Chapter 2
Differential Flatness Theory and Flatness-Based Control

2.1 Introduction

First, the chapter analyzes flatness-based control for lumped parameter systems, that is systems which are described by ordinary differential equations. The chapter overviews the definition and properties of differential flatness and presents basic classes of differentially flat systems. It is explained that all dynamical systems which satisfy differential flatness properties can be transformed through a change of variables into the linear canonical form. The first section of the chapter presents examples of single-input dynamical systems which are written into the linear canonical form by using the differential flatness theory diffeomorphism and the design of the associated feedback control loop is explained. The case of MIMO differentially flat dynamical system is also examined. It is shown that differentially flat systems which admit static feedback linearization can be transformed into the linear canonical form. Moreover, it is shown that for MIMO differentially flat systems, that admit only dynamic feedback linearization, it is again possible to succeed transformation to the linear canonical form and subsequently to design state feedback controllers.

Next, the chapter examines flatness-based control for distributed parameter systems, that is, systems which are described by partial differential equations. Unlike control of lumped parameter systems, distributed parameter systems control has been less investigated. Such systems are described by partial differential equations and the associated boundary conditions and play a critical role in several engineering problems, such as vibrating structures, flexible-link robots, waveguides and optical fibers, heat conduction, etc. Differential flatness theory enables the solution of such control problems. A flatness-based control method for distributed parameter systems proposes the decomposition of the desirable trajectory into a series of a reference flat output and its derivatives, and enables to compute control commands that succeed trajectory tracking. One can also consider flatness-based control for PDE systems which are based on the transformation of the PDE model into a finite differences models and the associated state-space description in a canonical form. The chapter

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overviews main findings and methods on flatness-based control of systems exhibiting a spatiotemporal (2D) dynamics.

2.2 Definition of Differentially Flat Systems

2.2.1 The Background of Differential Flatness Theory

Differential flatness theory and flatness-based control were introduced in the late 1980s by Michel Fliess and coresearchers and since then they keep on being developed and on providing efficient solutions to advanced control and state estimation problems [153].

The definition of a differentially flat system is as follows: A system $\dot{x} = f(x, u)$ with state vector $x \in R^n$, input vector $u \in R^m$ where f is a smooth vector field, is differentially flat if there exists a vector $y \in R^m$ in the form

$$y = h(x, u, \dot{u}, \ldots, u^{(r)}) \tag{2.1}$$

such that

$$\begin{aligned} x &= \phi(y, \dot{y}, \ldots, y^{(q)}) \\ u &= \alpha(y, \dot{y}, \ldots, y^{(q)}) \end{aligned} \tag{2.2}$$

where h, ϕ and α are smooth functions. This means that the new system's description is given by the m algebraic variables $y_i, \ i = 1, 2, \ldots, m$. The definition of the flat output given above was $y = h(x, u, \dot{u}, \ldots, u^{(r)})$. If the flat output is exclusively a function of the state vector x then the system is a 0-flat one. However, there may be a need to express the flat output as a function of not only the state vector x but also as a function of the control u and of its derivatives. For instance, the latter holds in the case of dynamic feedback linearization and in the application of the so-called dynamic extension. This means that the state vector of the system is extended by considering as additional state variables the control inputs and its derivatives.

Equation (2.2) shows that the state vector of the differentially flat system and its control inputs can be expressed as function of the flat output and of the flat output's derivatives. The basic question that arises in the study of differential flatness is whether, given the differential equations that describe the nonlinear system dynamics $\dot{x} = f(x, u)$, there exists a function $y = h()$ given by $y = h(x, u, \dot{u}, \ldots, u^{(r)})$, such that the state vector of the system x and the control input u can be expressed as functions of y and of its derivatives, as in Eq. (2.2).

This problem was initially set by D. Hilbert in 1912, for the second-order Monge's equation

$$\frac{d^2y}{dx^2} = F(x, y, z, \frac{dy}{dx}, \frac{dz}{dx}) \tag{2.3}$$

which is an underdetermined differential equation system (there is one differential equation with respect to x, connecting two functions of x, namely $y(x)$ and $z(x)$). Hilbert speaks about a solution that is not based on the computation of integrals. He has shown that this problem is related to the classification of underdetermined differential equation systems through a group of transformations called "invertible without integral."

Soon afterwards, Èlie Cartan, reworks on the question set by Hilbert and shows how the calculations on the Pfaff systems, permit to classify second-order Monge equations which admit a solution without integral. He deduced an explicit description for all curves in the Euclidean space R^3 for which the curvature ratio and the torsion are constant. The results of Èlie Cartan are complete for Pfaff systems of codimension 2, which are underdetermined differential equation systems of codimension 1, i.e., systems in which only one arbitrary function intervenes (otherwise stated, systems having only one control input). Èlie Cartan also suggested the notion of absolute equivalence; however, he did not define it with precision. He also noted that for Pfaff systems of codimension equal or greater than 2 (which means systems having at least two control inputs) the solution of the above problem becomes extremely complicated.

Next a formal definition will be given about the analogy in terms of the control theory, of what was described above as underdetermined differential equation systems which can be solved without integration.

2.2.2 *Differential Flatness for Finite Dimensional Systems*

As noted in Eqs. (2.1) and (2.2) differential flatness is a structural property of a class of nonlinear dynamical systems, denoting that all system variables (such as state vector elements and control inputs) can be written in terms of a set of specific variables (the so-called flat outputs) and their derivatives. The following nonlinear system is considered:

$$\dot{x}(t) = f(x(t), u(t)) \tag{2.4}$$

The time variable is $t \in R$, the state vector is $x(t) \in R^n$ with initial conditions $x(0) = x_0$, and the input variable is $u(t) \in R^m$. Next, the main principles of differentially flat systems are given [465, 535]:

The finite dimensional system of Eq. (2.4) can be written in the general form of an ordinary differential equation (ODE), i.e., $S_i(w, \dot{w}, \ddot{w}, \ldots, w^{(i)}),\ \ i = 1, 2, \ldots, q$. The entity w is a generic notation for the system variables (these variables are, for instance, the elements of the system's state vector $x(t)$ and the elements of the control input $u(t)$) while $w^{(i)}$, $i = 1, 2, \ldots, q$ are the associated derivatives. Such a system is said to be differentially flat if there is a collection of m functions $y = (y_1, \ldots, y_m)$ of the system variables and of their time derivatives, i.e., $y_i = \phi(w, \dot{w}, \ddot{w}, \ldots, w^{(\alpha_i)}),\ i = 1, \ldots, m$ satisfying the following two conditions [152, 340, 362, 364, 422]:

1. There does not exist any differential relation of the form $R(y, \dot{y}, \ldots, y^{(\beta)}) = 0$ which implies that the derivatives of the flat output are not coupled in the sense of an ODE, or equivalently it can be said that the flat output is differentially independent.
2. All system variables (i.e., the elements of the system's state vector w and the control input) can be expressed using only the flat output y and its time derivatives $w_i = \psi_i(y, \dot{y}, \ldots, y^{(\gamma_i)}),\ i = 1, \ldots, s$. An equivalent definition of differentially flat systems is as follows:

Definition: The system $\dot{x} = f(x, u)$, $x \in R^n$, $u \in R^m$ is differentially flat if there exist relations

$$\begin{aligned} &h:\ R^n \times (R^m)^{r+1} \rightarrow R^m, \\ &\phi:\ (R^m)^r \rightarrow R^n \text{ and} \\ &\psi:\ (R^m)^{r+1} \rightarrow R^m \end{aligned} \tag{2.5}$$

such that

$$\begin{aligned} &y = h(x, u, \dot{u}, \ldots, u^{(r)}), \\ &x = \phi(y, \dot{y}, \ldots, y^{(r-1)}),\ \text{and} \\ &u = \psi(y, \dot{y}, \ldots, y^{(r-1)}, y^{(r)}). \end{aligned} \tag{2.6}$$

This means that all system dynamics can be expressed as a function of the flat output and its derivatives; therefore, the state vector and the control input can be written as

$$\begin{aligned} &x(t) = \phi(y(t), \dot{y}(t), \ldots, y^{(r-1)}(t)), \text{ and} \\ &u(t) = \psi(y(t), \dot{y}(t), \ldots, y^{(r)}(t)) \end{aligned} \tag{2.7}$$

Next, an example is given to explain the design of a differentially flat controller for finite dimensional systems of known parameters.

Example 1: Flatness-based control for a nonlinear system of known parameters [263]. Consider the following model:

$$\begin{aligned} &\dot{x}_1 = x_3 - x_2 u \\ &\dot{x}_2 = -x_2 + u \\ &\dot{x}_3 = x_2 - x_1 + 2x_2(u - x_2) \end{aligned} \tag{2.8}$$

The flat output is chosen to be $y_1 = x_1 + \frac{x_2^2}{2}$. Thus one gets:

$$\begin{aligned} &y_1 = x_1 + \frac{x_2^2}{2} \\ &y_2 = \dot{y}_1 = (x_3 - x_2 u) + x_2(u - x_2) = x_3 - x_2^2 \\ &y_3 = \dot{y}_2 = \ddot{y}_1 = x_2 - x_1 + 2x_2(u - x_2) - 2x_2(u - x_2) = -x_1 + x_2 \\ &v = \dot{y}_3 = y_1^{(3)} = -x_3 + x_2 u - x_2 + u = -x_2 - x_3 + u(1 + x_2) \end{aligned} \tag{2.9}$$

It can be verified that property (1) holds, i.e., there does not exist any differential relation of the form $R(y, \dot{y}, \ldots, y^{(\beta)}) = 0$, and this implies that the derivatives of the flat output are not coupled. Moreover, it can be shown that property (2) also holds i.e., the components w of the system (elements of the system's state vector and control input) can be expressed using only the flat output y and its time derivatives $w_i = \psi_i(y, \dot{y}, \ldots, y^{(\gamma_i)}), i = i, \ldots, s$.

For instance to calculate x_1 with respect to y_1, $\dot{y}_1$, $\ddot{y}_1$ and $y_1^{(3)}$ the relation of $\ddot{y}_1$ is used, i.e.,

$$x_1^2 + 2x_1(1 + \ddot{y}_1) + \ddot{y}_1^2 - 2y_1 = 0 \tag{2.10}$$

from which two possible solutions are derived, i.e., $x_1 = -(1+\ddot{y}_1 - \sqrt{1 + 2(y_1 + \ddot{y}_1)})$ and $x_1 = -(1 + \ddot{y}_1 + \sqrt{1 + 2(y_1 + \ddot{y}_1)})$. Keeping the biggest out of these two solutions one obtains:

$$\begin{aligned} x_1 &= -(1 + \ddot{y}_1) + \sqrt{1 + 2(y_1 + \ddot{y}_1)} \\ x_2 &= \ddot{y}_1 + x_1 \\ x_3 &= \dot{y}_1 + \ddot{y}_1^2 + 2x_1\ddot{y}_1 + x_1^2 \\ u &= \frac{y_1^{(3)} + \ddot{y}_1^2 + \ddot{y}_1 + \dot{y}_1 + x_1 + 2x_1\ddot{y}_1 + x_1^2}{1 + x_1 + \ddot{y}_1} \end{aligned} \tag{2.11}$$

The computation of the equivalent model of the system in the linear canonical form is summarized as follows: By finding the derivatives of the flat output one gets a set of equations which can be solved with respect to the state variables and the control input of the initial state-space description of the system. First, the binomial of variable x_1 given in Eq. (2.10) is solved providing x_1 as a function of the flat output and its derivatives. Next, using the expression for x_1 and Eq. (2.11), state variable x_2 is also written as a function of the flat output and its derivatives. Finally, using the expressions for both x_1 and x_2 and Eq. (2.11), state variable x_3 is written as a function of the flat output and its derivatives. Thus one can finally express the state vector elements and the control input as function of the flat output and its derivatives, which completes the proof about differential flatness of the system.

From Eq. (2.11) it can be concluded that the initial system of Eq. (2.8) is indeed differentially flat. Using the flat output and its derivatives, the system of Eq. (2.8) can be written in Brunovsky (canonical) form:

$$\frac{d}{dt}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} v \tag{2.12}$$

where the new control input is $v = f(x) + g(x)u$. Therefore, a transformation of the system into a linear equivalent description is obtained and then it is straightforward to design a controller based on linear control theory. Thus, given the reference trajectory $[x_1^*, x_2^*, x_3^*]^T$, one can find the transformed reference trajectory $[y_1^*, \dot{y}_1^*, \ddot{y}_1^*]^T$ and select the appropriate control input v that succeeds tracking of the reference setpoints. Knowing v, the control u of the initial system can be found. Knowing v the control input that is actually applied to the system is $u = g^{-1}(x)[v - f(x)]$.

It is noted that for linear systems, the property of differential flatness is equivalent to that of controllability. Next, two examples of differentially flat MIMO dynamical systems are given. It is shown that the definition of the differential flat outputs also enables to transform the system into the Brunovksy (canonical) form:

Example 2: Differential flatness of a nonlinear spring–damper–mass system which consists of two masses.

The spring–damper–mass model is described in Fig. 2.1. The dynamic equations of the model are given by [89]

$$\begin{aligned} M_1\ddot{x}_1 &= -f_{K_1}(x) - f_{B_1}(x) + \\ &+ f_{K_2}(x) + f_{B_2}(x) + u_1 + 0.2u_2 \\ M_2\ddot{x}_2 &= -f_{K_2}(x) - f_{B_2}(x) + 0.25u_1 + u_2 \end{aligned} \tag{2.13}$$

where M_1 and M_2 are the masses of the system, $x(t) = [x_1, \dot{x}_1, x_2, \dot{x}_2]^T$ is the state vector which has as elements the positions and the velocities of the two masses, and $f_{K_1}(x)$ and $f_{K_2}(x)$ are spring forces defined by the following equations

$$\begin{aligned} f_{K_1}(x) &= K_{10} + \Delta K_1 x_1^3 \\ f_{K_2}(x) &= K_{20} + \Delta K_2 (x_2 - x_1)^3 \end{aligned} \tag{2.14}$$

$f_{B_1}(x)$ and $f_{B_2}(x)$ are friction forces which are defined as

$$\begin{aligned} f_{B_1}(x) &= b_{10}\dot{x}_1 + \Delta b_1 \dot{x}_1^2 \\ f_{B_2}(x) &= b_{20}(\dot{x}_2 - \dot{x}_1) + \Delta b_2 (\dot{x}_2 - \dot{x}_1)^2 \end{aligned} \tag{2.15}$$

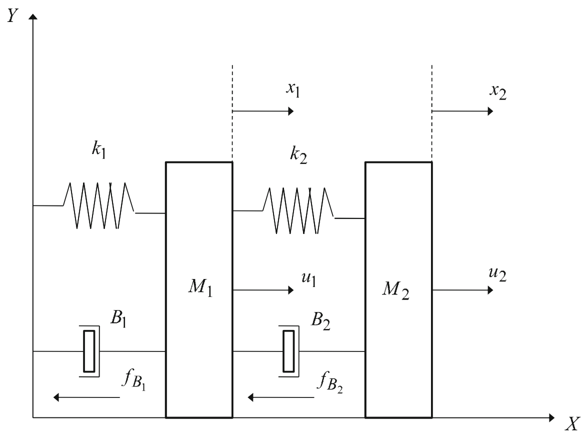

Fig. 2.1 A spring–damper–mass system consisting of two masses

The following flat outputs are defined

$$y_1 = x_1 \;\; y_2 = x_2 \tag{2.16}$$

Obviously, it holds

$$\begin{array}{c} x_1 = y_1 \;\; x_2 = y_2 \\ \dot{x}_1 = \dot{y}_1 \;\; \dot{x}_2 = \dot{y}_2 \end{array} \tag{2.17}$$

Thus, it is observed that the state variables of the spring-damper-mass model can be expressed as functions of the flat outputs and of the associated derivatives. Moreover, the following relations can be obtained about the control inputs of the model

$$\begin{array}{l} M_1\ddot{x}_1 = \\ -(K_{10}x_1 + \Delta K_1 x_1^3) - (b_{10}\dot{x}_1 + \Delta b_1 \dot{x}_1^2) + \\ +(K_{20}(x_2 - x_1) + \Delta K_2(x_2 - x_1)^3) + \\ +(b_{20}(\dot{x}_2 - \dot{x}_1) + \Delta b_2(\dot{x}_2 - \dot{x}_1)^2) + u_1 + 0.2u_2 \end{array} \tag{2.18}$$

$$\begin{array}{l} M_2\ddot{x}_2 = \\ -(K_{20}(x_2 - x_1) + \Delta K_2(x_2 - x_1)^3) - (b_{20}(\dot{x}_2 - \dot{x}_1) + \\ \Delta b_2(\dot{x}_2 - \dot{x}_1)^2 + 0.25u_1 + u_2 \end{array} \tag{2.19}$$

Using the definition of the flat outputs in Eqs. (2.18) and (2.19), one obtains

$$\begin{array}{l} M_1\ddot{y}_1 = \\ -(K_{10}y_1 + \Delta K_1 y_1^3) - (b_{10}\dot{y}_1 + \Delta b_1 \dot{y}_1^2) + \\ +(K_{20}(y_2 - y_1) + \Delta K_2(y_2 - y_1)^3) + \\ +(b_{20}(\dot{y}_2 - \dot{y}_1) + \Delta b_2(\dot{y}_2 - \dot{y}_1)^2) + u_1 + 0.2u_2 \end{array} \tag{2.20}$$

$$\begin{array}{l} M_2\ddot{y}_2 = \\ -(K_{20}(y_2 - y_1) + \Delta K_2(y_2 - y_1)^3) - (b_{20}(\dot{y}_2 - \dot{y}_1) + \\ +\Delta b_2(\dot{y}_2 - \dot{y}_1)^2 + 0.25u_1 + u_2 \end{array} \tag{2.21}$$

or equivalently

$$\begin{array}{l} M_1\ddot{y}_1 + (K_{10}y_1 + \Delta K_1 y_1^3) + (b_{10}\dot{y}_1 + \\ +\Delta b_1 \dot{y}_1^2) - (K_{20}(y_2 - y_1) + \Delta K_2(y_2 - y_1)^3) - \\ -(b_{20}(\dot{y}_2 - \dot{y}_1) + \Delta b_2(\dot{y}_2 - \dot{y}_1)^2) = u_1 + 0.2u_2 \end{array} \tag{2.22}$$

$$\begin{array}{l} M_2\ddot{y}_2 + (K_{20}(y_2 - y_1) + \Delta K_2(y_2 - y_1)^3) + \\ +(b_{20}(\dot{y}_2 - \dot{y}_1) + \Delta b_2(\dot{y}_2 - \dot{y}_1)^2 = \\ = 0.25u_1 + u_2 \end{array} \tag{2.23}$$

Defining the matrix of the coefficients of the control inputs and its inverse as

$$A = \begin{pmatrix} 1 & 0.2 \\ 0.25 & 1 \end{pmatrix} \ A^{-1} = \begin{pmatrix} 1.0526 & -0.2105 \\ -0.2632 & 1.0526 \end{pmatrix} \tag{2.24}$$

one obtains the following relation about the control inputs and the flat outputs defined in Eq. (2.16)

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1.0526 & -0.2105 \\ -0.2632 & 1.0526 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \end{pmatrix} \tag{2.25}$$

where $q_1 = M_1\ddot{y}_1 + (k_{10}y_1 + \Delta K_1 y_1^3) + (b_{10}\dot{y}_1 + \Delta b_1\dot{y}_1^2) + (K_{20}(y_2 - y_1) + \Delta K_2(y_2 - y_1)^3) + (b_{20}(\dot{y}_2 - \dot{y}_1) + \Delta b_2(\dot{y}_2 - \dot{y}_1)^2)$ and $q_2 = M_2\ddot{y}_2 + (K_{20}(y_2 - y_1) + \Delta K_2(y_2 - y_1)^3) + (b_{20}(\dot{y}_2 - \dot{y}_1) + \Delta b_2(\dot{y}_2 - \dot{y}_1)^2)$.

The previous relation between the state variables of the system and the flat outputs enables to write the system in the Brunovsky (canonical) form

$$\begin{aligned} &\ddot{y}_1 = \\ &\tfrac{1}{M_1}\{-(K_{10}y_1 + \Delta K_1 y_1^3) - (b_{10}\dot{y}_1 + \Delta b_1\dot{y}_1^2)\} + \\ &\tfrac{1}{M_1}\{(K_{20}(y_2 - y_1) + \Delta K_2(y_2 - y_1)^3) + \\ &+b_{20}(\dot{y}_2 - \dot{y}_1) + \Delta b_2(\dot{y}_2 - \dot{y}_1^2)\} + \tfrac{1}{M_1}u_1 + 0.2\tfrac{1}{M_1}u_2 \end{aligned} \tag{2.26}$$

$$\begin{aligned} &\ddot{y}_2 = \\ &\tfrac{1}{M_2}\{-(K_{20}(y_2 - y_1) + \Delta K_2(y_2 - y_1)^3) - \\ &-(b_{20}(\dot{y}_2 - \dot{y}_1) + \Delta b_2(\dot{y}_2 - \dot{y}_1)^2)\} + 0.25\tfrac{1}{M_2}u_1 + \tfrac{1}{M_2}u_2 \end{aligned} \tag{2.27}$$

Consequently, one has

$$\ddot{y}_1 = f_1(y_1, \dot{y}_1, y_2, \dot{y}_2) + g_{11}(x)u_1 + g_{12}(x)u_2 \tag{2.28}$$

$$\ddot{y}_2 = f_2(y_1, \dot{y}_1, y_2, \dot{y}_2) + g_{21}(x)u_1 + g_{22}(x)u_2 \tag{2.29}$$

with

$$\begin{aligned} &f_1(y_1, \dot{y}_1, y_2, \dot{y}_2) = f_1(x) = \\ &\tfrac{1}{M_1}\{-(K_{10}y_1 + \Delta K_1 y_1^3) - (b_{10}\dot{y}_1 + \Delta b_1\dot{y}_1^2)\} + \\ &\tfrac{1}{M_1}\{(K_{20}(y_2 - y_1) + \Delta K_2(y_2 - y_1)^3) + \\ &+b_{20}(\dot{y}_2 - \dot{y}_1) + \Delta b_2(\dot{y}_2 - \dot{y}_1)^2\} \end{aligned} \tag{2.30}$$

$$\begin{aligned} g_{11}(y_1, \dot{y}_1, y_2, \dot{y}_2) &= g_{12}(x) = \tfrac{1}{M_1} \\ g_{12}(y_1, \dot{y}_1, y_2, \dot{y}_2) &= g_{12}(x) = \tfrac{0.2}{M_1} \end{aligned} \tag{2.31}$$

$$\begin{array}{l} f_2(y_1, \dot{y}_1, y_2, \dot{y}_2) = f_2(x) = \\ \frac{1}{M_2}\{-(K_{20}(y_2 - y_1)) + \Delta K_2(y_2 - y_1)^3 - \\ -(b_{20}(\dot{y}_2 - \dot{y}_1) + \Delta b_2(\dot{y}_2 - \dot{y}_1)^2)\} \end{array} \tag{2.32}$$

$$\begin{array}{c} g_{21}(y_1, \dot{y}_1, y_2, \dot{y}_2) = g_{21}(x) = \frac{0.25}{M_2} \\ g_{22}(y_1, \dot{y}_1, y_2, \dot{y}_2) = g_{22}(x) = \frac{1}{M_2} \end{array} \tag{2.33}$$

Thus, one obtains the following canonical form description for the nonlinear system

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.34}$$

where

$$\begin{array}{c} v_1 = f_1(x) + g_{11}(x)u_1 + g_{12}(x)u_2 \\ v_2 = f_2(x) + g_{21}(x)u_1 + g_{22}(x)u_2 \end{array} \tag{2.35}$$

or equivalently

$$v = F(x) + G(x)u \tag{2.36}$$

where

$$\begin{array}{c} v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}, \quad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, F = \begin{pmatrix} f_1(x) \\ f_2(x) \end{pmatrix}, \\ G = \begin{pmatrix} g_{11}(x) & g_{12}(x) \\ g_{21}(x) & g_{22}(x) \end{pmatrix} \end{array} \tag{2.37}$$

Example 3: Differential flatness of the VTOL aircraft (vertically take-off and landing aircraft).

The dynamic model of the vertically take-off and landing aircraft (Fig. 2.2) is described by the following set of equations [561]

$$\begin{array}{c} \ddot{x} = u_1 sin(\theta) - \varepsilon u_2 cos(\theta) \\ \ddot{z} = u_1 cos(\theta) + \varepsilon u_2 sin(\theta) - 1 \\ \ddot{\theta} = u_2 \end{array} \tag{2.38}$$

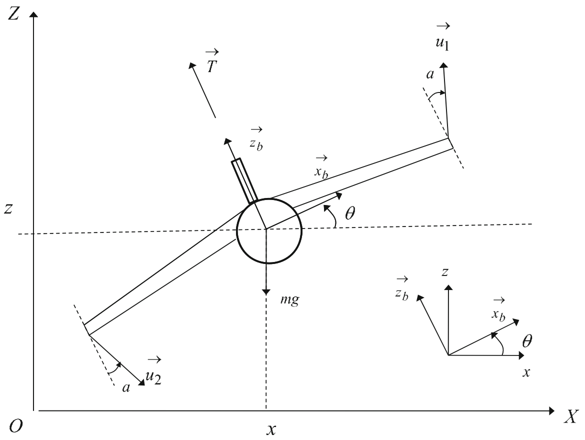

Fig. 2.2 The vertically take-off and landing aircraft model

where ε is a small parameter. Defining y_1 and y_2 two smooth functions satisfying the condition $\ddot{y}_1^2 + (\ddot{y}_2 + 1)^2 \neq 0$, e.g., $y_1 = x - \varepsilon sin(\theta)$, $y_2 = z + \varepsilon cos(\theta)$

$$
\begin{aligned}
x &= y_1 - \frac{\varepsilon \ddot{y}_1}{\sqrt{\ddot{y}_1^2 + (\ddot{y}_2+1)^2}}, \\
z &= y_2 - \frac{\varepsilon \ddot{y}_2 + 1}{\sqrt{\ddot{y}_1^2 + (\ddot{y}_2+1)^2}}, \\
\theta &= tan^{-1}\left(\frac{\ddot{y}_1}{\ddot{y}_2+1}\right)
\end{aligned}
\tag{2.39}
$$

Since all state variables are expressed as functions of the flat outputs and their derivatives, one can also write the control inputs u_1 and u_2 as functions of the flat outputs and their derivatives. This confirms that the model of the vertically take-off and landing aircraft is differentially flat. The system can be written in the Brunovsky form

$$
y_1^{(4)} = v_1, \quad y_2^{(4)} = v_2 \tag{2.40}
$$

where the control inputs v_1 and v_2 are defined as

$$
\begin{aligned}
v_1 &= \dot{\eta}_2 sin(\theta) + 2\eta_2 \dot{\theta} cos(\theta) + \\
&\quad + \eta_1 u_2 cos(\theta) - \eta_1 \dot{\theta}^2 sin(\theta) \\
v_2 &= \dot{\eta}_2 cos(\theta) - 2\eta_2 \dot{\theta} sin(\theta) - \\
&\quad - \eta_1 u_2 sin(\theta) - \eta_1 \dot{\theta}^2 cos(\theta)
\end{aligned}
\tag{2.41}
$$

with variables η_1 and η_2 being defined as

$$
\eta_1 = u_1 - \varepsilon \dot{\theta}^2, \quad \eta_2 = \dot{\eta}_1 \tag{2.42}
$$

The previous Brunovsky-form model of the vertically take-off and landing aircraft can be also written using state-space equations

$$\begin{pmatrix} \dot{y}_{1,1} \\ \dot{y}_{1,2} \\ \dot{y}_{1,3} \\ \dot{y}_{1,4} \\ \dot{y}_{2,1} \\ \dot{y}_{2,2} \\ \dot{y}_{2,3} \\ \dot{y}_{2,4} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} y_{1,1} \\ y_{1,2} \\ y_{1,3} \\ y_{1,4} \\ y_{2,1} \\ y_{2,2} \\ y_{2,3} \\ y_{2,4} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.43}$$

2.3 Properties of Differentially Flat Systems

2.3.1 Equivalence and Differential Flatness

2.3.1.1 Representation of System Dynamics as Vector Fields of Infinite Dimension

A basic property of differentially flat systems is that through a change of variables (diffeomorphism) they can be transformed to an equivalent description, which is the linear canonical (Brunovsky) form. This is analyzed next and stands for the Lie-Backlünd isomorphism to equivalence and differential flatness [153].

First a dynamical system of the form

$$\dot{x} = f(x)\ x \in X \subset R^n \tag{2.44}$$

is considered. This is described by the couple (X, f), where X is defined in R^n and f is a vector field on X. A *trajectory* is considered to be the function $t \to x(t)$ such that $\dot{x}(t) = f(x(t))\ \forall t \geq 0$. One can consider also the output mapping $x \to h(x)$ for which holds

$$\frac{d}{dt}h(x(t)) = \frac{dh}{dt}(x(t)) \cdot \dot{x}(t) = \frac{dh}{dt}(x(t)) \cdot f(x(t))\ \forall\ t \geq 0 \tag{2.45}$$

Equation (2.45) gives the total derivative, i.e., $\frac{dh}{dt}(x(t)) \cdot f(x(t))$ is also called *time derivative* of the function h, and is noted by $\dot{h}$.

Next, the notions of the total derivative and of the time derivative of function h can be generalized to the case of a nonlinear system with control input

$$\dot{x} = f(x, u) \tag{2.46}$$

where $X \times U \in R^n \times R^m$. Here, f is no longer a vector field but is an infinite collection of vector fields parameterized by u. Actually, for every u the function $x \rightarrow f_u(x) = f(x,u)$ is a vector field on X. One can also consider the case of dynamic feedback in which the system's state vector is extended by defining as state vector element the control input (and its derivatives), while the equations describing the system's dynamics are extended as follows:

$$\begin{aligned} \dot{x} &= f(x,u) \\ \dot{u} &= v \end{aligned} \tag{2.47}$$

In the latter case, in place of the state-space X one has the state-space $X \times U$. Assume now that the solution of $\dot{x} = f(x,u)$ is the function $t \rightarrow (x(t,u(t)))$ which has values in $X \times U$, such that $\dot{x}(t) = f(x(t),u(t))\ \forall\ t \geq 0$. Moreover, one can consider the infinite function

$$t \rightarrow \xi(t) = (x(t), u(t), \dot{u}(t), \ldots) \tag{2.48}$$

which takes values in $X \times U \times R_m^\infty$, where $R_m^\infty = R^m \times R^m \times \cdots$ represents an infinite product formed by vectors defined in R^m. A point of R_m^∞ is of the form $u^{(1)}, u^{(2)}, \ldots$ with $u^i \in R^m$, using that $\dot{x} = f(x,u)$. This function also satisfies the relation

$$\dot{\xi}(t) = (f(x(t), u(t)), \dot{u}(t), \ddot{u}(t), \ldots)\ \forall\, t \geq 0 \tag{2.49}$$

and this can be interpreted as a trajectory of an infinite vectors field

$$(x, u, u^{(1)}, \ldots) \rightarrow F(x, u, u^{(1)}, \ldots) = (f(x,u), u^{(1)}, u^{(2)}, \ldots,) \tag{2.50}$$

According to the above, the dynamical system $\dot{x} = f(x,u)$ is defined by the space $X \times U \times R_m^\infty$ and an infinite number of vector fields F on that space. As in the case of an autonomous dynamical system $\dot{x} = f(x)$, for the nonautonomous dynamical system $\dot{x} = f(x,u)$ one can also define the time derivative of the smooth output function $(x, u, u^{(1)}, \ldots) \rightarrow h(x, u, u^{(1)}, \ldots, u^{(k)})$, which is written as

$$\begin{aligned} &\dot{h}(x, u, u^{(1)}, \ldots, u^{(k+1)}) := Dh \cdot F = \\ &= \tfrac{\partial h}{\partial x} f(x,u) + \tfrac{\partial h}{\partial u} u^{(1)} + \tfrac{\partial h}{\partial u^1} u^{(2)} + \cdots \end{aligned} \tag{2.51}$$

If $h()$ depends on a finite number of variables then the above sum becomes finite too. For the above type of functions, one can define the so-called Fréchet topology which implies computations on smooth functions which are defined in k copies of R^m, for k being sufficiently large.

After a change of variables, a differentially flat system is written in the canonical Brunovsky form. Thus, it becomes equivalent to the trivial system R_m^∞, F_m with coordinates $y, y^{(1)}, y^{(2)}, \ldots,$ and the vector fields

$$(y^{(1)}, y^{(2)}, y^{(3)}, \ldots) = F_m(y, y^{(1)}, y^{(2)}, \ldots) \tag{2.52}$$

which describes systems composed by m chains of integrators (this is the canonical Brunovsky form for linear controllable systems).

2.3.1.2 Equivalence of Systems

Two dynamical systems are considered to be equivalents if there exists an invertible relationship which exchanges their trajectories. The definition of equivalent systems is as follows:

Definition: Two systems (M, F) and (N, G) (notation referring to state vector and vector field, respectively) are equivalent in $(p, q) \in (M, N)$, if and only if, there exists an endogenous transformation from a neighborhood of p to a neighborhood of q. The two systems (M, F) and (N, G) are equivalent if the equivalence holds for all pairs of points (p, q) of the space $M \times N$.

Using coordinates, the previous notions are expressed as follows: Considering the two systems $(X \times U \times R_m^{\infty}, F)$ and $(Y \times V \times R_s^{\infty}, G)$ describing the initial system dynamics

$$\dot{x} = f(x, u) \text{ with } (x, u) \in X \times U \subset R^n \times R^m \tag{2.53}$$

and the equivalent system dynamics

$$\dot{y} = g(y, v) \text{ with } (y, v) \in Y \times V \subset R^r \times R^s \tag{2.54}$$

The vector fields F, G are defined as

$$\begin{aligned} F(x, u, u^{(1)}, \ldots) &= (f(x, u), u^{(1)}, u^{(2)}, \ldots) \\ G(y, v, v^{(1)}, \ldots) &= (g(y, v), v^{(1)}, v^{(2)}, \ldots) \end{aligned} \tag{2.55}$$

If the two systems are equivalent, the endogenous transformation takes the form

$$\Psi(x, u, u^{(1)}, \ldots) = (\psi(x, \bar{u}), \beta(x, \bar{u}), \dot{\beta}(x, \bar{u}), \ldots) \tag{2.56}$$

where $\bar{u}$ is the abbreviated notation $\bar{u} = (u, u^{(1)}, \ldots, u^{(k)})$. Similarly, one can define the inverse endogenous transformation

$$\Phi(y, v, v^{(1)}, \ldots) = (\phi(y, \bar{v}), \alpha(y, \bar{v}), \dot{\alpha}(y, \bar{v}), \ldots) \tag{2.57}$$

Since Φ and Ψ are inverse to each other, one has

$$\begin{aligned} \psi(\phi(y, \bar{v}), \bar{\alpha}(y, \bar{v})) = y \text{ and } \phi(\psi(x, \bar{u}), \bar{\beta}(x, \bar{u})) = x \\ \beta(\phi(y, \bar{v}), \bar{\alpha}(y, \bar{v})) = v \text{ and } \alpha(\psi(x, \bar{u}), \bar{\beta}(x, \bar{u})) = u \end{aligned} \tag{2.58}$$

About vector fields F and G which are related to the derivatives of x and y, respectively, one has

$$f(\phi(y,\bar{v}),\alpha(y,\bar{v})) = D\phi(y,\bar{v})\cdot\bar{g}(y,\bar{v}) \tag{2.59}$$

where $\bar{g}$ corresponds to $(g, v^{(1)},\dots,v^{(k)})$

$$g(\psi(x,\bar{u}),\beta(x,\bar{u})) = D\psi(x,\bar{u})\cdot\bar{f}(x,\bar{u}) \tag{2.60}$$

where $\bar{f}$ corresponds to $(f, u^{(1)},\dots,u^{(k)})$. Additionally, it can be stated that if the trajectory of the first system is denoted as

$$t \rightarrow (x(t),u(t)) \tag{2.61}$$

then the trajectory of the second system becomes

$$t \rightarrow (y(t),v(t)) = (\psi(x(t),\bar{u}(t)),\beta(x(t),\bar{u}(t))) \tag{2.62}$$

Example 1: The VTOL aircraft (Vertical Take-off and Landing Aircraft) studied in Sect. 2.2.2 is revisited. State-space transformation for the VTOL model provides an example of equivalence between dynamical systems descriptions.

The initial dynamic model of the VTOL aircraft is given by

$$\begin{aligned}\ddot{x} &= -u_1 sin(\theta) + \varepsilon u_2 cos(\theta)\\ \ddot{z} &= u_1 cos(\theta) + \varepsilon u_2 sin(\theta) - 1\\ \dot{\theta} &= u_2\end{aligned} \tag{2.63}$$

This system is first shown to be globally equivalent to the transformed model

$$\begin{aligned}\ddot{y}_1 &= -\xi sin(\theta)\\ \ddot{y}_2 &= \xi cos(\theta) - 1\\ \dot{\theta} &= u_2\end{aligned} \tag{2.64}$$

where ξ and θ stand for the new control inputs. Indeed, choosing

$$\begin{aligned}X &:= (x,z,\dot{z},\dot{z},\theta,\dot{\theta}) \text{ and } Y := (y_1,y_2,\dot{y}_1,\dot{y}_2)\\ U &:= (u_1,u_2) \qquad\qquad V = (\xi,\theta)\end{aligned} \tag{2.65}$$

and with the abbreviated notations given above one can define the functions of the transformed state vector and of the transformed control input $Y = \psi(X,\bar{U})$ and $V = \beta(X,\bar{U})$, as

$$\psi(X,U) = \begin{pmatrix} x - \varepsilon sin(\theta)\\ z + \varepsilon cos(\theta)\\ \dot{x} - \varepsilon\dot{\theta}cos(\theta)\\ \dot{z} - \varepsilon\dot{\theta}sin(\theta)\end{pmatrix} \text{ and } \beta(X,\bar{U}) = \begin{pmatrix} u_1 - \varepsilon\dot{\theta}^2\\ \theta\end{pmatrix} \tag{2.66}$$

which finally enable to obtain the endogenous transformation Ψ. The inverse transformation Φ is given by $X = \phi(Y, \bar{V})$ and $U = \alpha(Y, \bar{V})$ according to the following

$$\phi(Y, \bar{V}) = \begin{pmatrix} y_1 + \varepsilon sin(\theta) \\ y_1 - \varepsilon cos(\theta) \\ \dot{y}_1 + \varepsilon\dot{\theta}cos(\theta) \\ \dot{y}_2 - \varepsilon\dot{\theta}sin(\theta) \\ \theta \\ \dot{\theta} \end{pmatrix} \text{ and } \alpha(Y, \bar{V}) = \begin{pmatrix} \xi + \varepsilon\dot{\theta}^2 \\ \ddot{\theta} \end{pmatrix} \quad (2.67)$$

Continuing with the vertical take-off and landing aircraft (VTOL) and using the dynamics of the aircraft which has been defined in Eq. (2.63), it is found that this system admits as flat output

$$y = (x - \varepsilon sin(\theta), z + \varepsilon cos(\theta)) \quad (2.68)$$

To express the state variables and the control input as functions of the flat output and its derivatives (this enables to find also the elements of the inverse transformation of Φ, that is $X = \phi(\bar{y})$ and $U = \alpha(y)$), the following implicit relations are used:

$$\begin{array}{c} (y_1 - x)^2 + (y_2 - z)^2 = \varepsilon^2 \\ (y_1 - x)(\ddot{y}_2 + 1)(y_2 - z)\ddot{y}_1 = 0 \\ (\ddot{y}_2 + 1)sin(\theta) + \ddot{y}_1 cos(\theta) = 0 \end{array} \quad (2.69)$$

Solving the above set of equations with respect to x, z, and θ, one gets

$$\begin{array}{c} x = y_1 \pm \varepsilon \frac{\ddot{y}_1}{\sqrt{\ddot{y}_1^2 + (\ddot{y}_2+1)^2}} \\ z = y_2 \pm \varepsilon \frac{(\ddot{y}_2+1)}{\sqrt{\ddot{y}_1^2 + (\ddot{y}_2+1)^2}} \\ \theta = tan^{-1}(\ddot{y}_2 + \frac{1}{\ddot{y}_1}) \end{array} \quad (2.70)$$

One has simply to differentiate so as to obtain $\dot{x}$, $\dot{z}$, $\dot{\theta}$, and u as functions of the derivatives of the flat output y. Singularity of the system is avoided if $\ddot{y}_1^2 + (\ddot{y}_2 + 1)^2 \neq 0$.

Next, the system is transformed to the linear canonical form described in Eq. (2.43) through dynamic feedback linearization. The feedback control consists of the following elements

$$\begin{array}{c} \dot{\xi} = -v_1 sin(\theta) + v_2 cos(\theta) + \xi\dot{\theta}^2 \\ u_1 = \xi + \varepsilon\dot{\theta}^2 \\ u_2 = -\frac{1}{\xi}(v_1 cos(\theta) + v_2 sin(\theta) + 2\dot{\xi}(\dot{\theta})) \end{array} \quad (2.71)$$

which transforms the system into the linear canonical form

$$\begin{aligned} y_1^{(4)} &= v_1 \\ y_2^{(4)} &= v_2 \end{aligned} \tag{2.72}$$

Thus, now one has the change of coordinates

$$(x, z, \theta, \dot{x}, \dot{z}, \dot{\theta}, \xi, \dot{\xi}) \rightarrow (y, \dot{y}, \ddot{y}, y^{(3)}) \tag{2.73}$$

The only singularity which may appear in this feedback control loop is when $\xi = 0$ that is $\ddot{y}_1^2 + (\ddot{y}_2 + 1)^2 = 0$ (which practically means that the aircraft is in free fall).

Example 2: The underactuated vessel dynamics. The model of an autonomous hovercraft provides another example about equivalence between an initial complicated nonlinear description of its dynamics and the linear canonical (Brunovsky form) [461].

The hovercraft model is obtained from the generic ship's model, after setting specific values for the elements of the inertia and Coriolis matrix and after reducing the number of the available control inputs. The state-space equation of the nonlinear underactuated hovercraft model (Fig. 2.3) is given by

$$\begin{aligned} \dot{x} &= ucos(\psi) - vsin(\psi) \\ \dot{y} &= usin(\psi) + vcos(\psi) \\ \dot{\psi} &= r \\ \dot{u} &= v \cdot r + \tau_u \\ \dot{v} &= -u \cdot r - \beta v \\ \dot{r} &= \tau_r \end{aligned} \tag{2.74}$$

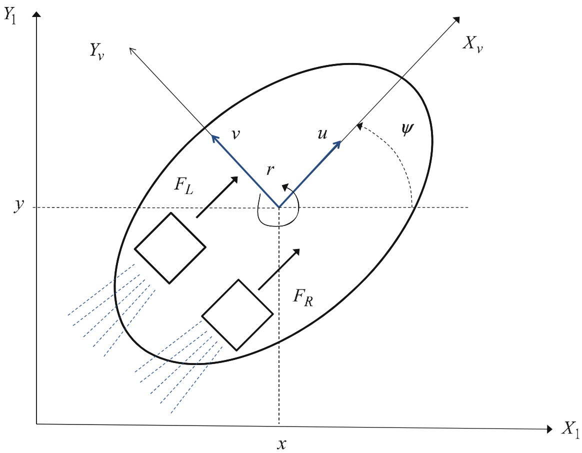

Fig. 2.3 Diagram of the underactuated hovercraft's kinematic model

where x and y are the cartesian coordinates of the vessel, ψ is the orientation angle, u is the surge velocity, v is the sway velocity, and r is the yaw rate. Coefficient β is a function of elements of the inertia matrix and hydrodynamic damping matrix of the vessel. The control inputs are the surge force τ_u and the yaw torque τ_r. The hovercraft's model is also written in the matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{u} \\ \dot{v} \\ \dot{r} \end{pmatrix} = \begin{pmatrix} ucos(\psi) - vsin(\psi) \\ usin(\psi) + vcos(\psi) \\ r \\ vr \\ -ur - \beta v \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_u \\ \tau_r \end{pmatrix} \tag{2.75}$$

or equivalently, one has the description

$$\dot{\tilde{x}} = \tilde{f}(\tilde{x}) + \tilde{g}(\tilde{x})\tilde{v} \tag{2.76}$$

The system's state vector is denoted as $\tilde{x} = [x, y, \psi, u, v, r]^T$, $f(\tilde{x}) \in R^{6\times 1}$, and $\tilde{g}(\tilde{x}) = [\tilde{g}_a, \tilde{g}_b] \in R^{6\times 2}$, while the control input is the vector $\tilde{v} = [\tau_u, \tau_r]^T$.

The system's state vector can be extended by including as additional state variables the control input τ_u and its first derivative $\dot{\tau}_u$. These are denoted as $z_1 = \tau_u$ and $z_2 = \dot{\tau}_u$. The extended state-space description of the system becomes

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{u} \\ \dot{v} \\ \dot{r} \\ \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} ucos(\psi) - vsin(\psi) \\ usin(\psi) + vcos(\psi) \\ r \\ vr + z_1 \\ -ur - \beta v \\ 0 \\ z_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \ddot{\tau}_u \\ \tau_r \end{pmatrix} \tag{2.77}$$

or equivalently, one has the description

$$\dot{z} = f(z) + g(z)\tilde{v} \tag{2.78}$$

The extended system's state vector is denoted as $z = [x, y, \psi, u, v, r, z_1, z_2]^T$. Moreover, one has $f(z) \in R^{8\times 1}$, and $g(z) = [g_a, g_b] \in R^{8\times 2}$, while the control input is the vector $\tilde{v} = [\ddot{\tau}_u, \tau_r]^T$.

It can be proven that the model of the underactuated vessel given in Eq. (2.74) is a differentially flat one. This means that all its state variables and the control inputs can be written as functions of a single variable, which is the flat output. In the hovercraft's case, the flat output is the vector of the vessel's cartesian coordinates, that is

$$\tilde{y} = [y_1, y_2] = [x, y] \tag{2.79}$$

It holds that

$$\begin{aligned}\ddot{x} &= \dot{u}cos(\psi) - u \cdot sin(\psi) \cdot \dot{\psi} - \dot{v}sin(\psi) - v \cdot cos(\psi)\dot{\psi} \\ \ddot{y} &= \dot{u}sin(\psi) + u \cdot cos(\psi) \cdot \dot{\psi} + \dot{v}cos(\psi) - v \cdot cos(\psi)\dot{\psi}\end{aligned} \tag{2.80}$$

Moreover, it holds that

$$\begin{aligned}\ddot{x} + \beta\dot{x} &= cos(\psi)(\dot{u} - v\dot{\psi} + \beta u) + sin(\psi)(-u\dot{\psi} - \dot{v} - \beta v) \\ \ddot{y} + \beta\dot{y} &= cos(\psi)(\dot{v} + u\dot{\psi} + \beta v) + sin(\psi)(-v\dot{\psi} + \dot{u} + \beta u)\end{aligned} \tag{2.81}$$

Using Eqs. (2.81) and (6.166), and after computing that

$$\begin{aligned}u\dot{\psi} + \dot{v} + \beta v &= u \cdot r - ur - \beta v + \beta v = 0 \\ \dot{u} - v\dot{\psi} + \beta u &= vr + \tau_u - vr + \beta u = \tau_u + \beta u\end{aligned} \tag{2.82}$$

one obtains

$$\begin{aligned}&\tfrac{\ddot{y}+\beta\dot{y}}{\ddot{x}+\beta\dot{x}} = \tfrac{cos(\psi)0+sin(\psi)(\tau_u+\beta u)}{\cos(\psi)(\tau_u+\beta u)-sin(\psi)0} \rightarrow \\ &\tfrac{\ddot{y}+\beta\dot{y}}{\ddot{x}+\beta\dot{x}} = tan(\psi) \rightarrow \psi = atan^{-1}(\tfrac{\ddot{y}+\beta\dot{y}}{\ddot{x}+\beta\dot{x}})\end{aligned} \tag{2.83}$$

Thus, through Eq. (2.83) it is proven that the state variable ψ (heading angle of the vessel) is a function of the flat output and of its derivatives.

From Eq. (2.81) one also has that

$$(\ddot{x} + \beta\dot{x})^2 + (\ddot{y} + \beta\dot{y})^2 = (\tau_u + \beta u)^2 \tag{2.84}$$

Moreover, it holds that

$$\begin{aligned}\dot{x}(\ddot{x} + \beta\dot{x}) &= (ucos(\psi) - vsin(\psi))cos(\psi)(\tau_u + \beta u) \\ \dot{y}(\ddot{y} + \beta\dot{y}) &= (usin(\psi) + vcos(\psi))sin(\psi)(\tau_u + \beta u)\end{aligned} \tag{2.85}$$

while using Eq. (2.84) and after intermediate computations one finally obtains

$$\dot{x}(\ddot{x} + \beta\dot{x}) + \dot{y}(\ddot{y} + \beta\dot{y}) = u(\tau_u + \beta u) \tag{2.86}$$

Dividing Eq. (2.86) with the square root of Eq. (2.84) one obtains

$$\tfrac{\dot{x}(\ddot{x}+\beta\dot{x})+\dot{y}(\ddot{y}+\beta\dot{y})}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}} = \tfrac{u(\tau_u+\beta u)}{(\tau_u+\beta u)} \tag{2.87}$$

which finally give

$$u = \tfrac{\dot{x}(\ddot{x}+\beta\dot{x})+\dot{y}(\ddot{y}+\beta\dot{y})}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}} \tag{2.88}$$

It also holds that

$$\dot{y}\ddot{x} - \dot{x}\ddot{y} = (usin(\psi) + vcos(\psi))(\dot{u}cos(\psi) - usin(\psi)\dot{\psi} - \\ -\dot{v}sin(\psi) - vcos(\psi)\dot{\psi}) - (ucos(\psi) - vsin(\psi))(\dot{u}sin(\psi) + ucos(\psi)\dot{\psi} + \\ +\dot{v}cos(\psi) - vsin(\psi)\dot{\psi}) \tag{2.89}$$

which after intermediate computations and substitution of the derivative variables from Eq. (2.75) give

$$\dot{y}\ddot{x} - \dot{x}\ddot{y} = v(\beta u + \tau_u) \tag{2.90}$$

From Eqs. (2.90) and (2.84) one obtains

$$v = \frac{\dot{y}\ddot{x} - \dot{x}\ddot{y}}{\sqrt{(\ddot{x} + \beta\dot{x})^2 + \ddot{y} + \beta\dot{y})^2}} \tag{2.91}$$

From the state-space equations it holds that

$$r = \dot{\psi} \tag{2.92}$$

where from Eq. (2.83) one has that

$$\psi = atan^{-1}(\frac{\ddot{y} + \beta\dot{y}}{\ddot{x} + \beta\dot{x}}) \tag{2.93}$$

which means that r is also a function of the flat output and of its derivatives. This can be also confirmed analytically. Indeed from Eq. (2.93) it holds that

$$\frac{cos^2(\psi)\dot{\psi} + sin^2(\psi)\dot{\psi}}{cos^2(\psi)} = \frac{(y^{(3)} + \beta\ddot{\psi})(\ddot{x} + \beta\dot{x}) - (\ddot{y} + \beta\dot{y})(x^{(3)} + \beta\ddot{x})}{(\ddot{x} + \beta\dot{x})^2} \tag{2.94}$$

which also gives

$$\frac{\dot{\psi}}{cos^2(\psi)} = \frac{(y^{(3)} + \beta\ddot{\psi})(\ddot{x} + \beta\dot{x}) - (\ddot{y} + \beta\dot{y})(x^{(3)} + \beta\ddot{x})}{(\ddot{x} + \beta\dot{x})^2} \tag{2.95}$$

while using that

$$\frac{1}{\cos^2\psi} = tan^2(\psi) + 1 \tag{2.96}$$

one obtains that

$$\cos^2\psi = \frac{(\ddot{x} + \beta\dot{x})^2}{(\ddot{x} + \beta\dot{x})^2 + (\ddot{y} + \beta\dot{y})^2} \tag{2.97}$$

Thus, from Eqs. (2.95) and (2.92) one has that

$$r=\dot{\psi}\Rightarrow r=cos^2(\psi)\frac{(y^{(3)}+\beta\ddot{\psi})(\ddot{x}+\beta\dot{x})-(\ddot{y}+\beta\dot{y})(x^{(3)}+\beta\ddot{x})}{(\ddot{x}+\beta\dot{x})^2} \tag{2.98}$$

which after intermediate operations gives

$$r=\frac{y^{(3)}(\ddot{x}+\beta\dot{x})-x^{(3)}(\ddot{y}+\beta\dot{y})-\beta^2(\ddot{x}\dot{y}-\ddot{y}\dot{x})}{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2} \tag{2.99}$$

Equivalently, from the state-space equations one has that

$$\tau_u=\dot{u}-v\cdot r\Rightarrow\tau_u=\frac{d}{dt}\left\{\frac{\dot{x}(\ddot{x}+\beta\dot{x})+\dot{y}(\ddot{y}+\beta\dot{y})}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}}\right\}- \\ -\frac{\dot{y}\ddot{x}-\dot{x}\ddot{y}}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}}\cdot\frac{y^{(3)}(\ddot{x}+\beta\dot{x})-x^{(3)}(\ddot{y}+\beta\dot{y})-\beta^2(\ddot{x}\dot{y}-\ddot{y}\dot{x})}{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2} \tag{2.100}$$

which after intermediate operations gives

$$\tau_u=\frac{\ddot{x}(\ddot{x}+\beta\dot{x})+\ddot{y}(\ddot{y}+\beta\dot{y})}{\sqrt{(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2}} \tag{2.101}$$

Finally, for the control input τ_r it holds that $\tau_r=\dot{r}$ and using Eq. (2.99) this implies that τ_r is also a function of the flat output and of its derivatives. This can be also shown analytically according to the following:

$$\tau_r=\dot{r}\Rightarrow\tau_r= \\ \frac{y^{(4)}(\ddot{x}+\beta x)-x^{(4)}(\ddot{y}+\beta\dot{y})+\beta(y^{(3)}\ddot{x}-x^{(3)}\ddot{y})-\beta^2(x^{(3)}\dot{y}-y^{(3)}\dot{x})}{[(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2]\cdot} \\ -2\frac{[y^{(3)}(\ddot{x}+\beta\dot{x})-x^3(\ddot{y}+\beta\dot{y})-\beta^2(\ddot{x}\dot{y}-\ddot{y}\dot{x})]}{[(\ddot{x}+\beta\dot{x})^2+(\ddot{y}+\beta\dot{y})^2]^2}\cdot \\ \cdot\{(\ddot{x}+\beta\dot{x})(x^{(3)}+\beta\ddot{x})+(\ddot{y}+\beta\dot{y})(y^{(3)}+\beta\ddot{y})\} \tag{2.102}$$

Through Eq. (2.102) it is confirmed that that all state variables and the control input of the hovercraft's model can be written as functions of the flat output and of its derivatives. Consequently, the vessel's model is a differential flat one.

Next, it will be shown that a flatness-based controller can be developed for the hovercraft's model. It has been shown that it holds

$\ddot{x}=\dot{u}cos(\psi)-usin(\psi)\dot{\psi}-\dot{v}sin(\psi)-vcos(\psi)\dot{\psi}\rightarrow\ddot{x}=(vr+\tau_u)cos(\psi)-usin(\psi)r-(-ur-\beta v)sin(\psi)-vcos(\psi)r\rightarrow\ddot{x}=\tau_u cos(\psi)+\beta vsin(\psi)$

By differentiating once more with respect to time and after intermediate operations one finally obtains

$$x^{(3)} = \dot{\tau}_u cos(\psi) - \tau_u sin(\psi)r + \\ +\beta(-ur - \beta v)sin(\psi) + \beta v cos(\psi)r \tag{2.103}$$

Similarly one has

$\ddot{y} = \dot{u}sin(\psi) + ucos(\psi)\dot{\psi} + \dot{v}cos(\psi) - vsin(\psi)\dot{\psi} \rightarrow \ddot{y} = (vr + \tau_u)sin(\psi) + ucos(\psi)r + (-ur - \beta v)cos(\psi) - vsin(\psi)r \rightarrow \ddot{y} = \tau_u sin(\psi) - \beta v cos(\psi)$

As explained in Eq. (2.104), the state vector of the system is extended so as to include as new state variables the control input τ_u and its first derivative $\dot{\tau}_u$. The new state variables are denoted as $z_1 = \tau_u$ and $\dot{z_1} = \dot{\tau}_u$. Using that the extended state-space description of the system is

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\psi} \\ \dot{u} \\ \dot{v} \\ \dot{r} \\ \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} ucos(\psi) - vsin(\psi) \\ usin(\psi) + vcos(\psi) \\ r \\ vr + z_1 \\ -ur - \beta v \\ 0 \\ z_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \ddot{\tau}_u \\ \tau_r \end{pmatrix} \tag{2.104}$$

or equivalently, one has the description

$$\dot{z} = f(z) + g(z)\tilde{v} \tag{2.105}$$

The system's state vector is again denoted as $z = [x, y, \psi, u, v, r, z_1, z_2]^T$, $f(z) \in R^{8\times 1}$, and $g(z) = [g_a, g_b] \in R^{8\times 2}$, while the control input is the vector $\tilde{v} = [\ddot{\tau}_u, \tau_r]^T$.

The extended state-space description of the system given in Eq. (2.104) or in its compact form described by Eq. (2.105), is used. By differentiating once more with respect to time and after intermediate operations one finally obtains

$$y^{(3)} = z_2 sin(\psi) + z_1 cos(\psi)r + \beta ur cos(\psi) + \\ +\beta^2 v cos(\psi) + \beta v sin(\psi)r \tag{2.106}$$

It can be noticed that in the equations of the third-order derivatives for both x and y only the control input τ_u and its derivative $\dot{\tau}_u$ appear, while the control input τ_r is missing. Therefore, differentiation of $x^{(3)}$ once more with respect to time is performed. This gives

$$x^{(4)} = \ddot{\tau}_u cos(\psi) - 2z_2 sin(\psi)r - z_1 cos(\psi)r^2 - z_1 sin(\psi)\tau_r - \beta vr^2 sin(\psi) - \beta z_1 r sin(\psi) - \beta u\tau_r sin(\psi) - \beta ur^2 cos(\psi) + \beta^2 ur sin(\psi) - \beta^3 v sin(\psi) - \beta^2 vr cos(\psi) - \beta ur^2 cos(\psi) + \beta^2 vr cos(\psi) - \beta vr^2 sin(\psi) + \beta v cos(\psi)\tau_r$$

while after substituting the time derivative according to Eq. (2.74) and after regrouping terms one obtains a description of the form

$$\begin{aligned} x^{(4)} = [&-2z_2 sin(\psi)r - z_1 cos(\psi)r^2 - \beta vr^2 sin(\psi) - \beta z_1 r sin(\psi) - \beta ur^2 cos(\psi)+ \\ &+\beta^2 ur sin(\psi) - \beta^3 v sin(\psi) - \beta^2 vr cos(\psi) - \beta ur^2 cos(\psi) + \beta^2 vr cos(\psi)- \\ &-\beta vr^2 sin\psi] + [cos(\psi)]\ddot{\tau}_u + [-z_1 sin(\psi) - \beta u sin(\psi) + \beta v cos(\psi)]\tau_r \end{aligned} \tag{2.107}$$

Consequently, the fourth derivative of x is finally written in the form

$$x^{(4)} = L_f^4 y_1 + L_{g_a} L_f^3 y_1 \ddot{\tau}_u + L_{g_b} L_f^3 y_1 \tau_r \tag{2.108}$$

where

$$\begin{aligned} L_f^4 y_1 = &-2z_2 sin(\psi)r - z_1 cos(\psi)r^2 - \beta vr^2 sin(\psi) - \beta z_1 r sin(\psi)- \\ &-\beta ur^2 cos(\psi) + \beta^2 ur sin(\psi) - \beta^3 v sin(\psi) - \beta^2 vr cos(\psi) - \beta ur^2 cos(\psi)+ \\ &+\beta^2 vr cos(\psi) - \beta vr^2 sin\psi \end{aligned} \tag{2.109}$$

$$L_{g_a} L_f^3 y_1 = cos(\psi) \tag{2.110}$$

$$L_{g_b} L_f^3 y_1 = -z_1 sin(\psi) - \beta u sin(\psi) + \beta v cos(\psi) \tag{2.111}$$

In a similar manner, differentiating once more with respect to time the expression about $y^{(3)}$ one gets

$$\begin{aligned} y^{(4)} = &\dot{z}_1 cos(\psi)r - z_1 sin(\psi)\dot{\psi} r + z_1 cos(\psi)\dot{r}- \\ &-\beta \dot{v} r sin(\psi) - \beta v\dot{r}\, \sin(\psi) - \beta vr cos(\psi)\dot{\psi}- \\ &-\beta \dot{u} r cos(\psi) - \beta u\dot{r} cos(\psi) + \beta ur sin(\psi)\dot{\psi}+ \\ &+\beta^2 \dot{v} cos(\psi) - \beta^2 v sin\psi \dot{\psi}+ \\ &+\dot{z}_2 sin(\psi) + z_2 cos(\psi)\dot{\psi} \end{aligned} \tag{2.112}$$

while after substituting the time derivative according to Eq. (6.166) and after regrouping terms one obtains a description of the form

$$\begin{aligned} y^{(4)} = &[z_2 r cos(\psi) - z_1 r^2 sin(\psi) + \beta ur^2 sin(\psi) + \beta^2 vr sin(\psi) - \beta vr^2 cos(\psi)]- \\ &-\beta vr^2 cos(\psi) - \beta z_1 r cos(\psi) + \beta ur^2 sin(\psi) - \beta ur cos(\psi) + \beta^2 v cos(\psi)- \\ &-\beta^2 vr sin(\psi) + z_2 r cos(\psi)] + [sin(\psi)]\ddot{\tau}_u + [z_1 cos(\psi) - \beta v sin(\psi) - \beta u cos(\psi)]\tau_r \end{aligned} \tag{2.113}$$

Thus $y^{(4)}$ can be also written in the form

$$y^{(4)} = L_f^4 y_2 + L_{g_a} L_f^3 y_2 \ddot{\tau}_u + L_{g_b} L_f^3 y_2 \tau_r \tag{2.114}$$

$$\begin{aligned} L_f^4 y_2 = [z_2 r cos(\psi) - z_1 r^2 sin(\psi) + \beta u r^2 sin(\psi) - \beta^2 v r sin(\psi) - \beta v r^2 cos(\psi)] - \\ -\beta v r^2 cos(\psi) - \beta z_1 r cos(\psi) + \beta u r^2 sin(\psi) - \beta u r cos(\psi) + \beta^2 v cos(\psi) - \\ -\beta^2 v r sin(\psi) + z_2 r cos(\psi)] \end{aligned} \tag{2.115}$$

and

$$L_{g_a} L_f^3 y_2 = sin(\psi) \tag{2.116}$$

$$L_{g_b} L_f^3 y_2 = z_1 cos(\psi) - \beta v sin(\psi) - \beta u cos(\psi) \tag{2.117}$$

Consequently, the aggregate input–output linearized description of the system becomes

$$\begin{aligned} x^{(4)} &= L_f^4 y_1 + L_{g_a} L_f^3 y_1 \ddot{\tau}_u + L_{g_b} L_f^3 y_1 \tau_r \\ y^{(4)} &= L_f^4 y_2 + L_{g_a} L_f^3 y_2 \ddot{\tau}_u + L_{g_b} L_f^3 y_2 \tau_r \end{aligned} \tag{2.118}$$

while by defining the new control inputs

$$\begin{aligned} v_1 &= L_f^4 y_1 + L_{g_a} L_f^3 y_1 \ddot{\tau}_u + L_{g_b} L_f^3 y_1 \tau_r \\ v_2 &= L_f^4 y_2 + L_{g_a} L_f^3 y_2 \ddot{\tau}_u + L_{g_b} L_f^3 y_2 \tau_r \end{aligned} \tag{2.119}$$

the following description for the input–output linearized hovercraft model is obtained

$$\begin{aligned} x^{(4)} &= v_1 \\ y^{(4)} &= v_2 \end{aligned} \tag{2.120}$$

For the dynamics of the linearized equivalent model of the system, the following new state variables can be defined

$$\begin{aligned} z_{1,1} = x \;\; z_{1,2} = \dot{x} \;\; z_{1,3} = \ddot{x} \;\; z_{1,4} = x^{(3)} \\ z_{2,1} = y \;\; z_{2,2} = \dot{y} \;\; z_{2,3} = \ddot{y} \;\; z_{2,4} = y^{(3)} \end{aligned} \tag{2.121}$$

and the state-space description of the system becomes

$$\begin{aligned} \dot{z} &= Az + Bv \\ z^m &= Cz \end{aligned} \tag{2.122}$$

or equivalently

$$\begin{pmatrix} \dot{z}_{1,1} \\ \dot{z}_{1,2} \\ \dot{z}_{1,3} \\ \dot{z}_{1,4} \\ \dot{z}_{2,1} \\ \dot{z}_{2,2} \\ \dot{z}_{2,3} \\ \dot{z}_{2,4} \end{pmatrix} = \begin{pmatrix} 0&1&0&0&0&0&0&0 \\ 0&0&1&0&0&0&0&0 \\ 0&0&0&1&0&0&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&0&0&0&1&0&0 \\ 0&0&0&0&0&0&1&0 \\ 0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&0&0&0 \end{pmatrix} \begin{pmatrix} z_{1,1} \\ z_{1,2} \\ z_{1,3} \\ z_{1,4} \\ z_{2,1} \\ z_{2,2} \\ z_{2,3} \\ z_{2,4} \end{pmatrix} + \begin{pmatrix} 0&0 \\ 0&0 \\ 0&0 \\ 1&0 \\ 0&0 \\ 0&0 \\ 0&0 \\ 0&1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \tag{2.123}$$

while the associated measurement equation is

$$\begin{pmatrix} z_1^m \\ z_2^m \end{pmatrix} = \begin{pmatrix} 1&0&0&0&0&0&0&0 \\ 0&0&0&0&1&0&0&0 \end{pmatrix} \begin{pmatrix} z_{1,1} \\ z_{1,2} \\ z_{1,3} \\ z_{1,4} \\ z_{2,1} \\ z_{2,2} \\ z_{2,3} \\ z_{2,4} \end{pmatrix} \tag{2.124}$$

This completes Example 2.
An important property of the endogenous transformations is that the number of control inputs in the initial and the final description of the system remains the same.

Theorem: If the two systems $(X \times U \times R_m^\infty, F)$ and $Y \times V \times R_s^\infty, G$ are equivalent, then they will have the same number of control inputs, that is $m = s$.

Proof: Consider the truncation Φ_m of Φ on $X \times U \times (R^m)^\mu$

$$\begin{array}{c} \Phi_\mu : \; X \times U \times (R^{m+k})^\mu \rightarrow Y \times V \times (R^s)^\mu \\ (x, u, u^1, \ldots, u^{k+\mu}) \rightarrow (\phi, \alpha, \dot{\alpha}, \ldots, \alpha^{(\mu)}) \end{array} \tag{2.125}$$

which means the first $\mu + 2$ blocks of the components of Ψ, while μ is a sufficiently large entity. Since Ψ is invertible, Ψ_{mu} is a submersion for all μ. Thus the dimension of the space from which Ψ has emanated is superior or equal to the dimension of the space of arrival. This is written as

$$n + m(k + \mu + 1) \geq s(\mu + 1) \; \forall \mu > 0 \tag{2.126}$$

This implies $m \geq s$. In a similar manner, it is shown that with the mapping Ψ it holds $s \geq m$. Consequently, one finally has $m = s$ which means that the number of control inputs in the two descriptions remains the same.

This definition of the equivalence is adapted to the equivalence between two so-called *diffieties*. As it will be explained in detail in Sect. 2.6, the concept of a diffiety is as follows: Variable (X, τ_x) comprising the state vector of the nonlinear system and its successive derivatives with respect to time, as well as the Lie derivatives of the mapping transformation along a vector field defined by the state vector elements and their derivatives is called a *manifold of jets* or a *diffiety*.

Given two diffieties (M, CTM) and (N, CTN) it is said that a function Ψ from M to N is Lie-Backlünd or a C-morphism, if the tangent (projection) function $T\Psi$ satisfies $T\Phi(CTM) \subset CTN$. Moreover, if Ψ has an inverse function Φ, such that $T\Psi(CTN) \subset CTM$, then Ψ is a Lie-Backlünd isomorphism or a C isomorphism. When such an isomorphism exists, the diffieties are called equivalent. Therefore, an endogenous transformation is a special type of a Lie-Bakclünd transformation which preserves the parametrization in time of the integral curves. It is also possible to introduce a more general concept which is the orbital equivalence, about the Lie-Bakclünd isomorphisms which preserve only the geometric locus of the integral curves.

2.3.1.3 Differentially Flat Systems and Equivalence to the Trivial System

Definition: The dynamical system (M, F) where M: is the state vector and F: is a vector field, is differentially flat in p (or respectively flat), if and only if it is equivalent in p (respectively equivalent) with the trivial system (therefore it can be described in the Brunovsky canonical form, as shown in the examples of Sect. 2.2.2).

The definition of equivalent systems can be used, and one can consider a differentially flat system $(X \times U \times R_m^\infty, F)$ associated with

$$\dot{x} = f(x, u) \text{ where } (x, u) \in X \times U \subset R^n \times R^m \tag{2.127}$$

By definition, the above system is equivalent to the trivial system R_n^∞, F_s, where the endogenous transformation Ψ takes the form

$$\Psi(x, u, u^{(1)}, \ldots) = (h(x, u), \dot{h}(x, u), \ddot{h}(x, u), \ldots) \tag{2.128}$$

In other terms, Ψ is the infinite extension of function $h()$. The inverse transformation of Ψ is denoted as Φ and holds

$$\Psi(\bar{y}) = (\psi(\bar{\psi}), \beta(\bar{\psi}), \dot{\beta}(\bar{\psi}) \ldots) \tag{2.129}$$

Since Φ and Ψ are inverse applications it holds that

$$\phi(\bar{h}(x, \bar{u})) = x \text{ and } \alpha(\bar{h}(x, \bar{u})) = u \tag{2.130}$$

Moreover, $t \rightarrow y(t)$ is a trajectory of $y = v$ and

$$t \rightarrow (x(t), u(t)) = (\psi(\bar{\psi}(t)), \beta(\bar{\psi}(t))) \tag{2.131}$$

is a trajectory of $\dot{x} = f(x, u)$.

Definition: Assume that $\{M, F\}$ is a system and that Ψ is the transformation that brings it to the canonical form. The first component of Ψ is considered to be the flat output.

Another important property of differentially flat systems is that the dimension of the flat output is equal to the number of the control inputs (and thus linearization and decoupling is succeeded). However, it is noted that the flat output is not unique. If y is a flat output, then $\gamma(y)$ is also a flat output, provided that γ is a smooth diffeomorphism. For single-input systems, the flat output is a scalar $s = m = 1$. It is easy to show that one can pass from one flat output to another by using only a static change of variables. The multivariable case is clearly more complicated. For example, for $s = m = 2$ it is easy to see that if y_1, y_2 is a flat output then $(y_1, y_2 + \gamma(y_1^{(r)}))$ is also a flat output, provided that $\gamma()$ is smooth function and $r \geq 0$. Since the flat output of a system is not unique, it is preferable to select the flat output which leads to simple computation for the linearization of the system and the design of the feedback controller.

2.3.2 *Differential Flatness and Trajectory Planning*

2.3.2.1 Applications of Differential Flatness in Trajectory Planning

Next, it will be shown how differential flatness can be used to solve the problem of trajectory planning (setpoints definition) for dynamical systems control. Assume the following nonlinear system in which the control input u does not implement feedback control

$$\dot{x} = f(x, u), \quad x \in R^n \;\; u \in R^m \tag{2.132}$$

with flat output $y = h(x, u, \dot{u}, \ldots, u^{(r)})$. In such a manner, the system's trajectories $(x(t), u(t))$ are written as a function of the flat output y and its derivatives $x = \phi(y, \dot{y}, \ldots, y^{(q)})$ and $u = \alpha(y, \dot{y}, \ldots, y^{(q)})$. The trajectory planning problem contains a start and an arrival state. It is assumed that the flat output is a vector with elements y_i which are written as

$$y_i(t) = \sum\nolimits_j A_{ij}\lambda_j(t) \tag{2.133}$$

where $\lambda_j, j = 1, \ldots, N$ are basis functions. The problem of definition of the flat output components y_i becomes equivalent to finding its projections in the space spanned by the basis functions λ_i.

Next, it is assumed that the initial state of the system is x_0 at t_0, while the final state of the system is x_f at t_f. The coefficients A_{ij} should satisfy the following

$$\begin{array}{cc} y_i(t_0) = \sum_j A_{ij}\lambda_j(t_0) & y_i(t_f) = \sum_j A_{ij}\lambda_j(t_f) \\ \cdots \quad\quad\quad \cdots & \cdots \\ y_i^{(q)}(t_0) = \sum_j A_{ij}\lambda_j^{(q)}(t_0) & y_i^{(q)}(t_f) = \sum_j A_{ij}\lambda_j^{(q)}(t_f) \end{array} \tag{2.134}$$

Without loss of generality it assumed that the vector of the flat output is of dimension $i = 1$, that is $y = [y_1]$ (these results can be also generalized to the multidimensional case). The following vectors are defined

$$\begin{array}{c} \tilde{y}_0 = (y_1(\tau_0), \ldots, y_1^{(q)}(\tau_0)) \\ \tilde{y}_f = (y_1(\tau_f), \ldots, y_1^{(q)}(\tau_f)) \\ \tilde{y} = (\tilde{y}_0, \tilde{y}_f) \end{array} \tag{2.135}$$

Using Eqs. (2.135) and (2.134) one has that

$$\tilde{y} = \begin{pmatrix} \Lambda(\tau_0) \\ \Lambda(\tau_j) \end{pmatrix} A = \Lambda A \tag{2.136}$$

The elements of A should satisfy Eq. (2.136). The only condition for Eq. (2.136) to have a solution, is that matrix Λ is a full rank one. This means that the space of functions λ_j has to be sufficiently rich. It can be concluded that the path planning problem for the case of differentially flat systems ends up to linear algebra theory.

2.3.2.2 Planning Under Constraints

In this case, the objective is to find a trajectory going from point a to point b, which will satisfy the constraints $K(x, u, \ldots, u^{(r)}) \leq 0$ at each time instant. In the flat coordinates, this consists in finding $T > 0$ and a function $[0, T] \ni t \rightarrow y(t)$ with $(y, \ldots, y^{(q)})$ to be given at $t = 0$ and T and to verify that $\forall t \in [0, T]$ the constraint $K(y, \ldots, y^{(v)})(t) \leq 0$ will hold, for a certain v and a specific function K, obtained from k, ϕ, and α, where k is the constraint condition and ϕ, α are the previously defined mappings $\dot{x} = \phi(y, \bar{v})$ and $u = \alpha(y, \bar{v})$. The difficulty of this problem increases when $q = v = 0$.

Next, it is simply assumed that the initial state Y_0 and the final state Y_f are equilibria. It is also assumed that the motion from the initial to the final state satisfies the following constraint: there exists a path $[0, 1] \ni \sigma \rightarrow Y(\sigma)$ such that $Y(0)$ and $Y(1)$ correspond to start and arrival equilibria , for all $\sigma \in [0, 1]$, $K(Y(\sigma), 0, \ldots, 0) < 0$. Therefore, there exists $T > 0$ and $[0, T] \ni t \rightarrow y(t)$ which is a solution to the problem. It suffices to take $Y(\eta(t/T))$ where T is sufficiently large, and with η to be a smooth increasing function $[0, 1] \ni s \rightarrow \eta(s) \in [0, 1]$ with $\eta(0) = 0$, $\eta(1) = 1$, and $\frac{d^i\eta}{ds^i} = 0$ for $i = 1, \ldots, max(q, v)$.

The problem becomes more complicated in the case of trajectory planning for systems subjected to nonholonomic constraints.

2.3.2.3 Planning of Trajectories with Singularities

With the previous analysis it has been shown that the endogenous transformation

$$\Psi(x, u, u^{(1)}, \ldots) = (h(x, \bar{u}), \dot{h}(x, \bar{u}), \ddot{h}(x, \bar{u}), \ldots) \tag{2.137}$$

associated with the flat output $y = h(x, \bar{u})$ is defined everywhere, is smooth and invertible in a manner that always enables state variable x and control input u to be expressed as functions of the flat output and its derivatives

$$(y, \dot{y}, \ldots, y^{(q)}) \rightarrow (x, u) = \phi(y, \dot{y}, \ldots, y^{(q)}) \tag{2.138}$$

However, a point of singularity may exist in the area of the state-space into which the control tries to bring the system's state vector (y is no longer invertible and Eq. (2.138)) cannot be solved. As function ϕ cannot be defined in such a point, the previous computations become meaningless. A manner to circumvent this problem and to avoid the singularity is to choose a trajectory for the system

$$t \rightarrow \phi(y(t), \dot{y}(t), \ldots, y^{(q)}(t)) \tag{2.139}$$

such that the point of potential singularity is bypassed. This procedure is depicted in the following example:

Example: The following system with differentially flat dynamics is considered:

$$\dot{x}_1 = u_1 \;\; \dot{x}_2 = u_2 u_1 \;\; \dot{x}_3 = x_2 u_1 \tag{2.140}$$

where the flat output is defined as $y = (x_1, x_3)$. In case that $u_1 = 0$ it holds that $\dot{x}_1 = \dot{y}_1 = 0$ and a singularity arises because it holds

$$(y, \dot{y}, \ddot{y}) \overset{\phi}{\rightarrow} (x_1, x_2, x_3, u_1, u_2) = \left(y_1, \frac{\dot{y}_2}{\dot{y}_1}, y_2, \dot{y}_1, \frac{\ddot{y}_2\dot{y}_1 - \ddot{y}_1\dot{y}_2}{\dot{y}_1^3}\right) \tag{2.141}$$

and the inverse transformation ϕ is not defined at the points of singularity. However, if one considers trajectories of the following form $t \rightarrow y(t) := (\sigma(t), p(\sigma(t)))$, with σ and π being smooth functions he obtains that

$$\frac{y_2(t)}{y_1(t)} = \frac{p(\sigma(t))}{\sigma(t)} \tag{2.142}$$

$$\frac{\dot{y}_2(t)}{\dot{y}_1(t)} = \frac{\frac{dp}{d\sigma}(\sigma(t)) \cdot \dot{\sigma}(t)}{\dot{\sigma}(t)} \tag{2.143}$$

and

$$\frac{\ddot{y}_2\dot{y}_1 - \ddot{y}_1\dot{y}_2}{\dot{y}_1^3} = \frac{\frac{d^2p}{d\sigma^2}(\sigma(t)) \cdot \dot{\sigma}^3(t)}{\dot{\sigma}^3(t)} \tag{2.144}$$

Therefore, one can extend $t \to \phi(y(t), \dot{y}(t), \ddot{y}(t))$ everywhere in time using

$$t \to \left(\sigma(t), \frac{dp}{d\sigma}(\sigma(t)), p(\sigma(t)), \dot{\sigma}(t), \frac{d^2p}{d\sigma^2}(\sigma(t))\right) \tag{2.145}$$

Trajectory planning is performed in a similar manner as before: actually, functions σ and p and their derivatives are constrained at the start and at the arrival point of the trajectory, while elsewhere they are free. In this example, the avoidance of the singularity is internally related with a symmetry of the system. Equations

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2u_1,\ \dot{x}_3 = x_2u_1 \tag{2.146}$$

are linear in u_1. The transformation, $t \to \tilde{t} = \sigma(t)$ and $u_1 \to \tilde{u}_1 = \frac{u_1}{\dot{\sigma}(t)}$, where only the time t and the control input u change, leaves the equations invariants.

2.3.3 *Differential Flatness, Feedback Control and Equivalence*

2.3.3.1 Closed-Loop Systems Under State Feedback Control and Equivalence

The analysis of Sect. 2.3.2 referred to a system's dynamics $\dot{x} = f(x, u)$ in which u was not specifically implementing a feedback control action. Next, the notion of equivalence will be extended to the case of closed-loop control. To this end, the initial system dynamics $\dot{x} = f(x, u)$ and its equivalent description $\dot{y} = g(y, v)$ are considered

$$\begin{aligned} \dot{x} = f(x, u),\ \ (x, u) \in X \times U \subset R^n \times R^m \\ \dot{y} = g(y, v),\ \ (y, v) \in Y \times V \subset R^r \times R^p \end{aligned} \tag{2.147}$$

These correspond to the systems $(X \times U \times R_m^\infty, F)$ and $(Y \times V \times R_s^\infty, G)$, where F and G are defined by

$$\begin{aligned} F(x, u, u^{(1)}, \ldots) := (f(x, u), u^{(1)}, u^{(2)}, \ldots) \\ G(y, v, v^{(1)}, \ldots) := (g(y, v), v^{(1)}, v^{(2)}, \ldots) \end{aligned} \tag{2.148}$$

Considering that these systems are equivalent, they have the same trajectories. A question that arises is if a transition is possible from the system $\dot{x} = f(x, u)$ to the system $\dot{y} = g(y, v)$ by a feedback loop (primarily dynamic or alternatively static), or inversely. For the transformed state vectors z and control inputs v one has

$$\begin{gathered}\dot{z} = \alpha(x, z, v) \;\; z \in Z \subset R^q \\ u = \kappa(x, z, v)\end{gathered} \tag{2.149}$$

Next, the following theorem is introduced [338]:

Theorem: Assume that the systems $\dot{x} = f(x, u)$ and $\dot{y} = g(y, v)$ are equivalent. Thus $\dot{x} = f(x, u)$ can be transformed by a dynamic loop and change of coordinates (dynamic feedback linearization) into

$$\dot{y} = g(y, v), \;\; \dot{v} = v^{(1)}, \;\; \dot{v}^{(1)} = v^{(2)}, \ldots, \dot{v}^{(\mu)} = w \tag{2.150}$$

for an entity μ being sufficiently large. Inversely, $\dot{y} = g(y, v)$ can be transformed by a feedback loop (dynamic) and change of coordinates into

$$\dot{x} = g(x, u), \;\; \dot{u} = u^{(1)}, \;\; \dot{u}^1 = u^{(2)}, \ldots, \dot{u}^{(\ni)} = w \tag{2.151}$$

for an entity v being sufficiently large.

Proof: The notations F and G are used for the infinite dimensional vector fields which represent the two systems. The equivalence shows that there exists an invertible function

$$\Phi(y, \bar{v}) = (\phi(y, \bar{v}), \alpha(y, \bar{v}), \dot{\alpha}(y, \bar{v}), \ldots) \tag{2.152}$$

such that

$$F(\Phi(y, \bar{v})) = D\Phi(y, \bar{v}) \cdot G(y, \bar{v}) \tag{2.153}$$

Next, one sets $\bar{y} = (y, v, v^{(1)}, \ldots, v^{(\mu)})$ and $w = v^{(\mu+1)}$. For μ sufficiently large, it holds that ϕ (respectively α) depends exclusively on $\bar{y}$ (respectively on $\bar{y}$ and w). With these notations Φ is written as

$$\phi(\tilde{y}, w) = (\phi(\tilde{y}), \alpha(\tilde{y}, \bar{w}), \dot{\alpha}(\tilde{y}, \bar{w}), \ldots) \tag{2.154}$$

and Eq. (2.153) implies in particular that

$$f(\phi(\tilde{y}, \alpha(\tilde{y}, w))) = D\phi(\tilde{y})\bar{g}(\tilde{y}, w) \tag{2.155}$$

where $\bar{g} := (g, v^{(1)}, \ldots, v^{(k)})$. Since Φ is invertible, and ϕ is of full rank it can thus be completed by a function π to obtain a change of coordinates

$$\tilde{y} \rightarrow \phi(\tilde{\psi}) = (\phi(\tilde{y}), \pi(\tilde{\psi})) \tag{2.156}$$

Considering now the dynamic feedback, one has

$$\begin{aligned} u &= \alpha(\phi^{-1}(x,z),w) \\ \dot{z} &= D\pi(\phi^{-1}(x,z))\cdot\tilde{g}(\phi^{-1}(x,z),w) \end{aligned} \tag{2.157}$$

which transforms $\dot{x} = f(x,u)$ into

$$\begin{pmatrix}\dot{x}\\ z\end{pmatrix} = \tilde{f}(x,z,w) := \begin{pmatrix} f(x,\alpha(\phi^{-1}(x,z),w)) \\ D\pi(\phi^{-1}(x,z))\cdot\tilde{g}(\phi^{-1}(x,z),w)\end{pmatrix} \tag{2.158}$$

Using the previous example one has

$$\tilde{f}(\phi(\tilde{y},w)) = \begin{pmatrix} f(\phi(\tilde{y},\alpha(\tilde{y},w))) \\ D\pi(\tilde{y})\cdot\tilde{g}(\tilde{y},w)\end{pmatrix} = \begin{pmatrix} D\phi(\tilde{y} \\ D\pi(\tilde{y}))\end{pmatrix}\tilde{g}(\tilde{y},w) = D\phi(\tilde{y})\cdot\tilde{g}(\tilde{y},w) \tag{2.159}$$

Thus, $\tilde{f}$ and $\tilde{g}$ are ϕ-conjugates which complete the proof.

A generic remark is that a differentially flat system is equivalent to a system in the linear canonical (Brunovsky) form, and thus one arrives immediately at the following conclusion: A flat dynamics is linearizable by dynamic feedback and change of coordinates.

2.3.3.2 Closed-Loop Systems and Equivalence Under Endogenous Feedback

The dynamical system $\dot{x} = f(x,u)$ is considered. The feedback loop

$$\begin{aligned} u &= \kappa(x,z,w) \\ \dot{z} &= \alpha(x,z,w) \end{aligned} \tag{2.160}$$

is called endogenous since the system's dynamics in open loop $\dot{x} = f(x,u)$ is equivalent to the dynamics of the system in closed loop, the latter given by

$$\begin{aligned} \dot{x} &= f(x,\kappa(x,z,w)) \\ \dot{z} &= \alpha(x,z,w) \end{aligned} \tag{2.161}$$

The term endogenous has the meaning that variables z and w are not included in the state vector of the system (case also known as "zero dynamics"). Using the above one arrives at a more refined definition of equivalence between systems and differential flatness.

Theorem: Two dynamics $\dot{x} = f(x, u)$ and $\dot{y} = g(y, v)$ are equivalent, if and only if, $\dot{x} = f(x, u)$ can be transformed by endogenous feedback and change of coordinates.

Corollary 1: A dynamics is flat, if and only if it is linearizable by dynamic endogenous feedback and change of coordinates.

Corollary 2: Consider the system description

$$\begin{aligned} \dot{x} &= f(x, \kappa(x, z, w)) \\ \dot{z} &= \alpha(x, z, w) \end{aligned} \tag{2.162}$$

where the constituents of the dynamic endogenous feedback are

$$\begin{aligned} \dot{u} &= \kappa(x, z, w) \\ \dot{z} &= \alpha(x, z, w) \end{aligned} \tag{2.163}$$

This system can be transformed by endogenous feedback into

$$\dot{x} = f(x, u), \text{ where } \dot{u} = u^{(1)}, \ldots, \dot{u}^{(\mu)} = w \tag{2.164}$$

for μ sufficiently large (thus finally the system's control input contains multiple integral terms). This shows clearly the properties which are preserved by equivalence. The property which is maintained by introducing integral terms and change of coordinates is controllability. An endogenous feedback is actually reversible to the effect of multiple integrators.

2.3.3.3 Trajectory Tracking in Feedback Control of Differentially Flat Systems

The problem of trajectory tracking for the system $\dot{x} = f(x, u)$ consists in finding a controller u which will permit to the system state variables to track the reference setpoint $t \rightarrow (x_r(t), u_r(t))$. It is convenient to add to the open loop control a correction term Δu, which is a function of the state vector error $\Delta x = x - x_r$. For differentially flat systems, there is a systematic method to calculate Δu from the tracking error Δx.

If the dynamics admits as flat output $y = h(x, \bar{u})$, one can use the corollary that a differentially flat system is linearizable by (dynamic) feedback and change of coordinates, into the linear canonical form $y^{(\mu+1)} = w$. For the linearized equivalent model, the control input is defined as $y_r^{(\mu+1)} - K\Delta\tilde{y}$, where K is a gains vector and Δy is a vector having as elements the flat output tracking error and its derivatives up to order μ. By applying such a feedback control input, the tracking error dynamics becomes

$$\Delta y^{(\mu+1)} = -K\Delta\tilde{y} \tag{2.165}$$

where $y_r(t) := (x_r(t), \bar{u}_r(t))$ and $\tilde{y} := (y, \dot{y}, \ldots, y^{\mu})$, and Δy represents the vector of the flat output tracking error. Such a feedback control law assures asymptotic tracking. There exists an invertible transformation

$$\Phi(\bar{y}) = (\phi(\bar{y}), \alpha(\bar{y}), \dot{\alpha}(\bar{y}), \ldots) \tag{2.166}$$

which connects the infinite vector fields $F(x, \bar{u}) := (f(x, u), u, u^{(1)}, \ldots)$ and $G(\bar{y}) := (y, y^{(1)}, \ldots)$. From the proof of the theorem on the equivalence of two systems (theorem given in Sect. 2.3.3.1), the above means that for the state vector one has

$$\begin{aligned} x = \phi(\tilde{y}_r(t) + \Delta\tilde{y}) = \\ = \phi(\tilde{y}_r(t)) + R_\phi(y_r(t), \Delta\tilde{y})\Delta\tilde{y} = \\ = x_r(t) + R_\phi(y_r(t), \Delta\tilde{y})\Delta\tilde{y} \end{aligned} \tag{2.167}$$

and for the control input one has

$$\begin{aligned} u = \alpha(\tilde{y}_r(t) + \Delta\tilde{y}, -K\Delta\tilde{y}) = \\ = \alpha(\tilde{y}_r(t)) + R_\alpha(y_r^{(\mu+1)}(t), \Delta\tilde{y})\begin{pmatrix} \Delta\tilde{y} \\ -K\Delta\tilde{y} \end{pmatrix} = \\ = u_r(t) + R_\alpha(\tilde{y}_r(t), y_r^{(\mu+1)}(t), \Delta\tilde{y}, \Delta w)\begin{pmatrix} \Delta\tilde{y} \\ -K\Delta\tilde{y} \end{pmatrix} \end{aligned} \tag{2.168}$$

where the following classical result of factorization has been used

$$\begin{aligned} R_\phi(Y, \Delta Y) := \int_0^1 D\phi(Y + t\Delta Y)dt \\ R_\alpha(Y, w, \Delta Y, \Delta w) := \int_0^1 D\alpha(Y + t\Delta Y, w + t\Delta w)dt \end{aligned} \tag{2.169}$$

Since $\Delta y \rightarrow 0$ for $t \rightarrow \infty$, this means that $x \rightarrow x_r(t)$ and $u \rightarrow u_r(t)$.

2.4 Flatness-Based Control and State Feedback for Systems with Model Uncertainties

Up to now, for the systems used for analyzing differential flatness properties and flatness-based control it has been assumed that an exact dynamic model was available. Flatness-based control can be also applied to systems characterized by model uncertainties and exogenous disturbances. Denoting the coefficients vector of the system as $\theta \in R^p$, the generalized system dynamics of Eq. (2.170) is written as

$$\dot{x}(t) = f(\theta, x(t), u(t)) \tag{2.170}$$

In the generic case, the system's coefficients θ are subject to uncertainty, i.e.,

$$\theta = \theta_0 + \tilde{\theta},\ \tilde{\theta}_i \in [\underline{\theta}_i, \bar{\theta}_i],\ i = 1, \ldots, p \tag{2.171}$$

where θ_0 is the nominal value of the parameter vector. In Eq. (2.170) the vector field $f : R^p \times R^n \times R^m \to R^n$ is smooth. Denoting the flat output $y(t) \in R$ and parameters vector θ, the definition of differential flatness given in Sect. 2.2.2 is rewritten as [189, 190]

$$\begin{aligned} y &= h(\theta, x, u, \dot{u}, \ldots, u^{(r)}) \\ x &= \phi(\theta, y, \dot{y}, \ldots, y^{(r-1)}) \\ u &= \psi(\theta, y, \dot{y}, \ldots, y^{(r)}) \end{aligned} \tag{2.172}$$

where h, ϕ, and ψ are smooth functions defining mappings $h : R^p \times R^n \times (R^m)^{r+1} \to R^m$, $\phi : R^p \times (R^m)^r \to R^n$ and $\psi : R^p \times (R^m)^{r+1} \to R^m$, respectively. It is also assumed that the flat output $y(t)$ is independent of the coefficients (parameters) vector θ. Equation (2.172) shows that for every given trajectory of the flat output $t \to y(t)$ the evolution of all other variables of the system $t \to x(t)$ and $t \to u(t)$ are given without integration of any differential equation. Moreover, for a sufficiently smooth desired trajectory of the flat output $t \to y^*(t)$ Eq. (2.172) can be used to design the corresponding feed-forward control $u^*(t)$ directly for the nominal system parameters θ_0. The trajectory y^* is called the nominal trajectory, while the trajectory u^* is called the nominal control. Thus, knowing the desirable system's output, one can also find the associated flat output y^* and subsequently the control input that makes the system track the desirable trajectory is given by

$$u^*(t) = \psi(\theta_0, y^*(t), \dot{y}^*(t), \ldots, y^{*(r)}(t)) \tag{2.173}$$

which means that for each admissible nominal trajectory $y^*(t)$ there is a nominal control input u^*. The initial condition of the desired trajectory of the flat output $t \to y^*(t)$ is defined as the vector $y_0^* = [y^*(0), \dot{y}^*(0), \ldots, y^{*(r-1)}(0)]^T$. The system is *consistent* with respect to the initial conditions x_0 if $x_0 = \phi(\theta, y_0^*)$. The following theorem has been proven [189]:

Theorem: If the desired trajectory of the flat output is consistent with the initial condition x_0 and $\theta = \theta_0$, then when applying the nominal control input of Eq. (2.173) the system becomes equivalent, by a change of coordinates, to a linear system in Brunovksy (canonical) form.

Without loss of generality, a single-input single-output flat system is considered. It is easy to show that such a flat system can be represented as follows: Setting $y = [y, \dot{y}, \ldots, y^{(n-1)}]^T = [y_1, y_2, \ldots, y_n]^T$ the system can be transformed via the diffeomorphism

$$y = h(\theta, x) \tag{2.174}$$

where $h = \phi^{-1}$ with ϕ defined in Eq. (2.172), into the control normal (canonical) form

$$\begin{aligned} \dot{y}_i(t) = y_{i+1}(t)\ i \in \{1, \ldots, n-1\} \\ \dot{y}_n(t) = \alpha(\theta, y(t), u(t)) \end{aligned} \tag{2.175}$$

where $y^{(i)}$ is the $i-1$th order derivative of the flat output. From the second row of Eq. (2.175), one can see that the nth order derivative of the flat output is finally written as a nonlinear function of the parameters vector θ, of the flat output's vector $y(t)$ and of the control input $u(t)$. The system in canonical form described by Eq. (2.175) and the system of Eqs. (2.170), (2.172), and (2.173) have the same solution $t \to y(t)$.

Defining the new control input $v(t) = \alpha(\theta, y(t), u(t))$, the Brunovsky form of the transformed initial system is written as

$$\begin{aligned} \dot{y}_i(t) = y_{i+1}(t)\ i \in \{1, \ldots, n-1\} \\ \dot{y}_n(t) = v(t) \end{aligned} \tag{2.176}$$

Once the system has been written in the Brunovksy (canonical) form of Eq. (2.176), a control input that can assure tracking of any desirable trajectory $\dot{y}_n^*$ is given by

$$v = \dot{y}_n^* + K^T \bar{e} \tag{2.177}$$

where the term $K^T \bar{e}$ corresponds to an error feedback controller with an integral term and thus processes the augmented error vector $\bar{e} = [\int_0^t e_1(\tau)d\tau, e_1, e_2, \ldots, e_n]$. The feedback control term can be also of the form of a proportional-derivative controller and in the latter case processes the error vector $e = [e_1, e_2, \ldots, e_n]$, thus taking the form

$$K^T e = -\sum_{i=1}^{n} k_{n+1-i} e_i(t),\ k_{n+1-i} > 0 \tag{2.178}$$

Thus, in case that the exact values of the model's parameters θ are known and these are equal to the nominal values θ_0, and no integration of the flat output error is used in the feedback loop, i.e. the feedback control signal is of the form $K^T e = -\sum_{i=1}^{n} k_{n+1-i} e_i(t)$, the closed-loop system error vector dynamics become

$$\begin{aligned} \dot{y}_n = v \Rightarrow \dot{y}_n = \dot{y}_n^* + K^T e \Rightarrow \\ \dot{y}_n = \dot{y}_n^* - k_n e_1 - k_{n-1} e_2 + \cdots - k_1 e_n \Rightarrow \\ \dot{y}_n - \dot{y}_n^* = -k_n e_1 - k_{n-1} e_2 + \cdots - k_1 e_n \Rightarrow \\ e^{(n)} + k_1 e^{(n-1)} + k_2 e^{(n-2)} + \cdots + k_{n-1} \dot{e} + k_n e = 0 \end{aligned} \tag{2.179}$$

where $e_i = y_i - y_i^* = y^{(i-1)} - y^{(i-1)*}$. The dynamics of the flat output tracking error can be also written in a matrix form

$$\begin{pmatrix} \dot{e}_1 \\ \dot{e}_2 \\ \cdots \\ \cdots \\ \dot{e}_n \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 1 \\ -k_n & -k_{n-1} & -k_{n-2} & \cdots & -k_1 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \cdots \\ \cdots \\ e_n \end{pmatrix} \tag{2.180}$$

In case of the error dynamics of Eq. (2.180) to succeed elimination of the tracking error it suffices to select the controller gains $k_1, k_2, \ldots, k_n$ so as the polynomial $e^{(n)} + k_1 e^{(n-1)} + k_2 e^{(n-2)} + \cdots + k_{n-1}\dot{e} + k_n e = 0$ to be a Hurwitz one.

When that the model's parameters θ are not precisely known, or have deviated from their nominal values θ_0, one can apply again a control input of the form described in Eqs. (2.177) and (2.178), and for specific magnitude of the exogenous disturbances or parametric uncertainties can expect again satisfactory tracking of the reference trajectory. In such a case, the robustness features of the closed loop can be analyzed with the use of results from interval polynomial stability theory (e.g. Kharitonov's theory) or other results on interval polynomial analysis (e.g. see [431]). The problem of robustness of a flatness-based controller in case of parametric uncertainties or exogenous perturbations has been studied in [189, 190, 433, 443].

2.5 Classification of Types of Differentially Flat Systems

2.5.1 Criteria About the Differential Flatness of a System

2.5.1.1 Definition of 0-Flat Systems

The existence of a computable criterion so as to decide if the dynamical system $\dot{x} = f(x, u),\ x \in R^n,\ u \in R^m$ is differentially flat remains open. This means that there is no systematic method to construct a flat output. This situation is somehow equivalent to the definition of Lyapunov functions for nonlinear dynamical systems. For even though Lyapunov functions are both theoretically and practically important for the study of the stability features of nonlinear dynamical systems, there is no systematic manner to define them.

The main difficulty in the computation of the flat output $y = h(x, u, \ldots, y^{(r)})$ remains in its dependence to the derivatives of the control input u, up to order r which can be arbitrarily large. To know if the order of r admits an upper bound (which can be dependent on n and m) remains a question. It is not known if for a certain dimension of the state vector and for a certain dimension of the control inputs vector such a finite boundary exists. In the following it will be noted that a dynamical system is r-flat if it admits a flat output depending up to the rth order derivative of the control input. Equivalently, in a 0-flat system its flat output depends exclusively on the elements of the system's state vector.

The next example shows that such an upper bound should depend, in the best case, linearly to the dimension of the state vector. Consider the dynamical system

$$x_1^{(a_1)} = u_1 \; x_1^{(a_2)} = u_2 \; \dot{x}_3 = u_1 u_2 \tag{2.181}$$

The flat output is taken to be

$$y_1 = x_3 + \sum_{i=1}^{\alpha_1} (-1)^i x_1^{(\alpha_1 - i)} u_2^{(i-1)}, \quad y_2 = x_2 \tag{2.182}$$

This system is r-flat, with $r = min(\alpha_1, \alpha_2) - 1$. There is no flat output depending on the derivatives of u which has an order equal or less than $r - 1$.

2.5.1.2 Systems Which Can Be Linearized Static and Dynamic State Feedback

It has been shown that dynamical systems which can be linearized with static state feedback can be also transformed to the linear canonical (Brunovsky) form. Thus, such systems are differentially flat, with the flat outputs to be exclusively functions of the state vector x (0-flat system). It is noted that a differentially flat system is not always a system which can be linearized with static state feedback. There exist systems which can be linearized through dynamic feedback linearization, that is their state vector is extended by including as additional state variables the control inputs and their derivatives (see previous examples on the linearization of the VTOL and the hovercraft model, given in Sect. 2.2.2). Such systems can be also written in the linear canonical form, and thus are also differentially flat.

2.5.1.3 Systems with Only One Control Input

Single-input dynamical systems, that is

$$\dot{x} = f(x, u) \; x \in R^n, \;\; u \in R \tag{2.183}$$

can be linearized with static state feedback linearization and can be finally written in the linear canonical form. Such systems are differentially flat.

2.5.1.4 Systems Affine-in-the-Input of Codimension 1

A system of the form

$$\dot{x} = f_0(x) + \sum_{j=1}^{n-1} g_j(x) u_j \;\; x \in R^n \tag{2.184}$$

that is systems in which the number of control inputs is by one less than the dimension of the system's state vector, are 0-flat, as long as they are controllable (strongly accessible in x). The problem becomes more complicated when the system is not an affine-in-the-input one. However, the latter systems can be finally transformed into affine-in-the-input ones too.

2.5.1.5 Systems Without Drift Term

These are systems of the form $\dot{x} = \sum_{i=1}^{m} f_i(x)u_i$. Within this class of systems, one can distinguish several cases. The following theorem holds [340]

Theorem 1: Systems without drift and with two control inputs

$$\dot{x} = f_1(x)u_1 + f_2(x)u_2 \tag{2.185}$$

are flat if and only if, the generic rank of E_k is equal to $k+2$, for $k = 0, \ldots, n-2$, where $E_0 = \{f_1, f_2\}$ and $E_{k+1} = \{E_k, [E_k, E_k]\}, \quad k > 0$. A dynamical system without drift and with two control inputs is always 0-flat. Such a system can be written in the chained form (which is a precursor of the linear canonical form), through static state feedback and a change of variables. For example, for a system of the form

$$\dot{x}_1 = u_1 \; \dot{x}_2 = u_2 \; \dot{x}_3 = x_2 u_1 \; \cdots \; \dot{x}_n = x_{n-1} u_1 \tag{2.186}$$

is a differentially flat one and the flat output is $y = (x_1, x_n)$.

2.5.1.6 Systems in Triangular Form

The state-space description of systems in triangular form is as follows:

$$\begin{aligned}
\dot{x}_1 &= f_1(x_1) + g_1(x_1)x_2 \\
\dot{x}_2 &= f_2(x_1, x_2) + g_2(x_1, x_2)x_3 \\
\dot{x}_3 &= f_3(x_1, x_2, x_3) + g_3(x_1, x_2, x_3)x_4 \\
&\cdots \\
\dot{x}_i &= f_i(x_1, x_2, \ldots, x_i) + g_i(x_1, x_2, \ldots, x_i)x_{i+1} \\
&\cdots \\
\dot{x}_{n-1} &= f_{n-1}(x_1, x_2, \ldots, x_{n-1}) + g_{n-1}(x_1, x_2, \ldots, x_{n-1})x_n \\
\dot{x}_n &= f_n(x_1, x_2, \ldots, x_n) + g_n(x_1, x_2, \ldots, x_n)u
\end{aligned} \tag{2.187}$$

The flat output for such systems is $y = x_1$.

Theorem 2: Driftless systems with a state vector of dimension n and $n-2$ control inputs. The system of the form

$$\dot{x} = \sum_{i=1}^{n-2} f_i(x)u_i \tag{2.188}$$

is a differentially flat one if it is controllable (strongly accessible for almost all x). More precisely, such a system is 0-flat is n is an odd number and is a 1-flat one if n is an even number.

2.5.2 A Sufficient Condition for Showing that a System Is Not Differentially Flat

The following theorem provides a sufficient condition for showing that dynamical systems are not differentially flat [340]:

Theorem: The criterion of the regulated variety. Assume that the dynamical system $\dot{x} = f(x, u)$ is a differentially flat one. Then, the projection to the p-dimensional space of its state-space equation $p = f(x, u)$ is a subvariety regulated for all x.

This criterion signifies that the elimination of the control input u from the n equations $\dot{x} = f(x, u)$ leads to $n - m$ equations of the form $F(x, \dot{x}) = 0$ with the following property: For all (x, p) such that $F(x, p) = 0$ there exists $\alpha \in R^n$, $\alpha \neq 0$ such that

$$\forall \lambda \in R \quad F(x, p + \lambda a) = 0 \tag{2.189}$$

Proof: Consider $(\bar{x}, \bar{u}, \bar{p})$ such that $\bar{p} = f(\bar{x}, \bar{u})$. Generally, f is of the order $m = dim(u)$ with respect to u. If x is removed from the state-space description then one arrives at $n-m$ equations which are based exclusively on x and p, that is $F(x, p) = 0$. Thus one has the constraint about the trajectory around $(\bar{x}, \bar{u}): \quad F(x, \dot{x}) = 0$. If the system is differentially flat it holds that

$$x = \phi(y, \dot{y}, \ldots, y^{(q)}) \tag{2.190}$$

where variable y is the flat output. Thus, there exists $(\bar{y}, \dot{\bar{y}}, \ldots, \bar{y}^{(q)})$ such that $\bar{x} = \phi(\bar{y}, \dot{\bar{y}}, \ldots, \bar{y}^{(q)})$. Then, the following identity holds

$$F(\phi(\bar{y}, \dot{\bar{y}}, \ldots, \bar{y}^{(q)}), \phi_y \dot{y} + \phi_{\dot{y}} \ddot{y} + \cdots + \phi_{y^{(q)}} y^{(q+1)}) = 0 \tag{2.191}$$

Taking now that the derivatives of function y at $t = 0$ up to order q are given by $y^{(r)}(0) = \bar{y}^{(r)}$ $r = 0, \ldots, q$ and $y^{\alpha}(0) = \bar{y}^{q} + \xi$ where ξ is an arbitrary vector in R^n, one has the following identity for all $\xi \in R^n$

$$F(\bar{x}, \bar{x} + \phi_{y^{\alpha}}\xi) = 0 \tag{2.192}$$

with $\phi_y^{(q)}$ to be evaluated at $\bar{y}, \dot{\bar{y}}, \ldots, \bar{y}^{(q)}$. From the point $(\bar{x}, \bar{p})$ of $F(x, p) = 0$ passes the affine space which is parallel to the image of $\phi_{y^{(q)}}$, that is a space of dimension at most equal to m and at least equal to 1.

Therefore by showing that there does not exist any flat output with derivatives which are not coupled in the sense of an ODE (or equivalently it can be said that there does not exist a flat output being differentially independent), one can prove that a dynamical system is not differentially flat. This result is particularly simple. It stands, however, for an efficient method to show that certain multi-input systems are not differentially flat.

Example 1: The system

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{x}_3 = (u_1)^2 + (u_2)^3 \tag{2.193}$$

is not a differentially flat one. This is because the subvariety $p_3 = p_1^2 + p_2^3$ is not a regulated one. There does not exist an $\alpha \in R^3$, $\alpha \neq 0$ such that

$$\forall \lambda \in R \ \ p_3 + \lambda\alpha_3 = (p_1 + \lambda\alpha_1)^2 + (p_2 + \lambda\alpha_2)^3 \tag{2.194}$$

Actually, the cubic term implies that $\alpha_2 = 0$ and the quadratic term implies that $\alpha_1 = 0$ and thus one has $\alpha_3 = 0$.

2.5.3 Liouvillian and Nondifferentially Flat Systems

There are some nonflat systems which do not satisfy differential flatness properties. Such systems cannot be linearized by state feedback (neither by static feedback linearization nor by dynamic feedback linearization). For such cases, the notion of Liouvillian systems has been introduced, which can be seen as an extension of differentially flat systems.

Nonflat systems are algebraically characterized by an integer called the *defect*. This integer is well defined by introducing the notion of maximal linearizing output. Using an informal definition, a set of variables $y = (y_1, \ldots, y_m)$ is a maximal linearizing output if the number of system variables which can be expressed as a differential function of y is maximal. Therefore, the maximal linearizing output characterizes the largest subsystem which satisfies the differential flatness property. This largest subsystem is called a flat subsystem.

The defect represents the number of the system's variables which do not belong to the flat subsystem. A nonlinear system is said to be *Liouvillian* or integrable by quadratures if the elements which do not belong to the flat subsystem can be obtained by elementary integrations of the elements of the flat subsystem. In the following examples, the notion of integrability by quadratures and that of Liouvillian systems is explained [340]

Example 1:

$$\dot{x}_1 = x_2 + x_3^2,\ \dot{x}_2 = x_3,\ \dot{x}_3 = u \tag{2.195}$$

This system is not linearizable by static or dynamic state feedback. The system is of defect equal to 1 and is found to be Liouvillian (there is a state variable which does not belong to the flat subsystem). A maximum linearizing output is given by $y = x_2$ and x_1 is the only variable that does not belong to the flat subsystem. However, x_1 can be obtained from the integration $x_1 = \int y + \dot{y}^2$, that is x_1 is given by a pure integral of a differential function of y.

Example 2: The following dynamic model is considered

$$\dot{x}_1 = u_1,\ \dot{x}_2 = u_2,\ \dot{x}_3 = x_1 u_2 - x_2 u_1,\ \dot{x}_4 = x_3 u_1\ \dot{x}_5 = x_3 u_2 \tag{2.196}$$

This system is nilpotent and completely controllable since it is driftless and satisfies the Lie algebra rank condition. However, the system is not flat and cannot be transformed into the canonical form. Nevertheless, there is a subsystem within it defined by the first four equations, which is differentially flat and consequently the system's defect is equal to 1. The maximal linearizing output of the system is given by $y = (y_1, y_2)$, where $y_1 = x_1$ and $y_2 = (zx_1 - x_4)/3$, with z to be defined by $z = (x_1 x_2 - x_3)/2$. Indeed $z = \dot{y}_2/\dot{y}_1$, thus $x_4 = y_1 z - 3y_2 = \alpha(y_1, \dot{y}_1, y_2, \dot{y}_2)$, $x_3 = \dot{x}_4/\dot{y}_1 = b(y_1, \dot{y}_1, \ddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2)$, $x_2 = (2z + x_3)/y_1 = c(y_1, \dot{y}_1, \ddot{y}_1, y_2, \dot{y}_2, \ddot{y}_2)$, $u_2 = \dot{x}_2 = q(y_1, \ldots, y_1^{(3)}, y_2, \ldots, y_2^{(3)})$. Since $x_5 = \int x_3 u_2 = \int b(y_1, \ldots, \ddot{y}_2) q(y_1, \ldots, y_2^{(3)})$ one gets that the above system of Eq. (2.196) is a Liouvillian one.

2.6 Elaborated Criteria for Checking Differential Flatness

2.6.1 Implicit Control Systems on Manifolds of Jets

Up to now the Lie-Backlünd approach to equivalence and flatness of nonlinear dynamical systems has been used. The Lie-Backlünd approach states that a system is differentially flat if and only if it is equivalent to (it can be transformed to) the trivial system, that is to a system in the linear canonical Brunovsky form. There are elaborated criteria that enable to determine if a system is differential flat [286]. These state that flatness is equivalent to the property of strong closedness of the ideal 1-form representing the differentials of all possible trivializations.

For a nonlinear dynamical system $\dot{x} = f(x, u)$ (that is a system in the explicit form) which is described by a set of ordinary differential equations, its implicit description is obtained after eliminating the control input from the state-space model. This takes the form:

$$F(x, \dot{x}) = 0 \tag{2.197}$$

Next, a manifold description is introduced. Considering the solutions of Eq. (2.197) the infinite dimensional manifold is considered: $X = X \times R^n_\infty$ or equivalently $X = X \times R^n \times R^n \cdots R^n$. It is assumed that the infinite set of global coordinates X is given by

$$X = (x_1, \ldots, x_n, \dot{x}_1, \ldots, \dot{x}_n, \ddot{x}_1, \ldots, \ddot{x}_n, \ldots, x_1^{(k)}, \ldots, x_n^{(k)}) \tag{2.198}$$

Moreover, the so-called Cartan vector field is defined

$$\tau_X = \sum_{i=1}^{n} \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial}{\partial x_i^{(j)}} \tag{2.199}$$

while also the Lie derivative of function $\phi \in C^\infty(X, R)$ along vector field τ_X is introduced

$$\frac{d\phi}{dt} = L_{\tau_X}\phi = \sum_{i=1}^{n} \sum_{j \geq 0} x_i^{(j+1)} \frac{\partial \phi}{\partial x_i^{(j)}} \tag{2.200}$$

Since it holds that $\frac{d}{dt} x_i^{(j)} = L_{\tau_X} x_i^{(j)} = \dot{x}_i^{(j)} = x_i^{(j+1)}$, the Cartan vector field acts on coordinates as a shift to the right. Variable (X, τ_X), comprising the state vector of the nonlinear system and its successive derivatives with respect to time, as well as the Lie derivatives of the mapping transformation function ϕ along a vector field defined by the state vector elements and their derivatives is called *manifold of jets* of infinite order or *diffiety*.

Thus, in regular implicit control systems, one has no longer the description of the system in the state-space form but uses a description based on the state vector and its derivatives, as well as the derivatives of the mapping ϕ. Next, the definition of a regular implicit control system is given:

Definition: A regular *implicit control system* is defined as a triple (X, τ_X, F) with $X = X \times R^n_\infty$, τ_X is the trivial Cartan field on X and $F \in C^\infty(TX; R^{n-m})$, satisfying $rank(\frac{\partial F}{\partial x} = n - m)$ in a suitable open dense subset of TX.

2.6.2 The Lie-Backlünd Equivalence for Implicit Systems

The notion of the Lie-Backlünd equivalence for explicit control systems is now extended to implicit control systems (systems described by manifolds of jets or diffieties). Two regular implicit control systems are defined [286].

Definition 1: Let X, τ_X, F and Y, τ_Y, G be two regular implicit systems. These are Lie-Backlünd equivalent at $(\bar{x}_0, \bar{y}_0) \in X_0 \times Y_0$ if and only if

(i) There exist neighborhoods X_0 of $\bar{x}_0 \in X_0$ and $\bar{y}_0 \in Y_0$ and a one-to-one mapping $\Phi = (\phi_0, \phi_1, \ldots) \in C^\infty(Y_0; X_0)$ with $C^\infty(Y_0; X_0)$ inverse Ψ, satisfying $\Phi(\bar{y}_0) = \bar{x}_0$ and such that the restrictions of the trivial Cartan fields $\tau_{Y|y_0}$ are Φ-related, namely $\Phi * \tau_{Y|y_0} = \tau_{X|x_0}$

(ii) The $C^\infty(Y_0|X_0)$ inverse mapping $\Psi = (\psi_0, \psi_1, \ldots)$ is such that $\Psi(\bar{x}_0) = \bar{y}_0$ and $\Psi * \tau_{X|x_0} = \tau_{Y|y_0}$.

The mappings Φ and Ψ are called mutually inverse Lie-Backlünd isomorphisms at (x_0, y_0).

It holds that two explicit control systems are Lie-Backlünd equivalent if and only if the associated implicit control systems are Lie-Backlünd equivalent. This is analytically stated as follows:

Two implicit systems (X, τ_X, F) and (Y, τ_Y, G) are given and two vector fields f and g compatible with (X, τ_X, F) and Y, τ_Y, G, respectively are defined. The corresponding explicit systems $\dot{x} = f(x, u)$ and $\dot{y} = g(y, v)$ are differentially equivalent or Lie-Backlünd equivalent at the pair $(x_0, u_0, \dot{u}_0), \ldots$, and $(y_0, v_0, \dot{v}_0), \ldots$ (with u_0 such that $\dot{x}_0 = f(x_0, u_0)$ and v_0 such that $\dot{y}_0 = g(x_0, v_0)$), if and only if (X, τ_X, F) and (Y, τ_Y, G) are Lie-Backlünd equivalent in accordance to the previous Definition 1, at $(x_0, \dot{x}_0, \ldots)$ and $(y_0, \dot{y}_0, \ldots)$.

Next, it is pointed out once more that an explicit system is flat, if and only if, it is equivalent to the trivial system (linear canonical Brunovsky form). This enables to define also necessary and sufficient condition for implicit systems to be differentially flat.

Definition 2: The implicit system (X, τ_X, F) is flat at $\bar{x}_0$ if and only if there exists $y_0 \in R^m_\infty$, such that (X, τ_X, F) is Lie-Backlünd equivalent at $(\bar{x}_0, \bar{y}_0) \in X_0 \times R^n_\infty$ to the m-dimensional trivial implicit system $(R^m_\infty, \tau_m, 0)$. In this case, the mutually inverse Lie-Backlünd isomorphisms Φ and Ψ are called inverse trivializations.

Having denoted the implicit system as $F(x, \dot{x})$, its differential can be defined next

$$dF = \sum_{j=1}^{n} \left(\frac{\partial F_i}{\partial x_j} dx_j + \frac{\partial F_i}{\partial \dot{x}_j} d\dot{x}_j \right) \quad i = 1, \ldots, n - m \tag{2.201}$$

Moreover, one defines the smooth mapping Φ of the 1-form (vector) ω to X as follows:

$$\Phi^* \omega(\bar{y}) = \sum_{k,i} \sum_{i,j} \omega^i_j (\Phi(\bar{\psi})) \frac{\partial \phi^j_i}{\partial y^{(l)}_k} (\bar{\psi}) dy^{(l)}_k \tag{2.202}$$

where it holds that $\phi_i^j(\bar{\psi}) = x_i^{(j)}$. Then the following theorem holds about the differential flatness of the implicit system

Theorem: The implicit system (X, τ_X, F) is flat at $\bar{x}_0$, with $\bar{x}_0 \in X_0$ if and only if there exists $\bar{y}_0 \in R_\infty^m$ and a local Lie-Backlünd isomorphism Φ from R_∞^m to X_0 satisfying $\Phi(\bar{y}_0) = \bar{x}_0$ and such that $\Phi^* \cdot dF = 0$.

2.6.3 Conditions for Differential Flatness of Implicit Systems

To formulate necessary and sufficient conditions for the differential flatness of implicit systems some preliminary definitions on polynomial matrices are used. The following polynomial matrices are introduced where the indeterminate is the differential operator $\frac{d}{dt}$

$$P(F) = \frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}}\frac{d}{dt} \quad P(\phi^0) = \sum_{j=0}^{ord(\phi^0)} \frac{\partial \phi^0}{\partial y^{(j)}}\frac{d^j}{dt^j} \tag{2.203}$$

with $P(F)$ (respectively $P(\phi^0)$) to be of size $(n-m) \times n$ (respectively $n \times m$). Next, the following theorem about the Smith decomposition (or diagonal reduction) of matrices is given [286]:

Theorem: Given a matrix $M \in M_{p,q}[\frac{d}{dt}]$ (the module of $p \times q$ matrices over $\Re[\frac{d}{dt}]$), there exist matrices $V \in U_p[\frac{d}{dt}]$, such that

$$\begin{aligned} VMU &= (\Delta_p, 0_{p,q-p}) \text{ if } p \leq q \\ VMU &= \begin{pmatrix} \Delta_q \\ 0_{p-q,q} \end{pmatrix} \text{ if } p > q \end{aligned} \tag{2.204}$$

where $0_{p,q-p}$ (respectively $0_{p-q,q}$) is the $q \times (q-p)$ (respectively $(p-q) \times q$) matrix with entries that are all zeros, and with Δ_p a $p \times p$ (respectively Δ_q a $q \times q$) diagonal matrix with elements $\delta_1, \ldots, \delta_\sigma, 0, \ldots, 0$ are such that δ_i is a nonzero $\frac{d}{dt}$ polynomial for $i = 1, \ldots, \sigma$ and is divisor of δ_j for all for all $\sigma \geq j \geq i$. Moreover, Δ_p (respectively Δ_q) is unique up to multiplication by a regular diagonal matrix in $\Re^{p \times p}$ (respectively $\Re^{q \times q}$).

Since matrix $P(F)$ of the differential description $P(F) \in M_{n-m,n}[\frac{d}{dt}]$ of the implicit system admits the Smith decomposition, it can be written as

$$VP(F)U = (\Delta, 0_{n-m,m}) \tag{2.205}$$

Moreover, the definition of *hyper-regular matrices* is given:

Definition: Given a matrix $M \in M_{p,q}[\frac{d}{dt}]$, it is said that M is hyper-regular if and only if its Smith decomposition gives $(\Delta_p, 0_{p,q-p}) = (I_p, 0_{p,q-p})$ if $p \leq q$ and

$$\begin{pmatrix} \Delta_q \\ 0_{p-q,q} \end{pmatrix} = \begin{pmatrix} I_q \\ 0_{p-q,q} \end{pmatrix} \text{ if } p \geq q \tag{2.206}$$

Other definitions associated with the Smith decomposition of the differential matrix $P(F)$ of the implicit system $F(x, \dot{x})$ are as follows:

It holds that $VP(F)U = (\Delta, 0_{n-m,m})$ and matrix U is denoted as R-Smith$(P(F))$. Next one denotes

$$\hat{U} = U \begin{pmatrix} 0_{n-m,m} \\ I_m \end{pmatrix} \tag{2.207}$$

Moreover, there exists matrix Q_0 defined as

$$Q_0 = \begin{pmatrix} 0_{m,n-m} & I_m \\ I_{n-m} & 0_{n-m,m} \end{pmatrix} U^{-1} \in L - \text{Smith}(\hat{U}) \tag{2.208}$$

and the associated matrices which satisfy

$$\tilde{Q}_0 = \begin{pmatrix} I_m & 0_{m,n-m} \end{pmatrix} \tag{2.209}$$

$$\hat{Q}_0 = (0_{n-m,m}, I_{n-m}) Q_0 \tag{2.210}$$

Next, it is considered that U is the right matrix in the Smith decomposition of the implicit system's differential matrix $P(F)$, that is $U \in R - \text{Smith}(P(F))$. Then the following m-dimensional vector 1-form ω is defined

$$\omega(\bar{x}) = \begin{pmatrix} \omega_1(\bar{x}) \\ \cdots \\ \omega_m(\bar{x}) \end{pmatrix} = \tilde{Q}_0 dx|_{X_0} = \begin{pmatrix} \sum_{j=1}^n \sum_{\alpha \geq 0} Q_{1,a}^j(\bar{x}) dx_j^\alpha|_{X_0} \\ \cdots \\ \sum_{j=1}^n \sum_{\alpha \geq 0} Q_{m,a}^j(\bar{x}) dx_j^\alpha|_{X_0} \end{pmatrix} \tag{2.211}$$

Using the above one arrives at a criterion for defining flat outputs.

Definition: Variational flat outputs. The vector 1-form $\omega = \tilde{Q}_0 dx|_{X_0}$, defined in Eq. (2.211), using matrices Q_0 and $\tilde{Q}_0$ defined in Eqs. (2.229) and (2.210), respectively, is a flat output of the variational system.

The following relations are defined next:

Definition: It is said that the $\Re[\frac{d}{dt}]$-ideal T generated by $\tau_1, \ldots, \tau_\Gamma$ is strongly closed, if there exists a matrix $M \in U_m[\frac{d}{d\tau}]$ such that $d(M_\tau) = 0$.

Theorem: A necessary and sufficient condition for systems X, τ_X, F to be flat at $(\bar{x}_0, \bar{y}_0)$ (over the class of meromorphic functions) is that there exist $U \in R -$ Smith$(P(F))$ and $Q \in L =$ Smith$(\hat{U})$, with $\hat{U}$ to be given by

$$\hat{U} = U \begin{pmatrix} 0_{n-m.m} \\ I_m \end{pmatrix} \tag{2.212}$$

such that the $\Re[\frac{d}{dt}]$-ideal Ω which is generated by the 1-forms $\omega_1, \ldots, \omega_m$ defined in Eq. (2.211) to be strongly closed in X_0.

The following operator is defined

$$D(H)\kappa = d(H\kappa) - Hd\kappa \tag{2.213}$$

for all m-dimensional vector p-form κ in $\Lambda^p((X))^m$, all $p \geq 1$ and all $H \in U_m[\frac{d}{dt}]$.

Then, the following theorem holds [286]:

Theorem: The $\Re[\frac{d}{dt}]$-ideal Ω generated by the components of the vector 1-form ω defined in Eq. (2.211) is strongly closed in X_0, or equivalently the system (X, τ_X, F) is flat, if and only if, there exists $\mu \in L_1(\Lambda((X))^m)$ and a matrix $M \in U_m[\frac{d}{dt}]$ such that

$$d\omega = \mu\omega \ D(\mu) = \mu^2 \ D(M) = -M\mu \tag{2.214}$$

The condition given in Eq. (2.214) may be seen as a generalization in the framework of manifolds of jets of infinite order of the well-known moving frame structure equations. Furthermore, the general matrix solution $\mu = (\mu_i^k)_{i,k=1,\ldots,m}$ of $d\omega = \mu\omega$, with ω defined in Eq. (2.211) is given by

$$\mu_i^k = \sum_{j=1}^{m} \sum_{\alpha,\beta=0}^{ord(\mu)} (\Gamma_{i,\alpha,\beta}^{j,k} + v_{i,\alpha,\beta}^{j,k})\omega_j^{(\alpha)} \wedge \frac{d^\beta}{d\tau^\beta} \tag{2.215}$$

with

$$\begin{cases} v_{i,\alpha,\beta}^{j,k} = v_{i,\alpha,\beta}^{k,j} \ \forall i, j, k = 1, \ldots, m \ \forall \alpha, \beta = 0, \ldots, ord(\mu), \ \alpha \neq \beta \text{ or } j \neq k \\ \qquad v_{i,\alpha,\alpha}^{k,k} \text{ arbitrary } \forall i, k = 1, \ldots, m \ \forall \alpha = 0 \cdots ord(\mu) \end{cases} \tag{2.216}$$

the integer ord(μ) being arbitrary but otherwise finite and satisfying $ord(\mu) \geq ord(\Gamma)$, the $\Gamma_{i,\alpha,\beta}^{j,k}$ being given by the following relation

$$d\omega_i = \sum_{\alpha,\beta=0}^{ord(\Gamma)} \sum_{j,k=1}^{m} \Gamma_{i,\alpha,\beta}^{j,k} \omega_j^{(\alpha)} \wedge \omega_k^{(\beta)} \tag{2.217}$$

and where $v_{i,\alpha,\beta}^{j,k}$'s are meromorphic functions depending on successive derivatives of x.

Moreover, the following hold:

The $\Re[\frac{d}{dt}]$-ideal Ω generated by the components of the vector 1-form ω defined in Eq. (2.211) is strongly closed in X_0, or equivalently the system (X, τ_X, F) is flat, if and only if, there exists $\mu \in L_1(\Lambda(X)^m)$, and two matrices $M \in M_{m,m}[\frac{d}{dt}]$ and $N \in M_{m,m}[\frac{d}{dt}]$, such that

$$d\omega = \mu\omega, \ D(\mu) = \mu^2, \ D(M) = -M\mu, \ D(N) = \mu N, \ MN = NM = I \tag{2.218}$$

The procedure for testing if a system is differentially flat and for computing the associated flat outputs is:

1. Write the system in the implicit form $F(x, \dot{x})$ by eliminating the control input from its state-space equations.
2. Compute the differential matrix of the system $P(F)$.
3. Perform Smith decomposition of $P(F)$ and retain the R-Smith matrix U.
4. Compute matrix $\hat{U}$ from Eq. (2.228).
5. Compute matrices Q_0 and $\tilde{Q}_0$ from Eqs. (2.210) and (2.231), respectively.
6. Compute the vector 1-form that is described in Eq. (2.211). The elements of the ω vector are the flat outputs of the system.

If for some $ord(\mu)$, the algorithm produces an invertible M, a flat output is obtained by integration of $dy = M\omega$, which is possible since $d(M\omega) = 0$. In the opposite case, the system is not differentially flat.

2.6.4 Example of Elaborated Differential Flatness Criteria to Nonlinear Systems

An example is given next, about the application of elaborated differential flatness criteria for implicit systems.

The unicycle robot: The kinematic model of the unicycle robot is considered [286]:

$$\begin{aligned} \dot{x} &= vcos(\theta) \\ \dot{y} &= vsin(\theta) \\ \dot{\theta} &= \tfrac{v}{l}tan(\phi) \end{aligned} \tag{2.219}$$

The system is written in implicit form $F(x, \dot{x}) = 0$. It has $n = 3$ states and $m = 2$ inputs, with $n - m = 1$ and thus it is equivalent to a single implicit equation which is obtained

$$F(x, y, \theta, \dot{x}, \dot{y}, \dot{\theta}) = \dot{x}sin(\theta) - \dot{y}cos(\theta) = 0 \tag{2.220}$$

Next, one computes

$$\begin{aligned} P(F) &= \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial \dot{x}}\frac{d}{dt}\ \ \frac{\partial F}{\partial y} + \frac{\partial F}{\partial \dot{y}}\frac{d}{dt}\ \ \frac{\partial F}{\partial \theta} + \frac{\partial F}{\partial \dot{\theta}}\frac{d}{dt}\right) \Rightarrow \\ P(F) &= \left(sin(\theta)\tfrac{d}{dt},\ -cos(\theta)\tfrac{d}{dt},\ \dot{x}cos(\theta) + \dot{\psi}sin(\theta)\right) \end{aligned} \tag{2.221}$$

Setting $E = \dot{x}cos(\theta) + \dot{y}sin(\theta)$ one applied the Smith decomposition algorithm to vector P. The Smith decomposition (or diagonal reduction) of a polynomial matrix is as follows:

Given a $\mu \times \ni$ polynomial matrix A over the noncumulative ring $\Re[\frac{d}{dt}]$, there exist matrices $V \in U_{\mu}[\frac{d}{dt}]$ and $U \in U_{\ni}[\frac{d}{dt}]$, such that $VAU = (\Delta, 0)$ if $\mu \geq \ni$ where Δ is a $\mu \times \mu$ (respectively $\ni \times \ni$) diagonal matrix, with diagonal elements $\delta_1, \ldots, \delta_n$,0, …, 0 such that δ_i is a nonzero $\frac{d}{dt}$-polynomial for $i = 1, \ldots, \sigma$ and is divisor of δ_j for all $\sigma \geq j \geq 1$.

$$VAU = \begin{pmatrix} \delta \\ 0 \end{pmatrix} \tag{2.222}$$

if $\mu \geq \ni$, where Δ is a $\mu \times \mu$ (respectively $\ni \times \ni$) diagonal matrix, whose diagonal elements $\delta_1, \ldots, \delta_n$,0, …, 0 are such that δ_i is a nonzero $\frac{d}{dt}$ polynomial

Moving the last column (of degree zero) to the first place by permutation with the two others one gets

$$P(F)U_0 = \left(E, \, -cos(\theta)\frac{d}{dt}, \, sin(\theta)\frac{d}{dt}\right) \tag{2.223}$$

with

$$U_0 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \tag{2.224}$$

and next by right-multiplying $P(F)U_0$ with U_1 where

$$U_1 = \begin{pmatrix} \frac{1}{E} & \frac{cos(\theta)}{E}\frac{d}{dt} & -\frac{sin(\theta)}{E}\frac{d}{dt} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.225}$$

finally gives

$$P(F)U = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} \tag{2.226}$$

with matrix U to be computed by

$$U = U_0 U_1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ \frac{1}{E} & \frac{cos(\theta)}{E}\frac{d}{dt} & -\frac{sin(\theta)}{E}\frac{d}{dt} \end{pmatrix} \tag{2.227}$$

It also holds that vector $P(F)$ is hyper-regular and that

$$\hat{U} = U \begin{pmatrix} 0_{1,2} \\ I_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \\ \frac{cos(\theta)}{E}\frac{d}{dt} & -\frac{sin(\theta)}{E}\frac{d}{dt} \end{pmatrix} \tag{2.228}$$

with I_2 being the identity matrix in R^2. Next by defining matrix Q_0 as

$$Q_0 = \begin{pmatrix} 0_{m,n-m} & I_m \\ I_{n=m} & 0_{n-m,m} \end{pmatrix} U^{-1} \ \in \ L - \mathrm{Smith}(\hat{U}) \tag{2.229}$$

and matrices $\tilde{Q}_0$, $\hat{Q}_0$ as

$$\tilde{Q}_0 = \begin{pmatrix} I_m \ 0_{m,n-m} \end{pmatrix} Q_0 \tag{2.230}$$

$$\hat{Q}_0 = \begin{pmatrix} 0_{n-m} \ I_{n-m} \end{pmatrix} Q_0 \tag{2.231}$$

one has

$$Q_0 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ \frac{sin(\theta)}{L}\frac{d}{dt} & -\frac{cos(\theta)}{L}\frac{d}{dt} & 1 \end{pmatrix} \tag{2.232}$$

$$\tilde{Q}_0 = \begin{pmatrix} 0 \ 1 \ 0 \\ 1 \ 0 \ 0 \end{pmatrix} \tag{2.233}$$

Multiplying Q_0 by the vector $dX = [dx, dy, d\theta]^T$ the last line of this product becomes

$$\begin{gathered} Q_0 dX = \frac{1}{E}(sin(\theta)d\dot{x} - cos(\theta)d\dot{\psi} + (\dot{x}cos(\theta) + \dot{\psi}sin(\theta)d\theta)) \Rightarrow \\ Q_0 dX = \frac{1}{E}d(\dot{x}sin(\theta) - \dot{y}cos(\theta)) \Rightarrow \\ Q_0 dX = 0 \end{gathered} \tag{2.234}$$

which comes from using Eq. (2.220) about the implicit system dynamics. Next matrix $\tilde{Q}_0$ is defined as

$$\tilde{Q}_0 = (I_m, 0_{m,n-m})Q_0 \tag{2.235}$$

and by multiplying $\tilde{Q}_0$ with vector $dX = [dx, dy, d\theta]^T$ one gets

$$\tilde{Q}_0 \begin{pmatrix} dx \\ dy \\ d\theta \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \tag{2.236}$$

which finally gives $\omega_1 = dy$ and $\omega_2 = dx$. Thus, the ideal Ω that is generated by Ω_1, Ω_2 is trivially strongly closed with $M = I_2$. Consequently, one finally concludes that the flat output of the system is $y_1 = y$ and $y_2 = x$.

2.7 Distributed Parameter Systems and Their Transformation into the Canonical Form

Up to now the analysis on differential flatness properties was focused on lumped parameter systems, that is systems described by ordinary differential equations. By applying differential flatness theory it is also possible to transform into the canonical form distributed parameter systems which are described by nonlinear partial differential equations of the parabolic, elliptic and hyperbolic type. This is particularly significant for developing efficient nonlinear control and filtering methods for such a type of systems. To demonstrate this, the model of nonlinear heat diffusion PDE will be used as an example.

2.7.1 State-Space Description of a Heat Diffusion Dynamics

The following nonlinear heat diffusion equation is considered, describing the spatiotemporal variations of temperature in a surface

$$\frac{\partial\phi}{\partial t} = K\frac{\partial^2\phi}{\partial x^2} + f(\phi) + u(x,t) \tag{2.237}$$

Using the approximation for the partial derivative in the partial differential equation given in Eq. (12.38) one has

$$\frac{\partial^2\phi}{\partial x^2} \simeq = \frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} \tag{2.238}$$

and considering spatial measurements of variable ϕ along axis x at points $x_0 + i\Delta x,\ i = 1, 2, \dots, N$ one has

$$\frac{\partial\phi_i}{\partial t} = \frac{K}{\Delta x^2}\phi_{i+1} - \frac{2K}{\Delta x^2}\phi_i + \frac{K}{\Delta x^2}\phi_{i-1} + f(\phi_i) + u(x_i,t) \tag{2.239}$$

By considering the associated samples of ϕ given by $\phi_0, \phi_1, \dots, \phi_N, \phi_{N+1}$ one has

$$\begin{aligned}
\frac{\partial\phi_1}{\partial t} &= \frac{K}{\Delta x^2}\phi_2 - \frac{2K}{\Delta x^2}\phi_1 + \frac{K}{\Delta x^2}\phi_0 + f(\phi_1) + u(x_1,t) \\
\frac{\partial\phi_2}{\partial t} &= \frac{K}{\Delta x^2}\phi_3 - \frac{2K}{\Delta x^2}\phi_2 + \frac{K}{\Delta x^2}\phi_1 + f(\phi_2) + u(x_2,t) \\
\frac{\partial\phi_3}{\partial t} &= \frac{K}{\Delta x^2}\phi_4 - \frac{2K}{\Delta x^2}\phi_3 + \frac{K}{\Delta x^2}\phi_2 + f(\phi_3) + u(x_3,t) \\
&\cdots \\
\frac{\partial\phi_{N-1}}{\partial t} &= \frac{K}{\Delta x^2}\phi_N - \frac{2K}{\Delta x^2}\phi_{N-1} + \frac{K}{\Delta x^2}\phi_{N-2} + f(\phi_{N-1}) + u(x_{N-1},t) \\
\frac{\partial\phi_N}{\partial t} &= \frac{K}{\Delta x^2}\phi_{N+1} - \frac{2K}{\Delta x^2}\phi_N + \frac{K}{\Delta x^2}\phi_{N-1} + f(\phi_N) + u(x_N,t)
\end{aligned} \tag{2.240}$$

By defining the following state vector

$$x^T = \left(\phi_1, \phi_2, \ldots, \phi_N\right) \tag{2.241}$$

one obtains the state-space description

$$\begin{array}{c} \dot{x}_1 = \frac{K}{\Delta x^2}x_2 - \frac{2K}{\Delta x^2}x_1 + \frac{K}{\Delta x^2}\phi_0 + f(x_1) + u(x_1) \\ \dot{x}_2 = \frac{K}{\Delta x^2}x_3 - \frac{2K}{\Delta x^2}x_2 + \frac{K}{\Delta x^2}x_1 + f(x_2) + u(x_2) \\ \dot{x}_3 = \frac{K}{\Delta x^2}x_4 - \frac{2K}{\Delta x^2}x_3 + \frac{K}{\Delta x^2}x_2 + f(x_3) + u(x_3) \\ \cdots \\ \dot{x}_{N-1} = \frac{K}{\Delta x^2}x_N - \frac{2K}{\Delta x^2}x_{N-1} + \frac{K}{\Delta x^2}x_{N-2} + f(x_{N-1}) + u(x_{N-1}) \\ \dot{x}_N = \frac{K}{\Delta x^2}\phi_{N+1} - \frac{2K}{\Delta x^2}x_N + \frac{K}{\Delta x^2}x_{N-1} + f(x_N) + u(x_N) \end{array} \tag{2.242}$$

Next, the following state variables are defined

$$\begin{array}{c} y_{1,i} = x_i \\ y_{2,i} = \dot{x}_i \end{array} \tag{2.243}$$

and the state-space description of the system becomes as follows:

$$\begin{array}{c} \dot{y}_{1,1} = \frac{K}{\Delta x^2}y_{1,2} - \frac{2K}{\Delta x^2}y_{1,1} + \frac{K}{\Delta x^2}\phi_0 + f(y_{1,1}) + u(y_{1,1}) \\ \dot{y}_{1,2} = \frac{K}{\Delta x^2}y_{1,3} - \frac{2K}{\Delta x^2}y_{1,2} + \frac{K}{\Delta x^2}y_{1,1} + f(y_{1,2}) + u(y_{1,2}) \\ \dot{y}_{1,3} = \frac{K}{\Delta x^2}y_{1,4} - \frac{2K}{\Delta x^2}y_{1,3} + \frac{K}{\Delta x^2}y_{1,2} + f(y_{1,3}) + u(y_{1,3}) \\ \cdots \\ \cdots \\ \dot{y}_{1,N-1} = \frac{K}{\Delta x^2}y_{1,N} - \frac{2K}{\Delta x^2}y_{1,N-1} + \frac{K}{\Delta x^2}y_{1,N-2} + f(y_{1,N-1}) \\ +u(y_{1,N-1}) \\ \dot{y}_{1,N} = \frac{K}{\Delta x^2}\phi_{N+1} - \frac{2K}{\Delta x^2}y_{1,N} + \frac{K}{\Delta x}y_{1,N-1} + f(y_{1,N}) + u(y_{1,N}) \end{array} \tag{2.244}$$

The dynamical system described in Eq. (13.52) is a differentially flat one with flat output defined as the vector $\tilde{y} = [y_{1,1}, y_{1,2}, \ldots, y_{1,N}]$. Indeed all state variables can be written as functions of the flat output and its derivatives. Moreover, by defining the new control inputs

$$\begin{array}{c} v_1 = \frac{K}{\Delta x^2}\phi_0 + f(y_{1,1}) + u(y_{1,1}) \\ v_2 = f(y_{1,2}) + u(y_{1,2}) \\ v_3 = f(y_{1,3}) + u(y_{1,3}) \\ \cdots \\ v_{N-1} = f(y_{1,N-1}) + u(y_{1,N-1}) \\ v_N = \frac{K}{\Delta x^2}\phi_{N+1} + f(y_{1,N}) + u(y_{1,N}) \end{array} \tag{2.245}$$

the following state-space description is obtained

$$\begin{pmatrix} \dot{y}_{1,1} \\ \dot{y}_{1,2} \\ \cdots \\ \dot{y}_{1,N-1} \\ \dot{y}_{1,N} \end{pmatrix} = \begin{pmatrix} -\frac{2K}{\Delta x^2} & \frac{K}{\Delta x^2} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \frac{K}{\Delta x^2} & -\frac{2K}{\Delta x^2} & \frac{K}{\Delta x^2} & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{K}{\Delta x^2} & -\frac{2K}{\Delta x^2} & \frac{K}{\Delta x^2} & \cdots & 0 & 0 & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & 0 & \frac{K}{\Delta x^2} & -\frac{2K}{\Delta x^2} & \frac{K}{\Delta x^2} \\ 0 & 0 & 0 & 0 & \cdots & 0 & 0 & \frac{K}{\Delta x^2} & -\frac{2K}{\Delta x^2} \end{pmatrix} \begin{pmatrix} y_{1,1} \\ y_{1,2} \\ \cdots \\ y_{1,N-1} \\ y_{1,N} \end{pmatrix} +$$

$$+ \begin{pmatrix} 1\,0\,0 & \cdots & 0\,0 \\ 0\,1\,0 & \cdots & 0\,0 \\ 0\,0\,1 & \cdots & 0\,0 \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0\,0\,0 & \cdots & 1\,0 \\ 0\,0\,0 & \cdots & 0\,1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \cdots \\ v_{N-1} \\ v_N \end{pmatrix} \tag{2.246}$$

Assuming that all measurements from the set of points $x_j \;\; j \in [1, 2, \ldots, m]$ are available, the associated observation (measurement) equation becomes

$$\begin{pmatrix} z_1 \\ z_2 \\ \cdots \\ z_m \end{pmatrix} = \begin{pmatrix} 1\,0\,0 & \cdots & 0\,0 \\ 0\,1\,0 & \cdots & 0\,0 \\ \cdots\cdots & \cdots & \cdots\cdots \\ 0\,0\,0 & \cdots & 1\,0 \\ 0\,0\,0 & \cdots & 0\,1 \end{pmatrix} \begin{pmatrix} y_{1,1} \\ y_{1,2} \\ y_{1,3} \\ \cdots \\ y_{1,N-1} \\ y_{1,N} \end{pmatrix} \tag{2.247}$$

Thus, in matrix form one has the following state-space description of the system

$$\begin{matrix} \dot{\tilde{y}} = A\tilde{y} + Bv \\ \tilde{z} = C\tilde{y} \end{matrix} \tag{2.248}$$

Moreover, denoting $a = \frac{K}{Dx^2}$ and $b = -\frac{2K}{Dx^2}$, the initial description of the system given in Eq. (13.54) is rewritten as follows

$$\begin{pmatrix} \dot{y}_{1,1} \\ \dot{y}_{1,2} \\ \cdots \\ \dot{y}_{1,N-1} \\ \dot{y}_{1,N} \end{pmatrix} = \begin{pmatrix} b\,a\,0\,0\,0 & \cdots & 0\,0\,0\,0\,0\,0 \\ a\,b\,a\,0\,0 & \cdots & 0\,0\,0\,0\,0\,0 \\ 0\,0\,a\,b\,a & \cdots & 0\,0\,0\,0\,0\,0 \\ \cdots & \cdots & \cdots \\ 0\,0\,0\,0\,0\,0\,0 & \cdots & a\,b\,a\,0 \\ 0\,0\,0\,0\,0\,0\,0 & \cdots & 0\,0\,a\,b \end{pmatrix} \begin{pmatrix} y_{1,1} \\ y_{1,2} \\ \cdots \\ y_{1,N-1} \\ y_{1,N} \end{pmatrix}$$

$$+ \begin{pmatrix} 1\,0\,0 & \cdots & 0\,0 \\ 0\,1\,0 & \cdots & 0\,0 \\ 0\,0\,1 & \cdots & 0\,0 \\ \cdots & \cdots & \cdots \\ 0\,0\,0 & \cdots & 1\,0 \\ 0\,0\,0 & \cdots & 0\,1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \cdots \\ v_{N-1} \\ v_N \end{pmatrix} \tag{2.249}$$

2.7.2 *Differential Flatness of the Nonlinear Heat Diffusion PDE*

The previously defined state vector of the PDE model is considered without the effect of the external control input $u(x,t)$.

$$\frac{\partial \phi}{\partial t} = K \frac{\partial^2 \phi}{\partial x^2} + f(\phi) \tag{2.250}$$

The state vector is again $\tilde{Y} = [y_{1,1}, y_{1,2}, \ldots, y1, i, \ldots, y_{1,N-1}, y_{1,N}]$, where $y_{1,1} = V_1$, $y_{1,2} = V_2, \ldots, y_{1,i} = V_i, \ldots, y_{1,N-1} = V_{N-1}$ and $y_{1,N} = V_N$. It will be shown in a different manner that the state-space description of the nonlinear PDE dynamics, which has as control input only the boundary condition ϕ_0 is a differentially flat one. One has

$$\dot{y}_{1,1} = -\frac{2K}{\Delta x^2} y_{1,1} + \frac{K}{\Delta x^2}\phi_0 + f(y_{1,1}) + \frac{K}{\Delta x^2} y_{1,2} \tag{2.251}$$

$$\dot{y}_{1,2} = -\frac{2K}{\Delta x^2} y_{1,2} + \frac{K}{\Delta x^2} y_{1,1} + f(y_{1,2}) + \frac{K}{\Delta x^2} y_{1,3} \tag{2.252}$$

$$\dot{y}_{1,3} = -\frac{2K}{\Delta x^2} y_{1,3} + \frac{K}{\Delta x^2} y_{1,2} + f(y_{1,3}) + \frac{K}{\Delta x^2} y_{1,4} \tag{2.253}$$

$$\cdots\ \cdots$$

$$\dot{y}_{1,i} = -\frac{2K}{\Delta x^2} y_{1,i} + \frac{K}{\Delta x^2} y_{1,i-1} + f(y_{1,i}) + \frac{K}{\Delta x^2} y_{1,i+1} \tag{2.254}$$

$$\cdots\ \cdots$$

$$\dot{y}_{1,N-1} = -\frac{2K}{\Delta x^2} y_{1,N-1} + \frac{K}{\Delta x^2} y_{1,N-2} + f(y_{1,N-1}) + \frac{K}{\Delta x^2} y_{1,N} \tag{2.255}$$

$$\dot{y}_{1,N} = -\frac{2K}{\Delta x^2} y_{1,N} + \frac{K}{\Delta x^2} y_{1,N-1} + f(y_{1,N}) + \frac{K}{\Delta x^2} \phi_{N+1} \tag{2.256}$$

The flat output is considered to be the state variable $y_{1,N}$, which is denoted as $y = y_{1,N}$. Next, it is shown that all state variables which stand also for virtual control inputs of the system $\alpha_i = y_{1,N-i}$, can be written as functions of the flat output $y = y_{1,N}$

From Eq. (13.62) and considering that ϕ_{N+1} is constant one obtains

$$\begin{array}{c} y_{1,N-1} = \alpha_1 = \frac{1}{K/\Delta x^2}[\dot{y}_{1,N} + \frac{2K}{\Delta x^2} y_{1,N} - \frac{K}{2\Delta x^2}\phi_{N+1} - f(y_{1,N})] \\ \Rightarrow y_{1,N-1} = h_1(y, \dot{y}, \cdots) \end{array} \tag{2.257}$$

and following a similar procedure, from Eq. (13.61) one gets

$$\begin{array}{c} y_{1,N-2} = \alpha_2 = \frac{1}{K/\Delta x^2}[\dot{y}_{1,N-1} + \frac{2K}{\Delta x^2} y_{1,N-1} - \frac{K}{2\Delta x^2} y_{1,N} - f(y_{1,N-1})] \\ \Rightarrow y_{1,N-2} = h_2(y, \dot{y}, \cdots) \end{array} \tag{2.258}$$

Continuing in a similar manner, from Eq. (13.60) one obtains

$$\begin{array}{c} y_{1,i-1} = \alpha_{N-i+1} = \frac{1}{K/\Delta x^2}[\dot{y}_{1,i} + \frac{2K}{\Delta x^2} y_{1,i} - \frac{K}{2\Delta x^2} y_{1,i+1} - f(y_{1,i})] \\ \Rightarrow y_{1,i-1} = h_{N-i+1}(y, \dot{y}, \cdots) \end{array} \tag{2.259}$$

From Eq. (13.59) one obtains

$$\begin{array}{c} y_{1,2} = \alpha_{N-2} = \frac{1}{K/\Delta x^2}[\dot{y}_{1,3} + \frac{2K}{\Delta x^2} y_{1,3} - \frac{K}{\Delta x^2} y_{1,4} - f(y_{1,3})] \\ \Rightarrow y_{1,2} = h_{N-2}(y, \dot{y}, \cdots) \end{array} \tag{2.260}$$

From Eq. (13.58) one obtains

$$\begin{array}{c} y_{1,1} = \alpha_{N-1} = \frac{1}{K/\Delta x^2}[\dot{y}_{1,2} + \frac{2K}{\Delta x^2} y_{1,2} - \frac{K}{\Delta x^2} y_{1,3} - f(y_{1,2})] \\ \Rightarrow y_{1,1} = h_{N-1}(y, \dot{y}, \cdots) \end{array} \tag{2.261}$$

Finally, From Eq. (13.57) one obtains

$$\begin{array}{c} \phi_0 = \alpha_N = \frac{1}{K/\Delta x^2}[\dot{y}_{1,1} + \frac{2K}{\Delta x^2} y_{1,1} - \frac{K}{\Delta x^2} y_2 - f(y_{1,1})] \\ \Rightarrow \phi_0 = h_N(y, \dot{y}, \cdots) \end{array} \tag{2.262}$$

The above procedure confirms that all state variables of the model

$$\begin{array}{cccc} y_{1,1} & y_{1,2} & y_{1,3} & \cdots \\ y_{1,i} & \cdots & y_{1,N-1} & y_{1,N} \end{array} \tag{2.263}$$

and the control input which is the boundary condition ϕ_0 can be written as functions of the flat output $y = y_{1,N}$ and of the flat output's derivatives. Consequently, differential flatness for the heat PDE is shown.

Additionally, one can consider decomposition of the PDE state-space equation into submodels, where at each submodel the virtual control input is $\alpha_i = y_{1,N-i}$

$$\begin{matrix} \dot{y}_{1,i} = -\frac{2K}{\Delta x^2} y_{1,i} + \frac{K}{\Delta x^2} y_{1,i-1} + f(y_{1,i}) + \frac{K}{\Delta x^2} y_{1,i+1} \\ \cdots \; \cdots \end{matrix} \tag{2.264}$$

and with local flat output $y_{1,i}$ one can confirm that all subsystems are differentially flat.

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