Chapter 2
Statistical Description of Turbulent Flows

This chapter focuses on the statistical approach to turbulence. On the one hand, it seeks to describe the evolution of mean and turbulent fields, and on the other, to highlight the transfer terms between these two fields. This splitting, introduced in 1883 by Reynolds, is not unique, nor the most satisfactory. Other flow decompositions are given in Chaps. 8 and 9. Nevertheless, the Reynolds approach is still to this date the only one allowing simple statistical assessments of fluid dynamic equations.

2.1 Method of Taking Averages

Any variable occurring in a turbulent field, such as velocity, pressure or temperature, is a random function of position $\boldsymbol{x}$ and time t. The first method to define an average is thus based on a probabilist approach. This means that the same experiment is repeated a larger number of times providing independent realizations of the field. The statistical mean $\bar{F}(\boldsymbol{x}, t)$ of a variable $f(\boldsymbol{x}, t)$ is then defined as

$$\bar{F}(\boldsymbol{x}, t) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^{N} f^{(i)}(\boldsymbol{x}, t)$$

where $f^{(i)}$ is the ith realization. This average will be the one employed throughout this chapter because of convenience when manipulating equations. It is however difficult to be implemented in concrete experiments. Two other methods are therefore involved in special cases. When the turbulent field is stationary, i.e. when time t does not enter into $\bar{F}$, a temporal average is possible, leading to

$$\bar{F}_T(\boldsymbol{x}) = \lim_{T \to \infty} \frac{1}{T} \int_{t_0}^{t_0+T} f\left(\boldsymbol{x}, t'\right) dt'$$

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where T is the observation time over only one realization. As an example, a sensor is placed at location $\boldsymbol{x}$ in a jet operated by a constant power supply. The duration T has to be large to permit $\bar{F}_T$ to approach $\bar{F}$. The exact demonstration relies on the hypothesis of ergodicity, which is developed at large by Lumley in his book on stochastic tools in turbulence [77]. Practically, a tangible requirement is that points at sufficiently large separation be uncorrelated. As a result, time T has to be much greater than the largest turbulent time scale encountered in the turbulent field. Moreover this time T depends on the nature of the considered variable f. Even simply for velocity, T differs for measurements dealing with $u(\boldsymbol{x}, t)$ and measurements involving $u'^2(\boldsymbol{x}, t)$ or $u'^4(\boldsymbol{x}, t)$.

When the turbulent field is homogeneous, i.e. when position $\boldsymbol{x}$ does not enter into $\bar{F}$, a spatial average is possible leading to

$$\bar{F}_V(t) = \lim_{V \to \infty} \frac{1}{V} \int_V f\left(\boldsymbol{x}', t\right) d\boldsymbol{x}'$$

As above, the volume V has to be large relative to the turbulent spatial scales involved in the variable f which is considered. The spatial average is generally employed when running computations which generally provide a knowledge of the turbulent field f at all points $\boldsymbol{x}$, and moreover at all times t in the volume V. An example is the turbulence decay in a large box after the initial excitation is turned off.

In the Reynolds decomposition, any physical variable f can be split into its mean part $\bar{F}$ and its fluctuation part f', i.e. $f = \bar{F} + f'$ with $\overline{f'} = 0$. As a convention for what follows, capital letters are employed as often as possible when designating mean physical quantities, in addition to the bar related to the mean operator. It is important to note that in this context, the mean part represents what is reasonably calculable, or at least the deterministic part, as opposed to the random or incoherent fluctuations which will be either modelled or measured.

Some important properties of the averaging operator are now listed. For two random variables $f = f(\boldsymbol{x}, t)$ and $g = g(\boldsymbol{x}, t)$ and a constant α, one easily establishes,

(i) $\overline{f + g} = \overline{F} + \overline{G}$

(ii) $\overline{\alpha f} = \alpha \overline{F}$

(iii) $\overline{\overline{F} g} = \overline{F}\,\overline{G}$

(iv) $\overline{\dfrac{\partial f}{\partial t}} = \dfrac{\partial \overline{F}}{\partial t} \qquad \overline{\dfrac{\partial f}{\partial x_i}} = \dfrac{\partial \overline{F}}{\partial x_i}$

(v) $\overline{\displaystyle\int f dt} = \displaystyle\int \overline{F} dt \qquad \overline{\displaystyle\int f dx_i} = \displaystyle\int \overline{F} dx_i$

Moreover, an important practical rule which will often be employed later on, concerns the product of two variables f and g,

$$\overline{fg} = \bar{F}\,\bar{G} + \overline{f'g'} \tag{2.1}$$

The alert reader may have noticed that the presence of a nonlinear term in the equations does not allow expression of the product as a function of $\bar{F}$ and $\bar{G}$ only, but introduces a new second-moment term $\overline{f'g'}$.

2.2 Reynolds Averaged Navier-Stokes Equations

Prior to considering the implications for turbulence of the Reynolds decomposition in a mean part and a fluctuating part, the fluid dynamics equations must be clearly stated. The reader can refer to classical textbooks in order to review the different local forms of these equations.

2.2.1 The Fluid Dynamics Equations

The fluid dynamics equations take the following forms whether they concern mass, momentum or energy conservation [5],

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{u}) = 0 \tag{2.2}$$

$$\frac{\partial (\rho \boldsymbol{u})}{\partial t} + \nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u}) = -\nabla p + \nabla \cdot \boldsymbol{\tau} \tag{2.3}$$

$$\frac{\partial (\rho h)}{\partial t} + \nabla \cdot (\rho h \boldsymbol{u}) = -\nabla \cdot \boldsymbol{q} + \frac{\partial p}{\partial t} + \boldsymbol{u} \cdot \nabla p + \nabla \boldsymbol{u} : \boldsymbol{\tau} \tag{2.4}$$

where ρ, $\boldsymbol{u}$, p and h designate density, velocity, pressure and enthalpy, respectively. No volume force such as gravity is considered here. The viscous stress tensor $\boldsymbol{\tau}$ is expressed for a Newtonian fluid as,

$$\begin{aligned} \boldsymbol{\tau} &= \mu \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t \right] + \lambda_2 \nabla \cdot \boldsymbol{u} \\ &= \mu \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t - \frac{2}{3} (\nabla \cdot \boldsymbol{u})\, \boldsymbol{I} \right] + \mu_b (\nabla \cdot \boldsymbol{u})\, \boldsymbol{I} \end{aligned} \tag{2.5}$$

where μ, λ_2 and $\mu_b = \lambda_2 + 2\mu/3$ designate the dynamic viscosity, the second viscosity and the bulk viscosity, respectively. To make clear the compact tensorial notations, let us explicit the convective term in (2.3) and the dissipation term in (2.4),

$$[\nabla \cdot (\rho \boldsymbol{u} \otimes \boldsymbol{u})]_i = \frac{\partial}{\partial x_j} (\rho u_i u_j) \qquad \nabla \boldsymbol{u} : \boldsymbol{\tau} = \frac{\partial u_i}{\partial x_j} \tau_{ij}$$

By substituting expression (2.5) into the Navier-Stokes equation (2.3), the thermodynamic pressure is now distinct from the effective or mechanical pressure given by $p - \mu_b \nabla \cdot \boldsymbol{u}$. The classical approach consists in taking $\mu_b = 0$, according to the Stokes hypothesis, and the viscous stress tensor is then simply expressed as,

$$\boldsymbol{\tau} = \mu \left[\nabla \boldsymbol{u} + (\nabla \boldsymbol{u})^t \right] - \frac{2}{3} \mu \left(\nabla \cdot \boldsymbol{u} \right) \boldsymbol{I} \tag{2.6}$$

However, except for a monoatomic gas [71], this assumption is experimentally not satisfied [72, 74] when the determination of μ_b is carried out by measuring sound absorption. It should be noted that this discussion only occurs for compressible flows. Expression (2.5) directly provides the expression of $\boldsymbol{\tau}$ for an incompressible flow, thus satisfying $\nabla \cdot \boldsymbol{u} = 0$. In addition, if the fluid is not Newtonian, in such case as water flows containing polymers or bubbles for drag reduction, other constitutive equations have to be used.

The fluid is presumed to act as an ideal gas, i.e. $p = \rho r T$, and the conductive heat flux is supposedly described by the Fourier law $\boldsymbol{q} = -\lambda \nabla T$. In these relations, r is the ideal gas constant of the studied fluid and λ its thermal conductivity. The use of the enthalpy variable h to write the conservation of energy enables to obtain the temperature T directly by using the relation $dh = c_p dT$, where c_p is the specific heat for a constant pressure. The energy conservation equation can be written in many other forms such as [5],

$$\frac{\partial(\rho e)}{\partial t} + \nabla \cdot (\rho e \boldsymbol{u}) = -\nabla \cdot \boldsymbol{q} - p \nabla \cdot \boldsymbol{u} + \nabla \boldsymbol{u} : \boldsymbol{\tau} \tag{2.7}$$

$$\frac{\partial(\rho e_t)}{\partial t} + \nabla \cdot (\rho e_t \boldsymbol{u}) = \nabla \cdot \left[-\boldsymbol{q} - p\, \boldsymbol{u} + \boldsymbol{\tau} \cdot \boldsymbol{u} \right] \tag{2.8}$$

where $e = c_v T$ and $e_t = e + \boldsymbol{u}^2/2$ are the internal energy and the total energy respectively, with $\rho e_t = p/(\gamma - 1) + \rho \boldsymbol{u}^2/2$ for an ideal gas.

In addition, the following rule obtained by using Eq. (2.2) for any variable f, will also be frequently employed,

$$\frac{\partial (\rho f)}{\partial t} + \nabla \cdot (\rho f \boldsymbol{u}) = \rho \left(\frac{\partial f}{\partial t} + \boldsymbol{u} \cdot \nabla f \right) \tag{2.9}$$

In order to simplify algebra in this statistical study, the flow is assumed incompressible, i.e. $\nabla \cdot \boldsymbol{u} = 0$. Compressible flows, which are more complex, will be briefly described in Chap. 9 as applied to statistical modelling, see Sect. 9.3.1. Therefore, the following will now focus essentially on the Navier-Stokes equation. The case of a dilatable fluid including buoyancy effects, that is to say for which $\rho = \rho(T)$, can be treated as an exercise in order to practice the course.

Some results of continuum mechanics concerning the viscous stress tensor are briefly recalled as to conclude this section. The velocity gradient tensor can be decomposed in the sum of a symmetrical part, the deformation or rate-of-strain tensor e_{ij}, and an anti-symmetrical part, the vorticity tensor ω_{ij},

$$\frac{\partial u_i}{\partial x_j} = \underbrace{\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right)}_{e_{ij}} + \underbrace{\frac{1}{2} \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right)}_{\omega_{ij}}$$

A physical interpretation of the velocity gradient tensor is given in Sect. 5.3. The vorticity tensor is linked to the vorticity vector $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ through the two relations below,

$$\omega_{ij} = \frac{1}{2}\epsilon_{ijk}\omega_k \qquad \omega_k = \frac{1}{2}\epsilon_{ijk}\omega_{ij}$$

Moreover, it is convenient to split the deformation tensor e_{ij} into its isotropic and deviatoric parts by writing,

$$e_{ij} = e_{ij}^{\mathcal{I}} + e_{ij}^{\mathcal{D}} = \frac{1}{3}e_{kk}\delta_{ij} + \left(e_{ij} - \frac{1}{3}e_{kk}\delta_{ij}\right) \tag{2.10}$$

The isotropic or spherical part $e_{ij}^{\mathcal{I}}$ expresses the volume change since

$$\mathrm{tr}(\boldsymbol{e}^{\mathcal{I}}) = e_{ij}^{\mathcal{I}}\delta_{ij} = \frac{1}{3}e_{kk}\,(1+1+1) = \frac{\partial u_k}{\partial x_k} = \nabla\cdot\boldsymbol{u}$$

whereas the deviatoric part $e_{ij}^{\mathcal{D}}$ is such as $\mathrm{tr}(\boldsymbol{e}^{\mathcal{D}}) = 0$, and is also symmetric. From now on, the deviatoric part of the velocity gradient tensor will be denoted s_{ij},

$$s_{ij} \equiv e_{ij}^{\mathcal{D}} \equiv \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) - \frac{1}{3}(\nabla\cdot\boldsymbol{u})\delta_{ij} \tag{2.11}$$

As is clear from the constitutive relation (2.6), the viscous stress tensor is linearly linked to this tensor s_{ij} by,

$$\tau_{ij} = 2\mu s_{ij} \tag{2.12}$$

for a Newtonian fluid satisfying Stokes's hypothesis. At last, the dissipation term $\nabla\boldsymbol{u} : \boldsymbol{\tau}$ appearing in the energy conservation equation (2.4) is always positive, and this result is straightforward by noting that the viscous tensor is symmetric

$$\nabla\boldsymbol{u} : \boldsymbol{\tau} = \frac{\partial u_i}{\partial x_j}\tau_{ij} = \frac{\partial u_j}{\partial x_i}\tau_{ji} = e_{ij}\tau_{ij} = s_{ij}\tau_{ij}$$

and thus $\nabla\boldsymbol{u} : \boldsymbol{\tau} = 2\mu s_{ij}^2 \geq 0$. This dissipation term represents the quantity of mechanical energy transformed into thermal energy by viscous effects, and it is necessarily positive because of the second thermodynamic principle. By developing this expression, is obtained

$$\nabla\boldsymbol{u} : \boldsymbol{\tau} = 2\mu\left(e_{ij}^2 - \frac{1}{3}e_{kk}^2\right) = 2\mu\left[e_{ij}^2 - \frac{1}{3}(\nabla\cdot\boldsymbol{u})^2\right]$$

A final remark is to observe that there is no distinction between e_{ij} and $e_{ij}^{\mathcal{D}} \equiv s_{ij}$ for an incompressible flow of a Newtonian fluid.

2.2.2 Averaged Equations

To derive averaged equations, the Reynolds decomposition is applied to velocity $u_i = \bar{U}_i + u'_i$ and pressure $p = \bar{P} + p'$, as well as to the viscous tensor $\tau_{ij} = \bar{\tau}_{ij} + \tau'_{ij}$. The flow is considered incompressible and density is supposed to be constant. However, we will try to keep general expressions as far as possible in the development. Firstly, from the incompressibility condition,

$$\frac{\partial}{\partial x_i}\left(\bar{U}_i + u'_i\right) = 0$$

one obtains by averaging the equation,

$$\frac{\partial \bar{U}_i}{\partial x_i} = 0 \quad \text{and then,} \quad \frac{\partial u'_i}{\partial x_i} = 0$$

by substraction of the two first equations. Therefore, the instantaneous fluctuating velocity field is incompressible. This property will be employed very often later on. In the same way, from the mass conservation equation (2.2)

$$\frac{\partial}{\partial x_j}\left[\rho\left(\bar{U}_j + u'_j\right)\right] = 0$$

and from the momentum conservation equation (2.3)

$$\frac{\partial}{\partial t}\left[\rho(\bar{U}_i + u'_i)\right] + \frac{\partial}{\partial x_j}\left[\rho(\bar{U}_i + u'_i)(\bar{U}_j + u'_j)\right] = -\frac{\partial(\bar{P} + p')}{\partial x_i} + \frac{\partial(\bar{\tau}_{ij} + \tau'_{ij})}{\partial x_j} \tag{2.13}$$

the averaging operation leads to the Reynolds averaged equations,

$$\frac{\partial}{\partial x_j}\left(\rho \bar{U}_j\right) = 0 \tag{2.14}$$

$$\frac{\partial\left(\rho \bar{U}_i\right)}{\partial t} + \frac{\partial\left(\rho \bar{U}_i\, \bar{U}_j\right)}{\partial x_j} = -\frac{\partial \bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\bar{\tau}_{ij} - \rho\overline{u'_i u'_j}\right] \tag{2.15}$$

The new unknown $-\rho\overline{u'_i u'_j}$ issued from the convective nonlinear term in (2.3), is called the Reynolds stress tensor. Generally this term is larger than the mean viscous stress tensor except for wall-bounded flows, where viscosity effects become preponderant close to the wall. The no-slip boundary condition indeed requires that $\overline{u'_i u'_j} \to 0$ at the wall.

Due to the appearance of the new unknown $-\rho\overline{u'_i u'_j}$, the set of equations giving the mean velocity field $\bar{U}_i$ cannot be solved, and there are more unknowns than equations. In short, this is called a *closure problem*. Two strategies can then be pursued to surmount the difficulty. The first comes from Boussinesq, and consists of modelling the Reynolds tensor $-\rho\overline{u_i'u_j'}$. Details are given in Sect. 2.5. The second one consists in writing an equation for this unknown tensor, i.e. a transport equation for the Reynolds stress tensor $-\rho\overline{u_i'u_j'}$. However, as one can guess, this new equation will introduce third-moment terms in velocity fluctuations, as shown later with the triple correlation term in Eq. (2.19) for instance, which still requires a closure.

The mean kinetic energy of the fluctuating field k_t, also called the turbulent kinetic energy,

$$k_t \equiv \frac{\overline{u_i'u_i'}}{2} = \frac{\overline{u_1'^2} + \overline{u_2'^2} + \overline{u_3'^2}}{2} \tag{2.16}$$

can be introduced through the contraction of the Reynolds stress tensor

$$-\rho\overline{u_i'u_j'}\,\delta_{ij} = -2\rho k_t$$

It therefore appears instructive to continue the presentation by considering the kinetic energy budget of the mean flow as well as the kinetic energy budget of the fluctuating field.

2.3 Kinetic Energy Budget of the Mean Flow

The equation which describes the kinetic energy $\rho\bar{U}_i^2/2$ of the mean flow can be obtained by multiplying the averaged Navier-Stokes equation (2.15) in the ith direction by the mean velocity $\bar{U}_i$ component,

$$\bar{U}_i \times \left\{ \frac{\partial\left(\rho\bar{U}_i\right)}{\partial t} + \frac{\partial\left(\rho\bar{U}_i\bar{U}_j\right)}{\partial x_j} = -\frac{\partial\bar{P}}{\partial x_i} + \frac{\partial}{\partial x_j}\left[\bar{\tau}_{ij} - \rho\overline{u_i'u_j'}\right] \right\}$$

From the mass conservation equation (2.14), a rule similar to relation (2.9) can easily be derived for an arbitrarily mean flow quantity $\bar{F}$,

$$\frac{\partial\left(\rho\bar{F}\right)}{\partial t} + \frac{\partial}{\partial x_i}\left(\rho\bar{U}_i\bar{F}\right) = \rho\frac{\partial\bar{F}}{\partial t} + \rho\bar{U}_i\frac{\partial\bar{F}}{\partial x_i} \equiv \frac{\bar{d}}{\bar{d}t}(\rho\bar{F}) \tag{2.17}$$

and is here used to rearrange the left-hand side of the previous equation in a conservative form,

$$\bar{U}_i \times \left\{ \frac{\partial(\rho\bar{U}_i)}{\partial t} + \frac{\partial(\rho\bar{U}_i\bar{U}_j)}{\partial x_j} \right\} = \rho\bar{U}_i\frac{\partial\bar{U}_i}{\partial t} + \rho\bar{U}_i\bar{U}_j\frac{\partial\bar{U}_i}{\partial x_j}$$
$$= \rho\frac{\partial}{\partial t}\left(\frac{\bar{U}_i^2}{2}\right) + \rho\bar{U}_j\frac{\partial}{\partial x_j}\left(\frac{\bar{U}_i^2}{2}\right)$$
$$= \frac{\partial}{\partial t}\left(\frac{\rho\bar{U}_i^2}{2}\right) + \frac{\partial}{\partial x_j}\left(\bar{U}_j\frac{\rho\bar{U}_i^2}{2}\right)$$

The other terms can be rewritten thanks to the incompressibility condition for the mean flow, and as a result, the kinetic energy budget can be recast in the following form,

$$\frac{\partial}{\partial t}\left(\frac{\rho\bar{U}_i^2}{2}\right) + \frac{\partial}{\partial x_j}\left(\bar{U}_j\frac{\rho\bar{U}_i^2}{2}\right) = \underbrace{\rho\overline{u_i'u_j'}\frac{\partial\bar{U}_i}{\partial x_j}}_{(a)} - \underbrace{\bar{\tau}_{ij}\frac{\partial\bar{U}_i}{\partial x_j}}_{(b)} - \underbrace{\frac{\partial\left(\bar{U}_i\bar{P}\right)}{\partial x_i}}_{(c)} + \underbrace{\frac{\partial}{\partial x_j}\left(\bar{U}_i\bar{\tau}_{ij}\right)}_{(d)} - \underbrace{\frac{\partial}{\partial x_j}\left(\bar{U}_i\rho\overline{u_i'u_j'}\right)}_{(e)} \tag{2.18}$$

Intentionally, let us first consider the last three terms (c), (d) and (e). They respectively represent the power of pressure forces, viscous forces and Reynolds stress forces. It is important to observe that these terms are zero for a homogeneous mean flow. Indeed, they represent pure diffusion transfers and are actually written as a flux divergence form. As a result, the most important terms are term (a) which represent a transfer between the mean flow and the fluctuating flow, and the term (b) which represents the viscous dissipation of the mean flow.

In order to correctly understand the transfer term between the mean field and the turbulent field, it is useful to write down the equation governing the turbulent kinetic energy of the fluctuating field.

2.4 Kinetic Energy Budget of the Fluctuating Field

The turbulent kinetic energy k_t is defined by relation (2.16) and its governing equation can be obtained by writing the general equation for the Reynolds stress tensor, and then contracting indices, the minus sign of $-\rho\overline{u_i'u_j'}$ being left off.

2.4.1 Transport Equation of Reynolds Stresses

The Navier-Stokes equation governing the component u'_i is obtained by subtracting to the initial equation (2.13) its averaged equation (2.15), which leads to,

$$\frac{\partial\left(\rho u'_i\right)}{\partial t}+\frac{\partial}{\partial x_k}\left[\rho\left(u'_i\bar{U}_k+\bar{U}_i u'_k+u'_i u'_k\right)\right]=-\frac{\partial p'}{\partial x_i}+\frac{\partial}{\partial x_k}\left(\rho\overline{u'_i u'_k}+\tau'_{ik}\right)$$

and this equation is denoted as (Σ_i). The dummy summation index is now denoted as k to avoid any confusion with the index j considered in the Reynolds stress component $-\rho\overline{u'_i u'_j}$. Similarly, we can obtain the equation governing the component u'_j, denoted as (Σ_j). The transport equation is then obtained by forming $u'_j\,\Sigma_i+u'_i\,\Sigma_j$ and by applying the average operator. More specifically, the three following groups are first rearranged,

$$\begin{cases} u'_j\dfrac{\partial}{\partial x_k}\left(\rho u'_i\bar{U}_k\right)+u'_i\dfrac{\partial}{\partial x_k}\left(\rho u'_j\bar{U}_k\right)=\dfrac{\partial}{\partial x_k}\left(\rho u'_i u'_j\bar{U}_k\right)\\ u'_j\dfrac{\partial}{\partial x_k}\left(\rho\bar{U}_i u'_k\right)+u'_i\dfrac{\partial}{\partial x_k}\left(\rho\bar{U}_j u'_k\right)=\rho u'_j u'_k\dfrac{\partial\bar{U}_i}{\partial x_k}+\rho u'_i u'_k\dfrac{\partial\bar{U}_j}{\partial x_k}\\ u'_j\dfrac{\partial}{\partial x_k}\left(\rho u'_i u'_k\right)+u'_i\dfrac{\partial}{\partial x_k}\left(\rho u'_j u'_k\right)=\dfrac{\partial}{\partial x_k}\left(\rho u'_i u'_j u'_k\right) \end{cases}$$

then applying the averaging operator and using the incompressibility condition, one gets

$$\frac{\partial\left(\rho\overline{u'_i u'_j}\right)}{\partial t}+\frac{\partial}{\partial x_k}\left(\rho\overline{u'_i u'_j}\,\bar{U}_k\right)=\mathcal{P}_{ij}+T_{ij}+\Pi_{ij}+D_{ij}-\rho\epsilon_{ij} \qquad (2.19)$$

where the different terms in the right-hand side are defined as,

$$\mathcal{P}_{ij}=-\left(\rho\overline{u'_j u'_k}\,\frac{\partial\bar{U}_i}{\partial x_k}+\rho\overline{u'_i u'_k}\,\frac{\partial\bar{U}_j}{\partial x_k}\right)$$

$$\mathcal{T}_{ij}=-\frac{\partial}{\partial x_k}\left(\rho\overline{u'_i u'_j u'_k}\right)$$

$$\Pi_{ij}=-\left(\overline{u'_j\frac{\partial p'}{\partial x_i}}+\overline{u'_i\frac{\partial p'}{\partial x_j}}\right)$$

$$\mathcal{D}_{ij}=\frac{\partial}{\partial x_k}\left(\overline{u'_j\tau'_{ik}}+\overline{u'_i\tau'_{jk}}\right)$$

$$\rho\epsilon_{ij}=\overline{\tau'_{ik}\frac{\partial u'_j}{\partial x_k}}+\overline{\tau'_{jk}\frac{\partial u'_i}{\partial x_k}}$$

An interpretation of this budget is presented in Sect. 2.6 for the plane channel flow. This equation can be numerically solved, as an alternative to the modelling of Reynolds stress tensor $-\rho\overline{u_i'u_j'}$. However, and as already mentioned in Sect. 2.2.2, this equation contains a triple correlation term $\mathcal{T}_{ij}$, which requires a closure.

2.4.2 Budget of the Turbulent Kinetic Energy

The transport equation for k_t is directly deduced from (2.19) by contracting the indices and keeping the half-sum, namely $\bar{d}(\rho k_t)/\bar{d}t = (\mathcal{P}_{ii}+\mathcal{T}_{ii}+\Pi_{ii}+\mathcal{D}_{ii}-\rho\epsilon_{ii})/2$. It yields,

$$\frac{\bar{d}(\rho k_t)}{\bar{d}t} = \underbrace{-\,\rho\overline{u_i'u_k'}\,\frac{\partial\bar{U}_i}{\partial x_k}}_{(a)} \underbrace{-\,\overline{\tau_{ik}'\frac{\partial u_i'}{\partial x_k}}}_{(b)} \underbrace{-\,\frac{1}{2}\frac{\partial}{\partial x_k}\rho\overline{u_i'u_i'u_k'}}_{(c)} \underbrace{-\,\overline{u_i'\frac{\partial p'}{\partial x_i}}}_{(d)} + \underbrace{\frac{\partial}{\partial x_k}\overline{u_i'\tau_{ik}'}}_{(e)} \tag{2.20}$$

Moreover, the pressure term (d) can be rearranged as follows,

$$\Pi = \overline{u_i'\frac{\partial p'}{\partial x_i}} = \frac{\partial}{\partial x_i}\overline{u_i'p'} - \overline{p'\frac{\partial u_i'}{\partial x_i}} = \frac{\partial}{\partial x_i}\overline{u_i'p'} \tag{2.21}$$

and the three terms (c), (d) and (e) can then be rewritten as the divergence of a flux vector $\nabla\cdot\bar{\boldsymbol{J}} = \mathcal{T} + \Pi + \mathcal{D}$ to show that they are transport terms. The triple velocity correlation $\mathcal{T}$ represents the diffusion of the turbulent kinetic energy by the fluctuating velocity, and the terms Π and $\mathcal{D}$ can be associated with diffusion effects through the pressure and the viscous stresses. For homogeneous turbulence, $\nabla\cdot\bar{\boldsymbol{J}} \equiv 0$, the convection of the turbulent kinetic energy is then only balanced by,

$$\frac{\bar{d}(\rho k_t)}{\bar{d}t} = -\rho\overline{u_i'u_j'}\,\frac{\partial\bar{U}_i}{\partial x_j} - \overline{\tau_{ij}'\frac{\partial u_i'}{\partial x_j}} \tag{2.22}$$

namely the transfer term $\mathcal{P}$ between the mean field and the turbulent field on the one hand, see comments of Eq. (2.18), and the viscous dissipation per unit of mass of the turbulent kinetic energy on the other hand, denoted $\rho\epsilon$. This quantity is always positive,

$$\rho\epsilon = \overline{\tau_{ij}'\frac{\partial u_i'}{\partial x_j}} = 2\mu\overline{s_{ij}'^{\,2}} \geq 0 \tag{2.23}$$

A simplified expression of the dissipation ϵ, which is often used in practice, is briefly introduced in Eq. (2.26), and discussed later in Sect. 6.5.

By comparing Eqs. (2.18) and (2.20), the energy exchanged between the mean field and the turbulent field can only be carried on through the term $\mathcal{P}$. This term is generally positive, corresponding to an energy supply from the mean field to the turbulent one, and is called, by abuse of language, the production term. A simple heuristic argument shows that this is true for a sheared flow. Let us take, for example, the case of a boundary layer where $d\bar{U}_1/dx_2 > 0$, and imagine that a fluid particle gets through the grey line from bottom to top, as shown in Fig. 2.1. Thus $u'_2 > 0$, and this particle finds itself in the midst of a material animated by a greater mean velocity, thus having a deficit in longitudinal velocity, leading to $u'_1 < 0$ and $\overline{u'_1 u'_2} < 0$. The same reasoning can be applied to a fluid particle going through to the grey line from top to bottom, or to other flows with a negative mean velocity gradient such as jets. Finally, the $\mathcal{P}$ production term appears to be rather positive.

The principal terms that have been highlighted in the analysis of the kinetic energy budgets for the mean and the turbulent fields are resumed in Fig. 2.2. The rule (2.1) is again illustrated here with the dissipation induced by viscous effects,

$$\overline{\tau_{ij}\frac{\partial u_i}{\partial x_j}} = \bar{\tau}_{ij}\frac{\partial \bar{U}_i}{\partial x_j} + \overline{\tau'_{ij}\frac{\partial u'_i}{\partial x_j}} = \bar{\tau}_{ij}\frac{\partial \bar{U}_i}{\partial x_j} + \rho\epsilon$$

which is split into two contributions in the budget of the kinetic energy of the mean flow (2.18) and of the turbulent flow (2.20). These two terms are production terms in the transport equation for the mean temperature, as shown later in Sect. 9.2.3.

2.5 Turbulent Viscosity: The Boussinesq Model

The most famous and widely used closure for the Reynolds stress tensor is based on the concept of a turbulent viscosity, introduced by Boussinesq. This hypothesis involves expressing the Reynolds stress tensor by analogy with the viscous stress $\boldsymbol{\tau}$.

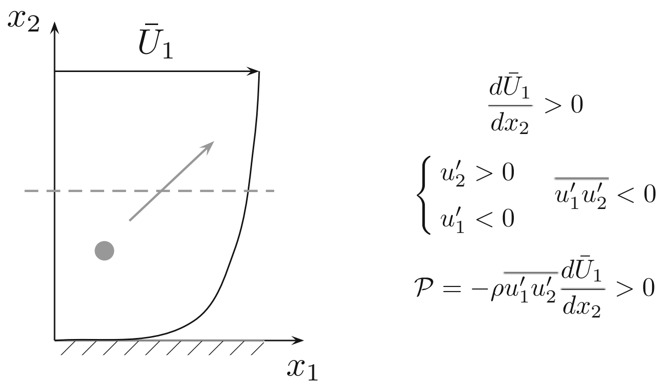


Fig. 2.1 Sketch of a fluid particle moving inside a boundary layer, and heuristic argument to estimate the sign of the production term $\mathcal{P}$

It is therefore assumed that

$$-\rho\overline{u_i'u_j'} = 2\mu_t\bar{S}_{ij} \underbrace{-\frac{2}{3}\rho k_t\delta_{ij}}_{(a)} \tag{2.24}$$

where μ_t is called the dynamic turbulent viscosity or the eddy viscosity. The isotropic term (a) is necessary to satisfy the condition $-\rho\overline{u_i'u_i'} = -2\rho k_t$ by contraction of the indices, that is for $i = j$ and taking the summation over i.

The turbulent viscosity seems on first approach to be a function of the flow $\mu_t = \mu_t(\boldsymbol{x}, t)$, as opposed to the molecular viscosity which is an intrinsic physical property of the fluid. A great challenge in turbulence is to derive a comprehensive formulation for μ_t, and examples will be given in the next chapters. Closure based on a turbulent viscosity model is presented here because it is the basis of most turbulent models used in numerical simulations of the Reynolds averaged Navier-Stokes equations. Chapter 9 treats in detail this issue, but it seems important to already note some consequences of such modelling for the Reynolds stress tensor since the reference to a turbulent viscosity is made throughout the textbook. Note also that approaches following in large eddy simulation are definitively different, and will be discussed in Chap. 8.

Replacing the six unknowns of the Reynolds stress tensor, three normal stresses $-\rho\overline{u_i'^2}$ and three shear stresses $-\rho\overline{u_i'u_j'}$ for $i \neq j$, by a unique unknown function μ_t induces a one-way energy transfer from the mean flow field towards the fluctuating field. Indeed, the term $\mathcal{P}$ takes the form

$$\mathcal{P} = -\rho\overline{u_i'u_j'}\frac{\partial\bar{U}_i}{\partial x_j} = \left(2\mu_t\bar{S}_{ij} - \frac{2}{3}\rho k_t\delta_{ij}\right)\frac{\partial\bar{U}_i}{\partial x_j} = 2\mu_t\overline{S}_{ij}^2 \geq 0$$

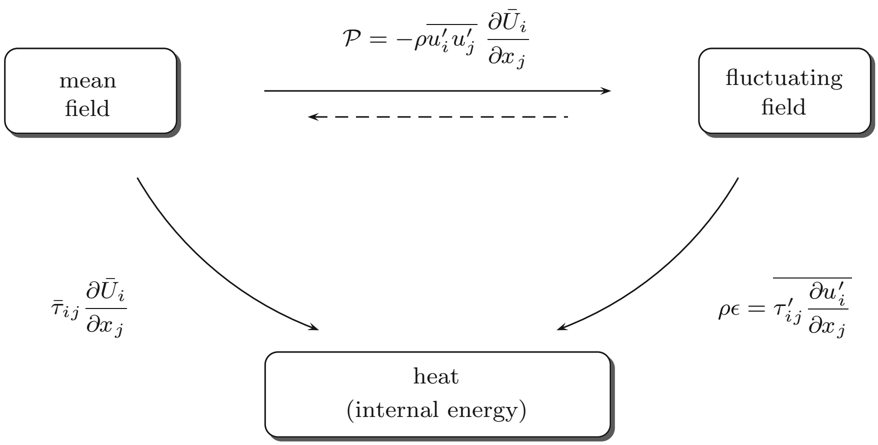


Fig. 2.2 Simplified diagram of energy transfers between the mean and turbulent fields, when neglecting all diffusion terms. The term $\mathcal{P}$ is generally a production term for the fluctuating field

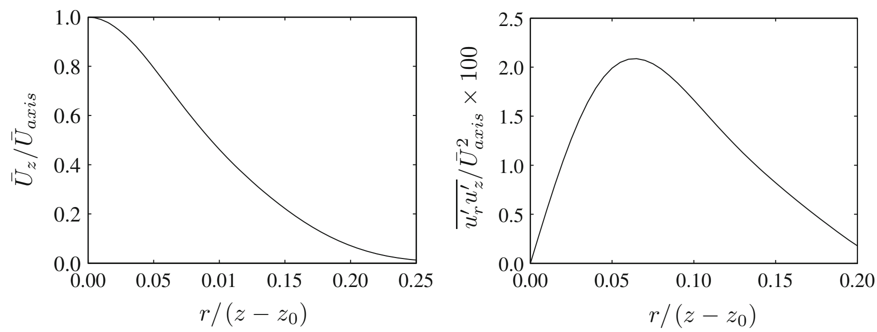


Fig. 2.3 Subsonic round jet at Mach M $= 0.16$ and Reynolds number $\mathrm{Re}_D = 9.5 \times 10^4$. Radial profiles of the mean velocity $\bar{U}_z/\bar{U}_{axis}$ and of the $\overline{u'_r u'_z}/\bar{U}^2_{axis}$ term against the radial distance $r/(z - z_0)$ in the fully developed region of the jet, at a distance larger than $25D$ from the nozzle exit. Data from Hussein et al. [539], see also Chap. 4

as $\mu_t > 0$, which is generally assumed in turbulent models. Moreover, in the case of a mean shear flow, with $\bar{U}_1 = \bar{U}_1(x_2)$ and $\bar{U}_2 = \bar{U}_3 = 0$, the model imposes the Reynolds stress to be zero for an extremum of the mean velocity, as the total strain τ_t seen by the fluid is given by

$$\tau_t = \bar{\tau}_{12} - \rho\overline{u'_1 u'_2} = \mu\frac{d\bar{U}_1}{dx_2} + \mu_t\frac{d\bar{U}_1}{dx_2} = (\mu + \mu_t)\frac{d\bar{U}_1}{dx_2} \tag{2.25}$$

There is no difficulty for symmetrical mean flow fields in 2-D and for axisymmetrical mean fields in 3-D. Figure 2.3 reproduces for example the $\bar{U}_z$ et $\overline{u'_r u'_z}$ profiles measured in a round jet by Hussein et al. [539]. Notice that $\overline{u'_r u'_z}$ is zero on the axis and that the production term $\mathcal{P}$ is positive.

However, at least two famous counterexamples exist regarding asymmetrical mean flows. The first is the wall jet, the experimental profiles of which appear in Fig. 2.4. Clearly $-\rho\overline{u'_1 u'_2}$ and $d\bar{U}_1/dx_2$ are zero for different x_2 positions. The second example concerns a channel flow with a smooth wall on one side and a rough wall on the other, studied by Hanjalić and Launder [618]. In both these cases, the location of the point where $-\rho\overline{u'_1 u'_2} = 0$ is displaced further than the point where $d\bar{U}_1/dx_2 = 0$ by the turbulent region which possesses the most intense velocity fluctuations, that is the jet compared to the wall in the first example, and the rough side compared to the smooth one in the second example.

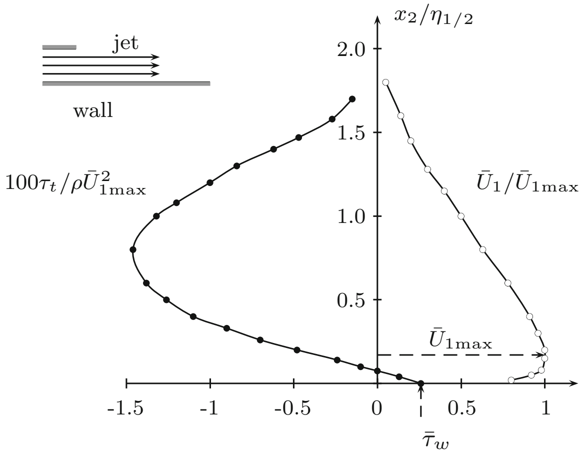


Fig. 2.4 Wall jet, Reynolds number at the nozzle exit Re $\simeq 1.8 \times 10^4$. Mean velocity $\bar{U}_1$ and total shear stress $\tau_t = \mu d\bar{U}_1/dx_2 - \rho\overline{u_1'u_2'}$ profiles as a function of the distance to the wall, $x_2/\eta_{1/2}$, where $\eta_{1/2}$ is the distance for which $\bar{U}_1 = 0.5 \times \bar{U}_{1max}$, and τ_w is the wall shear stress. By moving away from the wall, one first finds the point where $\tau_t = 0$, then the one where $d\bar{U}_1/dx_2 = 0$. Data from Tailland and Mathieu [557].

2.6 An Example: The Turbulent Channel Flow

In order to conclude this chapter and illustrate the turbulent kinetic energy assessment that has been established in a general case, the following section will focus on a fully developed stationary turbulent flow between two flat parallel walls, see Fig. 2.5. This turbulent channel flow is often used as a reference for direct numerical simulation, such for example the calculations of Kim et al. [629], Mansour et al. [640], Moser et al. [642] or Hoyas and Jiménez [621], because it is experimentally well documented, see Laufer [636], Comte-Bellot [604] or Johansson and Alfredsson [625].

A channel flow is found fully developed in experiments when the considered section is far enough from the entrance, $x_1 \geq 120h$, where h is the half-width of the channel. All mean quantities are then stationary and do not depend on x_1 except for the mean pressure. The mean velocity has the form $\bar{U}_1(x_2)$ and $\bar{U}_2 = \bar{U}_3 \equiv 0$. Moreover, the flow is statistically independent from x_3, meaning that x_3 is an homogeneous

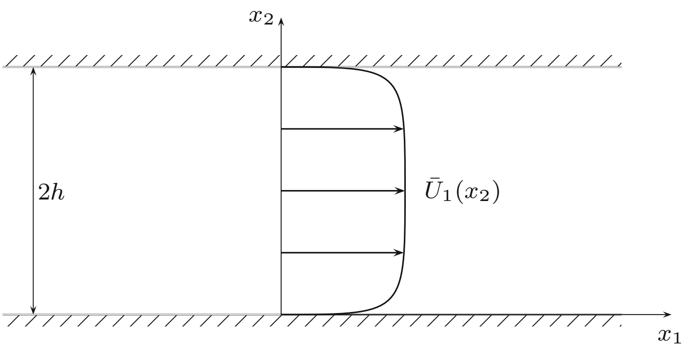


Fig. 2.5 Sketch of the geometry of a turbulent channel flow

direction for the turbulent field. The orientation of the x_3 axis has no influence, which leads to $\overline{u_1'u_3'} = \overline{u_2'u_3'} = 0$ and $\overline{p'u_3'} = 0$. Therefore, x_3 is a principal direction for the tensors, and the Reynolds tensor simplifies as,

$$-\rho\overline{u_i'u_j'} = -\rho \begin{pmatrix} \overline{u_1'^2} & \overline{u_1'u_2'} & 0 \\ \overline{u_1'u_2'} & \overline{u_2'^2} & 0 \\ 0 & 0 & \overline{u_3'^2} \end{pmatrix}$$

Equation (2.19) which governs the velocity correlations $\overline{u_i'u_j'}$, is written for the $i = j$ case, where the i indice is here replaced by the greek letter α so as to indicate that two repeated indices are not summed,

$$\frac{\bar{d}}{\bar{d}t}\left(\rho\overline{u_\alpha'^2}\right) = \mathcal{P}_{\alpha\alpha} + T_{\alpha\alpha} + \Pi_{\alpha\alpha} + \mathcal{D}_{\alpha\alpha} - \rho\epsilon_{\alpha\alpha}$$

The last two terms are often rearranged for incompressible flows. By noting that,

$$\mathcal{D}_{\alpha\alpha} = \mu\frac{\partial^2\overline{u_\alpha'^2}}{\partial x_k\partial x_k} + 2\mu\frac{\partial^2\overline{u_k'u_\alpha'}}{\partial x_k\partial x_\alpha}$$

$$\rho\epsilon_{\alpha\alpha} = 2\mu\overline{\frac{\partial u_\alpha'}{\partial x_k}\frac{\partial u_\alpha'}{\partial x_k}} + 2\mu\frac{\partial^2\overline{u_k'u_\alpha'}}{\partial x_k\partial x_\alpha}$$

the viscous diffusion and dissipation terms are thus combined as follows,

$$\mathcal{D}_{\alpha\alpha} - \rho\epsilon_{\alpha\alpha} = \mu\frac{\partial^2\overline{u_\alpha'^2}}{\partial x_k\partial x_k} - 2\mu\overline{\frac{\partial u_\alpha'}{\partial x_k}\frac{\partial u_\alpha'}{\partial x_k}} \equiv \mathcal{D}^h_{\alpha\alpha} - \rho\epsilon^h_{\alpha\alpha} \tag{2.26}$$

The new dissipation term $\epsilon^h_{\alpha\alpha}$ is not the correct thermodynamic expression of the dissipation, as formulated by Corrsin [68]. This term is sometimes called the isotropic dissipation, even though only homogeneity is required to have the equality between $\epsilon_{\alpha\alpha}$ and $\epsilon^h_{\alpha\alpha}$. This approximation of the exact dissipation is classically used in turbulence models since the transport equation of $\epsilon^h_{\alpha\alpha}$ is much simpler, see Sect. 6.5. At least for the case of the channel flow, the difference between the two expressions remains reasonably small [593].

The transport equation for the $\overline{u_\alpha'^2}$ component can then be recast as,

$$\begin{aligned}\frac{\bar{d}}{\bar{d}t}(\rho\overline{u_\alpha'^2}) &= -2\rho\overline{u_\alpha'u_k'}\frac{\partial\bar{U}_\alpha}{\partial x_k} - \frac{\partial}{\partial x_k}\rho\overline{u_\alpha'u_\alpha'u_k'} \\ &\quad - 2\overline{u_\alpha'\frac{\partial p'}{\partial x_\alpha}} + \mu\frac{\partial^2\overline{u_\alpha'^2}}{\partial x_k\partial x_k} - 2\mu\overline{\frac{\partial u_\alpha'}{\partial x_k}\frac{\partial u_\alpha'}{\partial x_k}} \\ &= \mathcal{P}_{\alpha\alpha} + \mathcal{T}_{\alpha\alpha} + \Pi_{\alpha\alpha} + \mathcal{D}^h_{\alpha\alpha} - \rho\epsilon^h_{\alpha\alpha}\end{aligned} \tag{2.27}$$

In this equation, $\mathcal{P}_{\alpha\alpha}$ is the production, $\mathcal{T}_{\alpha\alpha}$ is the turbulent diffusion, $\Pi_{\alpha\alpha}$ is the velocity pressure-gradient correlation, $\mathcal{D}^h_{\alpha\alpha}$ is the viscous diffusion and $\epsilon^h_{\alpha\alpha}$ is the dissipation. The term $\Pi_{\alpha\alpha}$ can also be decomposed as the sum of a pressure diffusion term and a pressure velocity-gradient correlation term, in a similar way as in expression (2.21),

$$\Pi_{\alpha\alpha} = 2\left(-\frac{\partial}{\partial x_\alpha}\overline{u'_\alpha p'} + \overline{p'\frac{\partial u'_\alpha}{\partial x_\alpha}}\right) = \Pi^d_{\alpha\alpha} + \Pi^s_{\alpha\alpha} \tag{2.28}$$

to highlight the specific role of the fluctuating pressure.

Equation (2.27) is now particularized for the case of a fully developed turbulent channel flow. For each normal stress component $\overline{u'^{\,2}_\alpha}$, it thus yields

$$\begin{aligned}
0 &= -2\overline{u'_1u'_2}\frac{d\bar{U}_1}{dx_2} + \frac{d}{dx_2}\left(-\overline{u'^2_1u'_2} + \nu\frac{d\overline{u'^2_1}}{dx_2}\right) + \frac{2}{\rho}\overline{p'\frac{\partial u'_1}{\partial x_1}} - 2\nu\overline{\frac{\partial u'_1}{\partial x_k}\frac{\partial u'_1}{\partial x_k}} \\
0 &= \quad 0 \quad + \frac{d}{dx_2}\left(-\overline{u'^2_2u'_2} + \nu\frac{d\overline{u'^2_2}}{dx_2} - \frac{2}{\rho}\overline{u'_2p'}\right) + \frac{2}{\rho}\overline{p'\frac{\partial u'_2}{\partial x_2}} - 2\nu\overline{\frac{\partial u'_2}{\partial x_k}\frac{\partial u'_2}{\partial x_k}} \\
0 &= \quad 0 \quad + \frac{d}{dx_2}\left(-\overline{u'^2_3u'_2} + \nu\frac{d\overline{u'^2_3}}{dx_2}\right) + \frac{2}{\rho}\overline{p'\frac{\partial u'_3}{\partial x_3}} - 2\nu\overline{\frac{\partial u'_3}{\partial x_k}\frac{\partial u'_3}{\partial x_k}}
\end{aligned} \tag{2.29}$$

In these equations, the terms are organized in order to have successively the production term, the diffusion and pressure–velocity correlation terms, the pressure–velocity gradient term and finally the viscous dissipation term. It is essential to note that only the longitudinal component u'_1 receives energy from the mean flow through the production term $\mathcal{P}_{11}$. The transverse velocity components u'_2 and u'_3 can therefore only receive energy from u'_1 through pressure fluctuations. The latter being scalar, there is no privileged direction, which explains why both u'_2 and u'_3 can receive energy. Pressure therefore has a redistribution role relative to the turbulent kinetic energy between the three components of velocity.

Finally, the half-sum of all three equations (2.29) provides the turbulent kinetic energy budget for the channel flow,

$$\begin{aligned}
0 &= -\rho\overline{u'_1u'_2}\frac{d\bar{U}_1}{dx_2} - \frac{1}{2}\frac{d}{dx_2}\rho\overline{u'_iu'_iu'_2} + \mu\frac{d^2k_t}{dx_2^2} - \frac{d}{dx_2}\overline{u'_2p'} - \mu\overline{\frac{\partial u'_i}{\partial x_k}\frac{\partial u'_i}{\partial x_k}} \\
0 &= \quad \mathcal{P} \quad + \quad \mathcal{T} \quad + \quad \mathcal{D}_h \quad + \quad \Pi^d \quad - \quad \rho\epsilon^h
\end{aligned} \tag{2.30}$$

Notice that pressure has an influence only through the transport term

$$\Pi^d \equiv \Pi^d_{ii}/2 = \Pi^d_{22}/2$$

in (2.30). Indeed, one has $\Pi = \Pi^d + \Pi^s$ with $\Pi^s = \Pi^s_{ii}/2 = 0$, and so it is expected that $\Pi^s_{11} = \overline{p'\partial u'_1/\partial x_1}$ is negative whereas Π^s_{22} and Π^s_{33} are expected to be rather positive terms.

These conjectures are confirmed by experiment, and results will be presented in detail in Chap. 3. It is however interesting to notice here that $\overline{u_1'^2}$ is greater than $\overline{u_2'^2}$ and $\overline{u_3'^2}$ in rms since $u_1' \simeq 2.5u_\tau$, $u_2' \simeq u_\tau$ and $u_3' \simeq 1.3u_\tau$, where u_τ is the friction velocity defined by $u_\tau = (\bar{\tau}_w/\rho)^{1/2}$ and $\bar{\tau}_w$ is the shear stress at the wall, namely $\bar{\tau}_w = \bar{\tau}_{12}(x_2 = 0)$.

Direct numerical simulations also complete these analyses, specifically for correlation terms involving pressure, which in general cannot be measured. The budgets of $\overline{u_1'^2}$, $\overline{u_2'^2}$, $\overline{u_3'^2}$ and $\overline{u_1'u_2'}$ computed by Hoyas and Jiménez [621, 622] are shown in Fig. 2.6. For $\overline{u_1'^2}$, the production term is found positive $\mathcal{P}_{11} > 0$. At the wall, the dissipation term is counterbalanced by the viscous diffusion. Moreover, the pressure–velocity correlation gradient term is negative as expected, $\Pi_{11}^s < 0$. The orders of magnitude of the terms involved in the budget of $\overline{u_2'^2}$, $\overline{u_3'^2}$ and $\overline{u_1'u_2'}$ are smaller. Concerning the analysis of the role of pressure, Π_{22}^s is found positive as soon as $x_2^+ > 10$, $\Pi_{33}^s > 0$ and $\Pi_{22}^d + \Pi_{22}^s > 0$. One also knows that $\Pi_{11}^d = \Pi_{33}^d = 0$. The production

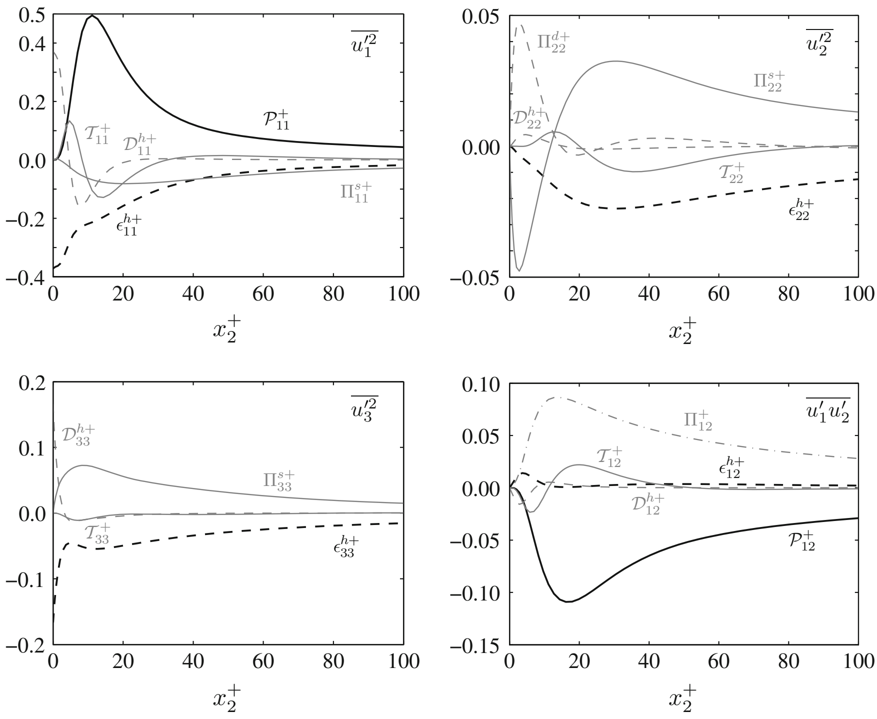


Fig. 2.6 Budgets of $\overline{u_1'^2}, \overline{u_2'^2}, \overline{u_3'^2}$ and $\overline{u_1'u_2'}$ for the channel flow at Reynolds number $\mathrm{Re}^+ = hu_\tau/\nu = 2003$, computed by Hoyas and Jiménez [621, 622]. See Eqs. (2.27) and (2.29) for the notations. Profiles are plotted as a function of the wall variable $x_2^+ = x_2 u_\tau/\nu$, where u_τ is the friction velocity, and all terms are made dimensionless with u_τ, ν and ρ

Fig. 2.7 Turbulent kinetic energy budget (2.30) for a channel flow at Reynolds number $\mathrm{Re}^+ = hu_\tau/\nu = 2003$, computed by Hoyas and Jiménez [621, 622]. Profiles are plotted as a function of the wall variable $x_2^+ = x_2 u_\tau/\nu$, and all terms are made dimensionless using u_τ, ν and ρ

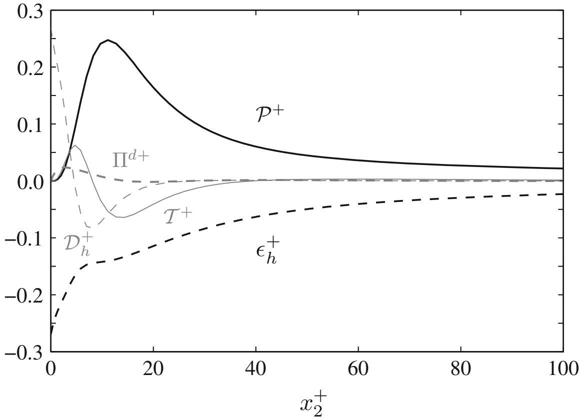
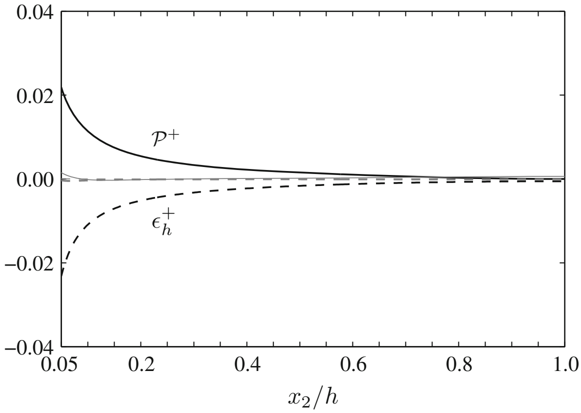


Fig. 2.8 Turbulent kinetic energy budget (2.30) for a channel flow at Reynolds number $\mathrm{Re}^+ = 2003$, computed by Hoyas and Jiménez [621, 622]. Profiles are plotted as a function of x_2/h, and all terms are made dimensionless using u_τ, ν and ρ



term $\mathcal{P}_{12}$ in the budget of $\overline{u_1'u_2'}$ is found negative, and according to (2.19), it can be written as,

$$\mathcal{P}_{12} = -\overline{u_2'^2}\frac{d\bar{U}_1}{dx_2} < 0$$

but this matches a positive production on the Reynolds stress $-\overline{u_1'u_2'}$.

Finally, the turbulent kinetic energy budget is represented in Fig. 2.7, according to Hoyas and Jiménez calculations [621, 622]. The role of pressure through the diffusion term is found to be very weak, even if pressure fluctuations are at the origin of the velocity components u_2' and u_3'. It can be also observed that for $x_2^+ \geq 30$, there is a quasi-equilibrium between dissipation and production. However this balance cannot extend to the center of the channel, where clearly the dissipation still exists, whereas the turbulent kinetic energy production tends to zero. In the budget of k_t shown in Fig. 2.8 as function of the distance x_2/h to the wall, the equilibrium subsists up to $x_2/h \simeq 0.5$. This interval over which production and dissipation are balanced, is characteristic of a wall flow, and this important result is developed in Chap. 3.

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