality relation between the mean absolute deviation and the standard deviation occurs $\delta(\mathbf{x}) = \sqrt{\frac{2}{\pi}}\sigma(\mathbf{x})$. As a consequence, minimizing the MAD is equivalent to minimizing the standard deviation, which means, in this specific case, the equivalence of the associated optimization problems. However, the MAD model does not require any specification of the return distribution.

The MAD accounts for all deviations of the rate of return of the portfolio from its expected value, both below and above the expected value. However, one may sensibly think that any rational investor would consider real risk only the deviations below the expected value. In other words, the variability of the portfolio rate of return above the mean should not be penalized since the investors are concerned with under-performance rather than over-performance of a portfolio. In terms of scenarios, the risky scenarios are those where the rate of return of the portfolio is below its expected value. We can modify the definition of the MAD in order to consider only the deviations below the expected value. We define the Semi Mean Absolute Deviation (Semi-MAD)

$$\bar{\delta}(\mathbf{x}) = \mathbb{E}\{\max\{0, \mathbb{E}\{\sum_{j=1}^{n} R_j x_j\} - \sum_{j=1}^{n} R_j x_j\}\}, \tag{2.9}$$

where the deviations above the expected value are not calculated. The portfolio optimization problem (2.8) presented for the MAD can be adapted to the Semi-MAD as follows:

$$\min \sum_{t=1}^{T} p_t d_t \tag{2.10a}$$

$$d_t \geq \mu - y_t \qquad t = 1, \ldots, T \tag{2.10b}$$

$$y_t = \sum_{j=1}^{n} r_{jt} x_j \qquad t = 1, \ldots, T \tag{2.10c}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.10d}$$

$$\mu \geq \mu_0 \tag{2.10e}$$

$$d_t \geq 0 \qquad t = 1, \ldots, T \tag{2.10f}$$

$$\mathbf{x} \in Q. \tag{2.10g}$$

The formulation for the Semi-MAD is the formulation of the MAD, from which inequalities (2.8b) have been dropped. If, for a given scenario t, $\mu - y_t > 0$, this means that under scenario t the rate of return of the portfolio y_t is below the expected value. In this case d_t in the optimum will be the difference $\mu - y_t$. If instead $\mu - y_t \leq 0$, constraint (2.10b) becomes redundant and in the optimum $d_t = 0$. Thus, the deviations above the expected value are not calculated in the objective function.

The Semi-MAD seems to be a very attractive measure, focusing on the downside risk only. However, it can be seen that it is equivalent to the MAD as the corresponding optimization models generate the same optimal portfolio. The intuition behind the equivalence, that is somehow surprising, is that the MAD is the sum of the deviations above and below the expected value. By definition of expected value, the sum of the deviations above the expected value is equal to the sum of the deviations below the expected value. Thus, the Semi-MAD is half the MAD. Minimizing the downside deviations is equivalent to minimizing the total deviations and equivalent to minimizing the deviations above the expected value as well. We make this equivalence formal.

Theorem 2.1 *Minimizing the MAD is equivalent to minimizing the Semi-MAD as* $\delta(\mathbf{x}) = 2\bar{\delta}(\mathbf{x})$.

Proof We first write the mean deviation of the portfolio rate of the return from its expected value and show that it is equal to 0:

$$\mathbb{E}\{R_{\mathbf{x}} - \mathbb{E}\{R_{\mathbf{x}}\}\} = \mathbb{E}\{R_{\mathbf{x}}\} - \mathbb{E}\{R_{\mathbf{x}}\} = 0$$

From this it immediately follows that the average positive deviation ($y_t - \mu(\mathbf{x}) > 0$ implies the rate of return of the portfolio in scenario t is above its expected value) is equal to the opposite of the average negative deviation ($y_t - \mu(\mathbf{x}) < 0$ implies the rate of return of the portfolio in scenario t is below its expected value). The absolute value of the average positive deviation is thus equal to the absolute value of average negative deviation, from which it follows that the MAD is twice the Semi-MAD.

□

Although the MAD has become a very popular risk measure, a different LP computable risk measure was earlier proposed, namely the Gini's mean difference. The variability of the portfolio return is captured here by the differences of the portfolio returns in different scenarios. For a discrete random variable represented by its realizations y_t, the *Gini's mean difference (GMD)* considers as risk the average absolute value of the differences of the portfolio returns y_t in different scenarios:

$$\Gamma(\mathbf{x}) = \frac{1}{2} \sum_{t'=1}^{T} \sum_{t''=1}^{T} |y_{t'} - y_{t''}| p_{t'} p_{t''}. \tag{2.11}$$

The risk function $\Gamma(\mathbf{x})$, to be minimized, is LP computable.

In Fig. 2.2, the values of the rate of return for a given portfolio under $T = 25$ scenarios are shown. The length of the vertical segment is the absolute value of the

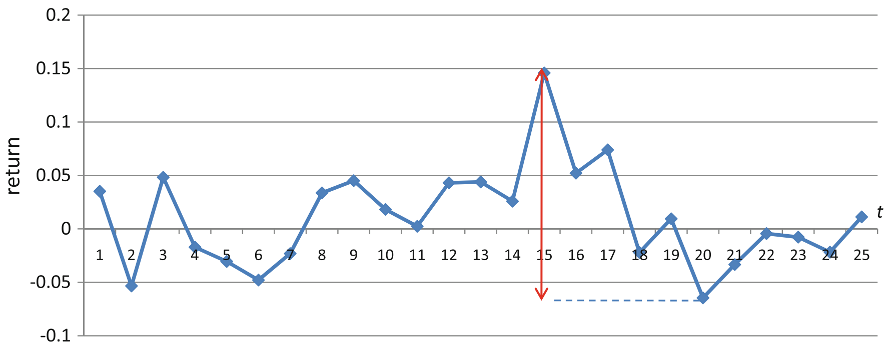

Fig. 2.2 The GMD measure

difference between the portfolio returns under scenarios 15 and 20, i.e. $d_{15,20} = |y_{15} - y_{20}|$.

The portfolio optimization model based on the GMD risk measure can be written as follows:

$$\min \sum_{t'=1}^{T} \sum_{t'' \neq t'} p_{t'} p_{t''} d_{t't''} \tag{2.12a}$$

$$d_{t't''} \geq y_{t'} - y_{t''} \qquad t', t'' = 1, \ldots, T;\ t'' \neq t' \tag{2.12b}$$

$$y_t = \sum_{j=1}^{n} r_{jt} x_j \qquad t = 1, \ldots, T \tag{2.12c}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.12d}$$

$$\mu \geq \mu_0 \tag{2.12e}$$

$$d_{t't''} \geq 0 \qquad t', t'' = 1, \ldots, T;\ t'' \neq t' \tag{2.12f}$$

$$\mathbf{x} \in Q. \tag{2.12g}$$

The model contains $T(T-1)$ non-negative variables $d_{t't''}$ and $T(T-1)$ inequalities to define them. The symmetry property $d_{t't''} = d_{t''t'}$ is here ignored. However, variables $d_{t't''}$ are associated with the singleton coefficient columns. Hence, while solving the dual instead of the original primal problem, the corresponding dual constraints take the form of simple upper bounds which are handled implicitly by the simplex method. In other words, the dual problem contains $T(T-1)$ variables but the number of constraints is then proportional to T. Such a dual approach may dramatically improve the required computational time in the case of large number of scenarios.

Similarly to MAD, in the case when the rates of return are multivariate normally distributed, the proportionality relation $\Gamma(\mathbf{x}) = \frac{2}{\sqrt{\pi}}\sigma(\mathbf{x})$ between the Gini's mean difference and the standard deviation occurs. As a consequence, minimizing the GMD is equivalent to minimizing the standard deviation, which means, in this specific case, the equivalence of the associated optimization problems. Albeit, the GMD model does not require any specific type of return distribution.

2.4 Basic LP Computable Safety Measures

In the previous chapter and in the previous section of this chapter, we have seen some specific risk measures, the variance (Markowitz model), the mean absolute deviation (MAD), the Gini's mean difference (GMD). These measures capture, in different ways, the variability of the rate of return of the portfolio. Given a required expected return of the portfolio μ_0, the investor may wish to reduce the variability of the portfolio rate of return, that is to minimize any of these risk measures. We analyze here different ways to measure the quality of a portfolio and define some specific safety measures, to be maximized. We do not consider the variability of the portfolio rate of return, neither the deviations from its expected value. In fact, we ignore the expected rate of return and try instead to protect the investor from the worst scenarios.

An appealing safety measure is the worst realization of the portfolio rate of return. We aim at maximizing the worst realization of the portfolio rate of return. The *worst realization* is defined as

$$M(\mathbf{x}) = \min_{t=1,\ldots,T} y_t = \min_{t=1,\ldots,T} \sum_{j=1}^{n} r_{jt}x_j, \tag{2.13}$$

and is LP computable. The portfolio optimization model with the worst realization as safety measure (the Minimax model) can be formulated as:

$$\max y \tag{2.14a}$$

$$\sum_{j=1}^{n} r_{jt}x_j \geq y \qquad t = 1,\ldots,T \tag{2.14b}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.14c}$$

$$\mu \geq \mu_0 \tag{2.14d}$$

$$\mathbf{x} \in Q. \tag{2.14e}$$

The variable y is an artificial variable that in the optimum takes the value of the portfolio rate of return in the worst scenario. In Fig. 2.3, the rates of return for a given portfolio over 25 scenarios are drawn. and the worst realization of the portfolio rate of return is emphasized.

Suppose that, among the feasible portfolios of the Minimax model, there are the two portfolios shown in Table 2.2. Suppose that the required expected rate of return is $\mu_0 = 2\,\%$. Both portfolios $\mathbf{x}'$ and $\mathbf{x}''$ guarantee an expected rate of return not worse than 2 %. Whereas portfolio $\mathbf{x}'$ has a larger expected rate of return, the model would prefer portfolio $\mathbf{x}''$ to portfolio $\mathbf{x}'$ because portfolio $\mathbf{x}''$ has the rate of return in the worst scenario, 2 %, larger than the worst rate of return of portfolio $\mathbf{x}'$, 1.8 %. The maximization of the worst realization somehow pushes all the realizations toward larger – and thus better – values, but at the same time focuses on the worst scenario only.

A natural generalization of the measure $M(\mathbf{x})$ is the statistical concept of *quantile*. In general, for given $\beta \in [0, 1]$, the β-quantile of a random variable R is the value q such that

$$\mathbb{P}\{R < q\} \leq \beta \leq \mathbb{P}\{R \leq q\}.$$

For $\beta \in (0, 1)$, it is known that the set of such β-quantiles is a closed interval (see Embrechts et al. 1997). Given a value of β, in order to formalize the quantile

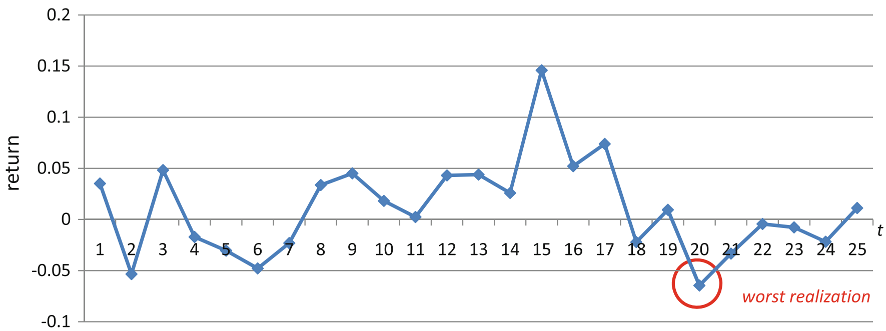

Fig. 2.3 The worst realization measure

Table 2.2 Optimal portfolio for the worst realization safety measure: An example

Scenario	Probability	Rates of return Portfolio $\mathbf{x}'$ (%)	Portfolio $\mathbf{x}''$ (%)
1	0.2	4.9	2.0
2	0.5	4.0	3.0
3	0.2	2.2	2.0
4	0.1	1.8	2.0
Mean	$\mu(\mathbf{x})$	3.6	2.5
Worst	$M(\mathbf{x})$	1.8	2.0

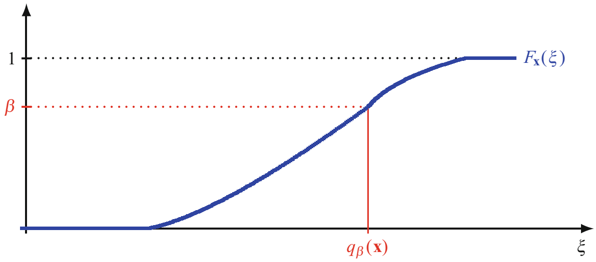

Fig. 2.4 VaR measure $q_\beta(\mathbf{x})$ and the cdf of portfolio returns

measures in the case of non-unique quantile, the left end of the entire interval of quantiles is taken. In our case, we denote by $q_\beta(\mathbf{x})$ the value of the β-quantile, that is the value of the rate of return defined as

$$q_\beta(\mathbf{x}) = \inf\{\eta : F_{\mathbf{x}}(\eta) \geq \beta\} \quad \text{for } 0 < \beta \leq 1, \tag{2.15}$$

where $F_{\mathbf{x}}(\cdot)$ is the cumulative distribution function defined in (2.1) (see Fig. 2.4).

In finance and banking literature, this quantile is usually called *Value-at-Risk* or simply *VaR* measure. Actually, for a given portfolio $\mathbf{x}$, its VaR depicts the worst (maximum) loss within a given confidence interval (see Jorion 2006). However, with a change of sign (losses as negative returns $-R_{\mathbf{x}}$), it is equivalent to the quantile $q_\beta(\mathbf{x})$.

Due to possible discontinuity of the cdf, the VaR measure is, generally, not an LP computable measure. The corresponding portfolio optimization model can be formulated as a MILP problem:

$$\max\ y \tag{2.16a}$$

$$\sum_{j=1}^{n} r_{jt}x_j \geq y - Mz_t \qquad t = 1,\ldots,T \tag{2.16b}$$

$$\sum_{t=1}^{T} p_t z_t \leq \beta - \pi,\ z_t \in \{0,1\} \qquad t = 1,\ldots,T \tag{2.16c}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.16d}$$

$$\mu \geq \mu_0 \tag{2.16e}$$

$$\mathbf{x} \in Q, \tag{2.16f}$$

where M is an arbitrary large constant (larger than any possible rate of return) while π is an arbitrary small positive constant ($\pi < p_t, t = 1,\ldots,T$). Note that, due to inequality (2.16b), binary variable z_t takes value 1 whenever variable y is greater than the portfolio return under scenario t ($y > y_t = \sum_{j=1}^{n} r_{jt}x_j$). Inequality (2.16c) guarantees that the probability of all scenarios such that $y > y_t$ is less than β. Therefore, the optimal value of the maximized variable y represents the optimal β-quantile value $q_\beta(\mathbf{x})$.

Recently, risk measures based on averaged quantiles have been introduced in different ways. The *tail mean* or *worst conditional expectation* $M_\beta(\mathbf{x})$, defined as the mean return of the portfolio taken over a given tolerance level (percentage) $0 < \beta \leq 1$ of the worst scenarios probability is a natural generalization of the measure $M(\mathbf{x})$. In finance literature, the tail mean quantity is usually called *Tail VaR*, *Average VaR* or *Conditional VaR* (where VaR reads after Value-at-Risk). Actually, the name CVaR is now the most commonly used and we adopt it.

For the simplest case of equally probable scenarios ($p_t = 1/T$) and proportional $\beta = k/T$, the CVaR measure $M_\beta(\mathbf{x})$ is defined as average of the k worst realizations

$$M_{\frac{k}{T}}(\mathbf{x}) = \frac{1}{k}\sum_{s=1}^{k} y_{t_s}, \tag{2.17}$$

where $y_{t_1}, y_{t_2}, \ldots, y_{t_k}$ are the k worst realizations for the portfolio rate of return.

In Fig. 2.5, we show an example of a portfolio whose CVaR value has been computed for $k = 3$ and $T = 25$.

For any probability p_t and tolerance level β, due to the finite number of scenarios, the CVaR measure $M_\beta(\mathbf{x})$ is well defined by the following optimization

$$M_\beta(\mathbf{x}) = \min_{u_t}\{\frac{1}{\beta}\sum_{t=1}^{T} y_t u_t : \sum_{t=1}^{T} u_t = \beta,\ 0 \leq u_t \leq p_t \ \ t = 1, \ldots, T\}, \tag{2.18}$$

where at optimality u_t is the percentage of the t-th worst return in $M_\beta(\mathbf{x})$. More precisely, $u_t = 0$ for any scenario t not included in the worst scenarios, $u_t = p_t$

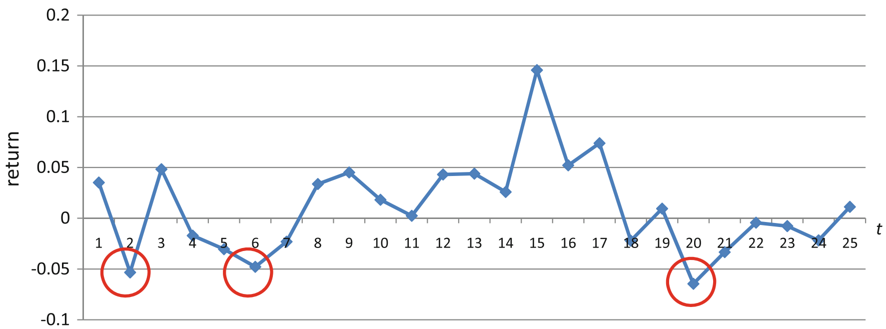

Fig. 2.5 The CVaR model $M_{\frac{k}{T}}(\mathbf{x})$

Table 2.3 Optimal portfolios for the CVaR measure: An example

Scenario	Probability	Rates of return	
		Portfolio $\mathbf{x}'$ (%)	Portfolio $\mathbf{x}''$ (%)
1	0.2	4.9	2.0
2	0.5	4.0	3.0
3	0.2	2.2	2.0
4	0.1	1.8	2.0
Worst	$M(\mathbf{x})$	1.8	2.0
CVaR	$M_{0.05}(\mathbf{x})$	1.8	2.0
CVaR	$M_{0.1}(\mathbf{x})$	1.8	2.0
CVaR	$M_{0.2}(\mathbf{x})$	2.0	2.0
CVaR	$M_{0.3}(\mathbf{x})$	2.07	2.0
CVaR	$M_{0.5}(\mathbf{x})$	2.84	2.0
CVaR	$M_{0.8}(\mathbf{x})$	3.28	2.38
CVaR	$M_{1.0}(\mathbf{x})$	3.6	2.5
Mean	$\mu(\mathbf{x})$	3.6	2.5

for any scenario t totally included in the worst scenarios, and $0 < u_t < p_t$ for one scenario t only.

When parameter β approaches 0 and becomes smaller than or equal to the minimal scenario probability ($\beta \leq \min_t p_t$), the measure becomes the worst return $M(\mathbf{x}) = \lim_{\beta \to 0_+} M_\beta(\mathbf{x})$. On the other hand, for $\beta = 1$ the corresponding CVaR becomes the mean ($M_1(\mathbf{x}) = \mu(\mathbf{x})$).

Recall the case of two portfolios shown in Table 2.2. In Table 2.3, we show their CVaR values for various tolerance levels. For $\beta = 0.05$ and $\beta = 0.1$ the CVaR values are equal to the corresponding return in the worst scenario, $M(\mathbf{x}') = 1.8\,\%$ and $M(\mathbf{x}'') = 2\,\%$, respectively. For $\beta = 0.2$ one gets equal CVaR values $M_{0.2}(\mathbf{x}') = M_{0.2}(\mathbf{x}'') = 2\,\%$, while for $\beta = 0.3$ one has $M_{0.3}(\mathbf{x}') = 2.07\,\%$ greater than $M_{0.3}(\mathbf{x}'') = 2\,\%$. The difference becomes larger for tolerance levels $\beta = 0.5$ and $\beta = 0.8$. Obviously, for $\beta = 1$ one gets the corresponding means as CVaR values.

Problem (2.18) is a linear program for a given portfolio $\mathbf{x}$, while it becomes nonlinear when the y_t are variables in the portfolio optimization problem. It turns out that this difficulty can be overcome by taking advantage of the LP dual problem to (2.18) leading to an equivalent LP dual formulation of the CVaR model that allows one to implement the optimization problem with auxiliary linear inequalities. Indeed, introducing dual variable η corresponding to the equation $\sum_{t=1}^{T} u_t = \beta$ and variables d_t^- corresponding to upper bounds on u_t one gets the LP dual problem:

$$M_\beta(\mathbf{x}) = \max_{\eta, d_t^-} \{\eta - \frac{1}{\beta} \sum_{t=1}^{T} p_t d_t^- : d_t^- \geq \eta - y_t,\ d_t^- \geq 0 \ \ t = 1, \ldots, T\}. \tag{2.19}$$

Due to the duality theory, for any given vector y_t the measure $M_\beta(\mathbf{x})$ can be found as the optimal value of the LP problem (2.19). Thus, the CVaR is a safety measure that is LP computable. The portfolio optimization model can be formulated as follows:

$$\max\ (\eta - \frac{1}{\beta}\sum_{t=1}^{T} p_t d_t^-) \tag{2.20a}$$

$$d_t^- \geq \eta - y_t \qquad t = 1,\dots,T \tag{2.20b}$$

$$y_t = \sum_{j=1}^{n} r_{jt} x_j \qquad t = 1,\dots,T \tag{2.20c}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.20d}$$

$$\mu \geq \mu_0 \tag{2.20e}$$

$$d_t^- \geq 0 \qquad t = 1,\dots,T \tag{2.20f}$$

$$\mathbf{x} \in Q, \tag{2.20g}$$

where η is an auxiliary (unbounded) variable that in the optimal solution will take the value of the β-quantile.

In the case of $\mathbb{P}\{R_{\mathbf{x}} \leq q_\beta(\mathbf{x})\} = \beta$, one gets $M_\beta(\mathbf{x}) = \mathbb{E}\{R_{\mathbf{x}} | R_{\mathbf{x}} \leq q_\beta(\mathbf{x})\}$. This represents the original concept of the CVaR measure. Although valid for many continuous distributions this formula, in general, cannot serve as a definition of the CVaR measure because a value ξ such that $\mathbb{P}\{R_{\mathbf{x}} \leq \xi\} = \beta$ may not exist. In general, $\mathbb{P}\{R_{\mathbf{x}} \leq q_\beta(\mathbf{x})\} = \beta' \geq \beta$ and $M_\beta(\mathbf{x}) \leq M_{\beta'}(\mathbf{x}) = \mathbb{E}\{R_{\mathbf{x}} | R_{\mathbf{x}} \leq q_\beta(\mathbf{x})\}$.

2.5 The Complete Set of Basic Linear Models

As shown in the previous sections several LP computable risk measures have been considered for portfolio optimization. Some of them were originally introduced rather as safety measures in our classification (e.g., CVaR measures). Nevertheless, all of them can be represented with positively homogeneous and shift independent risk measures ϱ of classical Markowitz type model. Simple as well as more complicated LP computable risk measures $\varrho(\mathbf{x})$ can be defined as

$$\varrho(\mathbf{x}) = \min\{\mathbf{a}^T\mathbf{v} : \quad \mathbf{A}\mathbf{v} = \mathbf{B}\mathbf{x},\ \mathbf{v} \geq \mathbf{0}, \mathbf{x} \in Q\}, \tag{2.21}$$

where $\mathbf{v}$ is a vector of auxiliary variables while the portfolio vector $\mathbf{x}$, apart from original portfolio constraints $\mathbf{x} \in Q$, only defines a parametric right hand side vector $\mathbf{b} = \mathbf{B}\mathbf{x}$. Obviously, the corresponding safety measures are given by a similar LP formula

$$\mu(\mathbf{x}) - \varrho(\mathbf{x}) = \max\{\sum_{j=1}^{n} \mu_j x_j - \mathbf{a}^T\mathbf{v} : \quad \mathbf{A}\mathbf{v} = \mathbf{B}\mathbf{x},\ \mathbf{v} \geq \mathbf{0}, \mathbf{x} \in Q\}. \tag{2.22}$$

For each model of type (2.21), the mean-risk bounding approach (1.10) leads to the LP problem

$$\min_{\mathbf{x},\mathbf{v}}\{\mathbf{a}^T\mathbf{v} :\ \mathbf{A}\mathbf{v} = \mathbf{B}\mathbf{x},\ \mathbf{v} \geq \mathbf{0},\ \sum_{j=1}^{n} \mu_j x_j \geq \mu_0,\ \mathbf{x} \in Q\}, \tag{2.23}$$

while the mean-safety bounding approach (1.12) applied to (2.22) results in

$$\max_{\mathbf{x},\mathbf{v}}\{\sum_{j=1}^{n} \mu_j x_j - \mathbf{a}^T\mathbf{v} :\ \mathbf{A}\mathbf{v} = \mathbf{B}\mathbf{x},\ \mathbf{v} \geq \mathbf{0},\ \sum_{j=1}^{n} \mu_j x_j \geq \mu_0,\ \mathbf{x} \in Q\}. \tag{2.24}$$

Similarly, the trade-off analysis approach (1.13) results in the LP model

$$\max_{\mathbf{x},\mathbf{v}}\{\sum_{j=1}^{n} \mu_j x_j - \lambda \mathbf{a}^T\mathbf{v} :\ \mathbf{A}\mathbf{v} = \mathbf{B}\mathbf{x},\ \mathbf{v} \geq \mathbf{0},\ \mathbf{x} \in Q\}. \tag{2.25}$$

2.5.1 Risk Measures from Safety Measures

Recall that, for a discrete random variable represented by its realizations y_t, the *worst realization* $M(\mathbf{x}) = \min_{t=1,\ldots,T}\{y_t\}$ is an appealing LP computable safety measure (see (2.13)). The corresponding (dispersion) risk measure $\varDelta(\mathbf{x}) = \mu(\mathbf{x}) - M(\mathbf{x})$, the *maximum (downside) semideviation*, is LP computable as

$$\varDelta(\mathbf{x}) = \min\{v\ :\ v \geq \sum_{j=1}^{n}(\mu_j - r_{jt})x_j, t = 1, \ldots, T\}. \tag{2.26}$$

The portfolio optimization model with the maximum semideviation as risk measure can be formulated as:

$$\min\ v \tag{2.27a}$$

$$\mu - \sum_{j=1}^{n} r_{jt} x_j \leq v \qquad t = 1, \ldots, T \tag{2.27b}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.27c}$$

$$\mu \geq \mu_0 \tag{2.27d}$$

$$\mathbf{x} \in Q. \tag{2.27e}$$

The variable v is an auxiliary variable that in the optimum will take the value of the maximum downside deviation of the portfolio rate of return from the mean return.

Similarly, the CVaR measure is a safety measure. The corresponding risk measure $\Delta_\beta(\mathbf{x}) = \mu(\mathbf{x}) - M_\beta(\mathbf{x})$ is called the (worst) *conditional semideviation* or *conditional drawdown* measure. For a discrete random variable represented by its realizations, due to (2.19), the conditional semideviations may be computed as the corresponding differences from the mean:

$$\Delta_\beta(\mathbf{x}) = \min\{\sum_{j=1}^{n} \mu_j x_j - \eta + \frac{1}{\beta}\sum_{t=1}^{T} d_t^- p_t : \; d_t^- \geq \eta - y_t, \; d_t^- \geq 0, \; t = 1,\ldots,T\}, \tag{2.28}$$

or, equivalently, setting $d_t^+ = d_t^- - \eta + y_t$, as:

$$\Delta_\beta(\mathbf{x}) = \min\{\sum_{t=1}^{T}(d_t^+ + \frac{1-\beta}{\beta}d_t^-)p_t : \; d_t^- - d_t^+ = \eta - y_t, \; d_t^-, d_t^+ \geq 0, \; t = 1,\ldots,T\}, \tag{2.29}$$

where η is an auxiliary (unbounded) variable that in the optimal solution will take the value of the β-quantile $q_\beta(\mathbf{x})$.

Thus, the conditional semideviation is an LP computable risk measure and the corresponding portfolio optimization model can be formulated as follows:

$$\min \sum_{t=1}^{T}(d_t^+ + \frac{1-\beta}{\beta}d_t^-)p_t \tag{2.30a}$$

$$d_t^- - d_t^+ = \eta - y_t, \; d_t^-, d_t^+ \geq 0 \qquad t = 1,\ldots,T \tag{2.30b}$$

$$y_t = \sum_{j=1}^{n} r_{jt} x_j \qquad t = 1,\ldots,T \tag{2.30c}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.30d}$$

$$\mu \geq \mu_0 \tag{2.30e}$$

$$\mathbf{x} \in Q. \tag{2.30f}$$

Note that for $\beta = 0.5$ one has $(1-\beta)/\beta = 1$. Hence, $\Delta_{0.5}(\mathbf{x})$ represents the mean absolute deviation from the median $q_{0.5}(\mathbf{x})$. The LP problem for computing this measure can be expressed in the form:

$$\varDelta_{0.5}(\mathbf{x}) = \min\{\sum_{t=1}^{T} d_t p_t : d_t \geq \eta - y_t,\ d_t \geq y_t - \eta,\ d_t \geq 0 \quad t = 1,\ldots,T\}.$$

One may notice that the above model differs from the classical MAD formulation (2.8) only due to replacement of $\mu(\mathbf{x})$ with (unrestricted) variable η.

2.5.2 Safety Measures from Risk Measures

Symmetrically, a safety measure can be obtained from a positively homogeneous and shift independent (deviation type) risk measure. For the Semi-MAD (2.9) the corresponding safety measure can be expressed as

$$\mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) = \mathbb{E}\{\mu(\mathbf{x}) - \max\{\mu(\mathbf{x}) - R_{\mathbf{x}}, 0\}\} = \mathbb{E}\{\min\{R_{\mathbf{x}}, \mu(\mathbf{x})\}\}, \tag{2.31}$$

thus representing the *mean downside underachievement*. The corresponding portfolio optimization problem can be written as follows:

$$\max \sum_{t=1}^{T} p_t v_t \tag{2.32a}$$

$$v_t \leq \sum_{j=1}^{n} r_{jt} x_j \qquad t = 1,\ldots,T \tag{2.32b}$$

$$v_t \leq \mu \qquad t = 1,\ldots,T \tag{2.32c}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.32d}$$

$$\mu \geq \mu_0 \tag{2.32e}$$

$$\mathbf{x} \in Q. \tag{2.32f}$$

The Gini's mean difference (2.11) has the corresponding safety measure defined as

$$\mu(\mathbf{x}) - \varGamma(\mathbf{x}) = \sum_{t'=1}^{T} \sum_{t''=1}^{T} \min\{y_{t'}, y_{t''}\} p_{t'} p_{t''}, \tag{2.33}$$

where the latter expression is obtained through algebraic calculations. Hence, (2.33) is the expectation of the minimum of two independent identically distributed random variables, thus representing the *mean worse return*.

This leads to the following LP formula

$$
\begin{aligned}
\mu(\mathbf{x}) - \Gamma(\mathbf{x}) = \max\{\sum_{t'=1}^{T}\ \sum_{t''=1}^{T} v_{t't''} p_{t'} p_{t''} :\\
v_{t't''} \le \sum_{j=1}^{n} r_{jt'} x_j,\ v_{t't''} \le \sum_{j=1}^{n} r_{jt''} x_j, t', t'' = 1, \dots, T\}.
\end{aligned}
\tag{2.34}
$$

The portfolio optimization model can be written as follows:

$$\max \sum_{t'=1}^{T}\ \sum_{t''=1}^{T} p_{t'} p_{t''} v_{t't''} \tag{2.35a}$$

$$v_{t't''} \le y_{t'} \qquad t', t'' = 1, \dots, T \tag{2.35b}$$

$$v_{t't''} \le y_{t''} \qquad t', t'' = 1, \dots, T \tag{2.35c}$$

$$y_t = \sum_{j=1}^{n} r_{jt} x_j \qquad t = 1, \dots, T \tag{2.35d}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.35e}$$

$$\mu \ge \mu_0 \tag{2.35f}$$

$$\mathbf{x} \in Q. \tag{2.35g}$$

2.5.3 Ratio Measures from Risk Measures

As mentioned in Chap. 1, an alternative approach to the bicriteria mean-risk approach to portfolio selection is based on maximizing the ratio $(\mu(\mathbf{x}) - r_0)/\varrho(\mathbf{x})$. The corresponding ratio optimization problem (1.14) can be converted into an LP form by the following transformation: introduce an auxiliary variable $z = 1/\varrho(\mathbf{x})$, then replace the original variables $\mathbf{x}$ and $\mathbf{v}$ with $\tilde{\mathbf{x}} = z\mathbf{x}$ and $\tilde{\mathbf{v}} = z\mathbf{v}$, respectively, getting the linear criterion and an LP feasible set. For risk measure $\varrho(\mathbf{x})$ defined by (2.21) one gets the following LP formulation of the corresponding ratio model

$$
\begin{aligned}
\max_{\tilde{\mathbf{x}}, \tilde{\mathbf{v}}, z}\{\sum_{j=1}^{n} \mu_j \tilde{x}_j - r_0 z\ : \mathbf{c}^T \tilde{\mathbf{v}} = z,\ \mathbf{A}\tilde{\mathbf{v}} = \mathbf{b}\tilde{\mathbf{x}},\ \tilde{\mathbf{v}} \ge \mathbf{0},\\
\sum_{j=1}^{n} \tilde{x}_j = z,\ \tilde{x}_j \ge 0, j = 1, \dots, n\},
\end{aligned}
\tag{2.36}
$$

where the second line constraints correspond to the simplest definition of set $Q = \{\mathbf{x} : \sum_{j=1}^{n} x_j = 1,\ x_j \geq 0, j = 1,\ldots,n\}$ and can be accordingly formulated for any other LP set. Once the transformed problem is solved, the values of the portfolio variables x_j can be found by dividing $\tilde{x}_j$ by the optimal value of z. Thus, the LP computable portfolio optimization models, we consider, remain within LP environment even in the case of ratio criterion used to define the tangency portfolio.

For the Semi-MAD model (2.10) with risk measure $\varrho(\mathbf{x}) = \bar{\delta}(\mathbf{x})$, the ratio optimization model can be written as

$$\max \left\{ \frac{\mu - r_0}{\sum_{t=1}^{T} p_t d_t} \ :\ (2.10b)\text{–}(2.10g) \right\}.$$

Introducing variables $z = 1/\sum_{t=1}^{T} p_t d_t$ and $\tilde{v} = z\mu$ we get the linear criterion $\tilde{v} - r_0 z$. Further, we multiply all the constraints by z and make the substitutions: $\tilde{d}_t = zd_t$, $\tilde{y}_t = zy_t$, for $t = 1,\ldots,T$, as well as $\tilde{x}_j = zx_j$, for $j = 1,\ldots,n$. Finally, we get the following LP formulation:

$$\max\ \tilde{v} - r_0 z \tag{2.37a}$$

$$\sum_{t=1}^{T} p_t \tilde{d}_t = 1 \tag{2.37b}$$

$$\tilde{d}_t \geq \tilde{v} - \tilde{y}_t,\ \tilde{d}_t \geq 0 \qquad t = 1,\ldots,T \tag{2.37c}$$

$$\tilde{y}_t = \sum_{j=1}^{n} r_{jt}\tilde{x}_j \qquad t = 1,\ldots,T \tag{2.37d}$$

$$\tilde{v} = \sum_{j=1}^{n} \mu_j \tilde{x}_j \tag{2.37e}$$

$$\sum_{j=1}^{n} \tilde{x}_j = z, \quad \tilde{x}_j \geq 0 \qquad j = 1,\ldots,n, \tag{2.37f}$$

where the last constraints correspond to the simplest definition of set Q.

Clear identification of dispersion type risk measures for all the LP computable safety measures allows us to define tangency portfolio optimization for all the models.

For the CVaR model with conditional semideviation as risk measure $\varrho(\mathbf{x}) = \Delta_\beta(\mathbf{x})$ (2.30) the ratio optimization model can be written as

$$\max \left\{ \frac{\mu - r_0}{\sum_{t=1}^{T}(d_t^+ + \frac{1-\beta}{\beta} d_t^-)p_t} \ :\ (2.30b)\text{–}(2.30f) \right\}.$$

Introducing variables $z = 1/\sum_{t=1}^{T}(d_t^+ + \frac{1-\beta}{\beta}d_t^-)p_t$ and $\tilde{v} = z\mu$ we get the linear criterion $\tilde{v} - r_0 z$. Further, we multiply all the constraints by z and make the substitutions: $\tilde{d}_t^+ = zd_t^+$, $\tilde{d}_t^- = zd_t^-$, $\tilde{y}_t = zy_t$ for $t = 1,\dots,T$, as well as $\tilde{x}_j = zx_j$, for $j = 1,\dots,n$. Then, we get the following LP formulation:

$$\max\ \tilde{v} - r_0 z \tag{2.38a}$$

$$\sum_{t=1}^{T}(\tilde{d}_t^+ + \frac{1-\beta}{\beta}\tilde{d}_t^-)p_t = 1 \tag{2.38b}$$

$$d_t^- - d_t^+ = \eta - y_t,\ d_t^-, d_t^+ \geq 0 \qquad t = 1,\dots,T \tag{2.38c}$$

$$\tilde{y}_t = \sum_{j=1}^{n} r_{jt}\tilde{x}_j \qquad t = 1,\dots,T \tag{2.38d}$$

$$\tilde{v} = \sum_{j=1}^{n} \mu_j \tilde{x}_j \tag{2.38e}$$

$$\sum_{j=1}^{n}\tilde{x}_j = z, \quad \tilde{x}_j \geq 0 \qquad j = 1,\dots,n. \tag{2.38f}$$

2.6 Advanced LP Computable Measures

The LP computable risk measures may be further extended to enhance the risk aversion modeling capabilities. First of all, the measures may be combined in a weighted sum which allows the generation of various mixed measures. Consider a set of, say m, risk measures $\varrho_k(\mathbf{x})$ and their linear combination with weights w_k:

$$\varrho_{\mathbf{w}}^{(m)}(\mathbf{x}) = \sum_{k=1}^{m} w_k \varrho_k(\mathbf{x}), \qquad \sum_{k=1}^{m} w_k \leq 1, \quad w_k \geq 0 \quad \text{for } k = 1,\dots,m. \tag{2.39}$$

Note that

$$\mu(\mathbf{x}) - \varrho_{\mathbf{w}}^{(m)}(\mathbf{x}) = w_0\mu(\mathbf{x}) + \sum_{k=1}^{m} w_k(\mu(\mathbf{x}) - \varrho_k(\mathbf{x})),$$

where $w_0 = 1 - \sum_{k=1}^{m} w_k \geq 0$.

In particular, one may build a multiple CVaR measure by considering, say m, tolerance levels $0 < \beta_1 < \beta_2 < \ldots < \beta_m \leq 1$ and using the weighted sum of the conditional semideviations $\Delta_{\beta_k}(\mathbf{x})$ as a new risk measure

$$\Delta_{\mathbf{w}}^{(m)}(\mathbf{x}) = \sum_{k=1}^{m} w_k \Delta_{\beta_k}(\mathbf{x}), \qquad \sum_{k=1}^{m} w_k = 1, \quad w_k > 0 \quad \text{for } k = 1, \ldots, m, \tag{2.40}$$

with the corresponding safety measure

$$M_{\mathbf{w}}^{(m)}(\mathbf{x}) = \mu(\mathbf{x}) - \Delta_{\mathbf{w}}^{(m)}(\mathbf{x}) = \sum_{k=1}^{m} w_k M_{\beta_k}(\mathbf{x}). \tag{2.41}$$

The resulting Weighted CVaR (WCVaR) models use multiple CVaR measures, thus allowing for more detailed risk aversion modeling. The WCVaR risk measure is obviously LP computable as

$$\begin{aligned} M_{\mathbf{w}}^{(m)}(\mathbf{x}) = \max\{ &\sum_{k=1}^{m} w_k(\eta_k - \frac{1}{\beta_k}\sum_{t=1}^{T} d_{kt}^{-} p_t) : \ d_{kt}^{-} \geq 0, \\ &d_{kt}^{-} \geq \eta_k - \sum_{j=1}^{n} r_{jt}x_j, t = 1, \ldots, T; k = 1, \ldots, m\}. \end{aligned} \tag{2.42}$$

The corresponding portfolio optimization model can be formulated as follows:

$$\max \sum_{k=1}^{m} w_k(\eta_k - \frac{1}{\beta_k}\sum_{t=1}^{T} d_{kt}^{-} p_t) \tag{2.43a}$$

$$d_{kt}^{-} \geq \eta_k - y_t, \ d_{kt}^{-} \geq 0 \qquad t = 1, \ldots, T; k = 1, \ldots, m \tag{2.43b}$$

$$y_t = \sum_{j=1}^{n} r_{jt}x_j \qquad t = 1, \ldots, T \tag{2.43c}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.43d}$$

$$\mu \geq \mu_0 \tag{2.43e}$$

$$\mathbf{x} \in Q. \tag{2.43f}$$

For appropriately defined weights the WCVaR measures may be considered some approximations to the Gini's mean difference with the advantage of being computationally much simpler than the GMD model itself.

The risk measures introduced in the previous section are quite different in modeling the downside risk aversion. Definitely, the strongest in this respect is the maximum semideviation $\Delta(\mathbf{x})$ while the conditional semideviation $\Delta_{\beta}(\mathbf{x})$ (CVaR model) allows us to extend the approach to a specified β quantile of the worst returns which results in a continuum of models evolving from the strongest downside risk aversion (β close to 0) to the complete risk neutrality ($\beta = 1$). The mean (downside) semideviation from the mean, used in the MAD model, is formally a downside risk measure. However, due to the symmetry of mean semideviations from the mean it is equally appropriate to interpret it as a measure of the upside risk. Similarly, the Gini's mean difference, although related to all the conditional maximum semideviations, is a symmetric risk measure (in the sense that for $R_{\mathbf{x}}$ and $-R_{\mathbf{x}}$ it has exactly the same value). For better modeling of the risk averse preferences one may enhance the below-mean downside risk aversion in various measures. The below-mean risk downside aversion is a concept of risk aversion assuming that the variability of returns above the mean should not be penalized since the investors are concerned about the under-performance rather than the over-performance of a portfolio. This can be implemented by focusing on the distribution of *downside underachievements* $\min\{R_{\mathbf{x}}, \mu(\mathbf{x})\}$ instead of the original distribution of returns $R_{\mathbf{x}}$.

Applying the mean semideviation (2.9) to the distribution of downside underachievements $\min\{R_{\mathbf{x}}, \mu(\mathbf{x})\}$ one gets

$$\bar{\delta}_2(\mathbf{x}) = \mathbb{E}\{\max\{\mathbb{E}\{\min\{R_{\mathbf{x}}, \mu(\mathbf{x})\}\} - R_{\mathbf{x}}, 0\}\} = \mathbb{E}\{\max\{\mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) - R_{\mathbf{x}}, 0\}\}.$$

This allows us to define the enhanced risk measure for the original distribution of returns $R_{\mathbf{x}}$ as $\bar{\delta}^{(2)}(\mathbf{x}) = \bar{\delta}(\mathbf{x}) + \bar{\delta}_2(\mathbf{x})$ with the corresponding safety measure $\mu(\mathbf{x}) - \bar{\delta}^{(2)}(\mathbf{x}) = \mu(\mathbf{x}) - \bar{\delta}(\mathbf{x}) - \bar{\delta}_2(\mathbf{x})$. The above approach can be repeated recursively, resulting in m (defined recursively) distribution dependent targets $\mu_1(\mathbf{x}) = \mu(\mathbf{x})$, $\mu_k(\mathbf{x}) = \mathbb{E}\{\min\{R_{\mathbf{x}}, \mu_{k-1}(\mathbf{x})\}\}$ for $k = 2, \ldots, m$, and the corresponding mean semideviations $\bar{\delta}_1(\mathbf{x}) = \bar{\delta}(\mathbf{x})$, $\bar{\delta}_k(\mathbf{x}) = \mathbb{E}\{\max\{\mu_k(\mathbf{x}) - R_{\mathbf{x}}, 0\}\}$ for $k = 1, \ldots, m$. The measure

$$\bar{\delta}^{(m)}_{\mathbf{w}}(\mathbf{x}) = \sum_{k=1}^{m} w_k \bar{\delta}_k(\mathbf{x}) \qquad 1 = w_1 \geq w_2 \geq \ldots \geq w_m \geq 0 \tag{2.44}$$

gives rise to the so-called m-MAD model. Actually, the measure can be interpreted as a single mean semideviation (from the mean) applied with a penalty function: $\bar{\delta}^{(m)}_{\mathbf{w}}(\mathbf{x}) = \mathbb{E}\{u(\max\{\mu(\mathbf{x}) - R_{\mathbf{x}}, 0\})\}$, where u is an increasing and convex piecewise linear penalty function with breakpoints $b_k = \mu(\mathbf{x}) - \mu_k(\mathbf{x})$ and slopes $s_k = w_1 + \ldots + w_k$, $k = 1, \ldots, m$. Therefore, the measure $\bar{\delta}^{(m)}_{\mathbf{w}}(\mathbf{x})$ is referred to as the *mean penalized semideviation* and is obviously LP computable leading to the following LP form of the m-MAD portfolio optimization model:

$$\min \sum_{k=1}^{m} w_k v_k \tag{2.45a}$$

$$v_k - \sum_{t=1}^{T} p_t d_{kt} = 0 \qquad k = 1, \ldots, m \tag{2.45b}$$

$$d_{kt} \geq 0,\ d_{kt} \geq \mu - y_t - \sum_{i=1}^{k-1} v_i \qquad t = 1, \ldots, T;\ k = 1, \ldots, m \tag{2.45c}$$

$$y_t = \sum_{j=1}^{n} r_{jt} x_j \qquad t = 1, \ldots, T \tag{2.45d}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.45e}$$

$$\mu \geq \mu_0 \tag{2.45f}$$

$$\mathbf{x} \in Q. \tag{2.45g}$$

The Gini's mean difference is a symmetric measure, thus equally treating both under and over-achievements. The enhancement technique allows us to define the downside Gini's mean difference by applying the Gini's mean difference to the distribution of downside underachievements $\min\{R_{\mathbf{x}}, \mu(\mathbf{x})\}$

$$\Gamma_2(\mathbf{x}) = \sum_{t=1}^{T} \min\{y_t, \mu(\mathbf{x})\} p_t - \sum_{t'=1}^{T} \sum_{t''=1}^{T} \min\{\min\{y_{t'}, \mu(\mathbf{x})\}, \min\{y_{t''}, \mu(\mathbf{x})\}\} p_{t'} p_{t''}.$$

Hence, we get the *downside Gini's mean difference* defined as the enhanced risk measure:

$$\Gamma^d(\mathbf{x}) = \Gamma_2(\mathbf{x}) + \bar{\delta}(\mathbf{x}) = \mu(\mathbf{x}) - \sum_{t'=1}^{T} \sum_{t''=1}^{T} \min\{y_{t'}, y_{t''}, \mu(\mathbf{x})\} p_{t'} p_{t''}. \tag{2.46}$$

The downside Gini's safety measure takes the form:

$$\mu(\mathbf{x}) - \Gamma^d(\mathbf{x}) = \sum_{t'=1}^{T} \sum_{t''=1}^{T} \min\{y_{t'}, y_{t''}, \mu(\mathbf{x})\} p_{t'} p_{t''}, \tag{2.47}$$

which is obviously LP computable. The portfolio optimization model based on the downside Gini's safety measure can be written as follows:

$$\max \sum_{t'=1}^{T} \sum_{t''=1}^{T} p_{t'} p_{t''} v_{t't''} \tag{2.48a}$$

$$v_{t't''} \leq y_{t'} \qquad t', t'' = 1, \dots, T \tag{2.48b}$$

$$v_{t't''} \leq y_{t''} \qquad t', t'' = 1, \dots, T \tag{2.48c}$$

$$v_{t't''} \leq \mu \qquad t', t'' = 1, \dots, T \tag{2.48d}$$

$$y_t = \sum_{j=1}^{n} r_{jt} x_j \qquad t = 1, \dots, T \tag{2.48e}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.48f}$$

$$\mu \geq \mu_0 \tag{2.48g}$$

$$\mathbf{x} \in Q. \tag{2.48h}$$

The notion of risk may be related to a possible failure of achieving some targets instead of the mean. It was formalized by the concept of below-target risk measures or shortfall criteria. The simplest shortfall criterion for a specific target value τ is the mean below-target deviation (first Lower Partial Moment, LPM)

$$\bar{\delta}_\tau(\mathbf{x}) = \mathbb{E}\{\max\{\tau - R_{\mathbf{x}}, 0\}\}. \tag{2.49}$$

The mean below-target deviation is LP computable for returns represented by their realizations and the corresponding portfolio optimization model can be written as follows:

$$\min \sum_{t=1}^{T} p_t d_t \tag{2.50a}$$

$$d_t \geq \tau - y_t \qquad t = 1, \dots, T \tag{2.50b}$$

$$y_t = \sum_{j=1}^{n} r_{jt} x_j \qquad t = 1, \dots, T \tag{2.50c}$$

$$\mu = \sum_{j=1}^{n} \mu_j x_j \tag{2.50d}$$

$$\mu \geq \mu_0 \tag{2.50e}$$

$$d_t \geq 0 \qquad t = 1, \dots, T \tag{2.50f}$$

$$\mathbf{x} \in Q. \tag{2.50g}$$

The mean below-target deviation from a specific target (2.49) represents only a single criterion. One may consider several, say m, targets $\tau_1 > \tau_2 > \ldots > \tau_m$ and use weighted sum of the shortfall criteria as a risk measure:

$$\sum_{k=1}^{m} w_k \bar{\delta}_{\tau_k}(\mathbf{x}) = \sum_{k=1}^{m} w_k \mathbb{E}\{\max\{\tau_k - R_{\mathbf{x}}, 0\}\}, \tag{2.51}$$

where w_k (for $k = 1, \ldots, m$) are positive weights which maintain the measure LP computable (when minimized). Actually, the measure can be interpreted as a single mean below-target deviation applied with a penalty function: $\mathbb{E}\{u(\max\{\tau_1 - R_{\mathbf{x}}, 0\})\}$, where u is increasing and convex piece-wise linear penalty function with breakpoints $b_k = \tau_1 - \tau_k$ and slopes $\bar{s}_k = w_1 + \ldots + w_k$, $k = 1, \ldots, m$.

The below-target deviations are very useful in investment situations with clearly defined minimum acceptable returns (e.g. bankruptcy level). Otherwise, appropriate selection of the target value might be a difficult task. However, for portfolio optimization they are rather rarely applied. Recently, the so-called Omega ratio measure defined, for a given target, as the ratio of the mean over-target deviation by the mean below-target deviation was introduced:

$$\Omega_\tau(\mathbf{x}) = \frac{\mathbb{E}\{\max\{R_{\mathbf{x}} - \tau, 0\}\}}{\mathbb{E}\{\max\{\tau - R_{\mathbf{x}}, 0\}\}} = \frac{\int_\tau^\infty (1 - F_{\mathbf{x}}(\xi))\, d\xi}{\int_{-\infty}^\tau F_{\mathbf{x}}(\xi)\, d\xi}. \tag{2.52}$$

Since $\tau - \mathbb{E}\{\max\{\tau - R_{\mathbf{x}}, 0\}\} = \mu(\mathbf{x}) - \mathbb{E}\{\max\{R_{\mathbf{x}} - \tau, 0\}\}$, one gets

$$\Omega_\tau(\mathbf{x}) = \frac{\bar{\delta}_\tau(\mathbf{x}) - (\tau - \mu(\mathbf{x}))}{\bar{\delta}_\tau(\mathbf{x})} = 1 + \frac{\mu(\mathbf{x}) - \tau}{\bar{\delta}_\tau(\mathbf{x})}.$$

Thus, the portfolio optimization model based on the Omega ratio maximization is equivalent to the standard ratio (tangent portfolio) model (1.14) for the $\bar{\delta}_\tau(\mathbf{x})$ measure with target τ replacing the risk-free rate of return:

$$\max \left\{ \frac{\mu - \tau}{\sum_{t=1}^{T} p_t d_t} \;:\; (2.50b)\text{–}(2.50g) \right\}.$$

Similarly to the MAD ratio model it is easily transformed to an LP form. Introducing variables $z = 1/\sum_{t=1}^{T} p_t d_t$ and $\tilde{v} = z\mu$ we get the linear criterion $\tilde{v} - \tau z$. Further, we multiply all the constraints by z and make the substitutions: $\tilde{d}_t = z d_t$, $\tilde{y}_t = z y_t$ for

$t = 1, \ldots, T$, as well as $\tilde{x}_j = zx_j$, for $j = 1, \ldots, n$. Finally, we get the following LP formulation:

$$\max \tilde{v} - \tau z \tag{2.53a}$$

$$\sum_{t=1}^{T} p_t \tilde{d}_t = 1 \tag{2.53b}$$

$$\tilde{d}_t + \tilde{y}_t \geq \tau z, \ \tilde{d}_t \geq 0 \qquad t = 1, \ldots, T \tag{2.53c}$$

$$\sum_{j=1}^{n} \mu_j \tilde{x}_j = \tilde{v} \tag{2.53d}$$

$$\sum_{j=1}^{n} r_{jt} \tilde{x}_j = \tilde{y}_t \qquad t = 1, \ldots, T \tag{2.53e}$$

$$\sum_{j=1}^{n} \tilde{x}_j = z, \quad \tilde{x}_j \geq 0 \qquad j = 1, \ldots, n, \tag{2.53f}$$

where the last constraints correspond to the set Q definition.

2.7 Notes and References

Initial attempts to have portfolio optimization models depended on the piecewise linear approximation of the variance (see Sharpe 1971a; Stone 1973). Later, several LP computable risk measures were introduced. Yitzhaki (1982) proposed the LP solvable portfolio optimization mean-risk model using Gini's mean (absolute) difference as the risk measure (the GMD model). The mean absolute deviation was very early considered in portfolio analysis by Sharpe (1971b). The complete LP solvable portfolio optimization model based on this risk measure (the MAD model) was presented and analyzed by Konno and Yamazaki (1991). The MAD model was extensively tested on various stock markets (see Konno and Yamazaki 1991; Mansini et al. 2003a; Xidonas et al. 2010) including its application to portfolios of mortgage-backed securities by Zenios and Kang (1993) where the distribution of rates of return is known to be non-symmetric. The MAD model usually, similarly to the Markowitz one, generated the portfolios with the largest returns but also entailing the largest risk of underachievement. This model has generated interest in LP portfolio optimization resulting in many new developments. Young (1998) analyzed the LP solvable portfolio optimization model based on risk defined by the worst case scenario (Minimax model), while Ogryczak (2000) introduced the multiple criteria LP model covering all the above as special aggregation techniques.

The Semi-MAD was independently presented by Feinstein and Thapa (1993) and Speranza (1993). The m-MAD model were introduced by Michalowski and

Ogryczak (2001), while Krzemienowski and Ogryczak (2005) introduced the downside Gini's mean difference. The mean absolute deviation from the median was suggested as risk measure by Sharpe (1971a).

The quantile risk measures were introduced in different ways by many authors (see Artzner et al. 1999; Embrechts et al. 1997; Ogryczak 1999; Rockafellar and Uryasev 2000). The tail mean or worst conditional expectation, defined as the mean return of the portfolio taken over a given percentage of the worst scenarios is a natural generalization of the measure due to Young (1998). In financial literature, the tail mean quantity is usually called tail VaR, average VaR or Conditional VaR (CVaR) (see Pflug 2000). Actually, the name CVaR, after Rockafellar and Uryasev (2000), is now the most commonly used. The measure was studied in several applications (see Andersson et al. 2001; Krokhmal et al. 2002; Roman et al. 2007; Topaloglou et al. 2002), and expanded in various forms (see Acerbi 2002; Krzemienowski 2009; Mansini et al. 2007; Zhu and Fukushima 2009).

Formal classification into risk and safety measures and their complementary pairs was introduced in Mansini et al. (2003a). The maximum semideviation measure was introduced in Ogryczak (2000). The (deviation) risk measure corresponding to the CVaR was considered as the (worst) conditional semideviation (Ogryczak and Ruszczyński 2002a) or conditional drawdown measure (Chekhlov et al. 2005). General deviation risk measures were analyzed by Rockafellar et al. (2006). Linear formulations of the ratio optimization models for all the basic LP computable risk measures was introduced in Mansini et al. (2003b).

The notion of risk related to a possible failure of achieving some targets was introduced by Roy (1952) as the so-called safety-first strategy and later led to the concept of below-target risk measures (see Fishburn 1977; Nawrocki 1992) or shortfall criteria.

The Omega measure was introduced by Shadwick and Keating (2002), while the first LP portfolio optimization model with this measure was shown by Mausser et al. (2006).

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