Chapter 2
Waveguides. Radiation Principle. Scattering Matrices

First, we briefly outline the chapter content. Section 2.1 is devoted to the boundary value problem

\[
(-\Delta - \mu)u(y, t) = f(y, t), \quad (y, t) \in \Pi,
\]
\[u(y, t) = 0, \quad (y, t) \in \partial \Pi,
\]
and the Fourier transform

\[
\hat{v}(\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \exp(-i\lambda t) v(t) \, dt
\]

reduces the problem to the family of problems depending on the parameter \(\lambda\):

\[
(-\Delta_y + \lambda^2 - \mu)\hat{u}(y, \lambda) = \hat{f}(y, \lambda), \quad y \in \Omega,
\]
\[\hat{u}(y, \lambda) = 0, \quad y \in \partial \Omega.
\]

If the inverse operator \(\mathcal{A}(\lambda, \mu)^{-1}\) of problem (2.0.3) exists for all \(\lambda \in \mathbb{R}\), the \(\mu\) being fixed, we obtain a solution \(u\) to problem (2.0.1) of the form

\[
u(\cdot, t) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \exp(i\lambda t) \mathcal{A}(\lambda, \mu)^{-1} \hat{f}(:, \lambda) \, d\lambda.
\]

However, the spectrum of the pencil \(\lambda \mapsto \mathcal{A}(\lambda, \mu)\), that is, the set of numbers \(\lambda\) such that the operator \(\mathcal{A}(\lambda, \mu)\) is not invertible, consists of an imaginary number sequence accumulating at infinity and, for sufficiently large \(\mu\), additionally contains finitely many real numbers. Therefore, formula (2.0.4) can fail and we will use the complex Fourier transform

\[
\hat{v}(\lambda) = (2\pi)^{-1/2} \int_{\mathbb{R}} \exp(-i\lambda t) v(t) \, dt, \quad \lambda \in \mathbb{R} + i\beta,
\]
where \( \mathbb{R} + i\beta = \{ \lambda \in \mathbb{C} : \text{Im} \lambda = \beta \} \); there are the inversion formula

\[
v(t) = (2\pi)^{-1/2} \int_{\mathbb{R} + i\beta} \exp(i\lambda t) \hat{v}(\lambda) d\lambda
\]

and the Parseval equality

\[
\int_{\mathbb{R}} \exp(2\beta t)|v(t)|^2 dt = \int_{\mathbb{R} + i\beta} |\hat{v}(\lambda)|^2 d\lambda.
\]

Let us assume that the line \( \mathbb{R} + i\beta \) is free from the spectrum of \( \mathcal{A}(\cdot, \mu) \) and the \( f \) in (2.0.1) satisfies the condition

\[
\int_{\Pi} \exp(2\beta t)|f(y, t)|^2 dy dt = \int_{\mathbb{R} + i\beta} |\hat{f}(y, \lambda)|^2 dy d\lambda < \infty.
\]

Then, according to Theorem 2.1.4, there exists a unique solution \( u \) to problem (2.0.1) such that

\[
u(\cdot, t) = (2\pi)^{-1/2} \int_{\mathbb{R} + i\beta} \exp(i\lambda t) \mathcal{A}(\lambda, \mu)^{-1} \hat{f}(\cdot, \lambda) d\lambda \quad (2.0.5)
\]

and the inequality

\[
\sum_{|\alpha| + k \leq 2} \int_{\Pi} \exp(2\beta t)|\partial^k_x \partial^\alpha_y u(y, t)|^2 dy dt \leq C \int_{\Pi} \exp(2\beta t)|f(y, t)|^2 dy dt
\]

holds with a constant \( C \) independent of \( f \).

These considerations motivate the statement of the boundary value problem in the domain \( G \) with cylindrical ends

\[
- \Delta u(x) - \mu u(x) = f(x), \quad x \in G, \quad u(x) = 0, \quad x \in \partial G,
\]

in function spaces with weighted norms (see Fig. 1.2 and the definition of \( G \) just after the figure). For integer \( l \geq 0 \), we denote by \( H^l(G) \) the Sobolev space with norm

\[
\|v; H^l(G)\| = \left( \sum_{j=0}^{l} \int_G \sum_{|\alpha| = j} |D^\alpha_x v(x)|^2 dx \right)^{1/2}.
\]

For real \( \beta \), we denote by \( \rho_\beta \) a smooth positive function on \( \overline{G} \) given by the equality

\[
\rho_\beta(x) = \exp(\beta|x|) \text{ for large } |x|.
\]

We also introduce the space \( H^1_\beta(G) \) with norm \( \|u; H^1_\beta(G)\| = \|\rho_\beta u; H^1(G)\| \). Let \( \hat{H}^2_\beta(G) \) denote the closure in \( H^2_\beta(G) \) of the set of smooth functions in \( \overline{G} \) that have compact supports in \( \overline{G} \) and vanish on \( \partial G \). The
operator \( u \mapsto (-\Delta - \mu)u \) of problem (2.0.6) implements a continuous mapping

\[ A_\beta(\mu) : \dot{H}^2_\beta(G) \to \dot{H}^0_\beta(G). \]

We denote by ker\( A_\beta(\mu) \) the kernel of \( A_\beta(\mu) \), i.e. the space \( \{ u \in \dot{H}^2_\beta(G) : A_\beta(\mu)u = 0 \} \), and we denote by \( \text{Im} A_\beta(\mu) \) the range of \( A_\beta(\mu) \).

\[ \text{Im} A_\beta(\mu) = \{ f \in H^0_\beta(G) : f = A_\beta(\mu)u, u \in \dot{H}^2_\beta(G) \}. \]

The operator \( A_\beta(\mu) \) is called Fredholm if \( \text{Im} A_\beta(\mu) \) is closed, and ker\( A_\beta(\mu) \) and coker\( A_\beta(\mu) := H^0_\beta(G)/\text{Im} A_\beta(\mu) \) are finite-dimensional, where \( H^0_\beta(G)/\text{Im} A_\beta(\mu) \) is the factor space \( H^0_\beta(G) \) modulo \( \text{Im} A_\beta(\mu) \). From Theorem 2.2.2 it follows that \( A_\beta(\mu) \) is Fredholm for all \( \beta \in \mathbb{R} \) except a certain sequence accumulated at infinity. Moreover, \( \dim(H^0_\beta(G)/\text{Im} A_\beta(\mu)) = \dim \ker A_{-\beta}(\mu) \) and the index \( \text{Ind} A_\beta(\mu) \) of \( A_\beta(\mu) \) can be defined by

\[ \text{Ind} A_\beta(\mu) = \dim \ker A_\beta(\mu) - \dim \ker A_{-\beta}(\mu). \]

We describe the asymptotics at infinity of solutions to problem (2.0.6) and calculate the difference \( \text{Ind} A_\beta(\mu) - \text{Ind} A_\gamma(\mu) \). Then we make use of these results when defining the scattering matrix and proving the existence of a unique solution to problem (2.0.6) subject to radiation conditions at infinity (the radiation principle).

2.1 Boundary Value Problem in a Cylinder

2.1.1 Statement of the Problem. Operator Pencil

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with smooth boundary \( \partial \Omega \). In the cylinder \( \Pi = \{(y, t) : y = (y_1, \ldots, y_n) \in \Omega, t \in \mathbb{R} \} \), we consider the problem

\[
(-\Delta - \mu)u(y, t) = f(y, t), \quad (y, t) \in \Pi, \\
u(y, t) = 0, \quad (y, t) \in \partial \Pi,
\]

where

\[
\Delta = \Delta_y + \partial_t^2, \quad \Delta_y = \sum_{j=1}^n \partial_j^2, \quad \partial_j = \partial/\partial y_j.
\]

We apply to problem (2.1.1) the Fourier transform

\[
\hat{v}(\lambda) = (2\pi)^{-1/2} \int_{-\infty}^{+\infty} \exp(-i\lambda t)v(t)\, dt
\]
and obtain the family of boundary value problems depending on the parameter $\lambda$:

\[ (-\Delta_y + \lambda^2 - \mu)\hat{u}(y, \lambda) = \hat{f}(y, \lambda), \quad y \in \Omega, \]
\[ \hat{u}(y, \lambda) = 0, \quad y \in \partial\Omega. \]

(2.1.3)

Now, we introduce an operator-valued function $\mathbb{C} \ni \lambda \mapsto \mathcal{A}(\lambda, \mu)$ defined by the equality

\[ \mathcal{A}(\lambda, \mu)v(y) = (-\Delta_y + \lambda^2 - \mu)v(y), \quad y \in \Omega, \]

(2.1.4)

for the functions $v$, smooth in $\overline{\Omega}$ and equal to zero on $\partial\Omega$; for the time being, the parameter $\mu$ is fixed. The function $\mathcal{A}(\cdot, \mu)$ is called an operator pencil. A number $\lambda_0 \in \mathbb{C}$ is said to be an eigenvalue of $\mathcal{A}(\cdot, \mu)$ if there exists a nontrivial solution $\varphi_0$ (an eigenvector) to the equation $\mathcal{A}(\lambda_0, \mu)v = 0$, that is, the $\lambda_0$ and $\varphi_0$ satisfy the boundary value problem

\[ (-\Delta_y + \lambda_0^2 - \mu)\varphi_0(y) = 0, \quad y \in \Omega, \]
\[ \varphi_0(y) = 0, \quad y \in \partial\Omega. \]

We also consider the problem

\[ (-\Delta_y - \mu)v(y) = 0, \quad y \in \Omega, \]
\[ v(y) = 0, \quad y \in \partial\Omega, \]

(2.1.5)

with spectral parameter $\mu$. The eigenvalues of problem (2.1.5) are called the thresholds of problem (2.1.1). The thresholds form a positive sequence $\tau_1 < \tau_2 < \ldots$, which strictly increases to infinity. Any eigenvalue $\tau_l$ is of finite multiplicity, that is, there exist at most finitely many linearly independent eigenvectors corresponding to $\tau_l$. Let us introduce the non-decreasing sequence $\{\mu_k\}_{k=1}^{\infty}$ of the eigenvalues of problem (2.1.5) counted according to their multiplicity. Generally speaking, the numbering of $\tau_l$ and that of $\mu_k$ are different; every $\mu_k$ coincides with one of the thresholds $\tau_l$.

For any $\mu$, the eigenvalues of the pencil $\lambda \mapsto \mathcal{A}(\lambda, \mu)$ are defined by the equality $\lambda_{k}^{\pm}(\mu) = \pm (\mu - \mu_k)^{1/2}$; more precisely, we set $\lambda_{k}^{\pm}(\mu) = \pm i(\mu_k - \mu)^{1/2}$ for $\mu_k > \mu$ with $(\mu_k - \mu)^{1/2} > 0$ and $\lambda_{k}^{\pm}(\mu) = \pm i(\mu - \mu_k)^{1/2}$ for $\mu_k < \mu$ with $(\mu - \mu_k)^{1/2} > 0$. If $\mu = \mu_k$, we have $\lambda_{k}^{\pm}(\mu) = \lambda_{k}^{\pm}(\mu) = 0$; in such a case we will sometimes write $\lambda_{k}^{0}(\mu)$ instead of $\lambda_{k}^{\pm}(\mu)$. Moreover, we sometimes write simply $\lambda_{k}^{\pm}$ instead of $\lambda_{k}^{\pm}(\mu)$. For $\mu_{k-1} < \mu < \mu_k$, the $\lambda_{k}^{\pm}(\mu), \lambda_{k+1}^{\pm}(\mu), \ldots$ are imaginary and the $\lambda_{1}^{\pm}(\mu), \ldots, \lambda_{k-1}^{\pm}(\mu)$ are real. To the eigenvalues $\lambda_{k}^{\pm}$ there corresponds the same eigenvector $\varphi_k$, which is also an eigenvector of problem (2.1.5) corresponding to the eigenvalue $\mu_k$. Any eigenvalue of the pencil $\mathcal{A}(\cdot, \mu)$ coincides with one of the eigenvalues mentioned in this paragraph.

We denote by $H^l(\Omega)$ the Sobolev function space in $\Omega$ with norm
\[ \|u\|_l = \left( \int_{\Omega} \sum_{|\alpha| \leq l} \left| \partial_y^\alpha u(y) \right|^2 dy \right)^{1/2}, \]  
where \( l = 0, 1, \ldots \); in particular, \( H^0(\Omega) = L_2(\Omega) \). Besides, we denote by \( \dot{H}^2(\Omega) \) the closure in \( H^2(\Omega) \) of the set of smooth functions in \( \overline{\Omega} \) that vanish on \( \partial \Omega \).

Let us consider the problem

\[ \begin{aligned}
-\Delta_y + \lambda^2 - \mu v(y) &= f(y), & y \in \Omega, \\
v(y) &= 0, & y \in \partial \Omega.
\end{aligned} \]  

**Proposition 2.1.1** (e.g., see [1])

(i) Assume that \( \lambda \) is not an eigenvalue of the pencil \( \mathcal{A}(\cdot, \mu) \), the \( \mu \) being fixed. Then for any \( f \in L_2(\Omega) \) there exists a unique solution \( v \in \dot{H}^2(\Omega) \) to problem (2.1.7) and the inequality

\[ \sum_{j=0}^{2} |\lambda|^{2j} \|v\|_{2-j}^2 \leq C \|f\|_0^2 \]  
holds with a constant \( C \) independent of \( f \).

(ii) Let \( F \) be a closed subset in \( \mathbb{C} \) that belongs to a strip \( \{ \lambda \in \mathbb{C} : |\text{Im}\lambda| < h < +\infty \} \) and contains no eigenvalues of the pencil \( \mathcal{A}(\cdot, \mu) \). Then, for any \( \lambda \in F \), estimate (2.1.8) holds with a constant \( C = C(F) \) that depends on \( F \) and remains independent of \( \lambda \) and \( f \).

Let \( \lambda_0 \) be an eigenvalue of \( \mathcal{A}(\cdot, \mu) \), and let \( \varphi^0 \) be an eigenvector corresponding to \( \lambda_0 \). Smooth functions \( \varphi^1, \ldots, \varphi^{m-1} \) on \( \overline{\Omega} \) which vanish on \( \partial \Omega \) and satisfy

\[ \sum_{k=0}^{l} \frac{1}{k!} \partial_y^k \mathcal{A}(\lambda_0, \mu) \varphi^{l-k} = 0, \quad l = 1, \ldots, m - 1, \]  
are called generalized eigenvectors. The ordered collection \( \varphi^0, \varphi^1, \ldots, \varphi^{m-1} \) is said to be a Jordan chain corresponding to \( \lambda_0 \). Clearly, in view of (2.1.4), the relations (2.1.9) take the form

\[ \begin{aligned}
\mathcal{A}(\lambda_0, \mu) \varphi^0 &= 0, \\
\mathcal{A}(\lambda_0, \mu) \varphi^1 + 2 \lambda_0 \varphi^0 &= 0, \\
\mathcal{A}(\lambda_0, \mu) \varphi^l + 2 \lambda_0 \varphi^{l-1} + 2 \varphi^{l-2} &= 0, \quad l = 2, \ldots, m - 1.
\end{aligned} \]  

There are no generalized eigenvectors for \( \lambda_0^\pm \neq 0 \). Indeed, assuming, for example, that a Jordan chain \( \varphi^0_k, \varphi^1_k \) exists for \( \lambda_0^+ \neq 0 \), we obtain the equations

\[ \begin{aligned}
\mathcal{A}(\lambda_0^+, \mu) \varphi^0_k &= 0, \\
\mathcal{A}(\lambda_0^+, \mu) \varphi^1_k + 2 \lambda_0^+ \varphi^0_k &= 0.
\end{aligned} \]
which can be written in the form
\[
(-\Delta y - \mu_k)\varphi_k^0 = 0,
\tag{2.1.11}
\]
\[
(-\Delta y - \mu_k)\varphi_k^1 + 2\lambda_k^0 \varphi_k^0 = 0.
\]

The boundary value problem
\[
(-\Delta y - \mu_k)v(y) = f(y), \quad y \in \Omega; \quad v(y) = 0, \quad y \in \partial\Omega,
\tag{2.1.12}
\]
has a solution, if and only if \((f, \psi)_\Omega = 0\) for each eigenvector \(\psi\) of this problem that corresponds to the eigenvalue \(\mu_k\); the \((f, \psi)_\Omega\) denotes the inner product in \(L_2(\Omega)\).

Therefore, there is no solution \(\varphi_k^1\) to the equation (2.1.11) with \(\lambda_k^+ \neq 0\). In the case of \(\mu = \mu_k\), we have \(\lambda_k^+ = \lambda_k^- = 0\) and a Jordan chain \(\varphi_k^0, \varphi_k^1\). Both of these vectors satisfy the same homogeneous boundary value problem (2.1.12) with \(f = 0\); the \(\varphi_k^0\) must be nonzero, and the \(\varphi_k^1\) may equal 0. It is easy to see from the equation (2.1.10) with \(l = 0\) and \(\lambda_0 = 0\) that there is no generalized eigenvector \(\varphi_k^3\).

The operator function \(\lambda \mapsto \mathcal{A}(\lambda, \mu)^{-1} : L_2(\Omega) \to \hat{H}^2(\Omega)\) except the poles at the eigenvalues of the pencil \(\lambda \mapsto \mathcal{A}(\lambda, \mu)\) is holomorphic everywhere. To describe the behavior of \(\mathcal{A}(\lambda, \mu)^{-1}\) in a neighborhood of the poles, we specify the general Keldysh’s theorem for our problem. Let \(\tau\) be an eigenvalue of problem (2.1.5) and let \(J\) be the geometric multiplicity of \(\tau\). We introduce a basis \(\varphi(0,1), \ldots, \varphi(0,J)\) of the eigenspace corresponding to \(\tau\). For \(\mu > \tau\), we denote by \(\lambda^\pm = \lambda^\pm(\mu)\) the eigenvalues \(\pm(\mu - \tau)^{1/2}\) of the pencil \(\mathcal{A}(\cdot, \mu)\). The multiplicity of each of the \(\lambda^\pm\) is equal to \(J\), and the eigenspace is spanned by \(\varphi(0,1), \ldots, \varphi(0,J)\). According to the Keldysh theorem, in a neighborhood of \(\lambda^+\) there holds the representation
\[
\mathcal{A}(\lambda, \mu)^{-1} = (\lambda - \lambda^+)^{-1} \sum_{j=1}^J (\cdot, \psi(0,j))_\Omega \varphi(0,j) + \Gamma(\lambda),
\tag{2.1.13}
\]
where \((u,v)_\Omega\) denotes the inner product in \(L_2(\Omega)\), \(\Gamma(\lambda) : L_2(\Omega) \to \hat{H}^2(\Omega)\) is a holomorphic function, and the \(\psi(0,1), \ldots, \psi(0,J)\) are eigenvectors of the pencil \(\mathcal{A}(\cdot, \mu)\) that correspond to \(\lambda^+\) (and, simultaneously, to \(\lambda^-\)) and satisfy the conditions
\[
(\partial_\lambda \mathcal{A}(\lambda^+, \mu)\varphi(0,j), \psi(0,k))_\Omega = \delta_{jk}.
\]
Since \(\partial_\lambda \mathcal{A}(\lambda^+, \mu) = 2\lambda^+\), we have \(2\lambda^+ (\psi(0,j), \psi(0,k))_\Omega = \delta_{jk}\) and, assuming \(\|\varphi(0,j)\|_0 = 1\), obtain \(\psi(0,j) = (2\lambda^+)^{-1/2}\varphi(0,j)\). Therefore, representation (2.1.13) takes the form
\[
\mathcal{A}(\lambda, \mu)^{-1} = (\lambda - \lambda^+)^{-1} \sum_{j=1}^J (2\lambda^+)^{-1} (\cdot, \varphi(0,j))_\Omega \varphi(0,j) + \Gamma(\lambda).
\tag{2.1.14}
To obtain a representation for $\mathfrak{A}(\lambda, \mu)^{-1}$ in a neighborhood of the pole $\lambda^-$, it suffices to change $\lambda^+$ for $\lambda^-$ in (2.1.14).

For $\mu < \tau$, we set $\lambda^\pm(\mu) = i(\tau - \mu)^{1/2}$, where $(\tau - \mu)^{1/2} > 0$, and denote by $\varphi^{(0,1)}, \ldots, \varphi^{(0,J)}$ an orthonormal basis in the eigenspace corresponding to $\lambda^\pm$ (recall that $\lambda^+(\mu)$ and $\lambda^-(\mu)$ have the same eigenspace). We arrive at the following assertion.

**Proposition 2.1.2** For any real $\mu \neq \tau$, the operator function $\lambda \mapsto \mathfrak{A}(\lambda, \mu)^{-1}$ admits the representation

$$
\mathfrak{A}(\lambda, \mu)^{-1} = (\lambda - \lambda^\pm)^{-1} \sum_{j=1}^J (2\lambda^\pm)^{-1}(\cdot, \varphi^{(0,j)})(\Omega)\varphi^{(0,j)} + \Gamma(\lambda)
$$

in a neighborhood of $\lambda^\pm = \lambda^\pm(\mu)$ with holomorphic function $\lambda \mapsto \Gamma(\lambda) : L^2(\Omega) \to \dot{H}^2(\Omega)$.

Now, we suppose that $\mu = \tau$. Then $\lambda^0 = 0$ is an eigenvalue of the pencil $\lambda \mapsto \mathfrak{A}(\lambda, \mu)$; the geometric multiplicity of $\lambda^0$ is equal to $J$. Let $\varphi^{(0,1)}, \ldots, \varphi^{(0,J)}$ be a basis in the eigenspace corresponding to $\lambda^0$ and $\varphi^{(1,j)}$, $\varphi^{(1,j)}$ a Jordan chain, where $j = 1, \ldots, J$. By the Keldysh theorem, in a neighborhood of the eigenvalue $\lambda^0$ there holds the representation

$$
\mathfrak{A}(\lambda, \mu)^{-1} = \sum_{j=1}^J \sum_{k=1}^2 (\lambda - \lambda^0)^{-k} \sum_{q=0}^{2-k} (\cdot, \psi^{(q,j)})(\Omega)\psi^{(2-k-q,j)} + \Gamma(\lambda),
$$

where $\psi^{(0,j)}$, $\psi^{(1,j)}$, $j = 1, \ldots, J$, is a collection of Jordan chains of the pencil $\mathfrak{A}(\cdot, \mu)$ that correspond to the $\lambda^0$ and satisfy the conditions

$$
\sum_{p+q+r=2+v} \frac{1}{p!} (\partial_\lambda^p \mathfrak{A}(\lambda^0, \mu)\varphi^{(q,\sigma)}(\cdot, \psi^{(r,\xi)})(\Omega) = \delta_{\sigma,\xi}\delta_{0,v},
$$

with $\sigma, \xi = 1, \ldots, J$ and $v = 0, 1$; the operator function $\lambda \mapsto \Gamma(\lambda) : L^2(\Omega) \to \dot{H}^2(\Omega)$ is holomorphic in a neighborhood of $\lambda^0$. Let the basis $\varphi^{(0,1)}, \ldots, \varphi^{(0,J)}$ be orthonormal and let every generalized eigenvector $\varphi^{(1,j)}$ be zero. Then (2.1.17) reduces to the relations

$$
(\varphi^{(0,\sigma)}, \psi^{(0,\xi)})(\Omega) = \delta_{\sigma,\xi}, \quad (\varphi^{(0,\sigma)}, \psi^{(1,\xi)})(\Omega) + (\psi^{(1,\sigma)}, \psi^{(0,\xi)})(\Omega) = 0.
$$

The equalities $\varphi^{(0,\sigma)}, \psi^{(0,\xi)})(\Omega) = \delta_{\sigma,\xi}$ imply that $\psi^{(0,\sigma)} = \varphi^{(0,\sigma)}$ for $\sigma = 1, \ldots, J$. Since $\varphi^{(1,\sigma)} = 0$, we have $\varphi^{(0,\sigma)}, \psi^{(1,\xi)})(\Omega) = 0$, which leads to $\psi^{(1,\xi)} = 0$ for $\xi = 1, \ldots, J$. 


Proposition 2.1.3 For $\mu = \tau$, in a neighborhood of $\lambda^0 = 0$ the operator function $
abla(\lambda, \mu)^{-1}$ admits the representation

$$\nabla(\lambda, \mu)^{-1} = (\lambda - \lambda^0)^{-2} \sum_{j=1}^{J} (\cdot, \varphi^{(0,j)})(\Omega) \varphi^{(0,j)} + \Gamma(\lambda)$$

(2.1.18)

with holomorphic function $\lambda \mapsto \Gamma(\lambda) : L_2(\Omega) \rightarrow \dot{H}^2(\Omega)$.

2.1.2 The Solvability of the Problem in a Cylinder

Let $C^\infty_c(\Pi)$ denote the set of smooth functions with compact supports in $\Pi$; as before, $\Pi = \{y, t) : y \in \Omega, t \in \mathbb{R}\}$. For $l = 0, 1, \ldots$ and $\beta \in \mathbb{R}$, we introduce the space $H^l_\beta(\Pi)$ as the completion of $C^\infty_c(\Pi)$ in the norm

$$\|u; H^l_\beta(\Pi)\| = \left( \sum_{|\alpha|+k \leq l} \int_\Pi \exp(2\beta r)|\partial_t^k \partial_y^\alpha u(y, t)|^2 dydt \right)^{1/2}.$$  

(2.1.19)

We denote by $\dot{H}^2_\beta(\Pi)$ the closure in $H^2_\beta(\Pi)$ of the set of smooth functions in $\Pi$ that have compact supports in $\Pi$ and vanish on $\partial \Pi$. The operator of problem (2.1.1) implements a continuous mapping

$$A_\beta(\mu) : \dot{H}^2_\beta(\Pi) \ni u \mapsto (-\Delta - \mu)u \in H^0_\beta(\Pi).$$

(2.1.20)

We will use the complex Fourier transform

$$\hat{v}(\lambda) = (2\pi)^{-1/2} \int_\mathbb{R} \exp(-i\lambda t)v(t)dt, \quad \lambda \in \mathbb{R} + i\beta,$$

(2.1.21)

where $\mathbb{R} + i\beta = \{\lambda \in \mathbb{C} : \text{Im}\lambda = \beta\}$, the inversion formula

$$v(t) = (2\pi)^{-1/2} \int_{\mathbb{R} + i\beta} \exp(i\lambda t)\hat{v}(\lambda)d\lambda,$$

(2.1.22)

and the Parseval equality

$$\int_\mathbb{R} \exp(2\beta t)|v(t)|^2dt = \int_{\mathbb{R} + i\beta} |\hat{v}(\lambda)|^2d\lambda.$$  

(2.1.23)
Theorem 2.1.4. Let the line $\mathbb{R} + i\beta$ be free from the eigenvalues of the pencil $\lambda \mapsto A(\lambda, \mu)$. Then, for any $f \in H_\beta^0(\Pi)$ there exists a unique solution $u \in \dot{H}_\beta^2(\Pi)$ to problem (2.1.1). The estimate

$$\|u; H_\beta^2(\Pi)\| \leq C \|f; H_\beta^0(\Pi)\|$$

holds with a constant $C$ independent of $f$.

Proof. The Fourier transform (2.1.21) reduces problem (2.1.1) to the family of problems

$$(-\Delta_y + \lambda^2 - \mu)\hat{u}(y, \lambda) = \hat{f}(y, \lambda), \quad y \in \Omega, \quad \hat{u}(\lambda, y) = 0, \quad y \in \partial\Omega,$$

with $\lambda \in \mathbb{R} + i\beta$. This line contains no eigenvalues of the pencil $A(\cdot, \mu)$. Therefore, by Proposition 2.1.1, for any $\lambda \in \mathbb{R} + i\beta$ there exists a unique solution $\hat{u}(\cdot, \lambda) := A(\lambda, \mu)^{-1}\hat{f}(\cdot, \lambda)$ to problem (2.1.25), which subject to the inequality

$$\sum_{j=0}^{2}|\lambda|^{2j}\|\hat{u}(\cdot, \lambda)\|_{2-j}^2 \leq C \|\hat{f}(\cdot, \lambda)\|_0^2$$

and the constant $C$ is independent of $\lambda$ and $\hat{f}(\lambda, \cdot)$. Consequently,

$$\int_{\mathbb{R} + i\beta} \sum_{j=0}^{2}|\lambda|^{2j}\|\hat{u}(\cdot, \lambda)\|_{2-j}^2 d\lambda \leq C \int_{\mathbb{R} + i\beta} \|\hat{f}(\cdot, \lambda)\|_0^2 d\lambda.$$

By virtue of (2.1.23), the left-hand side is equivalent to $\|u; H_\beta^2(\Pi)\|^2$ and the right-hand side is equal to $C \|f; H_\beta^0(\Pi)\|^2$. Thus, the function

$$u(\cdot, t) = (2\pi)^{-1/2} \int_{\mathbb{R} + i\beta} \exp(i\lambda t)A(\lambda, \mu)^{-1}\hat{f}(\cdot, \lambda) d\lambda$$

satisfies problem (2.1.1) and admits estimate (2.1.24). \qed

2.1.3 Asymptotics of Solutions

Let us assume that $f$ is a smooth function with compact support in $\Pi$. Then the function $\lambda \mapsto \hat{f}(\cdot, \lambda)$ is analytic on $\mathbb{C}$ and rapidly decaying in any strip $\{|\lambda| \in \mathbb{C} :$
\(|\text{Im}\lambda| \leq h < \infty\) as \(\lambda \to \infty\). The function \(\hat{u}(\cdot, \lambda) = \mathcal{A}(\lambda, \mu) \hat{f}(\cdot, \lambda)\) is analytic everywhere except at the poles of the function \(\lambda \mapsto \mathcal{A}(\lambda, \mu)\). Moreover, in view of inequality (2.1.26), \(\hat{u}(\cdot, \lambda)\) is also rapidly decaying in the aforementioned strip as \(\lambda \to \infty\). Let \(\beta\) and \(\gamma\) be real numbers such that the lines \(\{\lambda \in \mathbb{C} : \text{Im}\lambda = \beta\}\) and \(\{\lambda \in \mathbb{C} : \text{Im}\lambda = \gamma\}\) contain no poles of \(\mathcal{A}(\cdot, \mu)\). Then, in a representation of the form (2.1.27), we can, using the residue theorem, change \(\beta\) for \(\gamma\).

We now calculate the residues of the function \(\lambda \mapsto F(\lambda) := (2\pi)^{-1/2} \exp (i\lambda t) \mathcal{A}(\lambda, \mu)^{-1} \hat{f}(\cdot, \lambda)\). (2.1.28)

By Proposition 2.1.2,

\[\text{res}\, F(\lambda)\big|_{\lambda = \lambda^\pm} = (2\pi)^{-1/2} \sum_{j=1}^{J} (2\lambda^\pm)^{-1} \int_{\Omega} \hat{f}(y, \lambda^\pm) \overline{\varphi^{(0,j)}(y)} \, dy \varphi^{(0,j)};\]

as before, \(\lambda^\pm = \lambda^\pm(\mu)\), where \(\lambda^\pm(\mu) = \pm (\mu - \tau)^{1/2}\) for \(\mu > \tau\) and \(\lambda^\pm(\mu) = \pm i(\tau - \mu)^{1/2}\) for \(\mu < \tau\). For the real \(\lambda^\pm\), we have

\[
\int_{\Omega} \hat{f}(y, \lambda^\pm) \varphi^{(0,j)}(y) \, dy = (2\pi)^{-1/2} \int_{\mathbb{R}} \int_{\Omega} \exp (-i\lambda^\pm s) f(y, s) \overline{\varphi^{(0,j)}(y)} \, dy \, ds
\]

\[
= (2\pi)^{-1/2} (f, Z^\pm_j)_{\Pi},
\]

where

\[
Z^\pm_j(y, s) = \exp (i\lambda^\pm s) \varphi^{(0,j)}(y)
\]

and \((\cdot, \cdot)_{\Pi}\) is the inner product in \(L_2(\Pi)\). Thus, for the real \(\lambda^\pm\),

\[\text{res}\, F(\lambda)\big|_{\lambda = \lambda^\pm} = (2\pi)^{-1} \sum_{j=1}^{J} (2\lambda^\pm)^{-1} (f, Z^\pm_j)_{\Pi} Z^\pm_j.\] (2.1.30)

For the imaginary \(\lambda^\pm\), we obtain

\[\text{res}\, F(\lambda)\big|_{\lambda = \lambda^\pm} = (2\pi)^{-1} \sum_{j=1}^{J} (2\lambda^\pm)^{-1} (f, Z^\pm_j^\mp)_{\Pi} Z^\pm_j,\] (2.1.31)

where \(Z^\pm_j(y, s)\) is defined by equality (2.1.29). By Proposition 2.1.3, for \(\lambda^0 = 0\) we have
\[ \text{res} F(\lambda)|_{\lambda=\lambda^0} = (2\pi)^{-1/2} \sum_{j=1}^{J} \varphi^{(0,j)} \int_{\Omega} (it \, \hat{f}(y, 0) + \partial_\lambda \hat{f}(y, 0)) \, \varphi^{(0,j)}(y) \, dy \]

(2.1.32)

\[ = (2\pi)^{-1} \sum_{j=1}^{J} \varphi^{(0,j)} \int_{\mathbb{R}} \int_{\Omega} (it - is) f(y, s) \varphi^{(0,j)}(y) \, dy \, ds \]

\[ = (2\pi)^{-1} \sum_{j=1}^{J} \left( (f, Z_j^0) \Pi Z_j^1 + (f, Z_j^1) \Pi Z_j^0 \right) \]

with

\[ Z_j^0(y, t) = \varphi^{(0,j)}(y), \quad Z_j^1(y, t) = it \varphi^{(0,j)}(y). \]  

(2.1.33)

**Lemma 2.1.5** Let \( \lambda^\pm \) be an eigenvalue of \( \mathcal{A}(\cdot, \mu) \), \( \text{Im} \lambda^\pm \neq 0 \), and \( \beta < \text{Im} \lambda^+ < \gamma \) (\( \beta < \text{Im} \lambda^- < \gamma \)). Then, for any \( Z = Z_j^- (Z = Z_j^+) \) in (2.1.29), the estimate

\[ |(f, Z)_\Pi| \leq C (\| f; H^0_{\beta}(\Pi) \| + \| f; H^0_{\gamma}(\Pi) \|) \]

holds for \( f \in H^0_{\beta}(\Pi) \cap H^0_{\gamma}(\Pi) \) with constant \( C \) independent of \( f \). If \( \beta < 0 < \gamma \) and \( \lambda^\pm \) is a real eigenvalue, this estimate is also valid for any \( Z_j^\pm \) in (2.1.29), \( Z = Z_j^0 \), and \( Z = Z_j^1 \) in (2.1.33).

**Proof** We choose \( \eta_1, \eta_2 \in C^\infty(\mathbb{R}) \) such that \( 0 \leq \eta_2(t) \leq 1, \eta_2(t) = 1 \) for \( t \geq 1, \eta_2(t) = 0 \) for \( t \leq -1, \) and \( \eta_1 + \eta_2 = 1 \). For instance, we assume that \( \text{Im} \lambda^+ \neq 0 \) and \( \beta < \text{Im} \lambda^+ < \gamma \). Then, for \( Z = Z_j^- \), we have

\[ |(f, Z)_\Pi| \leq C \int_{\mathbb{R}} |f(y, s)| \exp(-s \text{Im} \lambda^-) \, dy \, ds \]

\[ \leq \int_{\mathbb{R}} \eta_1(s) |f(y, s)| \exp(\beta s) \exp(-s (\text{Im} \lambda^- + \beta)) \, dy \, ds \]

\[ + \int_{\mathbb{R}} \eta_2(s) |f(y, s)| \exp(\gamma s) \exp(-s (\text{Im} \lambda^- + \gamma)) \, dy \, ds. \]

Since \( \text{Im} \lambda^+ = -\text{Im} \lambda^- \), we obtain \( \text{Im} \lambda^- + \gamma > 0 \) and \( \text{Im} \lambda^- + \beta < 0 \). Therefore,

\[ |(f, Z)_\Pi| \leq C \| f; H^0_{\beta}(\Pi) \| \left( \int_{-\infty}^{0} \exp(-2s (\text{Im} \lambda^- + \beta)) \, ds \right)^{1/2} \]

\[ + C \| f; H^0_{\gamma}(\Pi) \| \left( \int_{0}^{+\infty} \exp(-2s (\text{Im} \lambda^- + \gamma)) \, ds \right)^{1/2} \]

\[ \leq C (\| f; H^0_{\beta}(\Pi) \| + \| f; H^0_{\gamma}(\Pi) \|). \]

The next theorem describes the asymptotics of a solution to problem (2.1.1) at infinity.
\textbf{Theorem 2.1.6} Let the lines \( \{ \lambda \in \mathbb{C} : \Im \lambda = \beta \} \) and \( \{ \lambda \in \mathbb{C} : \Im \lambda = \gamma \} \) be free from the eigenvalues of the pencil \( \mathcal{A}(\cdot, \mu) \) and \( f \in H^0_\beta(\Pi) \cap H^0_\gamma(\Pi) \). Then

\[ u_\beta = u_\gamma + 2\pi i \mathcal{S}(\beta, \gamma), \tag{2.1.34} \]

where \( u_\beta \) and \( u_\gamma \) are solutions to problem (2.1.1) in \( \hat{H}^2_\beta(\Pi) \) and \( \hat{H}^2_\gamma(\Pi) \) respectively, \( \beta < \gamma \), and \( \mathcal{S}(\beta, \gamma) \) is the sum of the residues of function (2.1.28) in the strip \( \{ \lambda \in \mathbb{C} : \beta < \Im \lambda < \gamma \} \). All functions \( Z^\pm_j \) in (2.1.29), \( Z^0_j \), and \( Z^1_j \) in (2.1.33) satisfy homogeneous problem (2.1.1). Equality (2.1.34) can be taken as an asymptotics of \( u_\beta(y, t) \) for \( t \to +\infty \) and as an asymptotics of \( u_\gamma(y, t) \) for \( t \to -\infty \); the \( u_\gamma(u_\beta) \) plays the role of a remainder as \( t \) tends to \( +\infty \) (to \( -\infty \)).

\textit{Proof} For \( f \) in the set \( C^\infty(\Pi) \) of smooth functions with compact support in \( \Pi \), equality (2.1.34) was discussed at the beginning of Sect. 2.1.3. By Lemma 2.1.5, the functionals \( f \mapsto (f, Z)_\Pi \) in \( \mathcal{S}(\beta, \gamma) \) are continuous on \( H^0_\beta(\Pi) \cap H^0_\gamma(\Pi) \). Therefore, we can obtain (2.1.34) for \( f \in H^0_\beta(\Pi) \cap H^0_\gamma(\Pi) \) by closing \( C^\infty(\Pi) \) in the norm \( \| f \| : = \| f; H^0_\beta(\Pi) \| + \| f; H^0_\gamma(\Pi) \| \) of the space \( H^0_\beta(\Pi) \cap H^0_\gamma(\Pi) \).

Straightforward calculation shows that the functions \( Z^\pm \), \( Z^0 \), and \( Z^1 \) satisfy homogeneous problem (2.1.1) (it also follows from (2.1.34) and the fact that the difference \( u_1 - u_2 \) satisfies this problem).

We now rewrite (2.1.34) using a more detailed notation. Let \( \lambda^\pm_k = \lambda^\pm_k(\mu) \) be the eigenvalue notation defined in the paragraph before formula (2.1.6). Besides, we assume that \( Z^\pm_k \) corresponds to \( \lambda^\pm_k(\mu) \), i.e., \( Z^\pm_k(y, t) = \exp(i\lambda^\pm_k(\mu)t)\varphi_k(y) \), where \( \varphi_k \) is an eigenvector corresponding to \( \lambda^\pm_k(\mu) \) etc. [see (2.1.29), (2.1.32), and (2.1.33)]. Then (2.1.34) takes the form

\[ u_\beta - u_\gamma = \sum_{\max[0, \beta] < \Im \lambda^+_k < \gamma} i(2\lambda^+_k)^{-1}(f, Z^-_k) \Pi Z^+_k \]

\[ + \sum_{0 > \Im \lambda^-_k > \min[0, \beta]} i(2\lambda^-_k)^{-1}(f, Z^+_k) \Pi Z^-_k \]

\[ + \sum_{\lambda^+_k \in \mathbb{R}} i(2\lambda^+_k)^{-1}(f, Z^+_k) \Pi Z^+_k \]

\[ + \sum_{\lambda^-_k = 0} i \left( (f, Z^0_k) \Pi Z^1_k + (f, Z^1_k) \Pi Z^0_k \right) \]

and the two last sums (corresponding to the real eigenvalues) are absent if \( \beta \gamma \geq 0 \).

The right-hand side of (2.1.35) is a linear combination of the solutions \( Z^+_k \), \( Z^-_k \), and so on, to homogeneous problem (2.1.1) (where \( f = 0 \)). The coefficients \( i(2\lambda^+_k)^{-1}(f, Z^-_k) \Pi \), \( i(2\lambda^-_k)^{-1}(f, Z^+_k) \Pi \), and so on, of this linear combination are continuous functionals on the space \( H^0_\beta(\Pi) \cap H^0_\gamma(\Pi) \).
The following simplifications of equality (2.1.35) for some special cases are evident. For $0 < \beta < \gamma$, (2.1.35) reduces to the form

$$u_\beta - u_\gamma = \sum_{\beta < \text{Im} \lambda_k < \gamma} i(2\lambda_k^+)^{-1}(f, Z_k^+) \prod Z_k^+.$$ 

In the case of $\beta < \gamma < 0$,

$$u_\beta - u_\gamma = \sum_{\beta < \text{Im} \lambda_k < \gamma} i(2\lambda_k^-)^{-1}(f, Z_k^-) \prod Z_k^-.$$ 

If the strip $\beta < \text{Im} \lambda < \gamma$ contains no eigenvalues of the pencil $\mathcal{A}(\cdot, \mu)$, except the real ones, we have

$$u_\beta - u_\gamma = \sum_{\lambda_k^\pm \in \mathbb{R}} i(2\lambda_k^\pm)^{-1}(f, Z_k^\pm) \prod Z_k^\pm + \sum_{\lambda_k^0 = 0} i \left( (f, Z_k^0) \prod Z_k^1 + (f, Z_k^1) \prod Z_k^0 \right);$$ 

if, in addition, the $\mu$ is not a threshold, this equality takes the form

$$u_\beta - u_\gamma = \sum_{\lambda_k^\pm \in \mathbb{R}} i(2\lambda_k^\pm)^{-1}(f, Z_k^\pm) \prod Z_k^\pm,$$

which is the most important situation in the subsequent chapters.

2.2 Problem in a Domain $G$ with Cylindrical Ends

2.2.1 Statement and Fredholm Property of the Problem

Let $G$ be a domain in $\mathbb{R}^{n+1}$ with smooth boundary $\partial G$ coinciding, outside a large ball, with the union $\Pi_+ \cup \cdots \cup \Pi_+^{T}$ of finitely many non-overlapping semicylinders

$$\Pi_+^r = \{(y^r, t^r) : y^r \in \Omega^r, t^r > 0\},$$

where $(y^r, t^r)$ are local coordinates in $\Pi_+^r$ and $\Omega^r$ is a bounded domain in $\mathbb{R}^n$. We consider the problem

$$- \Delta u(x) - \mu u(x) = f(x), \quad x \in G,$$

$$u(x) = 0, \quad x \in \partial G.$$  \hspace{1cm} (2.2.1)
For integer \( l \geq 0 \), we denote by \( H^l(G) \) the Sobolev space with norm
\[
\|v; H^l(G)\| = \left( \sum_{j=0}^{l} \int_G \sum_{|\alpha|=j} |D_x^{\alpha} v(x)|^2 \, dx \right)^{1/2}.
\]

We assume that \( \beta = (\beta^1, \ldots, \beta^T) \) with real \( \beta^r \) and denote by \( \rho_\beta \) a smooth positive on \( G \) function given on \( \Pi^r_+ \) by the equality \( \rho_\beta(y^r, t^r) = \exp(\beta^r t^r) \).

We also introduce the space \( \hat{H}^l_\beta(G) \) with norm \( \|u; \hat{H}^l_\beta(G)\| = \|\rho_\beta u; H^l(G)\| \).

Let \( \hat{H}^2_\beta(G) \) denote the closure in \( H^2_\beta(G) \) of the set of smooth functions in \( G \) that have compact supports in \( G \) and vanish on \( \partial G \). The operator \( u \mapsto (-\Delta - \mu)u \) of problem (2.2.1) implements a continuous mapping
\[
\mathcal{A}_\beta(\mu) : \hat{H}^2_\beta(G) \to H^0_\beta(G). \tag{2.2.2}
\]

We denote by \( \ker \mathcal{A}_\beta(\mu) \) the kernel of \( \mathcal{A}_\beta(\mu) \), i.e. the space \( \{u \in \hat{H}^2_\beta(G) : \mathcal{A}_\beta(\mu)u = 0\} \), and denote by \( \text{Im} \mathcal{A}_\beta(\mu) \) the range of \( \mathcal{A}_\beta(\mu) \),
\[
\text{Im} \mathcal{A}_\beta(\mu) = \{f \in H^0_\beta(G) : f = \mathcal{A}_\beta(\mu)u, u \in \hat{H}^2_\beta(G)\}.
\]

**Definition 2.2.1** The operator \( \mathcal{A}_\beta(\mu) \) is called Fredholm if \( \text{Im} \mathcal{A}_\beta(\mu) \) is closed and \( \ker \mathcal{A}_\beta(\mu) \) and \( \text{coker} \mathcal{A}_\beta(\mu) := H^0_\beta(G)/\text{Im} \mathcal{A}_\beta(\mu) \) are finite-dimensional, where \( H^0_\beta(G)/\text{Im} \mathcal{A}_\beta(\mu) \) is the factor space \( H^0_\beta(G) \) modulo \( \text{Im} \mathcal{A}_\beta(\mu) \).

Let us introduce an operator pencil \( \lambda \to \mathfrak{K}(\lambda, \mu) \) for \( (\cdot, \cdot)_G \) on the domain \( \Omega^r, r = 1, \ldots, T \).

**Theorem 2.2.2** \( \lambda \to \mathfrak{K}(\lambda, \mu) \) is Fredholm if and only if the line \( \{\lambda \in \mathbb{C} : \text{Im} \lambda = \beta^r\} \) is free from the eigenvalues of the pencil \( \mathfrak{K}(\cdot, \mu) \) for every \( r = 1, \ldots, T \).

(i) \( \dim(\mathcal{H}^0_r(G)/\text{Im} \mathcal{A}_{-\beta}(\mu)) = \dim \ker \mathcal{A}_{-\beta}(\mu) \).

(ii) \( f \in \text{Im} \mathcal{A}_{\beta}(\mu) \) if and only if \( (f, v)_G = 0 \) for all \( v \in \ker \mathcal{A}_{-\beta}(\mu) \) where \( (\cdot, \cdot)_G \) means the extension of the inner product in \( L_2(G) \) by continuity to the pair \( \mathcal{H}^0_r(G), H^0_\beta(G) \).

**2.2.2 Asymptotics of Solutions**

**Theorem 2.2.3** Let \( u \) be a solution to problem (2.2.1) such that \( u \in \hat{H}^2_\beta(G) \) with \( \beta = (\beta^1, \ldots, \beta^T) \). Let \( \eta_r f \in H^0_{\gamma^r}((\Pi^r_+ \cap G) \) for a certain \( r \), where \( \beta^r < \gamma^r \), \( \eta_r \) denotes a smooth function with support in \( \overline{\Pi^r_+ \cap G} \), and \( \eta_r(y^r, t^r) = 1 \) for \( t^r > T \) with a large \( T \). We assume the lines \( \{\lambda \in \mathbb{C} : \text{Im} \lambda = \beta^r\} \) and \( \{\lambda \in \mathbb{C} : \text{Im} \lambda = \gamma^r\} \) to be free from the eigenvalues of the pencil \( \mathfrak{K}(\cdot, \mu) \).
Then in $\Pi'_r$ for $t' > T$ there holds the equality

$$u = \sum_{\max(0, \beta') < \Im \lambda^+_k < \gamma'} c^+_k Z^+_k + \sum_{0 > \Im \lambda^-_k > \min(0, \beta') \leq \gamma'} c^-_k Z^-_k$$

$$(2.2.3)$$

$$+ \sum_{\lambda^+_k \in \mathbb{R}} c^+ \pm \lambda^+_k \pm Z^+ \kappa + \sum_{\lambda^-_k = 0} c^- \pm \lambda^-_k \pm Z^- \kappa + v,$$

where the functions $Z^+_k, Z^-_k$, and so on, are defined in $\Pi := \Pi' = \Omega' \times \mathbb{R}$ like those in (2.1.35), the $c^+_k, c^-_k$, and so on, are some constant coefficients and $\eta_r, v \in H^2_{\gamma'}(\Pi')$.

(The two last sums (corresponding to the real eigenvalues) are absent if $\beta' \gamma' > 0$.)

Proof We have

$$(-\Delta - \mu)(\eta_r u)(x) = g(x), \quad x \in \Pi', \quad (\eta_r u)(x) = 0, \quad x \in \partial \Pi',$$

where $g = \eta_r f - 2\nabla \eta_r \nabla u - u \Delta \eta_r$. Because $\nabla \eta_r$ and $\Delta \eta_r$ have compact supports, the $g$ belongs to $H^0_{\beta_1}(\Pi') \cap H^0_{\beta_2}(\Pi')$. Applying Theorems 2.1.4 and 2.1.6, we obtain (2.1.35) with $f = g, u_1 = \eta_r u$, and $v = u_1$. This leads to equality (2.2.3), where

$$c^+_k = i(2\lambda^+_k)^{-1}(g, Z^+_k)_{\Pi'} \quad c^-_k = i(2\lambda^-_k)^{-1}(g, Z^-_k)_{\Pi'} \quad \ldots \quad c^0_k = i(g, Z^+_1)_{\Pi'}.$$  

$(2.2.5)$

□

Note that, in the proof, the function $g$ depends on $f, u$, and $\eta_r$. Therefore, formulas (2.2.5) do not present explicit expressions of the coefficients in (2.2.3) as functionals defined immediately for $f$ in (2.2.1). Such expressions are given in Sect. 2.2.4.

2.2.3 Properties of the Index $\text{Ind} \ A_{\beta}(\mu)$ and of the Spaces $\ker A_{\beta}(\mu)$ and $\text{coker} A_{\beta}(\mu)$

Let $A_{\beta}(\mu)$ be Fredholm (see Definition 2.2.1). The difference $\dim \ker A_{\beta}(\mu) - \dim \text{coker} A_{\beta}(\mu)$ is called the index of $A_{\beta}(\mu)$ and denoted by $\text{Ind} A_{\beta}(\mu)$. Assuming both of the operators $A_{\beta}(\mu)$ and $A_{\gamma}(\mu)$ to be Fredholm, we calculate, in particular, the difference $\text{Ind} A_{\beta}(\mu) - \text{Ind} A_{\gamma}(\mu)$ in terms of the spectrum of the pencils $A(\cdot, \mu)$.

Recall that, for any non-zero eigenvalue $\lambda_0$ of a pencil $A(\cdot, \mu)$, there are no generalized eigenvectors and, consequently, the full multiplicity of $\lambda_0$ is equal to its geometric multiplicity, i.e., the full multiplicity coincides with $\dim \ker A(\lambda_0, \mu)$. If $\lambda_0 = 0$ turns out to be an eigenvalue of a certain $A(\cdot, \mu)$, for any eigenvector $\varphi^0 \in \ker A(\lambda_0, \mu)$ there exists a generalized eigenvector and the full multiplicity of $\lambda_0$ equals its doubled geometric multiplicity.
Theorem 2.2.4 Let $\beta = (\beta^1, \ldots, \beta^{T_1}, \beta^{T_2}, \ldots, \beta^T)$ and $\gamma = (\gamma^1, \ldots, \gamma^{T_1}, \beta^{T_2}, \ldots, \beta^T)$, where $\beta^r < \gamma^r$ for $r = 1, \ldots, T_1$, and let the lines $\mathbb{R} + i\beta^r$ and $\mathbb{R} + i\gamma^r$ be free from the eigenvalues of the pencil $\mathcal{A}(\cdot, \mu)$ for $r = 1, \ldots, T$. We denote by $\gamma$ the sum of the full multiplicities of the eigenvalues of the pencil $\mathcal{A}(\cdot, \mu)$ in the strip $\{\lambda \in \mathbb{C} : \beta^r < \text{Im}\lambda < \gamma^r\}$ with $r = 1, \ldots, T_1$ and set $\gamma = \gamma^1 + \cdots + \gamma^{T_1}$.

Then

$$\dim(\ker A_\beta(\mu)/\ker A_\gamma(\mu)) + \dim(\ker A_\gamma(\mu)/\ker A_\beta(\mu)) = \gamma, \quad (2.2.6)$$

$$\text{Ind} A_\beta(\mu) = \text{Ind} A_\gamma(\mu) + \gamma. \quad (2.2.7)$$

Proof We number all functions of the form $\eta_r Z_k^\pm, \eta_r Z_k^0$, and $\eta_r Z_k^1$ that correspond to the eigenvalues of the pencil $\mathcal{A}(\cdot, \mu)$ in the strip $\{\lambda \in \mathbb{C} : \beta^r < \text{Im}\lambda < \gamma^r\}$, $r = 1, \ldots, T_1$, by the same index and obtain the collection $Z_1, \ldots, Z_\gamma$. According to Theorem 2.2.3, any function $u$ in $\ker A_\beta(\mu)$ admits the asymptotics

$$u = c_1 Z_1 + \cdots + c_\gamma Z_\gamma + v, \quad (2.2.8)$$

with constant coefficients $c_j$ and $v$ in $\dot{H}_\gamma^2(G)$. Therefore, there exist at most $\gamma$ vectors in the space $\ker A_\beta(\mu)$ linearly independent modulo $\ker A_\gamma(\mu)$; we set $d := \dim(\ker A_\beta(\mu)/\ker A_\gamma(\mu))$ and have $0 \leq d \leq \gamma$. Without loss of generality, we assume that there exist vectors $U_j$ in $\ker A_\beta(\mu)$ such that

$$U_j = Z_j + \sum_{k=d+1}^{\gamma} c_{jk} Z_k + v_j, \quad j = 1, \ldots, d, \quad (2.2.9)$$

where $c_{jk} = \text{const}$ and $v_j \in \dot{H}_\gamma^2(G)$. Clearly, the $U_1, \ldots, U_d$ are linearly independent modulo $\ker A_\gamma(\mu)$.

Let $D$ denote $\dim(\ker A_\gamma(\mu)/\ker A_\beta(\mu))$; we will now verify that $D = \gamma - d$.

We first assume that $D < \gamma - d$ and denote by $\varphi_1, \ldots, \varphi_D$ a collection of vectors in $\ker A_\gamma(\mu)$ linearly independent modulo $\ker A_\beta(\mu)$. Then there exists a non-trivial linear combination $Z = c_{d+1}^0 Z_{d+1} + \cdots + c_{\gamma}^0 Z_{\gamma}$ such that $f := A_\beta(\mu) Z \in H_\gamma^0(G)$ and, moreover, $(f, \varphi_j)_G = 0$ for $j = 1, \ldots, D$. This and Theorem 2.2.2(iii) imply the existence of a function $V$ satisfying $A_\gamma(\mu) V = f$. Therefore, we have $U_0 := V - Z \in \ker A_\beta(\mu)$ and the vectors $U_0, U_1, \ldots, U_d$ are linearly independent modulo $\ker A_\gamma(\mu)$, which contradicts $d = \dim(\ker A_\beta(\mu)/\ker A_\gamma(\mu))$. Thus, we obtained the inequality $D \geq \gamma - d$.

Now, we suppose that $D > \gamma - d$. Let $\Phi_1, \ldots, \Phi_D$ be a collection of elements in $\ker A_\gamma(\mu)$ linearly independent modulo $\ker A_\beta(\mu)$. We choose a collection $\Phi_1, \ldots, \Phi_D$ in $H_\gamma^0(G)$ such that

$$(\Phi_j, \varphi_k)_G = \delta_{jk}, \quad j, k = 1, \ldots, D,$$

$$(\Phi_j, \psi)_G = 0 \quad \text{for all} \quad \psi \in \ker A_\beta(\mu).$$
2.2 Problem in a Domain $G$ with Cylindrical Ends

Then there exists $\chi_j$ that satisfies $A_\beta(\mu)\chi_j = \Phi_j$, where $j = 1, \ldots, D$. If needed, we can subtract from the $\chi_j$ a linear combination of $U_1, \ldots, U_d$ in (2.2.9) to provide the inclusions

$$\chi_j - \sum_{h=d+1}^\infty d_{jh}Z_h \in \dot{H}_\gamma^2(G), \quad j = 1, \ldots, D. \quad (2.2.10)$$

No nontrivial linear combination of $\chi_1, \ldots, \chi_D$ belongs to $\dot{H}_\gamma^2$; otherwise there is a linear combination of $A_\gamma(\mu)\chi_j = \Phi_j$ orthogonal to all of the vectors $\phi_1, \ldots, \phi_D$, which is impossible in view of the choice of the $\Phi_1, \ldots, \Phi_D$. This and (2.2.10) imply that $D \leq \infty - D$. Therefore, we obtain the equality $D = \infty$ and, consequently, equality (2.2.6).

Let us verify formula (2.2.7). According to Theorem 2.2.2(ii), $\dim \ker A_\beta(\mu) = \dim \ker A_\beta(\mu) - \dim \ker A_{\beta - \beta}(\mu)$, and the same with $\beta$ replaced by $\gamma$. From (2.2.6) it follows that

$$\dim \ker A_\beta(\mu) = \dim \ker A_\gamma(\mu) + d,$$
$$\dim \ker A_{\beta - \beta}(\mu) = \dim \ker A_{\gamma - \gamma}(\mu) + d - \infty,$$

and therefore $\text{Ind} A_\beta(\mu) = \text{Ind} A_\gamma(\mu) + \infty$. \hfill $\square$

2.2.4 Calculation of the Coefficients in the Asymptotics

Now, we are in a position to obtain explicit expressions for the coefficients in (2.2.3). We will use the notation $Z_j$ with $j = 1, \ldots, \infty$, defined at the beginning of the proof of Theorem 2.2.4, and introduce also

$$Z_j^+ := (2\lambda_k^\pm)^{-1} \eta_r Z_k^\pm \quad \text{for} \quad Z_j = \eta_r Z_k^\pm \quad \text{and} \quad \lambda_k^\pm \not\in \mathbb{R};$$
$$Z_j^* := (2\lambda_k^\pm)^{-1} \eta_r Z_k^\pm \quad \text{for} \quad Z_j = \eta_r Z_k^\pm \quad \text{and} \quad \lambda_k^\pm \in \mathbb{R} \setminus 0; \quad (2.2.11)$$
$$Z_j^+ := \eta_r Z_k^0 \quad \text{for} \quad Z_j = \eta_r Z_k^0 \quad \text{and} \quad \lambda_k^0 = 0;$$
$$Z_j^* := \eta_r Z_k^0 \quad \text{for} \quad Z_j = \eta_r Z_k^0 \quad \text{and} \quad \lambda_k^0 = 0;$$

the connection between $Z_j$ and $Z_j^*$ has been stated in (2.1.35).

We assume the hypotheses of Theorem 2.2.3 to be fulfilled and write the asymptotics of a solution $u \in \dot{H}_\beta^2$ in the form (2.2.8).
Proposition 2.2.5  Let $V = Z^*_j + \eta_r v$, where $Z^*_j$ is a function in (2.2.11) with a certain $r$ and $v \in \dot{H}^2_{-\beta}(G)$. We suppose the $V$ satisfies the equations

\[
(-\Delta - \mu)V(x) = 0, \quad x = (y^r, t^r) \in \Pi'_+, \quad t^r > T, \\
V(x) = 0, \quad x \in \partial \Pi'_+ \cap \partial G, \quad t^r > T.
\]

Then, for the coefficient $c_j$ in (2.2.8) there holds the equality

\[
c_j = i(A_{\beta}(\mu)\eta_r u, V)_G. \tag{2.2.12}
\]

Proof  We set $\eta_{r, \varepsilon}(y^r, t^r) := \eta_r(y^r, \varepsilon t^r)$ with small positive $\varepsilon$ and obtain

\[
(A_{\beta}(\mu)\eta_r u, V)_G = (A_{\beta}(\mu)\eta_{r, \varepsilon} u, V)_G + (A_{\beta}(\mu)(\eta_r - \eta_{r, \varepsilon}) u, V)_G.
\]

The function $(\eta_r - \eta_{r, \varepsilon}) u$ vanishes on infinity, so we can integrate the second term on the right by parts:

\[
(A_{\beta}(\mu)(\eta_r - \eta_{r, \varepsilon}) u, V)_G = ((-\Delta - \mu)(\eta_r - \eta_{r, \varepsilon}) u, V)_{\Pi'_+} = 0.
\]

Therefore,

\[
(A_{\beta}(\mu)\eta_r u, V)_G = (A_{\beta}(\mu)\eta_{r, \varepsilon} u, V)_G = (A_{\beta}(\mu)\eta_{r, \varepsilon} u, Z^*_j)_G \tag{2.2.13}
\]

\[
+ (A_{\beta}(\mu)\eta_{r, \varepsilon} u, V - Z^*_j)_G.
\]

According to Theorem 2.2.3,

\[
c_j = i(A_{\beta}(\mu)\eta_{r, \varepsilon} u, Z^*_j)_G. \tag{2.2.14}
\]

Moreover,

\[
|(A_{\beta}(\mu)\eta_{r, \varepsilon} u, V - Z^*_j)_G| \leq C\|\eta_{r, \varepsilon} u; H^2_{\beta}(G)\|v; H^0_{-\beta} \tag{2.2.15}
\]

with a constant $C$ independent of $\varepsilon$. Since $\|\eta_{r, \varepsilon} u; H^2_{\beta}(G)\| \to 0$ as $\varepsilon \to 0$, relations (2.2.13), (2.2.14), and (2.2.15) lead to (2.2.12). \hfill \Box

Proposition 2.2.6  Let the hypotheses of Theorem 2.2.4 be fulfilled and let $U_1, \ldots, U_d$ be vectors in $\ker A_{\beta}(\mu)$ that satisfy (2.2.9), where $d = \dim(\ker A_{\beta}(\mu)/\ker A_{\gamma}(\mu))$. Then there exist vectors $U^*_d, \ldots, U^*_\kappa$ in $\ker A_{-\gamma}(\mu)$ such that
\[ U_k^* = Z_k^* - \sum_{j=1}^{d} c_{jk} Z_j^* + v_k^*, \quad k = d + 1, \ldots, \zeta, \quad (2.2.16) \]

and \( v_k^* \in \dot{H}^2_\beta(G) \).

**Proof** By virtue of Theorem 2.2.4, there exist vectors \( V_{d+1}, \ldots, V_{\zeta} \) in \( \ker \beta_{-\zeta} (\mu) \) linearly independent modulo \( \ker \beta_{-\zeta} (\mu) \). According to Theorem 2.2.3, \( V_k \) admits the representation

\[ V_k = \sum_{j=1}^{\zeta} b_{kj} Z_j^* + v_k, \quad v_k \in \dot{H}^2_\beta(G), \quad k = d + 1, \ldots, \zeta. \quad (2.2.17) \]

We have

\[ 0 = (A_{\beta}(\mu) U_h, V_k)_G = (A_{\beta}(\mu) \sum_{r=1}^{T_i} \eta_r U_h, V_k)_G + (A_{\beta}(\mu) (1 - \sum_{r=1}^{T_i} \eta_r) U_h, V_k)_G. \quad (2.2.18) \]

The function \( (1 - \sum_{r=1}^{T_i} \eta_r) U_h \) belongs to \( \dot{H}^2_\beta(G) \) hence

\[ A_{\beta}(\mu) (1 - \sum_{r=1}^{T_i} \eta_r) U_h = \gamma_{\beta}(\mu) (1 - \sum_{r=1}^{T_i} \eta_r) U_h. \]

Taking into account this equality, the relation \( V_k \in \ker \beta_{-\zeta} (\mu) \), and Theorem 2.2.2(iii), we obtain \( A_{\gamma}(\mu) (1 - \sum_{r=1}^{T_i} \eta_r) U_h = 0 \) and

\[ 0 = (A_{\beta}(\mu) U_h, V_k)_G = (A_{\beta}(\mu) \sum_{r=1}^{T_i} \eta_r U_h, V_k)_G. \]

To calculate the right-hand side, we employ Proposition 2.2.5 and, in view of (2.2.9), arrive at

\[ \overline{b}_{kh} + \sum_{j=d+1}^{\zeta} c_{hj} \overline{b}_{kj} = 0, \quad h = 1, \ldots, d, \quad k = d + 1, \ldots, \zeta. \quad (2.2.19) \]

Therefore, the first \( d \) columns of the \( (\zeta - d) \times d \)-matrix \( b = \| b_{kp} \| \) are linear combinations of the rest \( \zeta - d \) columns. The rank of the matrix \( b \) is equal to \( \zeta - d \) because \( V_{d+1}, \ldots, V_{\zeta} \) are linearly independent modulo \( \ker \beta_{-\zeta} (\mu) \). It follows that the matrix \( \| b_{kj} \|_{k,j=d+1}^{\zeta} \) is nonsingular. This allows us to assume in (2.2.17) that \( b_{kj} = \delta_{kj} \) for \( k, j = d + 1, \ldots, \zeta \). Then, by virtue (2.2.19), we obtain

\[ \overline{b}_{kh} = -c_{hk}, \quad h = 1, \ldots, d, \quad k = d + 1, \ldots, \zeta, \]
which completes the proof. □

We now pass on to the basic theorem of this section. As before, we suppose that \( \beta = (\beta^1, \beta^T_1, \beta^T_2, \ldots, \beta^T_T) \) and \( \gamma = (\gamma^1, \gamma^T_1, \beta^T_2, \ldots, \beta^T_T) \), where \( \beta^r < \gamma^r \) for \( r = 1, \ldots, T_1 \), and the lines \( \mathbb{R} + i\beta^r \) and \( \mathbb{R} + i\gamma^r \) contain no eigenvalues of the pencil \( \mathcal{A}^r(\cdot, \mu) \) for \( r = 1, \ldots, T \). We denote by \( \kappa_r \) the sum of the full multiplicities of the eigenvalues of the pencil \( \mathcal{A}^r(\cdot, \mu) \) in the strip \( \{ \lambda \in \mathbb{C} : \beta^r < \text{Im} \lambda < \gamma^r \} \) with \( r = 1, \ldots, T_1 \) and set \( \kappa = \kappa_1 + \cdots + \kappa_{T_1} \). We also keep the notation \( d := \dim(\ker \mathcal{A}_\beta(\mu)/\ker \mathcal{A}_\gamma(\mu)) \).

**Theorem 2.2.7** Let \( f \in H^0_\gamma(G) \) and let problem (2.2.1) have a solution in \( \dot{H}^2_\beta(G) \). Then, for any constant \( c_1, \ldots, c_d \), there exists a solution \( u \in \dot{H}^2_\beta(G) \) to problem (2.2.1) such that

\[
u = \sum_{j=1}^d c_j Z_j + \sum_{k=d+1}^\kappa b_k Z_k + v,
\]

where \( v \in \dot{H}^2_\gamma(G) \) and \( Z_1, \ldots, Z_\kappa \) are the same as in (2.2.8). The constant \( b_k \) with \( k = d+1, \ldots, \kappa \) is defined by

\[
b_k = i (f, U_k^*)_G + \sum_{h=1}^d c_h c_{hk},
\]

where \( U_k^* \) belongs to \( \ker \mathcal{A}_{-\gamma}(\mu) \) and satisfies (2.2.16), \( k = d+1, \ldots, \kappa \).

**Proof** Let \( w \in \dot{H}^2_\beta(G) \) be an arbitrary solution to problem (2.2.1) and let \( U_1, \ldots, U_d \) be the vectors in \( \ker \mathcal{A}_\beta(\mu) \) defined by (2.2.9). We choose a linear combination \( \mathcal{L} \) of \( U_1, \ldots, U_d \) such that \( v := w + \mathcal{L} \) admits the representation

\[
v = \sum_{j=d+1}^\kappa a_j Z_j + \rho, \quad \rho \in \dot{H}^2_\gamma(G)
\]

with constant coefficients \( a_j \). Let us calculate the \( a_k \). The function \( \nu(1 - \sum_{r=1}^{T_1} \eta_r) \) belongs to \( \dot{H}^2_\beta(G) \), which follows \( (\mathcal{A}_\beta(\mu) \nu(1 - \sum_{r=1}^{T_1} \eta_r), U_k^*)_G = 0 \) (compare with the proof of Proposition 2.2.6). Therefore,

\[
i (f, U_k^*)_G = i (\mathcal{A}_\beta(\mu) \nu, U_k^*)_G = \sum_{r=1}^{T_1} i (\mathcal{A}_\beta(\mu) \eta_r \nu, U_k^*)_G.
\]

From Proposition 2.2.5, it follows that the right-hand side is equal to \( a_k \). The solution \( u \) in the statement of this theorem is defined by the equality \( u = v + c_1 U_1 + \cdots + c_d U_d \). □
2.3 Waves and Scattering Matrices

2.3.1 Waves

We start with the boundary value problem
\[
(-\Delta - \mu)u(y, t) = 0, \quad (y, t) \in \Pi, \\
u(y, t) = 0, \quad (y, t) \in \partial \Pi,
\]
in the cylinder \(\Pi = \{(y, t) : y = (y_1, \ldots, y_n) \in \Omega, t \in \mathbb{R}\}\) and with the operator pencil
\[
\mathcal{A}(\lambda, \mu)v(y) = (-\Delta_y + \lambda^2 - \mu)v(y), \quad y \in \Omega; \quad v|_{\partial \Omega} = 0.
\]
Let \(\{\mu_k\}_{k=1}^\infty\) be the non-decreasing sequence of the eigenvalues of the problem (2.3.3) counted according to their multiplicity (see 2.1.1). We fix a real \(\mu \neq \mu_k, k = 1, 2, \ldots\), that is, the \(\mu\) is not a threshold, and introduce the functions
\[
u^\pm_k(y, t; \mu) = \left(2|\lambda_\mp^k|\right)^{-1/2} \exp(i\lambda_\mp^k t)\varphi_k(y)
\]
with real \(\lambda_\pm^k = \pm(\mu - \mu_k)^{1/2}\) in the cylinder \(\Pi\); these functions satisfy problem (2.3.3). The \(u^+_k(u^-_k)\) will be called a wave incoming from +\(\infty\) (outgoing to +\(\infty\)). The number of the waves is equal to twice the number of \(\mu_k\) (counted according to their multiplicities) such that \(\mu_k < \mu\). Recall that \(\lambda_\pm^k\) are eigenvalues of the pencil \(\mathcal{A}(\cdot, \mu)\) with the same eigenvector \(\varphi_k\), which is also an eigenvector of problem (2.3.3) corresponding to the eigenvalue \(\mu_k\). The eigenvectors are orthogonal and normalized by the condition
\[
(\varphi_j, \varphi_k)_\Omega = \delta_{jk}.
\]
Let \(G\) be a domain in \(\mathbb{R}^{n+1}\) introduced at the beginning of Sect. 2.2.1; we consider problem (2.2.1). With every \(\Pi^r_+\), we associate a problem of the form (2.3.1) in the cylinder \(\Pi^r = \{(y^r, t^r) : y^r \in \Omega^r, t^r \in \mathbb{R}\}\). A number \(\tau\) is called a threshold for problem (2.2.1) if the \(\tau\) is a threshold for at least one of the problems in the cylinders \(\Pi^r\). In this section, we consider problem (2.2.1) for a real \(\mu\) different from the thresholds.

Let \(\chi \in C^\infty(\mathbb{R})\) be a cut-off function, \(\chi(t) = 0\) for \(t < 0\) and \(\chi(t) = 1\) for \(t > 1\). We multiply each wave in \(\Pi^r\) by the function \(t \mapsto \chi(t^r - t^r_0)\) with a certain (sufficiently large) \(t^r_0 > 0\) and then extend the product by zero to the domain \(G\). We denote the obtained functions by \(v_1, \ldots, v_{2M}\), where \(2M\) is the number of all
real eigenvalues of the pencils $\mathfrak{A}^1(\cdot, \mu), \ldots, \mathfrak{A}^T(\cdot, \mu)$ counted according to their (geometric) multiplicity.

For $l = 0, 1, \ldots$ and $\delta \in \mathbb{R}$, we introduce the space $H^l_\delta(G)$ with norm $\|u; H^l_\delta(G)\| = \|\rho_\delta u; H^l(G)\|$, where $\rho_\delta$ denotes a smooth positive on $\bar{G}$ function given on $\Pi'_+ \cap G$ by the equality $\rho_\delta(y', t') = \exp(\delta t')$; unlike a similar definition in 2.2.1, from now on we choose the weight index $\delta$ to be the same in all cylindrical ends. Let $\hat{H}^2_\delta(G)$ denote the closure in $H^2_\delta(G)$ of the set of smooth functions in $\bar{G}$ that have compact supports in $\bar{G}$ and vanish on $\partial G$. We now assume that the $\delta$ is positive and small so that the strip $\{ \lambda \in \mathbb{C} : |\text{Im}\lambda| < \delta\}$ contains no eigenvalues of the pencils $\mathfrak{A}^r(\cdot, \mu)$, $r = 1, \ldots, T$, except the real ones.

We denote by $\mathfrak{M}$ the linear space spanned by the functions $v_1, \ldots, v_{2M}$ and introduce the quotient space $\mathcal{W}(\mu, G) := (\mathfrak{M} + \hat{H}^2_\delta(G))/\hat{H}^2_\delta(G)$. The elements in $\mathcal{W}(\mu, G)$ are called waves in $G$. We will often write $\mathcal{W}$ instead of $\mathcal{W}(\mu, G)$.

**Proposition 2.3.1** The bilinear form

$$q(u, v) := ((-\Delta - \mu)u, v)_G - (u, (-\Delta - \mu)v)_G$$ \hspace{1cm} (2.3.6)

makes sense for $u$ and $v$ in $\mathfrak{M} + \hat{H}^2_\delta(G)$. Moreover, if one of the elements $u$ and $v$ belongs to $\hat{H}^2_\delta(G)$, the $q(u, v)$ vanishes. Therefore, the form $q(\cdot, \cdot)$ is defined on $\mathcal{W} \times \mathcal{W}$. For any waves $U$ and $V$ in $\mathcal{W}$, there holds equality $q(U, V) = -q(V, U)$.

**Proof** Any function in $\mathfrak{M} + \hat{H}^2_\delta(G)$ is of the form $c_1 v_1 + \cdots + c_{2M} v_{2M} + w$, where $c_1, \ldots, c_{2M}$ are some constants and $w \in \hat{H}^2_\delta(G)$. The support of $(-\Delta - \mu)v_j$ is compact and $(-\Delta - \mu)w$ belongs to $H^0_\delta(G)$, so the right-hand side in (2.3.6) makes sense. Let us assume that $u$ or $v$ belongs to $\hat{H}^2_\delta(G)$; then, integrating by parts, we obtain the equality

$$((-\Delta - \mu)u, v)_G = (u, (-\Delta - \mu)v)_G.$$

Therefore, we can set

$$q(\tilde{u}, \tilde{v}) := q(u, v),$$ \hspace{1cm} (2.3.7)

for any $u$ and $v$ in $u \in \mathfrak{M} + \hat{H}^2_\delta(G)$, where $\tilde{u}$ and $\tilde{v}$ denote the classes of $u$ and $v$ in $\mathcal{W}$. The equality $q(U, V) = -q(V, U)$ follows immediately from (2.3.6) and (2.3.7). \hfill $\square$

The number $q(U, U)$ is imaginary for any $U \in \mathcal{W}$. We call wave $U$ outgoing (incoming) if $iq(U, U)$ is a positive (negative) number. Let $u_{\pm}$ be a wave of the form (2.3.4) in a cylinder $\Pi'_+$. We extend the function $(y', t') \mapsto \chi(t' - t'_0)u_{\pm}(y', t', \mu)$ by zero to $G$ and denote by $U_{\pm}$ the class in $\mathcal{W}$ of the obtained function in $G$. In this way, we define the waves $U_{1, \ldots, U_{M}}$; as before, $2M = 2M(\mu)$ is equal to the number of all real eigenvalues of the pencils $\mathfrak{A}^1(\cdot, \mu), \ldots, \mathfrak{A}^T(\cdot, \mu)$ counted according to their multiplicity. Integrating by parts in expressions of the form $q(u, v)$, where $u$ and $v$ are representatives of the waves $U_{\pm}$, we arrive at the following
Proposition 2.3.2 The $U_j^+ (U_j^-)$, $j = 1, \ldots, M$, are incoming (outgoing) waves. The collection $U_1^+, \ldots, U_M^+, U_1^-, \ldots, U_M^-$ forms a basis in the space $\mathcal{W}$ subject to the orthogonality and normalization conditions

$$q(U_j, U_k) = 0 \text{ for } j \neq k, \quad iq(U_j^+, U_j^+) = -1,$$

$$iq(U_j^-, U_j^-) = 1 \quad \text{for } j = 1, \ldots, M.$$  \hfill (2.3.8)

2.3.2 Continuous Spectrum Eigenfunctions. The Scattering Matrix

In this section, we consider the parameter $\mu$ in an interval $[\mu', \mu'']$ that contains no thresholds of problem (2.2.1), where $\mu' > \tau_1$ and $\tau_1$ is the first threshold. Therefore, there exists such a positive $\delta$ that, for all $\mu \in [\mu', \mu'']$, the strip $\{ \lambda \in \mathbb{C} : |\text{Im}\lambda| < \delta \}$ is free from the eigenvalues of the pencils $\mathcal{W}(\cdot, \mu)$, $r = 1, \ldots, T$, except the real ones.

Now, we introduce several definitions. If $u \in \ker \mathcal{A}_\delta(\mu_0)$ and $u \neq 0$, and $u \in L_2(G)$, the $u$ is called a continuous spectrum eigenfunction of the problem

$$-\Delta u(x) - \mu u(x) = 0, \quad x \in G,$$

$$u(x) = 0, \quad x \in \partial G,$$  \hfill (2.3.9)

at the point $\mu_0$. If $u \in \ker \mathcal{A}_\delta(\mu_0)$, $u \neq 0$, and $u \in L_2(G)$, the $u$ is said to be an eigenfunction and $\mu_0$ is an eigenvalue of problem (2.3.9) embedded in the continuous spectrum; in fact, any such eigenfunction belongs to $\ker \mathcal{A}_\delta(\mu_0)$ (this can be derived from Theorem 2.2.3). For problem (2.3.9), it is known that the eigenvalues do not accumulate at finite distance. Therefore, the interval $[\mu', \mu'']$ contains finitely many eigenvalues at most. The number $\dim(\ker \mathcal{A}_\delta(\mu)/\ker \mathcal{A}_\delta(\mu))$ is called the continuous spectrum multiplicity at $\mu$. Equality (2.2.6) for $\beta = -\delta$ and $\gamma = \delta$ takes the form

$$\dim(\ker \mathcal{A}_\delta(\mu)/\ker \mathcal{A}_\delta(\mu)) = M(\mu)$$  \hfill (2.3.10)

because, in this case, $\kappa = 2M(\mu)$. The interval $[\mu', \mu'']$ contains no thresholds and therefore the continuous spectrum multiplicity is constant on this interval.

Any element $v \in \ker \mathcal{A}_\delta(\mu)$ defines a certain class $\tilde{v}$ in the wave space $\mathcal{W}$; we let $\mathcal{R}$ denote the image of $\ker \mathcal{A}_\delta(\mu)$ in $\mathcal{W}$. The $\mathcal{R}$ is a subspace in $\mathcal{W}$.

Theorem 2.3.3 Let $U_1^+ (\mu), \ldots, U_M^+ (\mu), U_1^- (\mu), \ldots, U_M^- (\mu)$ be the same basis of $\mathcal{W}(\mu, G)$ as in Proposition 2.3.2. Then there exist bases $\tilde{\xi}_1(\mu), \ldots, \tilde{\xi}_M(\mu)$ and $\tilde{\eta}_1(\mu), \ldots, \tilde{\eta}_M(\mu)$ in $\mathcal{R}(\mu)$ such that
Waveguides. Radiation Principle. Scattering Matrices

\[ \tilde{\zeta}_j(\mu) = U_j^+ (\mu) + \sum_{k=1}^{M} S_{jk}(\mu) U_k^- (\mu), \quad (2.3.11) \]

\[ \tilde{\eta}_j(\mu) = U_j^- (\mu) + \sum_{k=1}^{M} T_{jk}(\mu) U_k^+ (\mu). \quad (2.3.12) \]

**The matrix** \( S(\mu) = \|S_{jk}(\mu)\| \) **is unitary** and \( S(\mu)^{-1} = T(\mu) = \|T_{jk}(\mu)\| \).

**Proof** Let \( v_1, \ldots, v_M \) be linear independent elements in \( \ker A_- \delta(\mu)/\ker A_\delta(\mu) \) and \( \tilde{v}_1, \ldots, \tilde{v}_M \) their classes in \( \mathcal{W} \). We have

\[ \tilde{v}_j = \sum_{k=1}^{M} m_{jk}^+ U_k^+ + \sum_{k=1}^{M} m_{jk}^- U_k^-, \quad j = 1, \ldots, M. \]

The matrices \( \mathcal{M}^+ = \|m_{jk}^+\| \) and \( \mathcal{M}^- = \|m_{jk}^-\| \) are nonsingular. Indeed, if, for instance, \( \det \mathcal{M}^+ = 0 \), there exists a nonzero \( \tilde{v} \in \mathcal{W} \) with \( v \in \ker A_- \delta(\mu) \) such that

\[ v = \sum_{k=1}^{M} a_k u_k + w, \]

where \( a_k \) is a constant, \( u_k = U_k^- \), and \( w \in \hat{H}_\delta^2(G) \). From (2.3.6) it follows that \( q(v, v) = 0 \). On the other hand,

\[ q(v, v) = q\left( \sum_{k=1}^{M} a_k u_k, \sum_{k=1}^{M} a_k u_k \right) = \sum_{k=1}^{M} |a_k|^2 q(U_k, U_k) = i \sum_{k=1}^{M} |a_k|^2 \neq 0, \]

which is a contradiction. Therefore, there exist linear combinations \( \tilde{\zeta}_j \) and \( \tilde{\eta}_j \) of the \( \tilde{v}_1, \ldots, \tilde{v}_M \) that satisfy (2.3.11) and (2.3.12), respectively.

We now pass to verifying the second part of this theorem. For \( \tilde{\zeta}_l \) and \( \tilde{\eta}_m \) in (2.3.11) and (2.3.12), we choose representatives \( \zeta_l \) and \( \eta_m \) in \( \ker A_- \delta(\mu) \). Then \( q(\zeta_l, \eta_m) = 0 \) and, moreover,

\[ q(\zeta_l, \eta_m) = q\left( U_l^+, \sum_{j=1}^{M} S_{lj} U_j^-, U_m^+ + \sum_{k=1}^{M} T_{mk} U_k^+ \right) \]

\[ = q\left( \sum_{j=1}^{M} S_{lj} U_j^-, U_m^+ \right) + q\left( U_l^+, \sum_{k=1}^{M} T_{mk} U_k^+ \right) = i S_{lm} - i T_{ml}, \]
hence \( S(\mu) = T^*(\mu) \). Let us consider
\[
\tilde{v} := \sum_{j=1}^{M} S_{lj} \tilde{\eta}_j = \sum_{j=1}^{M} S_{lj} U_j^- + \sum_{j=1}^{M} \sum_{k=1}^{M} S_{lj} T_{jk} U_k^+.
\]
The coefficients of the \( \tilde{v} \) and \( \tilde{\zeta}_j \) are the same at \( U_j^- \), \( j = 1, \ldots, M \). Therefore, the coefficients also coincide at \( U_k^+, \ k = 1, \ldots, M \); otherwise, for the \( \tilde{v} - \tilde{\zeta}_j \), we obtain a contradiction like that in the first part of the proof. Thus, we have \( S(\mu)^{-1} = T(\mu) \). □

**Definition 2.3.4** The matrix \( S(\mu) = ||S_{jk}(\mu)||_{j,k=1}^{M} \) with entries in (2.3.11) is called the scattering matrix.

### 2.3.3 The Intrinsic Radiation Principle

Let \( U_1^+(\mu), \ldots, U_M^+(\mu), U_1^- (\mu), \ldots, U_M^- (\mu) \) be the same basis of \( \mathcal{W}(\mu, G) \) as in Proposition 2.3.2 and in Theorem 2.3.3. We choose any representatives \( u_j^- \) of \( U_j^- \), \( j = 1, \ldots, M \) and denote by \( \mathfrak{N} \) the linear hull of \( u_1^-, \ldots, u_M^- \). We define the norm of \( u = \sum c_j u_j^- + v \in \mathfrak{N} + \hat{H}^2_\delta(G) \) with \( c_j \in \mathbb{C} \) and \( v \in \hat{H}^2_\delta(G) \) by
\[
\|u\| = \sum |c_j| + \|v; \hat{H}^2_\delta(G)\|.
\]
Let \( A(\mu) \) be the restriction of the operator \( A_{-\delta}(\mu) \) to the space \( \mathfrak{N} + \hat{H}^2_\delta(G) \). The map
\[
A(\mu) : \mathfrak{N} + \hat{H}^2_\delta(G) \rightarrow \hat{H}^0_\delta(G)
\]
is continuous. The following theorem provides the statement of problem (2.2.1) with intrinsic radiation conditions at infinity (the numbers \( \mu \) and \( \delta \) are supposed to satisfy the requirements given at the beginning of 2.3.2.

**Theorem 2.3.5** Let \( z_1, \ldots, z_d \) be a basis in the space \( \ker A_\delta(\mu) \). \( f \in \hat{H}^0_\delta(G) \) and \((f, z_j)_G = 0, \ j = 1, \ldots, d\). Then:

1. There exists a solution \( u \in \mathfrak{N} + \hat{H}^2_\delta(G) \) of the equation \( A(\mu)u = f \) determined up to an arbitrary term in \( \ker A_\delta(\mu) \).
2. The inclusion
\[
v \equiv u - c_1 u_1^- - \cdots - c_M u_M^- \in \hat{H}^2_\delta(G) \quad (2.3.13)
\]
holds with $c_j = i(f, \tilde{\eta}_j)_G$.

3. The inequality

$$\|v; H^2_\delta(G)\| + |c_1| + \cdots + |c_M| \leq \text{const} \left( \|f; H^0_\delta(G)\| + \|v; L_2(G^c)\| \right)$$

(2.3.14)

holds with $v$ and $c_1, \ldots, c_M$ in (2.3.13), while $G^c$ is a compact subset of $G$. A solution $u_0$ that is subject to the additional conditions $(u_0, z_j)_G = 0$ for $j = 1, \ldots, d$ is unique and satisfies (2.3.14) with right-hand side changed for $\text{const}\|f; H^0_\delta(G)\|$.

Proof Let us outline the proof. The operator $A_{-\delta}(\mu)$ is Fredholm. Therefore, the orthogonality conditions $(f, z_j)_G = 0$, $j = 1, \ldots, d$, provide the existence of a solution $u \in \dot{H}^2_{-\delta}(G)$ to the equation $A_{-\delta}(\mu)u = f$. Since $z_1, \ldots, z_d, \eta_1, \ldots, \eta_M$ form a basis in $\ker A_{-\delta}(\mu)$, the general solution is of the form

$$u = u^0 + \sum_{j=1}^M a_j \eta_j + \sum_{k=1}^d b_k z_k$$

(2.3.15)

with a particular solution $u^0 \in \dot{H}^2_{-\delta}(G)$ and arbitrary constants $a_k$ and $b_k$. According to Theorem 2.3.3, we have $\det \|T_{jk}\| \neq 0$. Therefore, in view of (2.3.12), we can obtain (2.3.13) by choosing the coefficients $a_k$. The equality $c_j = i(f, \tilde{\eta}_j)_G$ now follows from Theorem 2.2.7 and the relations $q(U_j^-, \eta_k) = -i\delta_{jk}$, $j, k = 1, \ldots, M$.

In connections with estimate (2.3.14) see Theorems 5.3.5 and 5.1.4 in [37].
Resonant Tunneling
Quantum Waveguides of Variable Cross-Section, Asymptotics, Numerics, and Applications
Baskin, L.; Neittaanmaki, P.; Plamenevskii, B.A.; Sarafanov, O.
2015, XI, 275 p. 65 illus., Hardcover
ISBN: 978-3-319-15104-5