Chapter 2
Fractional Calculus

Abstract A brief exposition of fractional order operators and their properties is given. After that, we introduce the notion of generalized fractional operators.

Keywords Fractional derivatives and integrals · Generalized fractional derivatives and integrals · Fractional derivatives and integrals of variable order · Riemann-Liouville, Hadamard and Caputo operators · Fractional integration by parts · Multi-dimensional generalized fractional calculus

Fractional calculus was introduced on September 30, 1695. On that day, Leibniz wrote a letter to L'Hôpital, raising the possibility of generalizing the meaning of derivatives from integer order to noninteger order derivatives. L'Hôpital wanted to know the result for the derivative of order $n = 1/2$. Leibniz replied that "*one day, useful consequences will be drawn*" and, in fact, his vision became a reality. However, the study of noninteger order derivatives did not appear in the literature until 1819, when Lacroix presented a definition of fractional derivative based on the usual expression for the nth derivative of the power function (Lacroix 1819). Within years the fractional calculus became a very attractive subject to mathematicians, and many different forms of fractional (i.e., noninteger) differential operators were introduced: the Grunwald–Letnikow, Riemann–Liouville, Hadamard, Caputo, Riesz (Hilfer 2000; Kilbas et al. 2006; Podlubny 1999; Samko et al. 1993) and the more recent notions of Cresson (2007), Katugampola (2011), Klimek (2005), Kilbas and Saigo (2004) or variable order fractional operators introduced by Samko and Ross (1993).

In 2010, an interesting perspective to the subject, unifying all mentioned notions of fractional derivatives and integrals, was introduced in Agrawal (2010) and later studied in Bourdin et al. (2014), Klimek and Lupa (2013), Odzijewicz et al. (2012a, b, 2013a, b, c). Precisely, authors considered general operators, which by choosing special kernels, reduce to the standard fractional operators. However, other nonstandard kernels can also be considered as particular cases.

This chapter presents preliminary definitions and facts of classical, variable order, and generalized fractional operators.

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2.1 One-Dimensional Fractional Calculus

We begin with basic facts on the one-dimensional classical, variable order, and generalized fractional operators.

2.1.1 Classical Fractional Operators

In this section, we present definitions and properties of the one-dimensional fractional integrals and derivatives under consideration. The reader interested in the subject is refereed to the books (Kilbas et al. 2006; Klimek 2009; Podlubny 1999; Samko et al. 1993).

Definition 2.1 (*Left and right Riemann–Liouville fractional integrals*) We define the left and the right Riemann–Liouville fractional integrals ${}_aI_t^\alpha$ and ${}_tI_b^\alpha$ of order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ by

$$
{}_aI_t^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)\,\mathrm{d}\tau}{(t-\tau)^{1-\alpha}}, \quad t \in (a,b], \tag{2.1}
$$

and

$$
{}_tI_b^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)\,\mathrm{d}\tau}{(\tau-t)^{1-\alpha}}, \quad t \in [a,b), \tag{2.2}
$$

respectively. Here $\Gamma(\alpha)$ denotes Euler's Gamma function. Note that, ${}_aI_t^\alpha[f]$ and ${}_tI_b^\alpha[f]$ are defined a.e. on (a,b) for $f \in L^1(a,b;\mathbb{R})$.

One can also define fractional integral operators in the frame of Hadamard setting. In the following, we present definitions of Hadamard fractional integrals.

Definition 2.2 (*Left and right Hadamard fractional integrals*) Let $0 \leq a < b < \infty$. We define the left-sided and right-sided Hadamard integrals of fractional order $\alpha \in \mathbb{R}$ $(\alpha > 0)$ by

$$
{}_aJ_t^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_a^t \left(\log \frac{t}{\tau}\right)^{\alpha-1} \frac{f(\tau)\,\mathrm{d}\tau}{\tau}, \quad t > a
$$

and

$$
{}_tJ_b^\alpha[f](t) := \frac{1}{\Gamma(\alpha)} \int_t^b \left(\log \frac{\tau}{t}\right)^{\alpha-1} \frac{f(\tau)\,\mathrm{d}\tau}{\tau}, \quad t < b,
$$

respectively.

Since it is enough for the purposes of this book, we define Riemann–Liouville fractional derivatives of order α with $0 < \alpha < 1$. A more general definition for any α with $Re(\alpha) > 0$ can be found in Kilbas et al. (2006).

Definition 2.3 (*Left and right Riemann–Liouville fractional derivatives*) The left Riemann–Liouville fractional derivative of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ of a function f, denoted by ${}_aD_t^\alpha[f]$, is defined by

$$\forall t \in (a, b], \quad {}_aD_t^\alpha[f](t) := \frac{d}{\mathrm{d}t}{}_aI_t^{1-\alpha}[f](t).$$

Similarly, the right Riemann–Liouville fractional derivative of order α of a function f, denoted by ${}_tD_b^\alpha[f]$, is defined by

$$\forall t \in [a, b), \quad {}_tD_b^\alpha[f](t) := -\frac{d}{\mathrm{d}t}{}_tI_b^{1-\alpha}[f](t).$$

As we can see below, Riemann–Liouville fractional integral and differential operators of power functions return power functions.

Property 2.4 (Property 2.1 (Kilbas et al. 2006)) *Let* $\alpha, \beta > 0$. *Then the following identities hold:*

$${}_aI_t^\alpha[(\tau - a)^{\beta-1}](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(t-a)^{\beta+\alpha-1},$$

$${}_aD_t^\alpha[(\tau - a)^{\beta-1}](t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(t-a)^{\beta-\alpha-1},$$

$${}_tI_b^\alpha[(b - \tau)^{\beta-1}](t) = \frac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(b-t)^{\beta+\alpha-1},$$

and

$${}_tD_b^\alpha[(b - \tau)^{\beta-1}](t) = \frac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(b-t)^{\beta-\alpha-1}.$$

Definition 2.5 (*Left and right Caputo fractional derivatives*) The left and the right Caputo fractional derivatives of order $\alpha \in \mathbb{R}$ $(0 < \alpha < 1)$ are given by

$$\forall t \in (a, b], \quad {}_a^CD_t^\alpha[f](t) := {}_aI_t^{1-\alpha}\left[\frac{d}{\mathrm{d}t}f\right](t)$$

and

$$\forall t \in [a, b), \quad {}_t^CD_b^\alpha[f](t) := -{}_tI_b^{1-\alpha}\left[\frac{d}{\mathrm{d}t}f\right](t),$$

respectively.

Let $0 < \alpha < 1$ and $f \in AC([a,b];\mathbb{R})$, where AC denotes the class of absolutely continuous functions. Then the Riemann–Liouville and Caputo fractional derivatives satisfy relations

$$ {}_a^C D_t^\alpha[f](t) = {}_a D_t^\alpha[f](t) - \frac{f(a)}{(t-a)^\alpha \Gamma(1-\alpha)}, \tag{2.3}$$

$$ {}_t^C D_b^\alpha[f](t) = -{}_t D_b^\alpha[f](t) + \frac{f(b)}{(b-t)^\alpha \Gamma(1-\alpha)}, \tag{2.4}$$

that can be found in Kilbas et al. (2006). Moreover, for Riemann–Liouville fractional integrals and derivatives, the following composition rules hold:

$$ \left({}_a I_t^\alpha \circ {}_a D_t^\alpha\right)[f](t) = f(t), \tag{2.5}$$

$$ \left({}_t I_b^\alpha \circ {}_t D_b^\alpha\right)[f](t) = f(t), \tag{2.6}$$

provided that $f \in L^1(a,b;\mathbb{R})$, ${}_a I_t^\alpha[f], {}_t I_b^\alpha[f] \in AC([a,b];\mathbb{R})$ and ${}_a I_t^\alpha f(a) = 0$, ${}_t I_b^\alpha f(b) = 0$. Note that, if $f(a) = 0$, then (2.3) and (2.5) give

$$ \left({}_a I_t^\alpha \circ {}_a^C D_t^\alpha\right)[f](t) = \left({}_a I_t^\alpha \circ {}_a D_t^\alpha\right)[f](t) = f(t), \tag{2.7}$$

and if $f(b) = 0$, then (2.4) and (2.6) imply that

$$ \left({}_t I_b^\alpha \circ {}_t^C D_b^\alpha\right)[f](t) = \left({}_t I_b^\alpha \circ {}_t D_b^\alpha\right)[f](t) = f(t). \tag{2.8}$$

The following assertion shows that Riemann–Liouville fractional integrals satisfy semigroup property.

Property 2.6 (Lemma 2.3 (Kilbas et al. 2006)) *Let* $\alpha, \beta > 0$ *and* $f \in L^r(a,b;\mathbb{R})$ $(1 \le r \le \infty)$. *Then, equations*

$$ \left({}_a I_t^\alpha \circ {}_a I_t^\beta\right)[f](t) = {}_a I_t^{\alpha+\beta}[f](t) $$

and

$$ \left({}_t I_b^\alpha \circ {}_t I_b^\beta\right)[f](t) = {}_t I_b^{\alpha+\beta}[f](t) $$

are satisfied a.e. in (a,b).

Next results show that, for certain classes of functions, Riemann–Liouville fractional derivatives and Caputo fractional derivatives are left inverse operators of Riemann–Liouville fractional integrals.

Property 2.7 (cf. Lemma 2.4 (Kilbas et al. 2006)) *If* $0 < \alpha < 1$ *and* $f \in L^r(a, b; \mathbb{R})$ $(1 \le r \le \infty)$*, then the following is true:*

$$\left({}_aD_t^{\alpha} \circ {}_aI_t^{\alpha}\right)[f](t) = f(t),$$

$$\left({}_tD_b^{\alpha} \circ {}_tI_b^{\alpha}\right)[f](t) = f(t),$$

a.e. in (a, b).

Property 2.8 (cf. Lemma 2.21 (Kilbas et al. 2006)) *Let* $0 < \alpha < 1$. *If* f *is continuous on the interval* $[a, b]$*, then*

$$\left({}_a^CD_t^{\alpha} \circ {}_aI_t^{\alpha}\right)[f](t) = f(t),$$

$$\left({}_t^CD_b^{\alpha} \circ {}_tI_b^{\alpha}\right)[f](t) = f(t).$$

For r-Lebesgue integrable functions, Riemann–Liouville fractional integrals and derivatives satisfy the following composition properties:

Property 2.9 (cf. Property 2.2 (Kilbas et al. 2006)) *Let* $0 < \beta < \alpha < 1$ *and* $f \in L^r(a, b; \mathbb{R})$ $(1 \le r \le \infty)$. *Then, relations*

$$\left({}_aD_t^{\beta} \circ {}_aI_t^{\alpha}\right)[f](t) = {}_aI_t^{\alpha-\beta}[f](t)$$

and

$$\left({}_tD_b^{\beta} \circ {}_tI_b^{\alpha}\right)[f](t) = {}_tI_b^{\alpha-\beta}[f](t)$$

are satisfied a.e. in (a, b).

In classical calculus, integration by parts formula relates the integral of a product of functions to the integral of their derivative and antiderivative. As we can see below, this formula works also for fractional derivatives, however, it changes the type of differentiation: left Riemann–Lioville fractional derivatives are transformed to right Caputo fractional derivatives.

Property 2.10 (cf. Lemma 2.19 (Klimek 2009)) *Assume that* $0 < \alpha < 1$, $f \in AC([a, b]; \mathbb{R})$ *and* $g \in L^r(a, b; \mathbb{R})$ $(1 \le r \le \infty)$. *Then, the following integration by parts formula holds:*

$$\int_a^b f(t)\,{}_aD_t^{\alpha}[g](t)\,\mathrm{d}t = \int_a^b g(t)\,{}_t^CD_b^{\alpha}[f](t)\,\mathrm{d}t + f(t)\,{}_aI_t^{1-\alpha}[g](t)\Big|_{t=a}^{t=b}. \quad (2.9)$$

Let us recall the following property yielding boundedness of Riemann–Liouville fractional integral in the space $L^r(a, b; \mathbb{R})$ (cf. Lemma 2.1, formula 2.1.23, from the monograph by Kilbas et al. (2006)).

Property 2.11 *The fractional integral* ${}_aI_t^\alpha$ *is bounded in the space* $L^r(a, b; \mathbb{R})$ *for* $\alpha \in (0, 1)$ *and* $r \geq 1$*:*

$$||{}_aI_t^\alpha[f]||_{L^r} \leq K_\alpha ||f||_{L^r}, \quad K_\alpha = \frac{(b-a)^\alpha}{\Gamma(\alpha+1)}. \tag{2.10}$$

2.1.2 Variable Order Fractional Operators

In 1993, Samko and Ross (1993) proposed an interesting generalization of fractional operators. They introduced the study of fractional integration and differentiation when the order is not a constant but a function. Afterwards, several works were dedicated to variable order fractional operators, their applications and interpretations (Almeida and Samko 2009; Coimbra 2003; Lorenzo and Hartley 2002). In particular, Samko's variable order fractional calculus turns out to be very useful in mechanics and in the theory of viscous flows (Coimbra 2003; Diaz and Coimbra 2009; Lorenzo and Hartley 2002; Pedro et al. 2008; Ramirez and Coimbra 2010, 2011). Indeed, many physical processes exhibit fractional order behavior that may vary with time or space (Lorenzo and Hartley 2002). The paper (Coimbra 2003) is devoted to the study of a variable order fractional differential equation that characterizes some problems in the theory of viscoelasticity. In Diaz and Coimbra (2009) the authors analyze the dynamics and control of a nonlinear variable viscoelasticity oscillator, and two controllers are proposed for the variable order differential equations that track an arbitrary reference function. The work (Pedro et al. 2008) investigates the drag force acting on a particle due to the oscillatory flow of a viscous fluid. The drag force is determined using the variable order fractional calculus, where the order of derivative vary according to the dynamics of the flow. In Ramirez and Coimbra (2011) a variable order differential equation for a particle in a quiescent viscous liquid is developed. For more on the application of variable order fractional operators to the modeling of dynamic systems, we refer the reader to the review article (Ramirez and Coimbra 2010).

Let us introduce the following triangle:

$$\Delta := \left\{(t, \tau) \in \mathbb{R}^2 : \ a \leq \tau < t \leq b\right\},$$

and let $\alpha(t, \tau) : \Delta \to [0, 1]$ be such that $\alpha \in C^1\left(\bar{\Delta}; \mathbb{R}\right)$, where $\bar{\Delta}$ denotes the closure of the set Δ.

Definition 2.12 (*Left and right Riemann–Liouville integrals of variable order*) Operator

$${}_aI_t^{\alpha(\cdot,\cdot)}[f](t) := \int_a^t \frac{1}{\Gamma(\alpha(t, \tau))}(t-\tau)^{\alpha(t,\tau)-1} f(\tau)\,\mathrm{d}\tau \quad (t > a)$$

is the left Riemann–Liouville integral of variable fractional order $\alpha(\cdot, \cdot)$, while

$$ {}_tI_b^{\alpha(\cdot,\cdot)}[f](t) := \int_t^b \frac{1}{\Gamma(\alpha(\tau, t))}(\tau - t)^{\alpha(\tau,t)-1} f(\tau)\,\mathrm{d}\tau \quad (t < b) $$

is the right Riemann–Liouville integral of variable fractional order $\alpha(\cdot, \cdot)$.

The following example gives a variable order fractional integral for the power function $(t - a)^\gamma$.

Example 2.13 (*cf. Equation 4 of* (Samko and Ross 1993)) Let $\alpha(t, \tau) = \alpha(t)$ be a function depending only on variable t, $0 < \alpha(t) < 1$ for almost all $t \in (a, b)$ and $\gamma > -1$. Then,

$$ {}_aI_t^{\alpha(\cdot)}(t - a)^\gamma = \frac{\Gamma(\gamma + 1)(t - a)^{\gamma+\alpha(t)}}{\Gamma(\gamma + \alpha(t) + 1)}. \tag{2.11} $$

Next we define two types of variable order fractional derivatives.

Definition 2.14 (*Left and right Riemann–Liouville derivatives of variable order*) The left Riemann–Liouville derivative of variable fractional order $\alpha(\cdot, \cdot)$ of a function f is defined by

$$ \forall t \in (a, b], \quad {}_aD_t^{\alpha(\cdot,\cdot)}[f](t) := \frac{d}{\mathrm{d}t}{}_aI_t^{1-\alpha(\cdot,\cdot)}[f](t), $$

while the right Riemann–Liouville derivative of variable fractional order $\alpha(\cdot, \cdot)$ is defined by

$$ \forall t \in [a, b), \quad {}_tD_b^{\alpha(\cdot,\cdot)}[f](t) := -\frac{d}{\mathrm{d}t}{}_tI_b^{1-\alpha(\cdot,\cdot)}[f](t). $$

Definition 2.15 (*Left and right Caputo derivatives of variable fractional order*) The left Caputo derivative of variable fractional order $\alpha(\cdot, \cdot)$ is defined by

$$ \forall t \in (a, b], \quad {}_a^CD_t^{\alpha(\cdot,\cdot)}[f](t) := {}_aI_t^{1-\alpha(\cdot,\cdot)}\left[\frac{d}{\mathrm{d}t}f\right](t), $$

while the right Caputo derivative of variable fractional order $\alpha(\cdot, \cdot)$ is given by

$$ \forall t \in [a, b), \quad {}_t^CD_b^{\alpha(\cdot,\cdot)}[f](t) := -{}_tI_b^{1-\alpha(\cdot,\cdot)}\left[\frac{d}{\mathrm{d}t}f\right](t). $$

2.1.3 Generalized Fractional Operators

This section presents definitions of one-dimensional generalized fractional operators. In special cases, these operators simplify to the classical Riemann–Liouville fractional integrals, and Riemann–Liouville and Caputo fractional derivatives. As before,

$$\Delta := \left\{(t,\tau) \in \mathbb{R}^2 : \ a \leq \tau < t \leq b\right\}.$$

Definition 2.16 (*Generalized fractional integrals of Riemann–Liouville type*) Let us consider a function k defined almost everywhere on Δ with values in $\mathbb{R}$. For any function f defined almost everywhere on (a,b) with value in $\mathbb{R}$, the generalized fractional integral operator K_P is defined for almost all $t \in (a,b)$ by:

$$K_P[f](t) = \lambda \int_a^t k(t,\tau) f(\tau)\,\mathrm{d}\tau + \mu \int_t^b k(\tau,t) f(\tau)\,\mathrm{d}\tau, \tag{2.12}$$

with $P = \langle a, t, b, \lambda, \mu \rangle$, $\lambda, \mu \in \mathbb{R}$.

In particular, for suitably chosen kernels $k(t,\tau)$ and sets P, kernel operators K_P reduce to the classical or variable order fractional integrals of Riemann–Liouville type, and classical fractional integrals of Hadamard type.

Example 2.17 (a) Let $k^\alpha(t-\tau) = \frac{1}{\Gamma(\alpha)}(t-\tau)^{\alpha-1}$ and $0 < \alpha < 1$. If $P = \langle a, t, b, 1, 0 \rangle$, then

$$K_P[f](t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\,\mathrm{d}\tau =: {}_aI_t^\alpha[f](t)$$

is the left Riemann–Liouville fractional integral of order α; if $P = \langle a, t, b, 0, 1 \rangle$, then

$$K_P[f](t) = \frac{1}{\Gamma(\alpha)} \int_t^b (\tau-t)^{\alpha-1} f(\tau)\,\mathrm{d}\tau =: {}_tI_b^\alpha[f](t)$$

is the right Riemann–Liouville fractional integral of order α.

(b) For $k^\alpha(t,\tau) = \frac{1}{\Gamma(\alpha(t,\tau))}(t-\tau)^{\alpha(t,\tau)-1}$ and $P = \langle a, t, b, 1, 0 \rangle$,

$$K_P[f](t) = \int_a^t \frac{1}{\Gamma(\alpha(t,\tau))}(t-\tau)^{\alpha(t,\tau)-1} f(\tau)\,\mathrm{d}\tau =: {}_aI_t^{\alpha(\cdot,\cdot)}[f](t)$$

is the left Riemann–Liouville fractional integral of order $\alpha(\cdot,\cdot)$ and for $P = \langle a, t, b, 0, 1 \rangle$

$$K_P[f](t) = \int_t^b \frac{1}{\Gamma(\alpha(\tau,t))}(\tau - t)^{\alpha(t,\tau)-1} f(\tau)\,\mathrm{d}\tau =: {}_tI_b^{\alpha(\cdot,\cdot)}[f](t)$$

is the right Riemann–Liouville fractional integral of order $\alpha(\cdot,\cdot)$.

(c) For any $0 < \alpha < 1$, kernel $k^\alpha(t,\tau) = \frac{1}{\Gamma(\alpha)}\left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{1}{\tau}$ and $P = \langle a, t, b, 1, 0\rangle$, the general operator K_P reduces to the left Hadamard fractional integral:

$$K_P[f](t) = \frac{1}{\Gamma(\alpha)}\int_a^t \left(\log\frac{t}{\tau}\right)^{\alpha-1}\frac{f(\tau)\,\mathrm{d}\tau}{\tau} =: {}_aJ_t^\alpha[f](t);$$

and for $P = \langle a, t, b, 0, 1\rangle$ operator K_P reduces to the right Hadamard fractional integral:

$$K_P[f](t) = \frac{1}{\Gamma(\alpha)}\int_t^b \left(\log\frac{\tau}{t}\right)^{\alpha-1}\frac{f(\tau)\,\mathrm{d}\tau}{\tau} =: {}_tJ_b^\alpha[f](t).$$

(d) Generalized fractional integrals can be also reduced to, e.g., Riesz, Katugampola or Kilbas fractional operators. Their definitions can be found in Katugampola (2011), Kilbas and Saigo (2004), Kilbas et al. (2006).

The generalized differential operators A_P and B_P are defined with the help of the operator K_P.

Definition 2.18 (*Generalized fractional derivative of Riemann–Liouville type*) The generalized fractional derivative of Riemann–Liouville type, denoted by A_P, is defined by

$$A_P = \frac{d}{\mathrm{d}t} \circ K_P.$$

The next differential operator is obtained by interchanging the order of the operators in the composition that defines A_P.

Definition 2.19 (*Generalized fractional derivative of Caputo type*) The general kernel differential operator of Caputo type, denoted by B_P, is given by

$$B_P = K_P \circ \frac{d}{\mathrm{d}t}.$$

Example 2.20 The standard Riemann–Liouville and Caputo fractional derivatives (see, e.g., (Kilbas et al. 2006; Klimek 2009; Podlubny 1999; Samko et al. 1993)) are easily obtained from the general kernel operators A_P and B_P, respectively. Let $k^\alpha(t-\tau) = \frac{1}{\Gamma(1-\alpha)}(t-\tau)^{-\alpha}$, $\alpha \in (0,1)$. If $P = \langle a, t, b, 1, 0\rangle$, then

$$A_P[f](t) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{\mathrm{d}t} \int_a^t (t-\tau)^{-\alpha} f(\tau)\, \mathrm{d}\tau =: {}_aD_t^{\alpha}[f](t)$$

is the standard left Riemann–Liouville fractional derivative of order α, while

$$B_P[f](t) = \frac{1}{\Gamma(1-\alpha)} \int_a^t (t-\tau)^{-\alpha} f'(\tau)\, \mathrm{d}\tau =: {}_a^CD_t^{\alpha}[f](t)$$

is the standard left Caputo fractional derivative of order α; if $P = \langle a, t, b, 0, 1\rangle$, then

$$-A_P[f](t) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{\mathrm{d}t} \int_t^b (\tau-t)^{-\alpha} f(\tau)\, \mathrm{d}\tau =: {}_tD_b^{\alpha}[f](t)$$

is the standard right Riemann–Liouville fractional derivative of order α, while

$$-B_P[f](t) = -\frac{1}{\Gamma(1-\alpha)} \int_t^b (\tau-t)^{-\alpha} f'(\tau)\, \mathrm{d}\tau =: {}_t^CD_b^{\alpha}[f](t)$$

is the standard right Caputo fractional derivative of order α.

2.2 Multidimensional Fractional Calculus

In this section, we introduce notions of classical, variable order, and generalized partial fractional integrals and derivatives in a multidimensional finite domain. They are natural generalizations of the corresponding fractional operators of Sect. 2.1.1. Furthermore, similarly as in the integer order case, computation of partial fractional derivatives and integrals is reduced to the computation of one-variable fractional operators. Along the work, for $i = 1, \ldots, n$, let a_i, b_i and α_i be numbers in $\mathbb{R}$ and $t = (t_1, \ldots, t_n)$ be such that $t \in \Omega_n$, where $\Omega_n = (a_1, b_1) \times \cdots \times (a_n, b_n)$ is a subset of $\mathbb{R}^n$. Moreover, let us define the following sets: $\Delta_i := \left\{(t_i, \tau) \in \mathbb{R}^2 : a_i \le \tau < t_i \le b_i\right\}, i = 1, \ldots, n$.

2.2.1 *Classical Partial Fractional Integrals and Derivatives*

In this section we present definitions of classical partial fractional integrals and derivatives. Interested reader can find more details in Sect. 24.1 of the book (Samko et al. 1993).

Definition 2.21 (*Left and right Riemann–Liouville partial fractional integrals*) Let $t \in \Omega_n$. The left and the right partial Riemann–Liouville fractional integrals of order $\alpha_i \in \mathbb{R}$ ($\alpha_i > 0$) with respect to the ith variable t_i are defined by

$$ {}_{a_i}I_{t_i}^{\alpha_i}[f](t) := \frac{1}{\Gamma(\alpha_i)} \int_{a_i}^{t_i} \frac{f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)\,\mathrm{d}\tau}{(t_i - \tau)^{1-\alpha_i}}, \quad t_i > a_i, \tag{2.13} $$

and

$$ {}_{t_i}I_{b_i}^{\alpha_i}[f](t) := \frac{1}{\Gamma(\alpha_i)} \int_{t_i}^{b_i} \frac{f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)\,\mathrm{d}\tau}{(\tau - t_i)^{1-\alpha}}, \quad t_i < b_i, \tag{2.14} $$

respectively.

Definition 2.22 (*Left and right Riemann–Liouville partial fractional derivatives*) Let $t \in \Omega_n$. The left partial Riemann–Liouville fractional derivative of order α_i, $0 < \alpha_i < 1$, of a function f with respect to the ith variable t_i, is defined by ${}_{a_i}D_{t_i}^{\alpha_i}[f](t) := \frac{\partial}{\partial t_i} {}_{a_i}I_{t_i}^{1-\alpha_i}[f](t)$ for all $t_i \in (a_i, b_i]$. Similarly, the right partial Riemann–Liouville fractional derivative of order α_i of a function f, with respect to the ith variable t_i, is defined by ${}_{t_i}D_{b_i}^{\alpha_i}[f](t) := -\frac{\partial}{\partial t_i} {}_{t_i}I_{b_i}^{1-\alpha_i}[f](t)$ for all $t_i \in [a_i, b_i)$.

Definition 2.23 (*Left and right Caputo partial fractional derivatives*) Let $t \in \Omega_n$. The left and right partial Caputo fractional derivatives of order α_i, $0 < \alpha_i < 1$, of a function f with respect to the ith variable t_i, are given by

$$ {}_{a_i}^{C}D_{t_i}^{\alpha_i}[f](t) := {}_{a_i}I_{t_i}^{1-\alpha_i}\left[\frac{\partial}{\partial t_i} f\right](t), \quad \forall t_i \in (a_i, b_i], $$

and

$$ {}_{t_i}^{C}D_{b_i}^{\alpha_i}[f](t) := -{}_{t_i}I_{b_i}^{1-\alpha_i}\left[\frac{\partial}{\partial t_i} f\right](t), \quad \forall t_i \in [a_i, b_i), $$

respectively.

2.2.2 *Variable Order Partial Fractional Integrals and Derivatives*

In this section, we introduce the notions of partial fractional operators of variable order. In the following let us assume that $\alpha_i : \Delta_i \to [0, 1]$, $\alpha_i \in C^1\left(\bar{\Delta}; \mathbb{R}\right)$, $i = 1, \ldots, n$, $t \in \Omega_n$ and $f : \Omega_n \to \mathbb{R}$.

Definition 2.24 The left Riemann–Liouville partial integral of variable fractional order $\alpha_i(\cdot,\cdot)$ with respect to the ith variable t_i, is given by

$$ {}_{a_i}I_{t_i}^{\alpha_i(\cdot,\cdot)}[f](t) := \int\limits_{a_i}^{t_i} \frac{1}{\Gamma(\alpha_i(t_i,\tau))}(t_i-\tau)^{\alpha_i(t_i,\tau)-1} f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)\,\mathrm{d}\tau, $$

$t_i > a_i$, while

$$ {}_{t_i}I_{b_i}^{\alpha_i(\cdot,\cdot)}[f](t) := \int\limits_{t_i}^{b_i} \frac{1}{\Gamma(\alpha_i(\tau,t_i))}(\tau-t_i)^{\alpha_i(\tau,t_i)-1} f(t_1,\ldots,t_{i-1},\tau,t_{i+1},\ldots,t_n)\,\mathrm{d}\tau, $$

$t_i < b_i$, is the right Riemann–Liouville partial integral of variable fractional order $\alpha_i(\cdot,\cdot)$ with respect to variable t_i.

Definition 2.25 The left Riemann–Liouville partial derivative of variable fractional order $\alpha_i(\cdot,\cdot)$, with respect to the ith variable t_i, is given by

$$ \forall t_i \in (a_i,b_i], \quad {}_{a_i}D_{t_i}^{\alpha_i(\cdot,\cdot)}[f](t) = \frac{\partial}{\partial t_i}\,{}_{a_i}I_{t_i}^{1-\alpha_i(\cdot,\cdot)}[f](t) $$

while the right Riemann–Liouville partial derivative of variable fractional order $\alpha_i(\cdot,\cdot)$, with respect to the ith variable t_i, is defined by

$$ \forall t_i \in [a_i,b_i), \quad {}_{t_i}D_{b_i}^{\alpha_i(\cdot,\cdot)}[f](t) = -\frac{\partial}{\partial t_i}\,{}_{t_i}I_{b_i}^{1-\alpha_i(\cdot,\cdot)}[f](t). $$

Definition 2.26 The left Caputo partial derivative of variable fractional order $\alpha_i(\cdot,\cdot)$, with respect to the ith variable t_i, is defined by

$$ \forall t_i \in (a_i,b_i], \quad {}_{a_i}^{C}D_{t_i}^{\alpha_i(\cdot,\cdot)}[f](t) = {}_{a_i}I_{t_i}^{1-\alpha_i(\cdot,\cdot)}\left[\frac{\partial}{\partial t_i}f\right](t), $$

while the right Caputo partial derivative of variable fractional order $\alpha_i(\cdot,\cdot)$, with respect to the ith variable t_i, is given by

$$ \forall t_i \in [a_i,b_i), \quad {}_{t_i}^{C}D_{b_i}^{\alpha_i(\cdot,\cdot)}[f](t) = -{}_{t_i}I_{b_i}^{1-\alpha_i(\cdot,\cdot)}\left[\frac{\partial}{\partial t_i}f\right](t). $$

Note that, if $\alpha_i(\cdot,\cdot)$ is a constant function, then the partial operators of variable fractional order are reduced to corresponding partial integrals and derivatives of constant order introduced in Sect. 2.2.1.

2.2.3 Generalized Partial Fractional Operators

Let us assume that $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ are in $\mathbb{R}^n$. We shall present definitions of generalized partial fractional integrals and derivatives. Let $k_i : \Delta_i \to \mathbb{R}$, $i = 1, \ldots, n$ and $t \in \Omega_n$.

Definition 2.27 (*Generalized partial fractional integral*) For any function f defined almost everywhere on Ω_n with value in $\mathbb{R}$, the generalized partial integral K_{P_i} is defined for almost all $t_i \in (a_i, b_i)$ by:

$$K_{P_i}[f](t) := \lambda_i \int_{a_i}^{t_i} k_i(t_i, \tau) f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)\, d\tau$$
$$+ \mu_i \int_{t_i}^{b_i} k_i(\tau, t_i) f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)\, d\tau,$$

where $P_i = \langle a_i, t_i, b_i, \lambda_i, \mu_i \rangle$.

Definition 2.28 (*Generalized partial fractional derivative of Riemann–Liouville type*) The generalized partial fractional derivative of Riemann–Liouville type with respect to the ith variable t_i is given by $A_{P_i} := \frac{\partial}{\partial t_i} \circ K_{P_i}$.

Definition 2.29 (*Generalized partial fractional derivative of Caputo type*) The generalized partial fractional derivative of Caputo type with respect to the ith variable t_i is given by $B_{P_i} := K_{P_i} \circ \frac{\partial}{\partial t_i}$.

Example 2.30 Similarly, as in the one-dimensional case, partial operators K, A and B reduce to the standard partial fractional integrals and derivatives. The left- or right-sided Riemann–Liouville partial fractional integral with respect to the ith variable t_i is obtained by choosing the kernel $k_i^\alpha(t_i, \tau) = \frac{1}{\Gamma(\alpha_i)}(t_i - \tau)^{\alpha_i - 1}$. That is,

$$K_{P_i}[f](t) = \frac{1}{\Gamma(\alpha_i)} \int_{a_i}^{t_i} (t_i - \tau)^{\alpha_i - 1} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)\, d\tau =: {}_{a_i}I_{t_i}^{\alpha_i}[f](t),$$

for $P_i = \langle a_i, t_i, b_i, 1, 0 \rangle$, and

$$K_{P_i}[f](t) = \frac{1}{\Gamma(\alpha_i)} \int_{t_i}^{b_i} (\tau - t_i)^{\alpha_i - 1} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)\, d\tau =: {}_{t_i}I_{b_i}^{\alpha_i}[f](t),$$

for $P_i = \langle a_i, t_i, b_i, 0, 1 \rangle$. The standard left- and right-sided Riemann–Liouville and Caputo partial fractional derivatives with respect to ith variable t_i are received by choosing the kernel $k_i^\alpha(t_i, \tau) = \frac{1}{\Gamma(1-\alpha_i)}(t_i - \tau)^{-\alpha_i}$. If $P_i = \langle a_i, t_i, b_i, 1, 0 \rangle$, then

$$A_{P_i}[f](t) = \frac{1}{\Gamma(1-\alpha_i)} \frac{\partial}{\partial t_i} \int\limits_{a_i}^{t_i} (t_i - \tau)^{-\alpha_i} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)\, \mathrm{d}\tau$$
$$=: {}_{a_i}D_{t_i}^{\alpha_i}[f](t),$$

$$B_{P_i}[f](t) = \frac{1}{\Gamma(1-\alpha_i)} \int\limits_{a_i}^{t_i} (t_i - \tau)^{-\alpha_i} \frac{\partial}{\partial \tau} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)\, \mathrm{d}\tau$$
$$=: {}_{a_i}^{C}D_{t_i}^{\alpha_i}[f](t).$$

If $P_i = \langle a_i, t_i, b_i, 0, 1\rangle$, then

$$-A_{P_i}[f](t) = \frac{-1}{\Gamma(1-\alpha_i)} \frac{\partial}{\partial t_i} \int\limits_{t_i}^{b_i} (\tau - t_i)^{-\alpha_i} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)\, \mathrm{d}\tau$$
$$=: {}_{t_i}D_{b_i}^{\alpha_i}[f](t),$$

$$-B_{P_i}[f](t) = \frac{-1}{\Gamma(1-\alpha_i)} \int\limits_{t_i}^{b_i} (\tau - t_i)^{-\alpha_i} \frac{\partial}{\partial \tau} f(t_1, \ldots, t_{i-1}, \tau, t_{i+1}, \ldots, t_n)\, \mathrm{d}\tau$$
$$=: {}_{t_i}^{C}D_{b_i}^{\alpha_i}[f](t).$$

Moreover, one can easily check that also variable order partial fractional integrals and derivatievs are particular cases of operators K_{P_i}, A_{P_i} and B_{P_i}.

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