Mean-Variance Portfolio Analysis: The Markowitz Model

2.1 Basic Notions

The Markowitz model\(^1\) describes a market with \(N\) assets characterized by a random vector of returns

\[ R = (R_1, \ldots, R_N). \]

The following data are assumed to be given:

- The expected value (mean) \(m_i = ER_i\) of each random variable \(R_i\), \(i = 1, 2, \ldots, N\);
- The covariances \(\sigma_{ij} = \text{Cov}(R_i, R_j)\) for all pairs of random variables \(R_i\) and \(R_j\).

The covariance of two random variables, \(X\) and \(Y\), is defined by

\[ \text{Cov}(X, Y) = E[X - EX][Y - EY] = E(XY) - (EX)(EY). \]

We will denote by \(m\) the vector of the expected returns

\[ m = (m_1, \ldots, m_N) \]

and by \(V\) the covariance matrix

\[ V = (\sigma_{ij}), \quad \sigma_{ij} = \text{Cov}(R_i, R_j) \]

\(^1\)Markowitz, H., Portfolio Selection, Journal of Finance 7, 77–91, 1952. Markowitz was awarded a Nobel Prize in Economics in 1990, jointly with W. Sharpe and M. Miller.
of the random vector $R = (R_1, \ldots, R_N)$. (The expectations and the covariances are assumed to be well-defined and finite.) The matrix $V$ has $N$ rows and $N$ columns. The element at the intersection of $i$th row and $j$th column is $\sigma_{ij}$.

**Expectations and Covariances of Returns** Consider a portfolio $x = (x_1, \ldots, x_N)$, where $x_i$ is the amount of money invested in asset $i$. Recall that the return on the portfolio $x$ is computed according to the formula

$$R_x = \sum_{i=1}^{N} x_i R_i.$$  

Consequently, the expected return $m_x = ER_x$ on the portfolio $x$ is given by

$$m_x = \sum_{i=1}^{N} x_i m_i = \langle m, x \rangle$$

where

$$m_i = ER_i$$

and

$$m = (m_1, \ldots, m_N).$$

The variance $VarR_x$ of the portfolio return $R_x$ can be computed as follows:

$$\sigma_x^2 = Var(R_x) = E(R_x - m_x)^2$$

$$= E \left( \sum_{i=1}^{N} x_i R_i - \sum_{i=1}^{N} x_i m_i \right)^2 = E \left[ \sum_{i=1}^{N} x_i (R_i - m_i) \right]^2$$

$$= E \left[ \sum_{i=1}^{N} x_i (R_i - m_i) \right] \left[ \sum_{j=1}^{N} x_j (R_j - m_j) \right]$$

$$= E \left[ \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j (R_i - ER_i)(R_j - ER_j) \right]$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} x_i x_j Cov(R_i, R_j)x_j$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \sigma_{ij} x_j = \langle x, Vx \rangle.$$
Thus we have the following formulas for the expectation and the variance of the return $R_x$ on the portfolio $x$:

$$m_x = ER_x = \langle m, x \rangle,$$  \hspace{1cm} (2.1)

$$\sigma_x^2 = Var(R_x) = \langle x, Vx \rangle.$$  \hspace{1cm} (2.2)

**Markowitz’s Approach to Portfolio Selection** This approach is often used in practical decisions. Given the constraint $\sum x_i = 1$ on the portfolio weights, investors choose a portfolio $x$, having two objectives:

- Maximization of the expected value $m_x = ER_x$ of the portfolio return;
- Minimization of the portfolio risk, which is measured by $\sigma_x^2 = Var_R x$ or $\sigma_x$.

We denote by $\sigma_x$ the *standard deviation* of the random variable $R_x$:

$$\sigma_x = \sqrt{Var_R x} = \sqrt{E(R_x - m_x)^2}.$$  

It is the fundamental assumption of the Markowitz approach that only two numbers characterize the portfolio: the expectation and the variance of the portfolio return. The variance is used as a very simple measure of risk: the more “variable” the random return $R_x$ on the portfolio $x$, the higher the variance of $R_x$. If the return $R_x$ is certain, its variance is equal to zero, and so such a portfolio is *risk-free*.

### 2.2 Optimization Problem: Formulation and Discussion

**The Markowitz Optimization Problem** According to individual preferences, an investor puts weights on the conflicting objectives $m_x$ and $\sigma_x^2$ and maximizes

$$\tau m_x - \sigma_x^2$$

given the parameter $\tau \geq 0$. This parameter is called *risk tolerance*. Hence, according to Markowitz, the optimization problem to be solved is as follows:

$$\max_{x \in R^N} \{ \tau m_x - \sigma_x^2 \}$$

subject to

$$x_1 + \ldots + x_N = 1.$$
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More explicitly, the above problem can be written

\[
\max_{x \in \mathbb{R}^N} \left\{ \tau \sum_{i=1}^{N} m_i x_i - \sum_{i=1}^{N} \sum_{j=1}^{N} x_i \sigma_{ij} x_j \right\}
\]

subject to

\[x_1 + \ldots + x_N = 1.\]

Using the notation

\[e = (1, 1, \ldots, 1)\]

for the vector whose all coordinates are equal to one and writing \(\langle \cdot, \cdot \rangle\) for the scalar product, we can represent the Markowitz optimization problem as follows:

\[
\max_{x \in \mathbb{R}^N} \{ \tau \langle m, x \rangle - \langle x, Vx \rangle \}
\]

subject to

\[\langle e, x \rangle = 1.\]

**Advantages and Disadvantages of the Markowitz Approach** The Markowitz approach has the following important **advantages**:

- The preferences of the investor are described in a most simple way. Only one positive number, the risk tolerance \(\tau\), has to be determined.
- Only the expectations \(m_i = E R_i\) and the covariances \(\sigma_{ij} = Cov(R_i, R_j)\) of asset returns are needed.
- The optimization problem is quadratic concave, and powerful numerical algorithms exist for finding its solutions.
- Most importantly, the Markowitz optimization problem admits an explicit analytic solution, which makes it possible to examine its quantitative and qualitative properties in much detail.

The main **drawback** of the Markowitz approach is its inability to cover situations in which the distribution of the portfolio return cannot be fully characterized by such a scarce set of data as \(m_i\) and \(\sigma_{ij}\).

**Efficient Portfolios** Portfolios obtained by using the Markowitz approach are termed **efficient**.
Definition A portfolio $x^*$ is called (mean-variance) efficient if it solves the optimization problem

$$(M_\tau) \quad \max_{x \in \mathbb{R}^N} \{ \tau m_x - \sigma_x^2 \}$$

subject to: $x_1 + \ldots + x_N = 1$

for some $\tau \geq 0$.

### 2.3 Assumptions

**Basic Assumptions** We will start the analysis of the Markowitz model under the following assumptions (later, an alternative set of assumptions will be considered).

**Assumption 1** The covariance matrix $V$ is positive definite.

This assumption means that

$$\langle x, Vx \rangle \left( = \sum_{i,j=1}^N x_i \sigma_{ij} x_j \right) > 0 \text{ for each } x \neq 0.$$  

Since $\langle x, Vx \rangle = \text{Var}(R_x)$, we always have $\langle x, Vx \rangle \geq 0$. The above assumption requires that $\langle x, Vx \rangle = 0$ only if $x = 0$. As a consequence of Assumption 1, we obtain $\text{Var}R_i > 0$, i.e., all the assets $i = 1, 2, \ldots, N$ are risky.

If Assumption 1 is satisfied, then the objective function in the Markowitz problem $(M_\tau)$ is strictly concave and the solution to $(M_\tau)$ exists and is unique.$^2$

The set of efficient portfolios is a one-parameter family with parameter $\tau$ ranging through the set $[0, \infty)$ of all non-negative numbers.

The efficient portfolio $x^{\text{MIN}}$ corresponding to $\tau = 0$ is termed the minimum variance portfolio. It minimizes $\text{Var}R_x = \langle x, Vx \rangle$ over all normalized portfolios $x$.

**What Happens If Assumption 1 Fails to Hold?** Then there is a portfolio $y \neq 0$ with $\langle y, Vy \rangle = 0$. Hence

$$\text{Var}(R_y) = \text{Var}(y_1 R_1 + \ldots + y_N R_N) = 0.$$

$^2$For details see Mathematical Appendix A.
Thus $R_y$ is equal to a constant, $c$, with probability one. If $c \neq 0$, we can assume without loss of generality that $c > 0$ (replace $y$ by $-y$ if needed!). The property

$$y_1R_1 + \ldots + y_NR_N = c > 0$$

with probability 1 means the existence of a risk-free investment strategy with strictly positive return (which is ruled out in the present context).

If $c = 0$, then the equality $y_1R_1 + \ldots + y_NR_N = 0$, holding for some $(y_1, \ldots, y_N) \neq 0$, means that the random variables $R_1, \ldots, R_N$ are linearly dependent. Then at least one of them (any one for which $y_i \neq 0$) can be expressed as a linear combination of the others, which means the existence of a redundant asset.

In addition to Assumption 1, we will need the following

**Assumption 2** There are at least two assets $i$ and $j$ with expected returns $m_i \neq m_j$.

**What If Assumption 2 Does Not Hold?** If Assumption 2 is not satisfied, then there is only one efficient portfolio, $x^{MIN}$. Indeed, if Assumption 2 does not hold, then all the numbers $m_1, \ldots, m_N$ are the same and are equal, say, to some number $\theta$. Then we have $m = \theta e$, i.e., the vectors $m$ and $e = (1, 1, \ldots, 1)$ are collinear. In the Markowitz problem $(M_\tau)$, we have to maximize

$$\tau \langle m, x \rangle - \langle x, Vx \rangle$$

under the constraint

$$\langle e, x \rangle = 1.$$

If $m = \theta e$, then for every $x$ satisfying the constraint $\langle e, x \rangle = 1$, the value of the objective function is equal to

$$\tau \langle m, x \rangle - \langle x, Vx \rangle = \tau \theta \langle e, x \rangle - \langle x, Vx \rangle = \tau \theta - \langle x, Vx \rangle.$$

For each $\tau$, the maximum value of this function is attained at $x = x^{MIN}$ because $x^{MIN}$ minimizes $\langle x, Vx \rangle$ on the set of all normalized portfolios.

### 2.4 Efficient Portfolios and Efficient Frontier

**Efficient Frontier** We can draw a diagram depicting the set of all points $(\sigma^2_x, m_x)$ in the plane corresponding to all efficient portfolios $x$. This set is called the efficient frontier. The efficient frontier is a curve of the following typical form (Fig. 2.1):
The point $M$ of the curve in the above diagram corresponds to the minimum variance efficient portfolio (for which $\tau = 0$). All the other points $(\sigma^2_x, m_x)$ of the curve represent the variances and the expectations of the returns on efficient portfolios $x$ with $\tau > 0$.

**Efficient Portfolios: An Equivalent Definition** We give an equivalent definition of an efficient portfolio (which is often used in the literature).

**Proposition 2.1** A normalized portfolio $x^* \in R^N$ is efficient if and only if there exists no normalized portfolio $x \in R^N$ such that

$$m_x \geq m_{x^*} \text{ and } \sigma_x^2 < \sigma_{x^*}^2.$$  

The last two inequalities mean that $x^*$ solves the optimization problem

$$\min_{x \in R^N} \sigma_x^2$$

subject to

$$m_x \geq \mu \text{ and } \sum x_i = 1,$$

where $\mu = m_{x^*}$ and $x = (x_1, \ldots, x_N)$.

**Proof** “Only if”: We have to show that if $x^*$ is a solution to $(M_{\tau})$, then $x^*$ is a solution to $(M^\mu)$ with $\mu = m_{x^*}$. Suppose the contrary: $x^*$ is a solution to $(M_{\tau})$, but not to $(M^\mu)$, i.e., there is a normalized portfolio $x$ for which $m_x \geq \mu = m_{x^*}$ and $\sigma_x^2 < \sigma_{x^*}^2$. Then $\tau m_x - \sigma_x^2 > \tau m_{x^*} - \sigma_{x^*}^2$, which means that $x^*$ is not a solution to $(M_{\tau})$. A contradiction.
“If”: We have to show that if \( x^* \) is a solution to \((M^\mu)\) with \( \mu = m_{x^*} \), then \( x^* \) is a solution to \((M_\tau)\) for some \( \tau \geq 0 \). It can be shown that there exists a Lagrange multiplier \( \gamma \geq 0 \) relaxing the constraint \( m_x \geq \mu \) in \((M^\mu)\):

\[
-\sigma_x^2 + \gamma (m_x - \mu) \leq -\sigma_{x^*}^2 + \gamma (m_{x^*} - \mu)
\]

for each normalized portfolio \( x \). This implies

\[
\gamma m_x - \sigma_x^2 \leq \gamma m_{x^*} - \sigma_{x^*}^2.
\]

By setting \( \tau = \gamma \), we obtain that \( x^* \) is a solution to \((M_\tau)\), which completes the proof. \( \square \)

Remark The above proof is based on a general result on the existence of Lagrange multipliers for convex optimization problems—the Kuhn–Tucker theorem. This theorem is presented in Mathematical Appendix B.
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