Chapter 2
Review of the Variational Asymptotic Method and the Intrinsic Equations of a Beam

2.1 Introduction

Helicopter rotor blades can be adequately modeled as thin-walled composite beams, which are laterally flexible and, as such, usually operate in the nonlinear range. To solve such problems, one method is to use conventional beam models that rely on ad hoc assumptions on displacement or stress fields. An example of such models is the Saint-Venant's theory of torsion, which assumes that a beam remains rigid in its cross section as it twists. While this assumption works fine in the linear range of behavior for isotropic prismatic beams, it results in serious error when it is extended to composite beams. Therefore, to get acceptable results, one should use alternative solution methods.

A very promising alternative that is mentioned in Chap. 1 is VAM. It is a powerful method for solving problems of composite thin-walled beams and is free from ad hoc assumptions. The applicability of VAM in elasticity is because the elasticity problem can be stated as obtaining the stationary points of the energy functional. VAM simplifies the procedure for finding these stationary points of the energy functional when this functional depends on one or more inherently small parameters. It is therefore the right tool for building accurate models for dimensionally reducible structures (e.g., beams, plates and shells), Hodges (2006). VAM has both the merits of variational methods (viz., systematic and easily implementable numerically) and asymptotic methods (viz., without ad hoc assumptions), Roy and Yu (2009).

VAM splits the 3-D geometrically nonlinear elasticity problem into a 2-D analysis and a nonlinear 1-D analysis along the beam. The 2-D analysis is aimed at determining the cross-sectional stiffness and inertia matrices as well as the warping functions. It requires details of the cross-sectional geometry, elastic properties of materials and material densities and can be performed by VABS. The results of this analysis are used in all further 3-D analyses without the need to repeat such a 2-D analysis over again.

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The necessary conditions for achieving a linear cross-sectional analysis from the starting point of the geometrically nonlinear 3-D elasticity include small strain, linearly elastic material, and the smallness of a relative to l and R ($a \ll 1$). Here, a is a typical cross-sectional dimension, l is the wavelength of deformation along the beam axis, and R is the characteristic radius of initial curvature and twist (Hodges 2006).

The solution of the 1-D problem is obtained by utilizing the outcome of the 2-D cross-sectional analysis and by solving the nonlinear intrinsic differential equations of motion of the beam. Combining these two solutions provides the complete 3-D structural solution by recovering the 3-D stress, strain, and displacement fields.

2.2 Classification of Beams

2.2.1 *Class T Beams*

These are thin-walled beams with open cross sections, as seen in Fig. 2.1. If the wall thickness is h and the main characteristic length within the cross-sectional plane is a, then $a \gg h$. For class T beams, the torsional stiffness of the beam is considerably less than either of the two bending stiffnesses, and hence, the beam is torsionally soft. Such open section beams can be analyzed using the generalized Vlasov model.

2.2.2 *Class S Beams*

Figure 2.2 illustrates two examples of class S beams. These are strip-like beams that are soft in torsion, and soft in bending in one direction. Therefore, one bending stiffness is significantly larger than the other one and the torsional stiffness. Examples include high-aspect-ratio wings and helicopter blades.

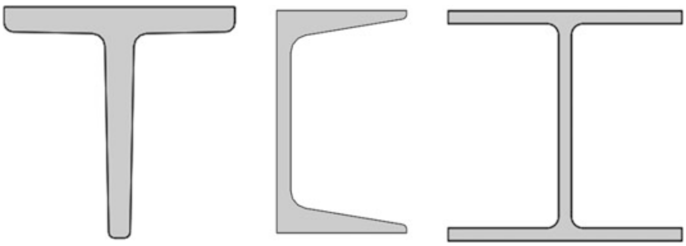

Fig. 2.1 Example cross sections of class T beams

Fig. 2.2 Example cross sections of class S beams

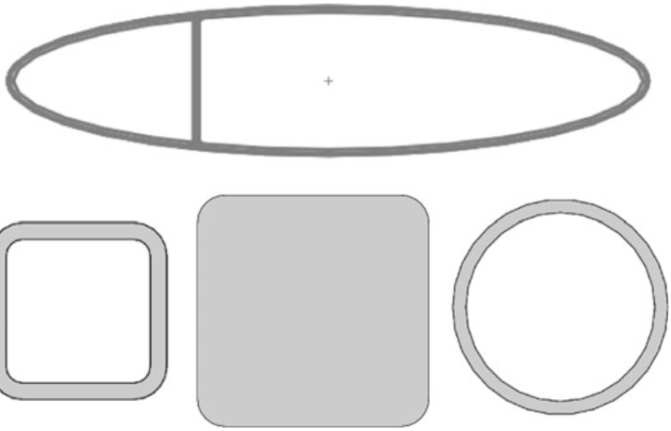

Fig. 2.3 Example cross sections of class R beams

2.2.3 Class R Beams

These are called regular beams. They are not class T or class S beams and may have solid or hollow sections that are closed (or closed cell) and are thin-walled. A few examples of class R beams can be seen in Fig. 2.3.

2.3 Cross-Sectional Modeling Using VAM

Accurate determination of the cross-sectional elastic constants of composite beams requires the presence of two distinct characteristics, Hodges (2006):

1. The theory behind the cross-sectional analysis must allow for elastic coupling in the 3-D material constants.
2. All six components of strain and stress and all possible components of displacement must be allowed, both in and out of the cross-sectional plane.

As seen in Fig. 2.4, two Cartesian coordinate systems are set up: the *b*-frame of the undeformed beam and the *B*-frame of the deformed beam. The origin of the undeformed system is usually put at the shear center (elastic center) of the cross section so that shear forces do not produce any twisting moments. Shear center of a cross section is a point in the cross section at which a shear force induces no twist.

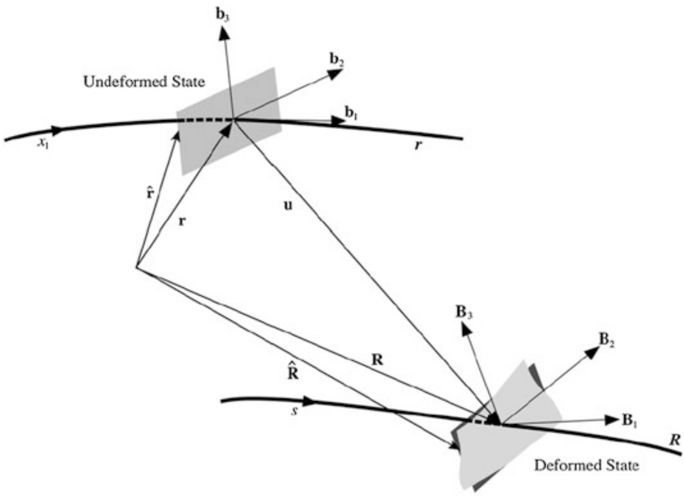

Fig. 2.4 Frames and reference lines of the beam model, Wang and Yu (2013), © Elsevier Ltd., reprinted with permission

In an isotropic beam, the shear center is the same as the center of twist. This is the point about which the cross section rotates under a pure twisting moment.

In addition, the tension center is the point in the cross section at which an axial force induces no bending (i.e., the centroid; the spanwise locus of centroids is called tension axis). Since in a cross section, the shear center, the center of gravity, and the centroid of the section are not necessarily identical, their spanwise counterparts, i.e., the elastic axis, the center of gravity axis, and the tension axis are not necessarily coincident either.

The unit vector b_1 of the undeformed b-frame is tangential to the undeformed reference line; b_2 and b_3 define the plane of the undeformed reference cross section. The origin of the deformed B-frame is the origin of the b-frame translated by the displacement components u_i. The unit vector B_1 is orthogonal to the non-warped but translated and rotated reference cross section. Note that B_1 is not necessarily tangential to the deformed reference line because the displaced cross section does not have to be orthogonal to the new reference line (i.e., Euler–Bernoulli approximation is not made; shear deformation is not neglected), Traugott et al. (2005).

Since the behavior of an elastic body is completely determined by its energy function, one may write the 3-D strain energy function, minimize it subjected to the warping constraints and then solve for the warping displacements to create an asymptotically correct 1-D energy function. In this way, by reproducing the 3-D energy in terms of 1-D quantities, a beam theory is derived. This dimensional reduction cannot be carried out exactly; however, VAM can find the 1-D energy that approximates the 3-D energy as closely as possible.

The strain energy of the cross section (per unit length) of the beam can be expressed as

$$U = \frac{1}{2}\langle\langle \Gamma^T D \Gamma \rangle\rangle \quad (2.1)$$

where Γ is the 3-D strain vector,

$$\Gamma = \lfloor \Gamma_{11} \quad 2\Gamma_{12} \quad 2\Gamma_{13} \quad \Gamma_{22} \quad 2\Gamma_{23} \quad \Gamma_{33} \rfloor^T \quad (2.2)$$

and

$$\langle\langle \bullet \rangle\rangle = \langle (\bullet) g \rangle = \int_A (\bullet)\sqrt{g}\,\mathrm{d}x_2\mathrm{d}x_3, \quad \langle \bullet \rangle = \int_A (\bullet)\mathrm{d}x_2\mathrm{d}x_3 \quad (2.3)$$

$$\sqrt{g} = 1 - x_2 k_3 - x_3 k_2 \quad (2.4)$$

A is the cross-sectional plane of the undeformed beam (the reference cross section), D is the 6×6 symmetric material matrix in the local Cartesian system and g is the determinant of the metric tensor for the undeformed state.

Equation (2.1) for strain energy density implies a stress–strain law of the form

$$\sigma = D\Gamma \quad (2.5)$$

where the 3-D stress and strain components are elements of the following vectors

$$\begin{aligned} \sigma &= \lfloor \sigma_{11} \quad \sigma_{12} \quad \sigma_{13} \quad \sigma_{22} \quad \sigma_{23} \quad \sigma_{33} \rfloor^{\mathrm{T}} \\ \Gamma &= \lfloor \Gamma_{11} \quad 2\Gamma_{12} \quad 2\Gamma_{13} \quad \Gamma_{22} \quad 2\Gamma_{23} \quad \Gamma_{33} \rfloor^{\mathrm{T}} \end{aligned} \quad (2.6)$$

The basic 2-D analysis problem is to minimize the strain energy functional U subject to four no rigid body motion conditions for the warping functions $w_i = w_i(x_1, x_2, x_3)$. The explicit form of these conditions of no rigid body translation and rotation are (Hodges 2006),

$$\langle w_i \rangle = 0; \; i = 1, 2, 3, \quad \langle x_2 w_3 - x_3 w_2 \rangle = 0 \quad (2.7)$$

These conditions guarantee the uniqueness of the definition of the warping field and are equivalent to removing the rigid body motion components (i.e., three translations and the in-plane rotation) of the warping field. The warping field is the solution to the mentioned minimization problem. Using matrices and the concept of orthogonality, Eq. (2.7) can be expressed as the orthogonality of the warping function w to the kernel matrix ψ constraint,

$$\langle w^{\mathrm{T}} \psi \rangle = 0 \quad (2.8)$$

where

$$w = \begin{Bmatrix} w_1 \\ w_2 \\ w_3 \end{Bmatrix} \quad \psi = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -x_3 \\ 0 & 0 & 1 & x_2 \end{bmatrix} \tag{2.9}$$

In order to solve the mentioned minimization problem and find the stationary value of the strain energy per unit length, U, given in Eq. (2.1) subjected to constraints (2.8), the warping vector is assumed as

$$w(x_1, x_2, x_3) = \underbrace{S(x_2, x_3)}_{\text{FEM Shape Functions}} \cdot \overbrace{V(x_1)}^{\text{Nodal Warping}} \tag{2.10}$$

where $S(x_2, x_3)$ represents the matrix of the FEM shape functions on the beam cross section and V is a column matrix of the nodal values of the warping displacement along the longitudinal axis of the beam. The use of shape functions reduces the 3-D problem (i.e., calculating w) to a 1-D problem (i.e., calculating V). So, now the strain energy per unit length, U, should be minimized with respect to V. Minimization of strain energy subjected to the orthogonality constraint (2.8) is obtained by using the method of Lagrangian multipliers. By ignoring the shear deformations, one obtains the classic approximation of the strain energy for anisotropic materials as follows (Hodges 2006):

$$2U_0 = \begin{Bmatrix} \bar{\gamma}_{11} \\ \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & \overline{S}_{14} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{23} & \overline{S}_{24} \\ \overline{S}_{13} & \overline{S}_{23} & \overline{S}_{33} & \overline{S}_{34} \\ \overline{S}_{14} & \overline{S}_{24} & \overline{S}_{34} & \overline{S}_{44} \end{bmatrix} \begin{Bmatrix} \bar{\gamma}_{11} \\ \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{Bmatrix} \tag{2.11}$$

In this quadratic form, the stiffness constants $\overline{S}_{ij}$ depend on the initial twist and curvature as well as on the geometry and material composition of the cross section. This 4 × 4 model is sufficiently accurate for the analysis of long-wavelength static or low-frequency dynamic behavior of slender initially curved and twisted composite beams (Hodges et al. 1992). Using the classic stiffness matrix S,

$$S = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & \overline{S}_{14} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{23} & \overline{S}_{24} \\ \overline{S}_{13} & \overline{S}_{23} & \overline{S}_{33} & \overline{S}_{34} \\ \overline{S}_{14} & \overline{S}_{24} & \overline{S}_{34} & \overline{S}_{44} \end{bmatrix} \tag{2.12}$$

and for small strain, the 1-D constitutive law would be linear and expressible as

$$\begin{Bmatrix} F_1 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{12} & \overline{S}_{13} & \overline{S}_{14} \\ \overline{S}_{12} & \overline{S}_{22} & \overline{S}_{23} & \overline{S}_{24} \\ \overline{S}_{13} & \overline{S}_{23} & \overline{S}_{33} & \overline{S}_{34} \\ \overline{S}_{14} & \overline{S}_{24} & \overline{S}_{34} & \overline{S}_{44} \end{bmatrix} \begin{Bmatrix} \bar{\gamma}_{11} \\ \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{Bmatrix} \tag{2.13}$$

For homogeneous prismatic beams made of isotropic materials, the expression for classical strain energy per unit length is

$$2U_0 = \begin{Bmatrix} \bar{\gamma}_{11} \\ \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathrm{EA} & 0 & 0 & 0 \\ 0 & \mathrm{GJ} & 0 & 0 \\ 0 & 0 & \mathrm{EI}_2 & 0 \\ 0 & 0 & 0 & \mathrm{EI}_3 \end{bmatrix} \begin{Bmatrix} \bar{\gamma}_{11} \\ \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{Bmatrix} \tag{2.14}$$

where the classical stiffness matrix is

$$S = \begin{bmatrix} \mathrm{EA} & 0 & 0 & 0 \\ 0 & \mathrm{GJ} & 0 & 0 \\ 0 & 0 & \mathrm{EI}_2 & 0 \\ 0 & 0 & 0 & \mathrm{EI}_3 \end{bmatrix} \tag{2.15}$$

Here, EA is the extensional stiffness, GJ is the Saint-Venant's torsional stiffness, EI_α is the bending stiffness about x_α ($\alpha = 2, 3$), E is the Young's modulus, G is the shear modulus, and the cross-sectional axes x_α are the principal axes of inertia originating at the centroid. Furthermore, $\bar{\gamma}_{11}$ is the extension of the reference line, $\bar{\kappa}_1$ is the elastic twist, and the elastic bending curvatures are denoted by $\bar{\kappa}_2$ and $\bar{\kappa}_3$. The zeros on the off-diagonal elements of the stiffness matrix are indications of the decoupled behavior of the structure in different directions. If one computes the results at the centroid (i.e., the origin of the reference frame is transferred to the centroid), the existence of off-diagonal elements in the stiffness matrix is impossible. In fact, for any homogeneous isotropic section, a 4 × 4 description, and the reference at the centroid, there is no coupling between extension and bending (Palacios 2008).

For thick beams or for beams in high-frequency vibrations, shear deformation cannot be ignored. So, the classic model should be replaced with the generalized Timoshenko model:

$$2U = \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} \tag{2.16}$$

or

$$2U = \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} A & B \\ B^{\mathrm{T}} & D \end{bmatrix} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} \quad (2.17)$$

Therefore, the 1-D constitutive law in generalized Timoshenko model is

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{12} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{13} & S_{23} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{14} & S_{24} & S_{34} & S_{44} & S_{45} & S_{46} \\ S_{15} & S_{25} & S_{35} & S_{45} & S_{55} & S_{56} \\ S_{16} & S_{26} & S_{36} & S_{46} & S_{56} & S_{66} \end{bmatrix} \begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} \quad (2.18)$$

or

$$\begin{Bmatrix} F \\ M \end{Bmatrix} = \begin{bmatrix} A & B \\ B^{\mathrm{T}} & D \end{bmatrix} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} \quad (2.19)$$

Alternatively,

$$\begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} = \begin{bmatrix} R & Z \\ Z^{\mathrm{T}} & T \end{bmatrix} \begin{Bmatrix} F \\ M \end{Bmatrix} \quad (2.20)$$

As it is mentioned in Chap. 1, the linear cross-sectional analysis of the VAM is performed by VABS. In order to run VABS, a 2-D meshed model of the cross section is constructed by a CAD or FEM software. It can then be transformed into an input file for VABS. To model initially twisted and curved beams, three real numbers for the twist, k_1, and the bending curvatures, k_2 and k_3, should be provided in the input file. Also layup parameters such as the layup angle should be provided. Finally, material properties including Young's moduli, E_i, shear moduli, G_{ij}, Poisson's ratios, υ_{ij}, and mass density, ρ, should be given.

After performing the solution, the results include scalar quantities such as the mass center, the principal axes of inertia, the centroid, and the neutral axes. Matrix results include the cross-sectional 6 × 6 mass (three translation and three rotations) and the 4 × 4 cross-sectional stiffness matrix of the classical model (extension, torsion, and two bending). There are also the 6 × 6 cross-sectional stiffness matrix of the generalized Timoshenko model (extension, two shears, torsion, and two bendings), and the 5 × 5 cross-sectional stiffness matrix of the generalized Vlasov model (extension, torsion, two bendings, and the derivative of torsion).

To obtain the generalized Vlasov model, first, the generalized Timoshenko model that works best at high frequencies is constructed and the position of the shear center is determined. Then, VABS moves the origin of the coordinate system to the shear center and repeats the calculations to obtain a generalized Vlasov model. This model is useful for analyzing thin-walled beams with open sections.

The cross-sectional modeling of smart composite beams has also been successfully performed (with distributed actuators embedded within the composite structure) using UM/VABS.

2.4 General Formulation of the 1-D Analysis

Having used the VAM logic and obtained the 2-D cross-sectional properties by VABS, a 1-D analysis along the longitudinal axis of the beam is now in order. The combination of the previously mentioned 2-D solution and the solution of the 1-D problem provide a complete 3-D picture of the mechanical quantities of interest along and across the beam. The 1-D analysis utilizes the intrinsic equations of motion, the intrinsic kinematical equations, the momentum–velocity equations, and the constitutive equations of the material of the beam. It utilizes the results of the cross-sectional analysis in order to calculate the generalized stress and strain resultants as well as the 1-D displacements. It should be noted that the intrinsic equations are not tied to a specific choice of displacement or rotation variables. Furthermore, there is only one set of intrinsic equations; all other correct and variationally consistent sets of beam equations are linear combinations of the correct intrinsic set of equations.

2.4.1 Intrinsic Equations of Motion

The internal force and moment vectors F and M are partial derivatives of the strain energy of the cross section (per unit length), U,

$$F = \left(\frac{\partial U}{\partial \gamma}\right)^{\mathrm{T}}, \quad M = \left(\frac{\partial U}{\partial \kappa}\right)^{T} \tag{2.21}$$

Similarly, the generalized sectional linear and angular momenta P and H are conjugate to motion variables by derivatives of the kinetic energy function,

$$P = \left(\frac{\partial K.E.}{\partial V}\right)^{\mathrm{T}}, \quad H = \left(\frac{\partial K.E.}{\partial \Omega}\right)^{\mathrm{T}} \tag{2.22}$$

Now, recalling Hamilton's principle,

$$\int_{t_1}^{t_2}\int_{0}^{L} \left\{\delta(K.E. - U) + \delta\overline{W}\right\} \mathrm{d}x_1 \mathrm{d}t = \delta\overline{A} \tag{2.23}$$

for the case of generalized Timoshenko beam without active elements, one obtains, Hodges (2006)

$$\int_{t_1}^{t_2}\int_0^L \left\{\delta\bar{q}^{\mathrm{T}}\left(F' + \tilde{K}F + f - \dot{P} - \widetilde{\Omega}P\right) + \delta\bar{\psi}^{\mathrm{T}}\left[M' + \tilde{K}M + (\tilde{e}_1 + \tilde{\gamma})F + m - \dot{H} - \widetilde{\Omega}H - \tilde{V}P\right]\right\}\mathrm{d}x_1\mathrm{d}t$$
$$= \int_0^L \left[\delta\bar{q}^T\left(\hat{P} - P\right) + \delta\bar{\psi}^{\mathrm{T}}\left(\hat{H} - H\right)\right]\Big|_{t_1}^{t_2}\mathrm{d}x_1 - \int_{t_1}^{t_2}\left[\delta\bar{q}^{\mathrm{T}}\left(\hat{F} - F\right) + \delta\bar{\psi}^{\mathrm{T}}\left(\hat{M} - M\right)\right]\Big|_0^L\mathrm{d}t \tag{2.24}$$

where the tilde notation has been used to express a cross product of two vectors in a concise matrix representation:

$$\tilde{K}F = \begin{bmatrix} 0 & -K_3 & K_2 \\ K_3 & 0 & -K_1 \\ -K_2 & K_1 & 0 \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} K_2F_3 - K_3F_2 \\ K_3F_1 - K_1F_3 \\ K_1F_2 - K_2F_1 \end{bmatrix} = \vec{K} \times \vec{F} \tag{2.25}$$

The corresponding Euler–Lagrange equations are

$$F' + \tilde{K}F + f = \dot{P} + \tilde{\Omega}P \tag{2.26}$$

$$M' + \tilde{K}M + (\tilde{e}_1 + \tilde{\gamma})F + m = \dot{H} + \tilde{\Omega}H + \tilde{V}P \tag{2.27}$$

Equations (2.26) and (2.27) are called the nonlinear intrinsic equations of motion of a beam. Here, F and M are column vectors of internal forces and moments, respectively. The first element of F is the axial force and the second and third elements are the shear forces, expressed in the deformed beam basis, B. Similarly, the first element of M is the twisting moment and its second and third elements are the bending moments, again in the deformed beam frame, B. The scalar form of the intrinsic equations of motion is

$$F_1' + K_2F_3 - K_3F_2 + f_1 = \dot{P}_1 + \Omega_2P_3 - \Omega_3P_2 \tag{2.28}$$

$$F_2' + K_3F_1 - K_1F_3 + f_2 = \dot{P}_2 + \Omega_3P_1 - \Omega_1P_3 \tag{2.29}$$

$$F_3' + K_1F_2 - K_2F_1 + f_3 = \dot{P}_3 + \Omega_1P_2 - \Omega_2P_1 \tag{2.30}$$

and

$$\begin{aligned} &M_1' + K_2M_3 - K_3M_2 + 2\gamma_{12}F_3 - 2\gamma_{13}F_2 + m_1 \\ &\quad = \dot{H}_1 + \Omega_2H_3 - \Omega_3H_2 + V_2P_3 - V_3P_2 \end{aligned} \tag{2.31}$$

$$\begin{aligned} &M_2' + K_3M_1 - K_1M_3 + 2\gamma_{13}F_1 - (1+\gamma_{11})F_3 + m_2 \\ &\quad = \dot{H}_2 + \Omega_3H_1 - \Omega_1H_3 + V_3P_1 - V_1P_3 \end{aligned} \tag{2.32}$$

$$\begin{aligned} &M_3' + K_1M_2 - K_2M_1 + (1+\gamma_{11})F_2 - 2\gamma_{12}F_1 + m_3 \\ &\quad = \dot{H}_3 + \Omega_1H_2 - \Omega_2H_1 + V_1P_2 - V_2P_1 \end{aligned} \tag{2.33}$$

These equations are the geometrically exact equations for the dynamics of a beam in an absolute frame of reference, A. They resemble Euler's dynamical equations, and their symmetric form enables one to write them in a compact matrix form.

For the special case of static behavior, Eqs. (2.26) and (2.27) reduce to those of Reissner (1973). In fact, by setting the left-hand side of these equations equal to zero, a generalized Euler–Kirchhoff–Clebsch theory is obtained. This static theory, when specialized for isotropic materials, is often called the elastica theory, Hodges (2006).

By ignoring the shear deformation components in Eqs. (2.26) and (2.27) and renaming $\kappa = \bar{\kappa}$ and $\gamma = \bar{\gamma}_{11}e_1$, the equations of motion for the classical theory of beams are obtained as (Hodges 2006)

$$F' + \tilde{K}F + f = \dot{P} + \tilde{\Omega}P \tag{2.34}$$

$$M' + \tilde{K}M + (1+\bar{\gamma}_{11})\tilde{e}_1F + m = \dot{H} + \tilde{\Omega}H + \tilde{V}P \tag{2.35}$$

where the total curvature and twist of the blade are the summation of their built-in values and the added curvature and twist as a result of elastic deformation (i.e., the summation of the initial and the elastic curvatures),

$$K = k + \kappa \tag{2.36}$$

The boundary conditions are another output of the application of the Hamilton's principle in which either force or moment can be specified or calculated at the two ends of the beam.

2.4.2 Intrinsic Kinematical Equations

The nonlinear intrinsic kinematical equations of a beam that should be solved together with the equations of motion are (Hodges 2006)

$$V' + \tilde{K}V + (\tilde{e}_1 + \tilde{\gamma})\Omega = \dot{\gamma} \tag{2.37}$$

$$\Omega' + \tilde{K}\Omega = \dot{\kappa} \tag{2.38}$$

The corresponding scalar equations are

$$V_1' + K_2V_3 - K_3V_2 + 2\gamma_{12}\Omega_3 - 2\gamma_{13}\Omega_2 = \dot{\gamma}_{11} \tag{2.39}$$

$$V_2' + K_3V_1 - K_1V_3 - (1 + \gamma_{11})\Omega_3 + 2\gamma_{13}\Omega_1 = 2\dot{\gamma}_{12} \tag{2.40}$$

$$V_3' + K_1V_2 - K_2V_1 + (1 + \gamma_{11})\Omega_2 - 2\gamma_{12}\Omega_1 = 2\dot{\gamma}_{13} \tag{2.41}$$

$$\Omega_1' + K_2\Omega_3 - K_3\Omega_2 = \dot{\kappa}_1 \tag{2.42}$$

$$\Omega_2' + K_3\Omega_1 - K_1\Omega_3 = \dot{\kappa}_2 \tag{2.43}$$

$$\Omega_3' + K_1\Omega_2 - K_2\Omega_1 = \dot{\kappa}_3 \tag{2.44}$$

2.4.3 *Momentum–Velocity Equations*

The four nonlinear intrinsic vector equations mentioned so far, i.e., Equations (2.26), (2.27), (2.37), and (2.38), are nonlinear partial differential equations. The momentum–velocity equations, however, are a set of linear algebraic equations,

$$\begin{Bmatrix} P \\ H \end{Bmatrix} = \begin{bmatrix} \mu\Delta & -\mu\tilde{\bar{\xi}} \\ \mu\tilde{\bar{\xi}} & i \end{bmatrix} \begin{Bmatrix} V \\ \Omega \end{Bmatrix} \tag{2.45}$$

where

$$\xi = \begin{Bmatrix} 0 \\ x_2 \\ x_3 \end{Bmatrix}, \quad \bar{\xi} = \begin{Bmatrix} 0 \\ \bar{x}_2 \\ \bar{x}_3 \end{Bmatrix}, \quad \tilde{\bar{\xi}} = \begin{bmatrix} 0 & -\bar{x}_3 & \bar{x}_2 \\ \bar{x}_3 & 0 & 0 \\ -\bar{x}_2 & 0 & 0 \end{bmatrix} \tag{2.46}$$

The expanded form of Eq. (2.45) is

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ H_1 \\ H_2 \\ H_3 \end{Bmatrix} = \begin{bmatrix} \mu & 0 & 0 & 0 & \mu\bar{x}_3 & -\mu\bar{x}_2 \\ 0 & \mu & 0 & -\mu\bar{x}_3 & 0 & 0 \\ 0 & 0 & \mu & \mu\bar{x}_2 & 0 & 0 \\ 0 & -\mu\bar{x}_3 & \mu\bar{x}_2 & i_2 + i_3 & 0 & 0 \\ \mu\bar{x}_3 & 0 & 0 & 0 & i_2 & i_{23} \\ -\mu\bar{x}_2 & 0 & 0 & 0 & i_{23} & i_3 \end{bmatrix} \begin{Bmatrix} V_1 \\ V_2 \\ V_3 \\ \Omega_1 \\ \Omega_2 \\ \Omega_3 \end{Bmatrix} \tag{2.47}$$

where the quantities with a bar refer to the location of the centroid with respect to the shear center of the cross section. Also,

$$i = \langle\langle \rho(\xi^T \xi \Delta - \xi \xi^T) \rangle\rangle = \begin{bmatrix} i_2 + i_3 & 0 & 0 \\ 0 & i_2 & i_{23} \\ 0 & i_{23} & i_3 \end{bmatrix} \tag{2.48}$$

where

$$i_2 = \rho \int_A x_3^2 \mathrm{d}x_2 \mathrm{d}x_3, \quad i_3 = \rho \int_A x_2^2 \mathrm{d}x_2 \mathrm{d}x_3, \quad i_{23} = -\rho \int_A x_2 x_3 \mathrm{d}x_2 \mathrm{d}x_3 \tag{2.49}$$

are the cross-sectional mass moments and the product of inertia measured with respect to the shear center. Finally, using Eq. (2.3),

$$\mu = \langle\langle \rho \rangle\rangle = \langle \rho \sqrt{g} \rangle \rightarrow \quad \mu = \int_A (\rho \sqrt{g})\, \mathrm{d}x_2 \mathrm{d}x_3 \tag{2.50}$$

Substitution of Eq. (2.4) in (2.50) gives

$$\mu = \int_A \rho(1 - x_2 k_3 - x_3 k_2) \mathrm{d}x_2 \mathrm{d}x_3 \tag{2.51}$$

If the initial curvature k is zero Eq. (2.51) simply reduces to

$$\mu = \int_A \rho \mathrm{d}x_2 \mathrm{d}x_3 \tag{2.52}$$

which for a homogeneous section results in the following familiar expression for mass per unit length,

$$\mu = \rho A \tag{2.53}$$

Now recalling Eqs. (2.45) and (2.46), since the reference frame has been put at the shear center, the location of the centroid which is shown by bar coordinates would be the position of the centroid with respect to the shear center of the section. If the centroid is close enough to the shear center, the formulation simplifies. Since then,

$$\bar{x}_2 = \bar{x}_3 = 0 \tag{2.54}$$

Therefore using Eq. (2.46),

$$\bar{\xi} = 0, \quad \tilde{\bar{\xi}} = 0 \tag{2.55}$$

Consequently, when the centroid and shear center of the cross section coincide, all off-diagonal (coupling) terms of Eq. (2.45) will vanish except for the ones that are due to the polar moment of inertia.

2.4.4 Constitutive Equations

The 2-D analysis mentioned before provides the warping functions required for the recovery of 3-D stress and strain, as well as the stiffness (or its inverse, i.e., flexibility) matrix used in the following constitutive equations:

$$\begin{Bmatrix} F \\ M \end{Bmatrix} = \underbrace{\begin{bmatrix} A & B \\ B^{\mathrm{T}} & D \end{bmatrix}}_{S} \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix}, \quad \begin{Bmatrix} \gamma \\ \kappa \end{Bmatrix} = \underbrace{\begin{bmatrix} R & Z \\ Z^{T} & T \end{bmatrix}}_{S^{-1}} \begin{Bmatrix} F \\ M \end{Bmatrix} \tag{2.56}$$

or in the scalar form,

$$\begin{Bmatrix} \gamma_{11} \\ 2\gamma_{12} \\ 2\gamma_{13} \\ \kappa_1 \\ \kappa_2 \\ \kappa_3 \end{Bmatrix} = \begin{bmatrix} R_{11} & R_{12} & R_{13} & Z_{11} & Z_{12} & Z_{13} \\ R_{21} & R_{22} & R_{23} & Z_{21} & Z_{22} & Z_{23} \\ R_{31} & R_{32} & R_{33} & Z_{31} & Z_{32} & Z_{33} \\ Z_{11} & Z_{21} & Z_{31} & T_{11} & T_{12} & T_{13} \\ Z_{12} & Z_{22} & Z_{32} & T_{21} & T_{22} & T_{23} \\ Z_{13} & Z_{23} & Z_{33} & T_{31} & T_{32} & T_{33} \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ M_1 \\ M_2 \\ M_3 \end{Bmatrix}$$

Such a linear structural law is valid only for small local strains which can, however, result in large global deformations as they occur in helicopter blades (Traugott et al. 2005).

In the general nonlinear elasto-dynamic case, Eqs. (2.26), (2.27), (2.36), (2.37), (2.38), (2.45), and (2.56) form a system of four nonlinear vector partial differential equations and five linear algebraic vector equations for a total of nine unknown vectors: F, M, V, Ω, P, H, γ, κ, and K, at every point along the beam and at every instant of time. These unknown vectors correspond to 27 scalar variables. Here, F and M are the internal force and moment (generalized forces), P and H are the linear and angular momentum (generalized momenta), V and Ω are the linear and angular velocity (generalized velocities), γ and κ are the beam strains and curvatures (generalized strains), and f and m are the applied external forces and moments per unit length. All quantities refer to the B-frame of the deformed cross section.

Solution of these equations requires the application of initial and boundary conditions. The boundary conditions are another output of the application of the

Hamilton's principle in which either force or moment can be specified or found at the two ends of the beam. The intrinsic equations of motion are not a stand-alone set of equations, and in general, they should be solved together with kinematical equations, constitutive relations, as well as the initial and boundary conditions. Having solved this system of equations for the mentioned unknowns, other variables of interest can be easily calculated.

2.4.5 Strain–Displacement Equations

The generalized strain–displacement equations are (Hodges 2006)

$$\gamma = C(e_1 + u' + \tilde{k}u) - e_1 \tag{2.57}$$

$$\tilde{\kappa} = -C'C^{\mathrm{T}} + C\tilde{k}C^{\mathrm{T}} - \tilde{k} \tag{2.58}$$

2.4.6 Velocity–Displacement Equations

The generalized velocity–displacement equations are

$$V = C(v + \dot{u} + \tilde{\omega}u) \tag{2.59}$$

$$\tilde{\Omega} = -\dot{C}C^{\mathrm{T}} + C\tilde{\omega}C^{\mathrm{T}} \tag{2.60}$$

where V and Ω are measured in the deformed frame, and v and ω are measured in the undeformed frame.

2.4.7 Rodrigues Parameters

One may define a rotation by four parameters: three direction cosine values e_i which define the unit vector $\vec{e} = e_i b_i$ in the direction of the axis of rotation, together with α which is the angle of rotation about this axis. Based on this logic, the Rodrigues parameters $\theta = [\,\theta_1 \quad \theta_2 \quad \theta_3\,]^T$ are defined as,

$$\theta_i = 2e_i \tan(\alpha/2) \tag{2.61}$$

Also, the rotation matrix is

$$C = \frac{\left[1 - (1/4)\theta^{\mathrm{T}}\theta\right]\Delta - \tilde{\theta} + (1/2)\theta\theta^{\mathrm{T}}}{1 + (1/4)\theta^{\mathrm{T}}\theta} \tag{2.62}$$

The associated curvature is related to the Rodrigues parameters in the following way:

$$\kappa = \left(\frac{\Delta - \frac{1}{2}\tilde{\theta}}{1 + \frac{1}{4}\theta^{\mathrm{T}}\theta}\right)\theta' + Ck - k \tag{2.63}$$

Finally, Ω, the column matrix of angular velocity components in the deformed system, B, can be related to the angular velocity vector, ω, in the undeformed system, b, using the Rodrigues parameters

$$\Omega = \left(\frac{\Delta - \frac{1}{2}\tilde{\theta}}{1 + \frac{1}{4}\theta^{T}\theta}\right)\dot{\theta} + C\omega \tag{2.64}$$

Having calculated γ and κ as a part of the solution of the system of equations mentioned before, Eq. (2.63) can now be solved for θ, [notice that C itself depends on θ as is seen in Eq. (2.62)]. Finally, Eq. (2.57) is solved for *u,* which is the displacement vector on the beam reference line.

2.5 Recovery Relations and Their Application in Stress Analysis

The suitability of a design can be evaluated using localized quantities like the 3-D stress and strain components. Nevertheless, the 1-D beam analysis only provides the global behavior of composite beams. Such a global outcome cannot replace a detailed 3-D analysis. In order to recover the complete 3-D components, the 1-D and the 2-D results should be combined properly.

Therefore, the next step is to calculate the 3-D strain and 3-D stress components. This step is usually referred to as recovering the 3-D response and in which the recovery relations are used. They include expressions for 3-D displacements, as well as strain and stress components in terms of the 1-D beam quantities and the local cross-sectional coordinates.

Referring to Fig. 2.4, one may express the position of a particle in the deformed configuration, i.e., $R(x_1, x_2, x_3)$, in terms of its position vector r in the undeformed beam,

$$R(x_1, x_2, x_3) = \left(r + \underbrace{u}_{\text{from } 1-\text{D}} \right) + \left(x_2 \underbrace{B_2}_{\text{from C got in } 1-\text{D}} + x_3 B_3 \right) + \underbrace{w_i}_{\text{from } 2-\text{D}} B_i \tag{2.65}$$

Having obtained the displacement of the reference line, u, from the 1-D analysis and the warping functions of the cross section, w_{i}, from the 2-D analysis, the geometry of the deformed beam is now fully known. The 3-D strain components can be written as (Hodges 2006)

$$\Gamma = \lfloor \Gamma_{11} \quad 2\Gamma_{12} \quad 2\Gamma_{13} \quad \Gamma_{22} \quad 2\Gamma_{23} \quad \Gamma_{33} \rfloor^T \tag{2.66}$$

$$\begin{aligned} &\Gamma = \Gamma(w, w', \bar{\varepsilon}) \\ &\Gamma = \Gamma_a \underbrace{w}_{2-\text{D}} + \Gamma_\varepsilon \underbrace{\bar{\varepsilon}}_{1-\text{D}} + \underbrace{\Gamma_R w}_{\text{initial curvature } 2-\text{D}} + \Gamma_l w' \end{aligned} \tag{2.67}$$

where the 1-D generalized strain is

$$\bar{\varepsilon} = \begin{Bmatrix} \bar{\gamma}_{11} \\ \bar{\kappa}_1 \\ \bar{\kappa}_2 \\ \bar{\kappa}_3 \end{Bmatrix} \tag{2.68}$$

and the operators are, for warping,

$$\Gamma_a = \begin{bmatrix} 0 & 0 & 0 \\ \frac{\partial}{\partial x_2} & 0 & 0 \\ \frac{\partial}{\partial x_3} & 0 & 0 \\ 0 & \frac{\partial}{\partial x_2} & 0 \\ 0 & \frac{\partial}{\partial x_3} & \frac{\partial}{\partial x_2} \\ 0 & 0 & \frac{\partial}{\partial x_3} \end{bmatrix} \tag{2.69}$$

for the 1-D strain,

$$\Gamma_\varepsilon = \frac{1}{\sqrt{g}} \begin{bmatrix} 1 & 0 & x_3 & -x_2 \\ 0 & -x_3 & 0 & 0 \\ 0 & x_2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.70}$$

for the initial curvature and twist,

$$\Gamma_R = \frac{1}{\sqrt{g}}\begin{bmatrix} \tilde{k} + \Delta k_1 \left(x_3 \frac{\partial}{\partial x_2} - x_2 \frac{\partial}{\partial x_3} \right) \\ O_3 \end{bmatrix} \tag{2.71}$$

and finally for the warping derivative,

$$\Gamma_l = \frac{1}{\sqrt{g}}\begin{bmatrix} \Delta \\ O_3 \end{bmatrix} \tag{2.72}$$

Once strain components are calculated, stresses can be computed by the use of the Hooke's law,

$$\sigma = D\Gamma \tag{2.73}$$

To sum up, in a typical problem first using the VAM logic and VABS, the cross-sectional properties are calculated. Then, the 1-D analysis is performed along the span of the beam that utilizes the intrinsic equations of motion, the intrinsic kinematical equations, the constitutive equations of the material of the beam, and the momentum–velocity equations. Finally, the results are combined as shown above to provide the 3-D stress and strain distributions.

2.6 Finite Difference Formulation in Time and Space

In order to solve the system of nonlinear partial differential Eqs. (2.34), (2.35), (2.37), and (2.38) numerically, or to calculate the steady-state solution of this system, the finite difference method (FDM) and the shooting method will be used in the following chapters. In this section, a few equations that will be used later are presented.

Figure 2.5 illustrates a beam that has been discretized by N nodes along its span. The corresponding finite difference space–time grid presentation has been depicted

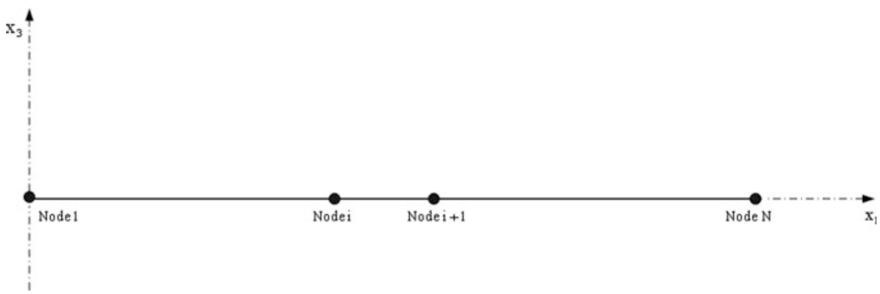

Fig. 2.5 Nodes along the blade and the coordinate system of the undeformed blade

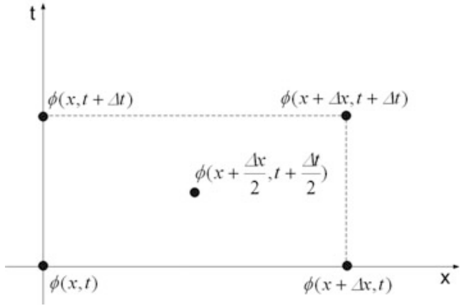

Fig. 2.6 The space–time grid for the numerical solution of a partial differential equation

in Fig. 2.6. Consider the value of a generic variable, $\phi(x,t)$, at position x and time t. In order to simplify the notation, the following convention is used to express this value:

$$\phi_i = \phi(x,t) \tag{2.74}$$

where the subscript i is the beam node number corresponding to position x. At points neighboring (x, t) on the space–time grid shown in Fig. 2.6, the same variable can be expressed as

$$\phi_{i+1} = \phi(x+\Delta x, t), \quad \phi_i^+ = \phi(x, t+\Delta t), \quad \phi_{i+1}^+ = \phi(x+\Delta x, t+\Delta t) \tag{2.75}$$

where the superscript '+' refers to the values at the next time step and i + 1 is the next spatial node (right-hand side of i). Using Taylor series expansion and a forward and a backward difference in space, one obtains, respectively,

$$\phi\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right) = \phi\left(x, t+\frac{\Delta t}{2}\right) + \phi'\left(x, t+\frac{\Delta t}{2}\right)\frac{\Delta x}{2} \tag{2.76}$$

$$\phi\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right) = \phi\left(x+\Delta x, t+\frac{\Delta t}{2}\right) - \phi'\left(x+\Delta x, t+\frac{\Delta t}{2}\right)\frac{\Delta x}{2} \tag{2.77}$$

Adding Eqs. (2.76) and (2.77) gives

$$\begin{aligned} 2\phi\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right) &= \phi\left(x, t+\frac{\Delta t}{2}\right) + \phi\left(x+\Delta x, t+\frac{\Delta t}{2}\right) \\ &\quad + \left[\phi'\left(x, t+\frac{\Delta t}{2}\right) - \phi'\left(x+\Delta x, t+\frac{\Delta t}{2}\right)\right]\frac{\Delta x}{2} \end{aligned} \tag{2.78}$$

Now, considering forward and backward differences in time,

$$\phi\left(x, t+\frac{\Delta t}{2}\right) = \phi(x,t) + \dot{\phi}(x,t)\frac{\Delta t}{2} \tag{2.79}$$

$$\phi\left(x, t+\frac{\Delta t}{2}\right) = \phi(x,t+\Delta t) - \dot{\phi}(x,t+\Delta t)\frac{\Delta t}{2} \tag{2.80}$$

Adding Eqs. (2.79) and (2.80) gives,

$$2\phi\left(x, t+\frac{\Delta t}{2}\right) = \phi(x,t+\Delta t) + \phi(x,t) + \left[\dot{\phi}(x,t) - \dot{\phi}(x,t+\Delta t)\right]\frac{\Delta t}{2} \tag{2.81}$$

Similarly, using forward and backward differences in time,

$$\phi\left(x+\Delta x, t+\frac{\Delta t}{2}\right) = \phi(x+\Delta x,t) + \dot{\phi}(x+\Delta x,t)\frac{\Delta t}{2} \tag{2.82}$$

$$\phi\left(x+\Delta x, t+\frac{\Delta t}{2}\right) = \phi(x+\Delta x,t+\Delta t) - \dot{\phi}(x+\Delta x,t+\Delta t)\frac{\Delta t}{2} \tag{2.83}$$

Adding Eqs. (2.82) and (2.83) results in

$$\begin{aligned} 2\phi\left(x+\Delta x, t+\frac{\Delta t}{2}\right) &= \phi(x+\Delta x,t+\Delta t) + \phi(x+\Delta x,t) \\ &\quad + \left[\dot{\phi}(x+\Delta x,t) - \dot{\phi}(x+\Delta x,t+\Delta t)\right]\frac{\Delta t}{2} \end{aligned} \tag{2.84}$$

Substitution of Eqs. (2.81) and (2.84) into Eq. (2.78) gives

$$\begin{aligned} 2\phi\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right) &= \frac{1}{2}[\phi(x,t) + \phi(x,t+\Delta t) + \phi(x+\Delta x,t) + \phi(x+\Delta x,t+\Delta t)] \\ &\quad + \left[\dot{\phi}(x,t) - \dot{\phi}(x,t+\Delta t)\right]\frac{\Delta t}{4} \\ &\quad + \left[\dot{\phi}(x+\Delta x,t) - \dot{\phi}(x+\Delta x,t+\Delta t)\right]\frac{\Delta t}{4} \\ &\quad + \left[\phi'\left(x, t+\frac{\Delta t}{2}\right) - \phi'\left(x+\Delta x, t+\frac{\Delta t}{2}\right)\right]\frac{\Delta x}{2} \end{aligned} \tag{2.85}$$

Taylor series expansions in time and in space can be used in Eq. (2.85) to give the generic function value at the center of the space–time grid,

$$\phi\left(x+\frac{\Delta x}{2},t+\frac{\Delta t}{2}\right)=\frac{1}{4}[\phi(x,t)+\phi(x,t+\Delta t)+\phi(x+\Delta x,t)+\phi(x+\Delta x,t+\Delta t)]$$
$$-\ddot{\phi}(x,t)\frac{(\Delta t)^2}{8}-\ddot{\phi}(x+\Delta x,t)\frac{(\Delta t)^2}{8}-\phi''\left(x,t+\frac{\Delta t}{2}\right)\frac{(\Delta x)^2}{4} \tag{2.86}$$

Using the notation given in Eq. (2.75), and by ignoring the higher order terms, Eq. (2.86) reduces to

$$\phi\left(x+\frac{\Delta x}{2},t+\frac{\Delta t}{2}\right)=\frac{1}{4}\left(\phi_{i+1}^{+}+\phi_i^{+}+\phi_{i+1}+\phi_i\right)+O\left(\Delta x^2,\Delta t^2\right) \tag{2.87}$$

According to Eq. (2.87), the function value at the center of a space–time grid may well be approximated by the average of its values at the neighboring grid nodes. Now, for the partial derivatives at the center point, using Taylor series expansion,

$$\phi\left(x+\Delta x,t+\frac{\Delta t}{2}\right)=\phi\left(x+\frac{\Delta x}{2},t+\frac{\Delta t}{2}\right)+\phi'\left(x+\frac{\Delta x}{2},t+\frac{\Delta t}{2}\right)\frac{\Delta x}{2}+O(\Delta x)^2 \tag{2.88}$$

$$\phi\left(x,t+\frac{\Delta t}{2}\right)=\phi\left(x+\frac{\Delta x}{2},t+\frac{\Delta t}{2}\right)-\phi'\left(x+\frac{\Delta x}{2},t+\frac{\Delta t}{2}\right)\frac{\Delta x}{2}+O(\Delta x)^2 \tag{2.89}$$

Subtracting Eq. (2.89) from Eq. (2.88) gives

$$\phi'\left(x+\frac{\Delta x}{2},t+\frac{\Delta t}{2}\right)\Delta x=\phi\left(x+\Delta x,t+\frac{\Delta t}{2}\right)-\phi\left(x,t+\frac{\Delta t}{2}\right)+O(\Delta x)^2 \tag{2.90}$$

Similarly,

$$\phi(x+\Delta x,t)=\phi\left(x+\Delta x,t+\frac{\Delta t}{2}\right)-\dot{\phi}\left(x+\Delta x,t+\frac{\Delta t}{2}\right)\frac{\Delta t}{2}+O(\Delta t)^2 \tag{2.91}$$

$$\phi(x+\Delta x,t+\Delta t)=\phi\left(x+\Delta x,t+\frac{\Delta t}{2}\right)+\dot{\phi}\left(x+\Delta x,t+\frac{\Delta t}{2}\right)\frac{\Delta t}{2}+O(\Delta t)^2 \tag{2.92}$$

Adding Eqs. (2.91) and (2.92) gives

$$2\phi\left(x+\Delta x, t+\frac{\Delta t}{2}\right) = \phi(x+\Delta x, t+\Delta t) + \phi(x+\Delta x, t) + O(\Delta t)^2 \tag{2.93}$$

Similarly,

$$\phi(x,t) = \phi\left(x, t+\frac{\Delta t}{2}\right) - \dot{\phi}\left(x, t+\frac{\Delta t}{2}\right)\frac{\Delta t}{2} + O(\Delta t)^2 \tag{2.94}$$

$$\phi(x, t+\Delta t) = \phi\left(x, t+\frac{\Delta t}{2}\right) + \dot{\phi}\left(x, t+\frac{\Delta t}{2}\right)\frac{\Delta t}{2} + O(\Delta t)^2 \tag{2.95}$$

Which result in

$$2\phi\left(x, t+\frac{\Delta t}{2}\right) = \phi(x,t) + \phi(x, t+\Delta t) + O(\Delta t)^2 \tag{2.96}$$

Substitution of Eqs. (2.93) and (2.96) into Eq. (2.90) gives

$$\begin{aligned}\phi'\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right)\Delta x = \frac{1}{2}[\phi(x+\Delta x, t+\Delta t) + \phi(x+\Delta x, t) - \phi(x,t) \\ -\phi(x, t+\Delta t)] + O\left[(\Delta x)^2, (\Delta t)^2\right]\end{aligned} \tag{2.97}$$

Using Eq. (2.97) and the notation introduced in Eq. (2.75), the generic form of the derivative with respect to x_1 becomes

$$\phi'\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right) = \frac{1}{2\Delta x}\left(\phi_{i+1}^{+} - \phi_i^{+} + \phi_{i+1} - \phi_i\right) + O\left(\Delta x^2, \Delta t^2\right) \tag{2.98}$$

For the generic form of the time derivative, one may start with

$$\phi\left(x+\frac{\Delta x}{2}, t+\Delta t\right) = \phi\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right) + \dot{\phi}\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right)\frac{\Delta t}{2} + O(\Delta t)^2 \tag{2.99}$$

$$\phi\left(x+\frac{\Delta x}{2}, t\right) = \phi\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right) - \dot{\phi}\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right)\frac{\Delta t}{2} + O(\Delta t)^2 \tag{2.100}$$

Subtracting Eq. (2.100) from Eq. (2.99) gives

$$\dot{\phi}\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right)\Delta t = \phi\left(x+\frac{\Delta x}{2}, t+\Delta t\right) - \phi\left(x+\frac{\Delta x}{2}, t\right) + O(\Delta t)^2 \quad (2.101)$$

Now,

$$\phi(x, t+\Delta t) = \phi\left(x+\frac{\Delta x}{2}, t+\Delta t\right) - \phi'\left(x+\frac{\Delta x}{2}, t+\Delta t\right)\frac{\Delta x}{2} + O(\Delta x)^2 \quad (2.102)$$

$$\phi(x+\Delta x, t+\Delta t) = \phi\left(x+\frac{\Delta x}{2}, t+\Delta t\right) + \phi'\left(x+\frac{\Delta x}{2}, t+\Delta t\right)\frac{\Delta x}{2} + O(\Delta x)^2 \quad (2.103)$$

Adding Eqs. (2.102) and (2.103) gives

$$2\phi\left(x+\frac{\Delta x}{2}, t+\Delta t\right) = \phi(x+\Delta x, t+\Delta t) + \phi(x, t+\Delta t) + O(\Delta x)^2 \quad (2.104)$$

Similarly,

$$\phi(x, t) = \phi\left(x+\frac{\Delta x}{2}, t\right) - \phi'\left(x+\frac{\Delta x}{2}, t\right)\frac{\Delta x}{2} + O(\Delta x)^2 \quad (2.105)$$

$$\phi(x+\Delta x, t) = \phi\left(x+\frac{\Delta x}{2}, t\right) + \phi'\left(x+\frac{\Delta x}{2}, t\right)\frac{\Delta x}{2} + O(\Delta x)^2 \quad (2.106)$$

Adding Eqs. (2.105) and (2.106) results in

$$2\phi\left(x+\frac{\Delta x}{2}, t\right) = \phi(x, t) + \phi(x+\Delta x, t) + O(\Delta x)^2 \quad (2.107)$$

Substitution of Eqs. (2.104) and (2.107) into Eq. (2.101) gives

$$\begin{aligned}\dot{\phi}\left(x+\frac{\Delta x}{2}, t+\frac{\Delta t}{2}\right)\Delta t =& \frac{1}{2}[\phi(x+\Delta x, t+\Delta t) + \phi(x, t+\Delta t) - \phi(x, t) \\ & -\phi(x+\Delta x, t)] + O\left[(\Delta x)^2, (\Delta t)^2\right]\end{aligned} \quad (2.108)$$

Finally, using Eq. (2.75) and Eq. (2.108), one obtains

$$\dot{\phi}\left(x+\frac{\Delta x}{2},t+\frac{\Delta t}{2}\right)=\frac{1}{2\Delta t}\left(\phi_{i+1}^{+}-\phi_{i+1}+\phi_{i}^{+}-\phi_{i}\right)+O\left(\Delta x^{2},\Delta t^{2}\right) \quad (2.109)$$

Therefore, the set of difference equations for a 1-D dynamic problem would be

$$\phi\left(x+\frac{\Delta x}{2},t+\frac{\Delta t}{2}\right)=\frac{1}{4}\left(\phi_{i+1}^{+}+\phi_{i}^{+}+\phi_{i+1}+\phi_{i}\right)+O\left(\Delta x^{2},\Delta t^{2}\right) \quad (2.87)$$

$$\phi'\left(x+\frac{\Delta x}{2},t+\frac{\Delta t}{2}\right)=\frac{1}{2\Delta x}\left(\phi_{i+1}^{+}-\phi_{i}^{+}+\phi_{i+1}-\phi_{i}\right)+O\left(\Delta x^{2},\Delta t^{2}\right) \quad (2.98)$$

$$\dot{\phi}\left(x+\frac{\Delta x}{2},t+\frac{\Delta t}{2}\right)=\frac{1}{2\Delta t}\left(\phi_{i+1}^{+}-\phi_{i+1}+\phi_{i}^{+}-\phi_{i}\right)+O\left(\Delta x^{2},\Delta t^{2}\right) \quad (2.109)$$

The quantities at the left-hand side of Eqs. (2.87), (2.98), and (2.109) are the elemental ones and are expressed in terms of the nodal values. Equations (2.87), (2.98), and (2.109) provide the second-order approximate finite difference expressions for a variable and its derivatives with respect to time and space. They were used in Ghorashi (1994) and in Esmailzadeh and Ghorashi (1997) to solve a moving load problem. In this research, by properly defining the initial and boundary conditions, these equations will be used to convert the discussed system of nonlinear partial differential equations into a set of linear algebraic difference equations.

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