# Chapter 2
# Classical Versus Quantal Features in a Projected Coherent State

## 2.1 Definitions and Preliminaries

Most properties of the low lying spectra can be described in terms of few collective degrees of freedom. These can be given either phenomenologically [RBD95, RBF11] or in terms of single particle motion. Thus, projecting out the collective degrees of freedom from a many body system was always a central topic of nuclear structure theory [BaVe78, Vil77]. Accounting for collective motion and moreover for the coupling of the mentioned degrees of freedom with the non-collective coordinates is not an easy task. A significant simplification is achieved when one defines a semiclassical approach by means of a variational principle. Transferring the quantum mechanical many body problem to a semiclassical picture allows us to use the classical mechanical tools which are more developed and efficient. This operation is conventionally called dequantization procedure which is most reliable if the variational state is of the coherent type. The meaning of this statement is that the quantized classical trajectories lead to a spectrum, which is close to that corresponding to the initial many body Hamiltonian. Due to the overcomplete property of coherent states a full account of the dynamic in the whole Hilbert space is possible. Indeed, by expanding the coherent state in a Hilbert space basis, no expansion coefficient is vanishing. For example if we treat a boson Hamiltonian with coherent states the matrix elements include the contribution from all states of the boson space, which is not the case when a diagonalization method is adopted.

As a matter of fact this is the property which was exploited in many publications about the Coherent State Model (CSM) [RD76, RCGD82, HHL70]. This model uses an axially quadrupole deformed coherent state of Glauber type as an intrinsic ground state. Moreover, other two deformed states are defined by lowest order polynomial excitations of ground state, the excitations being defined so that some experimental data are satisfied. These states are modeling the intrinsic beta and gamma bands states, respectively. By angular momentum projection three sets of states are obtained which are to describe the main properties of the ground, beta and gamma bands. By construction the polynomial excitations are chosen such that the three intrinsic states



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as well as the three sets of projected states are mutually orthogonal, respectively. Within the restricted boson space of projected states, obtained as described above, an effective Hamiltonian is defined such that the three bands are maximally decoupled. The CSM works especially well for high spin states. These states behave more or less semiclassically. This can be proved by the following simple reasoning. Suppose we have a spherical rigid rotor with the spectrum

$$E_J = \frac{J(J+1)\hbar^2}{2\mathcal{J}}. \tag{2.1.1}$$

Going with $\hbar$ to zero and with J to infinity like $k/\hbar$ with $k$ constant one obtains that $E_J$ is a constant with respect to $J$ which in fact reclaims a classical behavior.

To prove the fact that a coherent state is suitable for accounting the classical properties we have to study the behavior of both unprojected and angular momentum projected state, from the point of view of the Heisenberg uncertainty relations.

Let us consider the coherent state defined with the z-component of the quadrupole boson operators $b^{\dagger}_{2\mu}, b_{2\mu}$ with $-2 \le \mu \le 2$:

$$|\Psi\rangle = e^{(db^{\dagger}_{20} - d^* b_{20})}|0\rangle, \tag{2.1.2}$$

where $|0\rangle$ stands for the boson vacuum state while $d$ is a complex number. The coherent nature of this function is determined by:

$$b_{20}|\Psi\rangle = d|\Psi\rangle, \quad \langle\Psi|b^{\dagger}_{20} = d^*. \tag{2.1.3}$$

This equation can be proved by using the operator equation:

$$e^{\hat{A}}\hat{O}e^{-\hat{A}} = \sum_{n=0}^{\infty}\frac{1}{n!}[[A, \ldots [A, O]..]]. \tag{2.1.4}$$

Indeed, choosing $\hat{O} = b_{20}$ and $-\hat{A} = db^{\dagger}_{20} - d^* b_{20}$, one obtains:

$$b_{20}|\Psi\rangle = e^{-\hat{A}}e^{\hat{A}}b_{20}e^{-\hat{A}}|o\rangle = e^{-\hat{A}}(b_{20} + d)|0\rangle = d|\Psi\rangle \tag{2.1.5}$$

The second Eq. (2.1.3) is obtained from the first one by applying the Hermitian conjugation operation to it.

Using the Baker-Campbell-Hausdorff factorization:

$$e^{A+B} = e^A e^B e^{-\frac{1}{2}[A,B]}, \tag{2.1.6}$$

the coherent state is written in the form:

$$|\Psi\rangle = e^{-\frac{|d|^2}{2}} e^{db^{\dagger}_{20}}|0\rangle. \tag{2.1.7}$$

The average of the quadrupole operator:

$$Q_{20} = q_0 \left( b^\dagger_{20} + b_{20} \right), \tag{2.1.8}$$

with the coherent state has the expression:

$$\langle \Psi | Q_{20} | \Psi \rangle = 2q_0 Re\, d. \tag{2.1.9}$$

Thus the real part of $d$ has the significance of the quadrupole deformation. Similarly, averaging the quadrupole momentum, conjugate of $Q_{20}$, with the coherent state one finds that the imaginary part of $d$ is proportional with the classical quadrupole momentum. The function $|\Psi\rangle$ is a vacuum state for the shifted quadrupole boson operator:

$$(b_{20} - d)\, |\Psi\rangle = 0. \tag{2.1.10}$$

which is obvious from (2.1.5). The function $|\Psi\rangle$ has not a definite angular momentum, i.e. it is not eigenfunction of the angular momentum operator squared, $\hat{J}^2$. However it is eigenstate of $\hat{J}_z$. Due to this feature we say that $|\Psi\rangle$ is an axially deformed function. This statement can be easily proved by writing the r.h.s. of Eq. (2.1.7) as a boson power series:

$$|\Psi\rangle = e^{-\frac{|d|^2}{2}} \sum_{n=0} \frac{1}{n!} d^n \left( b^\dagger_{20} \right)^n |0\rangle. \tag{2.1.11}$$

The first term of the expansion is proportional to the vacuum state which is of zero angular momentum, the second term is a quadrupole state while the third one is proportional to

$$\sum_{J=0,2,4} C^{2\ 2\ J}_{0\ 0\ 0} \left( b^\dagger_2 b^\dagger_2 \right)_{J0} |0\rangle \equiv \sum_{J=0,2,4} \mathcal{A}_J |2, J, 0\rangle \tag{2.1.12}$$

where in the r.h.s. is a superposition of states of two bosons, angular momentum J (=0, 2, 4) and projection zero. Similarly, the three boson term is a superposition of states $|3, J, 0\rangle$ with $J = 0, 2, 3, 4, 6$. Therefore, the coherent state is a superposition of components with various angular momenta and vanishing projection. It is clear now that the state under consideration breaks two symmetries, namely the rotational and the gauge ones. The last symmetry breaking is evident since according to Eq. (2.1.11), $|\Psi\rangle$ is a sum of components with different number of bosons.

Since we want to discuss the classical features which might be described with the CSM we restrict our considerations to the case of real $d$. In terms of bosons, the quadrupole coordinate and its conjugate momentum can be defined as:

$$\alpha_{2\mu} = \frac{1}{k\sqrt{2}} \left( b^\dagger_{2\mu} + (-1)^\mu b_{2,-\mu} \right), \quad \pi_{2\mu} = \frac{ik}{\sqrt{2}} \left( (-1)^\mu b^\dagger_{2,-\mu} - b_{2\mu} \right). \tag{2.1.13}$$

The above transformation is canonical irrespective of the value of the real constant $k$. The vacuum state $|0\rangle$ is a function of coordinate only. Indeed, the equation

$$b_{20}|0\rangle = 0, \tag{2.1.14}$$

can be written as:

$$\frac{\partial}{\partial \alpha_{20}}|0\rangle = -k^2\alpha_{20} \tag{2.1.15}$$

This equation can be integrated with the result:

$$|0\rangle = Ce^{-\frac{k^2\alpha_{20}^2}{2}}. \tag{2.1.16}$$

with $C$ an integration constant. Therefore the vacuum state depends only on the quadrupole coordinate but not on the momenta. For the sake of simplicity let us denote by

$$F(\alpha_{2\mu}) = |0\rangle. \tag{2.1.17}$$

The coherent state can be written as:

$$\begin{aligned}\Psi &= e^{-i\frac{\sqrt{2}d}{k}\pi_{20}}F(\alpha_{2\mu}) = e^{-\frac{\sqrt{2}d}{k}\frac{\partial}{\partial\alpha_{20}}}F(\alpha_{2\mu})\\ &= \sum_n \frac{(\frac{-\sqrt{2}d}{k})^n}{n!}\left(\frac{\partial}{\partial\alpha_{20}}\right)^n F(\alpha_{2\mu}) = F(\alpha_{2,-2},\alpha_{2,-1},\alpha_{20}-\frac{\sqrt{2}d}{k},\alpha_{21},\alpha_{22}).\end{aligned} \tag{2.1.18}$$

This equation shows that the coherent state is just the vacuum state with the coordinate $\alpha_{20}$ shifted to $\alpha_{20} - \frac{d\sqrt{2}}{k}$. The shift operation is achieved by the displacement operator:

$$D(d) = e^{d(b_{20}^{\dagger}-b_{20})}. \tag{2.1.19}$$

The mentioned operator breaks the rotational symmetry. Indeed, let us consider the simplest invariant, namely a harmonic Hamiltonian:

$$H = \omega\sum_{\mu} b_{2\mu}^{\dagger}b_{2\mu} \tag{2.1.20}$$

Under the action of the operator D(d) one obtains a rotational non-invariant term:

$$D(d)HD(-d) = H - d\omega\left(b_{20}^{\dagger} + b_{20}\right) + \omega d^2. \tag{2.1.21}$$

Note that the displacement operator transforms the scalar operator $H$ into an operator which is not invariant to rotations. Let us consider now the Hermitian operator:

$$H^{'} = H + \lambda b_{20}^{\dagger} + \lambda^{*} b_{20} \tag{2.1.22}$$

Applying this equation on the coherent state $\Psi$, we obtain:

$$H^{'}\Psi = \left(\omega d b_{20}^{\dagger} + \lambda b_{20}^{\dagger} + \lambda^{*} d\right)\Psi \tag{2.1.23}$$

Choosing $d$ such that $\omega d + \lambda = 0$, the above equation becomes:

$$H^{'}\Psi = -\frac{|\lambda|^2}{\omega}\Psi \tag{2.1.24}$$

From this equation it results that the deformed Hamiltonian $H^{'}$ admits the coherent state as eigenstate. Consider now that the coefficients $\lambda$ and $\lambda^{*}$ are functions of time in the Hamiltonian:

$$H(t) = \omega b_{20}^{\dagger} b_{20} + \lambda(t) b_{20}^{\dagger} + \lambda^{*}(t) b_{20} \equiv H_0 + H_{int} \tag{2.1.25}$$

The Hamiltonian $H_0$ has the eigenstates:

$$H_0|n\rangle = n\omega|n\rangle, \quad |n\rangle = \frac{1}{\sqrt{n!}}\left(b_{20}^{\dagger}\right)^{n}|0\rangle. \tag{2.1.26}$$

On the other hand the coherent state can be written as:

$$|\Psi\rangle \equiv |d\rangle = e^{-\frac{|d|^2}{2}}\sum_{n}\frac{1}{n!}\left(d b_{20}^{\dagger}\right)^{n}|0\rangle = e^{-\frac{|d|^2}{2}}\sum_{n}\frac{d^n}{\sqrt{n!}}|n\rangle \tag{2.1.27}$$

Within the interaction representation we define the time dependent state:

$$|d(t)\rangle = e^{-iH_0 t/\hbar}|d\rangle = e^{-\frac{|d|^2}{2}}\sum_{n}\frac{\left(d e^{-i\omega t}\right)^{n}}{\sqrt{n!}}|n\rangle = |d e^{-i\omega t}\rangle \tag{2.1.28}$$

The time dependent Schrödinger equation associated to $H(t)$ reads:

$$i\frac{\partial}{\partial t}|\phi(t)\rangle = H(t)|\phi(t)\rangle = \left[\omega b_{20}^{\dagger} b_{20} + \lambda(t) b_{20}^{\dagger} + \lambda^{*}(t) b_{20}\right]|\phi(t)\rangle. \tag{2.1.29}$$

Supposing that at the initial moment of time the system stays in the vacuum state. Then the formal solution of the above equation is [ZH90]:

$$\begin{aligned}|\phi(t)\rangle &= e^{-\frac{i}{\hbar}\int\limits_0^t dt'\left(\hbar\omega b_{20}^\dagger b_{20}+\lambda(t')b_{20}^\dagger+\lambda^*(t')b_{20}\right)}|0\rangle \\ &= \exp\left[d(t)b_{20}^\dagger - d^*(t)b_{20}\right]|0\rangle e^{i\eta(t)} = |d(t)\rangle e^{i\eta(t)},\end{aligned} \qquad (2.1.30)$$

where the following notations have been used:

$$\begin{aligned} d(t) &= -ie^{-i\omega t}\int\limits_0^t \lambda^*(\tau)e^{i\omega\tau}d\tau, \\ \eta(t) &= -\frac{1}{2}\omega t - \int\limits_0^t Re\left[\lambda(\tau)d(\tau)\right]d\tau \end{aligned} \qquad (2.1.31)$$

This result says that the system remains always in a coherent state. Therefore, *if at the initial time the system is in a coherent state, including the extremal state, it will remain in a coherent state for ever.*

Now we shall present another definition of the coherent state based on the group theory. Let us note first that the set of operators $\{\hat{n} = b_{20}^\dagger b_{20}, b_{20}^\dagger, b_{20}, I\}$ where $I$ stands for the unity operator, form a Lie algebra denoted by $h_4$. The corresponding group is conventionally called the Heisenberg-Weyl group and denoted by $H_4$. The Hilbert space of $H_4$ is spanned by the eigenstate of the number operator:

$$b_{20}^\dagger b_{20}|n\rangle = n|n\rangle. \qquad (2.1.32)$$

Using the notation $H_0 = b_{20}^\dagger b_{20}$, we have:

$$H_0|n\rangle = n|n\rangle \qquad (2.1.33)$$

It results that the vacuum state is the ground state of $H_0$. In that respect the vacuum state is an extremal state for $H_0$. We shall call stability subgroup that subgroup which leaves the extremal state invariant. For $H_4$ the stability subgroup is $U(1) \otimes U(1)$ with an algebra spanned by $\{\hat{n}, I\}$. The stability group consists of all operations of the form:

$$h = e^{i(\delta\hat{n}+\varphi I)}. \qquad (2.1.34)$$

Thus

$$h|0\rangle = |0\rangle e^{i\varphi}. \qquad (2.1.35)$$

The coset $H_4/U(1) \otimes U(1)$ is a set of elements $\Omega$ which provides for any element g from $H_4$ a unique decomposition:

$$g = Dh \tag{2.1.36}$$

A typical representative in the coset space $H_4/U(1) \otimes U(1)$ is:

$$D(d) = exp(db_{20}^{\dagger} - d^* b_{20}). \tag{2.1.37}$$

By definition a coherent state is the action of the coset elements on the extremal state.

$$\Psi = D(d)|0\rangle. \tag{2.1.38}$$

Remarkably, the coherent state of the Glauber type and the one defined on group theory grounds are identical. This is not generally true. This will be explicitly shown for the case of the $SU(2)$ group.

The properties mentioned above for the coherent states are the basic ones, which will be used in various contexts along this book. In this chapter we shall focus on the semiclassical features of the coherent states with symmetries restored.

## 2.2 Unprojected State

### *2.2.1 The Quadrupole Coordinate and Momentum*

The conjugate coordinates:

$$\hat{\alpha}_{20} = \frac{1}{\sqrt{2}}\left(b_{20}^{\dagger} + b_{20}\right), \quad \hat{\pi}_{20} = \frac{i}{\sqrt{2}}\left(b_{20}^{\dagger} - b_{20}\right), \tag{2.2.1}$$

satisfy the equation:

$$[\hat{\alpha}_{20}, \hat{\pi}_{20}] = i, \tag{2.2.2}$$

where "$i$" denotes the imaginary unit. Conventionally, we use the units system where $\hbar = 1$.

The averages of $\hat{\alpha}$ and $\hat{\alpha}^2$ on $|\Psi\rangle$ are:

$$\langle\Psi|\hat{\alpha}_{20}|\Psi\rangle = \sqrt{2}d, \quad \langle\Psi|\hat{\alpha}_{20}^2|\Psi\rangle = 2d^2 + \frac{1}{2}. \tag{2.2.3}$$

The conjugate momentum and its square have the averages:

$$\langle\Psi|\hat{\pi}_{20}|\Psi\rangle = 0, \quad \langle\Psi|\hat{\pi}_{20}^2|\Psi\rangle = \frac{1}{2}. \tag{2.2.4}$$

Using these results, the uncertainty relation associated to the conjugate coordinates $\alpha$ and $\pi$ has the form:

$$\Delta\hat{\alpha}_{20}\Delta\hat{\pi}_{20} = \frac{1}{2}, \tag{2.2.5}$$

where by $\Delta x$ one denotes the dispersion of the coordinate $x$. Notice that the dispersion product reaches the minimum value of the set allowed by the Heisenberg uncertainty principle. Due to this feature one asserts that the coherent state $|\Psi\rangle$ is an optimal state to describe the properties which define the border of quantal and classical behavior.

### 2.2.2 The Boson Number and Its Conjugate Phase

In this subsection we consider again that $d$ is a complex number. Let us denote by $\hat{N}_0$ the boson number operator:

$$\hat{N}_0 = b_{20}^\dagger b_{20}. \tag{2.2.6}$$

Writing the operator $\hat{N}_0^2$ in a normal order, the expectation values for $\hat{N}_0$ and $\hat{N}_0^2$ can be easily calculated:

$$\begin{aligned}\langle\Psi|\hat{N}_0|\Psi\rangle &= |d|^2 \equiv N_0,\\ \langle\Psi|\hat{N}_0^2|\Psi\rangle &= |d|^2 + |d|^4 = N_0 + N_0^2.\end{aligned} \tag{2.2.7}$$

Thus, the dispersion of the boson number operator can be easily calculated:

$$(\Delta\hat{N})^2 = |d|^2 \equiv N_0. \tag{2.2.8}$$

Writing the complex number $d$ in the polar form

$$d = |d|e^{i\varphi} \tag{2.2.9}$$

and using Eq. (2.1.3) one obtains:

$$\begin{aligned}\langle\Psi|b_{20}|\Psi\rangle &= |d|e^{i\varphi} = N_0^{1/2}e^{i\varphi},\\ e^{i\varphi} &= \langle\Psi|b_{20}|\Psi\rangle\left(\langle\Psi|\hat{N}_0|\Psi\rangle\right)^{-1/2}.\end{aligned} \tag{2.2.10}$$

The question which arises is whether such a factorization holds also for the operators whose averages are involved in the above equation. Before dealing with the quantum mechanical problem of the boson number and its conjugate phase we would like to present first the classical counterpart of this long standing problem.

Let H be a Hamiltonian defined in terms of the quadrupole boson operators $b^\dagger_{2m}, b_{2m}$ with $-2 \le m \le 2$ and consider the Time Dependent Variational Principle (*TDVP*) equation:

$$\delta \int_0^t \langle\Psi|H - i\frac{\partial}{\partial t'}|\Psi\rangle dt' = 0, \tag{2.2.11}$$

where the variational state is the coherent state $|\Psi\rangle$ (2.1.2), with $d$ a complex number depending on time. Let us denote by

$$\mathcal{H} = \langle\Psi|H|\Psi\rangle. \tag{2.2.12}$$

The *TDVP* leads to the Hamilton equations of motion for the classical coordinates $d$ and $d^*$.

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial d} &= -i \overset{\bullet}{d}{}^*,\\ \frac{\partial\mathcal{H}}{\partial d^*} &= i \overset{\bullet}{d},\end{aligned} \tag{2.2.13}$$

where "•" indicates the time derivative. From these equations it results that $d$ and $d^*$ play the role of a classical coordinate and its conjugate momentum, respectively, while $\mathcal{H}$ is the classical energy function, or Hamilton function. Changing the classical coordinates $d, d^*$ to the polar coordinates by the transformation

$$(d, d^*) \rightarrow (r, \varphi), \tag{2.2.14}$$

with $r = |d|^2$, the equations of motion become:

$$\begin{aligned}\frac{\partial\mathcal{H}}{\partial r} &= -\overset{\bullet}{\varphi},\\ \frac{\partial\mathcal{H}}{\partial \varphi} &= \overset{\bullet}{r}.\end{aligned} \tag{2.2.15}$$

These equations suggest that the classical image (the average of $\hat{N}_0$ with $\Psi$) of the boson number operator, i.e. $r$, and the phase $\varphi$ are, indeed, conjugate classical coordinates, namely $r$ is a classical coordinate and $\varphi$ its conjugate momentum. One can check that their Poisson bracket is equal to unity. Certainly, it would be desirable that a pair of Hermitian operators whose commutator is the imaginary unity exists such that their averages with $\Psi$ are just the canonical conjugate classical coordinates $r$ and $\varphi$. In what follows we devote some space to the issue just formulated.

It is useful to introduce the off-diagonal operator

$$\hat{P}_0^\dagger = \hat{N}_0^{-1/2} b_{20}^\dagger \tag{2.2.16}$$

which is the quantal counterpart of Eq. (2.2.10). The operator $\hat{N}_0^{-1/2}$ is defined by the following equation:

$$\hat{N}_0^{-1/2} = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} dx \exp\left(-x^2 \hat{N}_0\right). \tag{2.2.17}$$

The Hermitian conjugate operator $\hat{P}_0$, satisfies the commutation relation:

$$\left[\hat{P}_0, \hat{N}_0\right] = \hat{P}_0. \tag{2.2.18}$$

The conjugate coordinate corresponding to the boson number operator is:

$$\hat{\Phi}_0 = -i \ln \hat{P}_0. \tag{2.2.19}$$

Indeed, considering the power expansion of the ln function in terms of $(\hat{P}_0 - 1)$, one checks that the operators $\hat{N}_0$ and $\hat{\Phi}_0$ satisfy the commutation relation:

$$\left[\hat{N}_0, \hat{\Phi}_0\right] = i. \tag{2.2.20}$$

For monopole bosons, the conjugate coordinates of boson number and phase were described in details in Ref. [Hol80]. By contradistinction to the monopole case, here the rotation symmetry is broken. Indeed, while the boson number operator is a scalar, the phase operator $\hat{P}$ is a tensor of rank two and projection zero. It is an open question whether a construction of a scalar phase operator $\hat{P}$ is possible or not.

We note that our derivation of the phase operator is based on Eq. (2.2.18). Actually, this equation may be looked at as a defining equation for $\hat{P}_0$. Certainly the solution of this equation is not unique. For example, a possible solution is:

$$\hat{P}_0 = b_{20}. \tag{2.2.21}$$

In this case we have:

$$\hat{\Phi}_0 = -i \ln b_{20}. \tag{2.2.22}$$

One can check that the following equations for the expectation values hold:

$$\langle\Psi|\hat{\Phi}_0|\Psi\rangle = -i\ln d, \quad \langle\Psi|\hat{\Phi}_0^2|\Psi\rangle = -(\ln d)^2\,. \tag{2.2.23}$$

Consequently, the corresponding dispersion is vanishing:

$$\Delta\hat{\Phi}_0 = 0, \tag{2.2.24}$$

which reflects the fact that $\Psi$ is an eigenfunction of $\hat{\Phi}_0$:

$$\hat{\Phi}_0|\Psi\rangle = -i\,(\ln d)\,|\Psi\rangle. \tag{2.2.25}$$

A direct use of Eq. (2.2.19) to calculate the uncertainty relation for the boson number and phase, is quite a cumbersome task especially due to the logarithm function. However, a considerable simplification is obtained by noticing that the deviation of $\hat{\Phi}_0$ from its expectation value can be expressed as:

$$\delta\hat{\Phi}_0 = -i\frac{\delta\hat{P}_0}{\hat{P}_0}. \tag{2.2.26}$$

We define a new dispersion of the phase operator by:

$$D\hat{\Phi}_0 = \frac{\Delta\hat{P}_0}{\langle\Psi|\hat{P}_0|\Psi\rangle}. \tag{2.2.27}$$

We shall prove [Hol80] that the newly defined quantity satisfies the Heisenberg uncertainty relation. Indeed, following the procedure of Ref. [CaNi68] one successively obtains:

$$\begin{aligned}\langle\Psi|\hat{P}_0|\Psi\rangle &= de^{-|d|^2}\sum_{k=0}^{\infty}\frac{|d|^{2k}}{k!\sqrt{k+1}},\\ \langle\Psi|\hat{P}_0^2|\Psi\rangle &= d^2e^{-|d|^2}\sum_{k=0}^{\infty}\frac{|d|^{2k}}{k!\sqrt{(k+1)(k+2)}}.\end{aligned} \tag{2.2.28}$$

In the asymptotic region of $|d|$, compact forms for the sums involved in the above equation were obtained in Ref. [CaNi65], such that the final expressions for the considered expectation values are:

$$\begin{aligned}\langle\Psi|\hat{P}_0|\Psi\rangle &= \frac{d}{|d|}\left[1-\frac{1}{8|d|^2}+\cdots\right],\\ \langle\Psi|\hat{P}_0^2|\Psi\rangle &= \frac{d^2}{|d|^2}\left[1-\frac{1}{2|d|^2}-\frac{3}{8|d|^4}+\cdots\right].\end{aligned} \tag{2.2.29}$$

With these results one finds that for large values of $|d|$, the following uncertainty relation holds:

$$\Delta\hat{N}_0 D\hat{\Phi}_0 = \frac{1}{2}. \tag{2.2.30}$$

As we shall show in what follows, this equation is valid in the region of large number of bosons. Indeed, in the region of large $|d|$ the $\Psi$ composing terms of maximal weights are those of large boson number.

The uncertainty relation of the boson number operator and its conjugate phase has been first studied for photons by Dirac [Dir27] and for oscillator by Susskind and Glover [SuGl64]. The above equations have been obtained by representing the photon annihilation operator as a product of a unitary operator, written as $U = e^{i\phi}$, and a selfadjoint function of the boson number operator $f(\hat{N})$. The solution is $f = \hat{N}^{1/2}$ and is based on the assumption that $\phi$ is a self-adjoint operator. Later on it was proved that the conjugate phase variable is not well defined and therefore the corresponding uncertainty relation is doubtful. Indeed, one can check that the operator U is not unitary and, consequently, $\phi$ is not a self-adjoint operator and thereby cannot be assigned to a physical observable [CaNi65]. The reason for non-unitary is the presence of a vanishing boson number in the spectrum of $\hat{N}$. Even if we exclude this value, which prevents $\hat{N}$ to be invertible, the phase is not well defined for small values of $N$ [Loui63]. Indeed, denoting by $|n_0\rangle$ the eigenstates of $\hat{N}_0$, the matrix elements of Eq. (2.2.20) lead to:

$$\langle n_0|\hat{\Phi}_0|m_0\rangle = i\frac{\delta_{n_0,m_0}}{n_0 - m_0}, \tag{2.2.31}$$

which doesn't make sense for small values of the boson number. However, for large values of the boson number this can be assimilated with a continuous variable and the ratio from the right hand side of Eq. (2.2.31) is just the first derivative of the Dirac $\delta$-function which is a well-defined entity.

Positive attempts to define Hermitian operators depending on the phase, which together with the boson operator $\hat{N}_0$ satisfy the uncertainty relation, have been made by several authors [CaNi65, Loui63, CaNi68, Lev65]. Thus, the operators:

$$\hat{C}_0 = \frac{1}{2}(\hat{P}_0 + \hat{P}_0^\dagger),$$
$$\hat{S}_0 = \frac{1}{2i}(\hat{P}_0 - \hat{P}_0^\dagger) \tag{2.2.32}$$

are Hermitian and satisfy the uncertainty relations [CaNi68]:

$$\Delta\hat{N}_0\Delta\hat{S}_0 \geq \frac{1}{2}\langle\hat{C}_0\rangle,$$
$$\Delta\hat{N}_0\Delta\hat{C}_0 \geq \frac{1}{2}\langle\hat{S}_0\rangle, \tag{2.2.33}$$

where $\langle..\rangle$ denotes the average of the operator involved, with the coherent state. The limitation on simultaneous measurement of observables $S_0$ and $C_0$ associated to the above mentioned Hermitian operators is expressed by the uncertainty product:

$$(\Delta\hat{S}_0)(\Delta\hat{C}_0) \geq \frac{1}{4}e^{-N_0}, \tag{2.2.34}$$

with $N_0$ denoting the square of the dispersion $\Delta\hat{N}_0$. A more symmetric uncertainty relation in the regime of large $|d|$ is obtained by combining the already obtained results:

$$(\Delta\hat{N}_0)^2\frac{(\Delta\hat{C}_0)^2 + (\Delta\hat{S}_0)^2}{(\langle\hat{C}_0\rangle)^2 + (\langle\hat{S}_0\rangle)^2} \geq \frac{1}{4}. \tag{2.2.35}$$

Although here we deal with quadrupole bosons the proof of the uncertainty relations mentioned above goes identically with those given in Refs. [CaNi65, CaNi68].

## 2.3 Projected Spherical States

As already mentioned, the coherent state has neither a definite angular momentum nor a definite number of bosons. The question is whether the classical features, revealed by $|\Psi\rangle$ and reflected in the Heisenberg uncertainty equations of pairs of conjugate coordinates, are preserved when the rotational symmetry is restored, i.e. from the deformed state one projects out the components of a definite angular momentum. The same question is valid also for the gauge invariance restoration. A measure of the deviation from the classical behavior is again the departure of the dispersion product from the classical value. In what follows we attempt to answer these questions for the two pairs of conjugate coordinates considered above.

### *2.3.1 The Case of $\alpha$, $\pi$ Coordinates*

As we proceeded above and, moreover, in order to keep close track to the CSM, which will be fully treated in one of the next chapters, for the case of $(\alpha, \pi)$ coordinates the parameter $d$ is taken real.

Through angular momentum projection one generates a set of orthogonal states:

$$\phi_{JM}^{(g)} = N_J^{(g)} P_{M0}^J|\Psi\rangle, \tag{2.3.1}$$

where $P_{MK}^J$ denotes the angular momentum projection operator

$$P_{MK}^J = \frac{2J+1}{8\pi^2}\int D_{MK}^{J*}(\Omega)\hat{R}(\Omega)d\Omega, \tag{2.3.2}$$

with $D^J_{MK}$ standing for the Wigner function, or rotation matrix, $\hat{R}(\Omega)$ is the rotation defined by the Euler angles $\Omega$, while $N^{(g)}_J$ is the normalization factor. The projected functions account for the main features of the rotational ground band [RCGD82]. For this reason the function is accompanied by the upper index "$^{(g)}$". The norms have been analytically studied for any deformation and moreover very simple formulas for near vibrational and well deformed regimes have been obtained [RBF12]. For the sake of completeness we give the necessary expressions:

$$\left(N^{(g)}_J\right)^{-2} = (2J+1)I^{(0)}_J e^{-d^2}, \tag{2.3.3}$$

with

$$I^{(k)}_J(x) = \int_0^1 P_J(y)\,(P_2(y))^k\, e^{xP_2(y)}dy, \quad x = d^2, \tag{2.3.4}$$

where $P_k(x)$ is the Legendre polynomial of rank $k$. Expectation values of the conjugate coordinates and their squares have the expressions:

$$\begin{aligned}
\langle\phi^{(g)}_{JM}|\hat{\alpha}|\phi^{(g)}_{JM}\rangle &= \sqrt{2}dC^{J\,2\,J}_{M\,0\,M}C^{J\,2\,J}_{0\,0\,0},\\
\langle\phi^{(g)}_{JM}|\hat{\alpha}^2|\phi^{(g)}_{JM}\rangle &= \frac{1}{2} + d^2\Bigg[\sum_{J'=0,2,4} C^{J\;J'\;J}_{M\,0\,M}C^{J\;J'\;J}_{0\;0\;0}\left(C^{2\,2\,J'}_{0\,0\,0}\right)^2\\
&\quad + \sum_{J'=0,2,4}\left(C^{J'\,2\;J}_{0\;0\;0}\right)^2\left(C^{J'2\;J}_{M\,0\,M}\right)^2\left(\frac{N^{(g)}_J}{N^{(g)}_{J'}}\right)^2\Bigg],\\
\langle\phi^{(g)}_{JM}|\hat{\pi}|\phi^{(g)}_{JM}\rangle &= 0,\\
\langle\phi^{(g)}_{JM}|\hat{\pi}^2|\phi^{(g)}_{JM}\rangle &= \frac{1}{2} + d^2\Bigg[-\sum_{J'=0,2,4} C^{J\;J'\;J}_{M\,0\,M}C^{J\;J'\;J}_{0\;0\;0}\left(C^{2\,2\,J'}_{0\,0\,0}\right)^2\\
&\quad + \sum_{J'=0,2,4}\left(C^{J'\,2\;J}_{0\;0\;0}\right)^2\left(C^{J'\,2\;J}_{M\,0\,M}\right)^2\left(\frac{N^{(g)}_J}{N^{(g)}_{J'}}\right)^2\Bigg].
\end{aligned} \tag{2.3.5}$$

Standard notation, $C^{j_1\,j_2\,j}_{m_1\,m_2\,m}$, for the Clebsch-Gordan coefficient is used. From here the dispersions of $\hat{\alpha}$ and $\hat{\pi}$ are readily obtained and then the dispersion product is analytically expressed.

Now we simultaneously restore the rotation and gauge symmetries. The boson number projection operator is defined by:

$$\hat{P}_N = \frac{1}{2\pi}\int_0^{2\pi} e^{i\phi(\hat{N}-N)}d\phi. \tag{2.3.6}$$

Applying successively the projection operators $P^J_{MK}$ and $\hat{P}_N$ on the coherent state $\Psi$, one obtains a state of good angular momentum and boson number:

$$|NJM\rangle = \mathcal{N}_{NJ}\hat{P}_N P^J_{MK}|\Psi\rangle. \tag{2.3.7}$$

Here $\mathcal{N}_{JN}$ denotes the normalization factor and has the expression:

$$(\mathcal{N}_{NJ})^{-2} = e^{-d^2}\frac{d^{2N}}{N!}(2J+1)\mathcal{S}_{NJ}, \tag{2.3.8}$$

where the matrix $\mathcal{S}_{mJ}$ is defined by:

$$\mathcal{S}_{mJ} = \int_0^1 (P_2(x))^m\, P_J(x)dx. \tag{2.3.9}$$

Following the path described in Ref. [RD76], one obtains:

$$\mathcal{S}_{lJ}(d) = \sum_{m=0}^{l}\frac{(-)^{l-m}3^m(l)!(2m)!(m+\frac{1}{2}J)!}{2^{l-J}m!(l-m)!(m-\frac{1}{2}J)!(2m+J+1)!}. \tag{2.3.10}$$

The overlap matrix elements given above satisfy the restriction: they are nonvanishing only if $l \le J/2$.

The explicit expression of the projected state is:

$$|NJM\rangle = \mathcal{N}_{NJ}e^{-d^2/2}\frac{d^N}{N!}\frac{2J+1}{8\pi^2}\int D^{J*}_{M0}(\Omega)\hat{R}(\Omega)\left(b^\dagger_{20}\right)^N d\Omega|0\rangle, \tag{2.3.11}$$

where $\hat{R}(\Omega)$ is the rotation defined by the set of Euler angles $\Omega$.

The expectation values of the conjugate variables $\hat{\alpha}_{20}$ and $\hat{\pi}_{20}$ are equal to zero since each of the composing terms changes the boson number by one unit. Therefore, the corresponding dispersions squared are just the average values of their squares. By direct calculations one finds:

$$\Delta\hat{\alpha}_{20}\Delta\hat{\pi}_{20} = \frac{1}{2} + \sum_{J'=0,2,4}\left(C^{J'\,2\,J}_{M\,0\,M}\right)^2\left(C^{J'\,2\,J}_{0\,0\,0}\right)^2 d^2\left(\frac{\mathcal{N}_{NJ}}{\mathcal{N}_{(N-1)J'}}\right)^2. \tag{2.3.12}$$

### 2.3.2 Dispersions of $\hat{N}$ and $\hat{P}$ on Projected States

Note that while averaging the boson number operator with the coherent state $|\Psi\rangle$ only the component $b^{\dagger}_{20}b_{20}$ gives a non-vanishing contribution, when the average is performed with the angular momentum projected state all the terms involved in the expression of the boson number operator, contribute. Therefore in this case the boson number $\hat{N}_0$ is to be replaced with the boson total number operator:

$$\hat{N} = \sum_{-2\le m\le 2} b^{\dagger}_{2m}b_{2m}. \quad (2.3.13)$$

The phase operator $\hat{P}$ satisfying the commutation relation

$$\left[\hat{P}, \hat{N}\right] = \hat{P}, \quad (2.3.14)$$

has the expression:

$$\hat{P} = \sum_{-2\le m\le 2} b_{2m}\hat{N}^{-1/2}, \quad (2.3.15)$$

where the reciprocal square root operator is defined as in Eq. (2.2.17). Within the same spirit and with similar caution as before the conjugate phase operator is:

$$\hat{\Phi} = -i \ln \hat{P}. \quad (2.3.16)$$

The expectation values of the boson number operator $\hat{N}$ and its square $\hat{N}^2$ have been analytically obtained Ref. [RD76].

$$\begin{aligned}\langle\phi^{(g)}_{JM}|\hat{N}|\phi^{(g)}_{JM}\rangle &= |d|^2\frac{I^{(1)}_J}{I^{(0)}_J},\\ \langle\phi^{(g)}_{JM}|\hat{N}^2|\phi^{(g)}_{JM}\rangle &= |d|^2\frac{I^{(1)}_J}{I^{(0)}_J} + |d|^4\frac{I^{(2)}_J}{I^{(0)}_J}.\end{aligned} \quad (2.3.17)$$

One can check that the overlap integral ratios involved in the above equations are related by the following equation [RCGD82, RBF12]:

$$x^2\frac{I^{(2)}_J}{I^{(0)}_J} = \frac{1}{2}x(x-3)\frac{I^{(1)}_J}{I^{(0)}_J} + \frac{1}{4}\left(2x^2 + J(J+1)\right), \quad x = |d|^2. \quad (2.3.18)$$

From these equations one obtains the dispersion of $\hat{N}$:

$$\left(\Delta\hat{N}\right)_J = -|d|^4\left(\frac{I_J^{(1)}}{I_J^{(0)}}\right)^2 + \frac{1}{2}|d|^2(|d|^2-1)\frac{I_J^{(1)}}{I_J^{(0)}} + \frac{1}{4}\left(2|d|^4 + J(J+1)\right). \tag{2.3.19}$$

The uncertainty relations will be calculated by choosing as conjugate operator the phase operator $\hat{P}$ divided by its average value [Hol80] and alternatively the Hermitian operators $\hat{C}$ and $\hat{S}$ defined as before [CaNi68]:

$$\hat{C} = \frac{1}{2}\left(\hat{P} + \hat{P}^{\dagger}\right),$$
$$\hat{S} = \frac{1}{2i}\left(\hat{P} - \hat{P}^{\dagger}\right). \tag{2.3.20}$$

In what follows we shall describe a method of calculating the dispersion of the associated phase operator $\hat{P}$. The average of $\hat{P}$ corresponding to the angular momentum projected state $|\phi_{JM}^{(g)}\rangle$ is:

$$\begin{aligned}\langle\phi_{JM}^{(g)}|\hat{P}|\phi_{JM}^{(g)}\rangle &= \left(N_J^{(g)}\right)^2 \langle\Psi|P_{M0}^{J\dagger}\sum_{\mu} b_{2\mu}P_{M0}^{J}\hat{N}^{-1/2}|\Psi\rangle \\ &= e^{-\frac{|d|^2}{2}}\left(N_J^{(g)}\right)^2 \langle\Psi|P_{M0}^{J\dagger}\sum_{\mu} b_{2\mu}P_{M0}^{J}\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty}\frac{(-)^m x^{2m}}{m!}\hat{N}^m d^n\frac{b_{20}^{\dagger n}}{n!}dx|0\rangle \\ &= e^{-\frac{|d|^2}{2}}\left(N_J^{(g)}\right)^2 \langle\Psi|P_{M0}^{J\dagger}\sum_{\mu} b_{2\mu}P_{M0}^{J}\frac{1}{\sqrt{n}}d^n\frac{b_{20}^{\dagger n}}{n!}|0\rangle \\ &= e^{-\frac{d^2}{2}}C_{M\,0\,M}^{J\,2\,J}C_{0\,0\,0}^{J\,2\,J}\left(N_J^{(g)}\right)^2\langle\Psi|P_{00}^{J}\sum_{n=1}^{\infty}\frac{d^n}{\sqrt{n}}\frac{\left(b_{20}^{\dagger}\right)^{n-1}}{(n-1)!}|0\rangle \\ &= de^{-|d|^2}C_{M\,0\,M}^{J\,2\,J}C_{0\,0\,0}^{J\,2\,J}\left(N_J^{(g)}\right)^2\frac{2J+1}{2}\int_{-1}^{+1}dx \\ &\quad\times\left[\sum_{m=0}^{\infty}\frac{1}{m!\sqrt{m+1}}\left(|d|^2P_2(x)\right)^m P_J(x)\right].\end{aligned} \tag{2.3.21}$$

The final result is:

$$\langle\phi_{JM}^{(g)}|\hat{P}|\phi_{JM}^{(g)}\rangle = C_{M\,0\,M}^{J\,2\,J}C_{0\,0\,0}^{J\,2\,J}d\frac{\mathcal{I}_J^{(0)}}{I_J^{(0)}}, \tag{2.3.22}$$

where we denoted:

$$\mathcal{I}_J^{(0)} = \sum_{m=0}^{\infty} \frac{|d|^{2m}}{m!\sqrt{m+1}} \mathcal{S}_{mJ}. \tag{2.3.23}$$

Applying a similar procedure as before but for $\hat{P}^2$, one obtains the final result:

$$\langle \phi_{JM}^{(g)} | \hat{P}^2 | \phi_{JM}^{(g)} \rangle = \mathcal{C}_M^J d^2 \frac{T_J^{(0)}}{I_J^{(0)}}, \tag{2.3.24}$$

with

$$\begin{aligned} \mathcal{C}_M^J &= \sum_{J'=0,2,4} C_{0\,0\,0}^{2\,2\,J'} C_{0\,\,0\,\,0}^{J\,J'\,J} C_{M\,0\,M}^{J\,J'\,J} \sum_{\mu} C_{\mu\,-\mu\,0}^{2\,2\,J'}, \\ T_J^0 &= \sum_{m=0}^{\infty} \frac{|d|^{2m}}{m!\sqrt{(m+1)(m+2)}} \mathcal{S}_{mJ}. \end{aligned} \tag{2.3.25}$$

Having the expressions of the expectation values of $\hat{P}$ and $\hat{P}^2$, the dispersion of $P$ is readily obtained.

$$\left(\Delta \hat{P}\right)_{JM}^2 = \langle \phi_{JM}^{(g)} | \hat{P}^2 | \phi_{JM}^{(g)} \rangle - \left( \langle \phi_{JM}^{(g)} | \hat{P} | \phi_{JM}^{(g)} \rangle \right)^2. \tag{2.3.26}$$

Although calculating the average of $\hat{\Phi}$ is quite a cumbersome task, that is possible. However according to Ref. [Hol80] the Heisenberg uncertainty inequality is satisfied by the dispersions of $\hat{N}$ and

$$(D\hat{P})_{JM} = \frac{(\Delta \hat{P})_{JM}}{|\langle \phi_{JM}^{(g)} | \hat{P} | \phi_{JM}^{(g)} \rangle|} \tag{2.3.27}$$

Therefore the departure from the classical limit is measured by $(\Delta \hat{N})_J (D\hat{P})_{JM}$.

The expectation values of the Hermitian operators $\hat{C}$ and $\hat{S}$, defined by

$$\begin{aligned} \hat{C} &= \frac{1}{2}\left(\hat{P} + \hat{P}^\dagger\right), \\ \hat{S} &= \frac{1}{2i}\left(\hat{P} - \hat{P}^\dagger\right) \end{aligned} \tag{2.3.28}$$

are easily obtained from Eqs. (2.3.22) and (2.3.24).

$$\langle \phi_{J0}^{(g)} | \hat{C} | \phi_{J0}^{(g)} \rangle = \left( C_{0\,0\,0}^{J\,2\,J} \right)^2 (Re\ d) \frac{\mathcal{I}_J^{(0)}}{I_J^{(0)}},$$

$$\langle\phi^{(g)}_{J0}|\hat{S}|\phi^{(g)}_{J0}\rangle = \left(C^{J\,2\,J}_{0\,0\,0}\right)^2 (Im\ d)\frac{\mathcal{I}^{(0)}_J}{I^{(0)}_J}. \tag{2.3.29}$$

Following the procedure described above we obtain:

$$\langle\phi^{(g)}_{J0}|\hat{C}^2+\hat{S}^2|\phi^{(g)}_{J0}\rangle = |d|^2\sum_{J'}\left(C^{J\,2\,J'}_{0\,0\,0}\right)^2\frac{\mathcal{U}^{(0)}_{J'}}{I^{(0)}_J}+\frac{5}{2}\frac{\mathcal{U}^{(0)}_J}{I^{(0)}_J}, \tag{2.3.30}$$

with

$$\mathcal{U}^{(0)}_J = \sum_{k=0}^{\infty}\frac{|d|^{2k}}{(k+1)!}S_{kJ}. \tag{2.3.31}$$

The normalized sum of dispersions associated to the two observables $\hat{C}$ and $\hat{S}$ is:

$$\begin{aligned}\frac{(\Delta\hat{C})^2_{J0}+(\Delta\hat{S})^2_{J0}}{\langle\hat{C}\rangle^2_{J0}+\langle\hat{S}\rangle^2_{J0}} &= \frac{1}{|d|^2\left(C^{J\,2\,J}_{0\,0\,0}\right)^4}\\ &\quad\times\left[|d|^2\sum_{J'}\left(C^{J\,2\,J'}_{0\,0\,0}\right)^2\frac{\mathcal{U}^{(0)}_{J'}I^{(0)}_J}{(\mathcal{I}^{(0)}_J)^2}+\frac{5}{2}\frac{\mathcal{U}^{(0)}_J I^{(0)}_J}{(\mathcal{I}^{(0)}_J)^2}\right]-1\\ &\equiv (\Delta R)^2_J,\end{aligned} \tag{2.3.32}$$

where the low index $J0$ suggests that the involved dispersions and average values correspond to the angular momentum projected state $\phi^{(g)}_{J0}$. Also, the notation $\langle\hat{O}\rangle_{J0}$ was used for the average value of $\hat{O}$ with the mentioned projected state.

The uncertainty relation associated to the two observables is obtained by equating

$$F_J = (\Delta\hat{N})_J\sqrt{\frac{(\Delta\hat{C})^2_{J0}+(\Delta\hat{S})^2_{J0}}{\langle\hat{C}\rangle^2_{J0}+\langle\hat{S}\rangle^2_{J0}}} \tag{2.3.33}$$

to the product of the right hand sides of Eq. (2.3.19) and $(\Delta R)_J$ given by (2.3.32). The departure of $F$ from the value of $1/2$ constitutes a measure for the quantal nature of the system behavior.

Concerning the $N$, $J$ projected states and the pair of coordinates $(\hat{N}, \hat{P})$ the results are as follows. The dispersion of $\hat{N}$ is vanishing since the projected state is eigenstate of $\hat{N}$. Also, the operators $\hat{P}$ and $\hat{P}^2$ violate the boson number conservation and therefore their corresponding averages are vanishing if the number of bosons $N$ is nonvanishing. The mentioned averages are undetermined for the state with $N = 0$.

How the projection of symmetries affect the uncertainty inequalities and therefore how far from the classical limit one could go by these operations, will be quantitatively analyzed in several numerical applications.

## 2.4 Numerical Analysis

We start by giving the expansion weights of $\Psi$ corresponding to various boson basis:

$$
\begin{aligned}
|\Psi\rangle &= \sum_n C_n |n\rangle, \\
|\Psi\rangle &= \sum_J C_{J0} |J0\rangle, \\
|\Psi\rangle &= \sum_{NJ} C_{NJ0} |NJ0\rangle.
\end{aligned} \tag{2.4.1}
$$

where $|n\rangle$ are eigenstates of the boson number operator $b^\dagger_{20}b_{20}$, $|J0\rangle$ denotes the eigenstates of angular momentum square, $\hat{J}^2$, and its projection on z-axis, $J_0$. The third basis $\{|NJ0\rangle\}$ is determined by the quantum numbers: the boson number $N$, the angular momentum $J$ and z-projection of the angular momentum, 0. Actually, these expansions correspond just to the boson bases defined above by the studied symmetry projection.

Using the results described before, one finds out that the expansion weights have the following analytical expressions:

$$
\begin{aligned}
C_n &= e^{-d^2/2}\frac{d^n}{\sqrt{n!}}, \\
C_{J0} &= \left(N_J^{(g)}\right)^{-1}, \\
C_{NJ} &= (\mathcal{N}_{NJ})^{-1}.
\end{aligned} \tag{2.4.2}
$$

These weights have been plotted in Fig. 2.1a–c, as function of the deformation parameter $d$. The curves have maxima for some deformations which indicate that for such deformations the corresponding states, showing up in the expansion, are the dominant components.

The dispersions product of the conjugate coordinates $\hat{\alpha}_{20}$ and $\hat{\pi}_{20}$ calculated with the $J-$projected states is presented as function of $d$ in Fig. 2.2c. It is well known that the classical limit of this quantity is 1/2 (units of $\hbar$). According to Fig. 2.2c the $J = 0$ projected state is the only projected state which behaves semiclassically in the region of small $d(\leq 1.5)$. The remaining states lay apart from the classical limit. The larger $J$, the larger the deviation from the classical limit. In the region of large $d(>3.)$ the deviation from the classical limit is an increasing function of $d$, irrespective the values

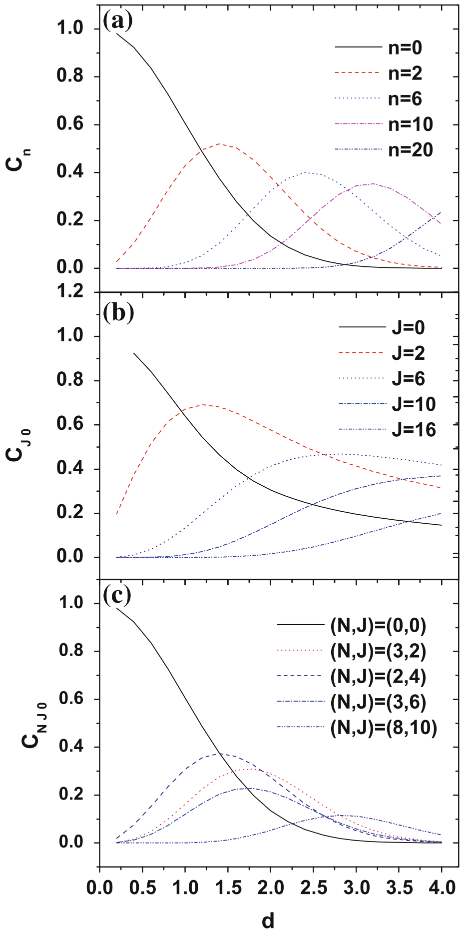


**Fig. 2.1** Expansion coefficients of the coherent state $|\Psi\rangle$ in three distinct basis, $|n\rangle$ (*panel* **a**), $|J0\rangle$ (*panel* **b**), and $|NJ0\rangle$ (*panel* **c**), are plotted as function of the deformation parameter $d$

of $J$. This behavior trend is determined by the increasing nature of the dispersion of $\hat{\alpha}_{20}$. Thus for both conjugate coordinates mentioned above, the nuclear deformation favors the quantal behavior of the system.

In Fig. 2.3 we represented the product of $\hat{\alpha}_{20}$ and $\hat{\pi}_{20}$ dispersions as function of $d$, using the states $|NJM\rangle$, which restore both symmetries mentioned above. As seen in Fig. 2.3, the dispersion product has a strong $J$-dependence for small values of $d$ while for large valued of $d$, i.e. in the rotational limit, this tends to the classical limit. In contrast to the case of the $J$-projected function here increasing $d$, favors the classical behavior. This is reflected in the decreasing $J$-dependence of the dispersion product as well as in approaching the classical value.

It is worth raising the question whether the features mentioned above depend on the chosen pair of conjugate coordinates. We do not attempt to give a general answer but analyze, for comparison, what happens when the pair of conjugate variables is

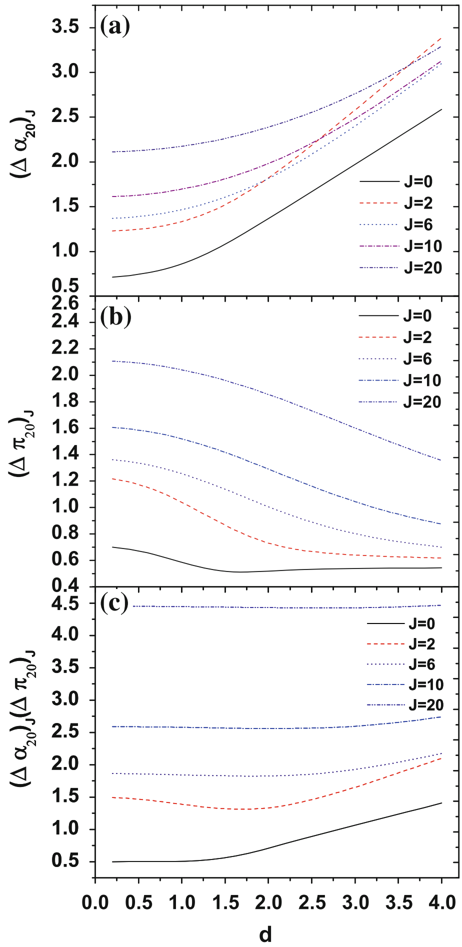


**Fig. 2.2** Dispersions of the conjugate coordinates $\alpha_{20}$ and $\pi_{20}$, as well as their product are given as functions of $d$ in *panels* **a**, **b** and **c** for angular momentum projected states

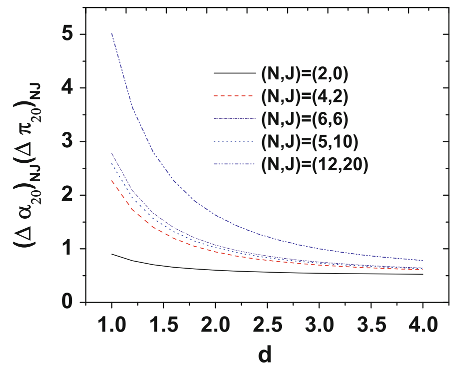


**Fig. 2.3** Dispersion product for the conjugate coordinates $\hat{\alpha}_{20}$ and $\hat{\pi}_{20}$, corresponding to the $NJ$-projected states are plotted as function of the deformation parameter $d$

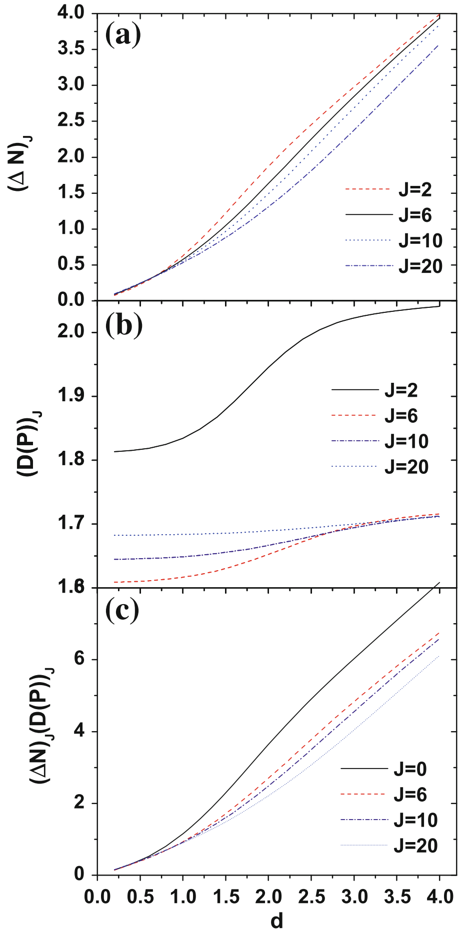


**Fig. 2.4** Dispersions for the boson number operator (*panel* **a**) and the corresponding phase operator $P$ (*panel* **b**) are plotted as function of $d$. Also, the dispersion product is given in *panel* **c**. Calculations are performed for angular momentum projected states

$(\hat{N}, \hat{P})$. The results are pictured in Fig. 2.4a–c. Dispersion of $N$ increases with $d$ and the split due to the $J$ dependence increases slowly from zero within a narrow interval. Note that for $d$ going to zero the projected function $\phi^{(g)}_{JM}$ goes to the state $|\frac{J}{2}, \frac{J}{2}, 0, J, M\rangle$ [RCGD82] with the standard notation $|N, v, \lambda, J, M\rangle$: $N$ being the number of bosons, $v$ the seniority, $\lambda$ the missing quantum number, $J$ the angular momentum and $M$ the projection on the laboratory z-axis. Therefore, in the spherical limit $N$ becomes a good quantum number and the dispersion is vanishing. By contrast, the normalized dispersion $D(P)$ has a large spread over $J$ for small values of $d$ but for large deformation the $J \neq 2$ dispersions attain a common value. Note that apart from the quantitative aspects, the dispersion product preserves the look of $\Delta N$. The behavior of the pair of coordinates $(N, R)$ is visualized in Fig. 2.5a–c. The split of R-dispersions due to their J-dependence is quite large for small deformation and is decreasing with $d$. The situation when dispersion of $R_J$ with $J \neq 2$ get a

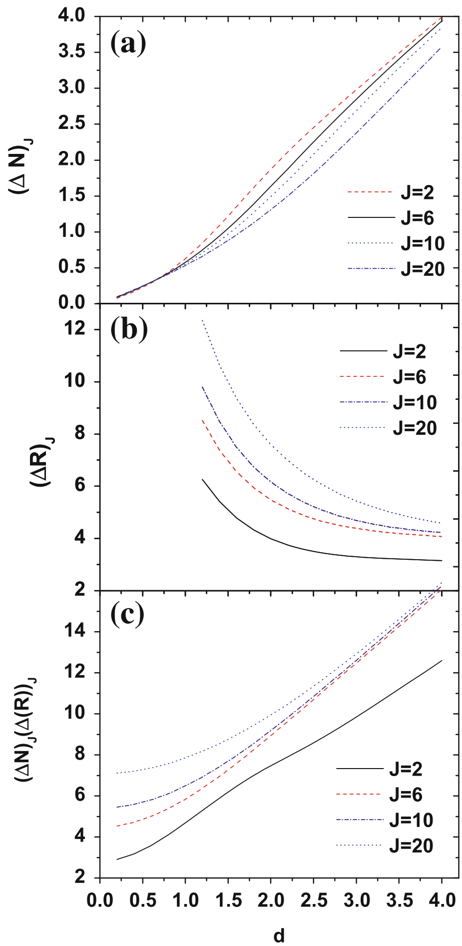


**Fig. 2.5** Dispersions for the boson number operator (*panel* **a**) and for the observable $\hat{R}$ defined by (2.3.32) (*panel* **b**) as well as their product (**c**) are plotted as function of $d$ for angular momentum projected states

common value is reached for $d$ larger than the maximum value shown in Fig. 2.4. The uncertainty relation for the pair $(N, R)$ is shown in the plot of $F_J$ (2.3.33) as a function of d. One notices that the departure from the classical limit is an increasing function of angular momentum. Also, this is increasing with the nuclear deformation. It is worth noticing that for large deformation, the $J \neq 2$ values become indistinguishable from each other.

The results obtained so far can be summarized as follows. The expansion weights of the coherent states in three distinct bases, exhibit a maximum when represented as function of the deformation parameter $d$. The larger are the selected quantum numbers the larger is the deformation for which the weight is maximum.

In the $(\alpha, \pi)$ representation only the $J = 0$ projected state behaves classically and that happens for small values of $d$. In the region of large $d$ the departure from the classical picture is slightly increasing with the deformation.

The behavior of the $(\alpha, \pi)$ pair of conjugate coordinates in a $NJ$-projected state is different from that described above for a $J$-projected state. Indeed, from Fig. 2.3 we notice that the quantal features prevail for small deformation, while for the rotational limit of large $d$ the associated Heisenberg relation approaches the classical limit. Also, in this limit the $J$-dependence of the uncertainty relations is very weak.

From Figs. 2.2c and 2.3 we conclude that in the $(\alpha, \pi)$ and J-projected states representation the system departs from the classical picture by increasing $d$ while for the $(N, J)$ projected states the larger the deformation the closer is the system to the classical behavior. In both cases the small deformation region is characterized by a quantal behavior reflected by the departure from the classical limits as well as by the split of the dispersion product due to the J-dependence. Comparing the figures referring to the uncertainty relations for the $(\alpha, \pi)$ in the J projected and $NJ$ projected states respectively, one may conclude that the share of classical and quantal features depends on the symmetry of the wave function. In the specific situations the more symmetric is the system the closer is its behavior to the classical picture.

The delicate problem of boson number and the conjugate phase was treated by two alternative choices for the conjugate phase-like operator. In the first case the operator is $\hat{P}$ with a proper normalization. Although this is not a Hermitian operator the Heisenberg uncertainty relation holds for a large number of bosons. The dispersion product is quickly increasing with $d$ starting with values close to zero ($\approx 0.162$). The behavior for small deformation is justified by the fact that the $J$-projected state becomes eigenstate of $\hat{N}$. For larger $d$ the system behaves in a classical manner while for small deformation the quantal features prevail. The split of the dispersion product due to its $J$-dependence is not significant for $J \neq 2$. One may say that although the departure from the classical limit of the dispersion product is large the classical feature reclaiming a weak $J$-dependence, still persists. The dispersion product is increasing with $d$ and is also $J$ independent for large values of $J$.

For the second alternative situation the phase like operators $\hat{C}$ and $\hat{S}$ were used to define, for the sake of having a symmetrical form, the dispersion of the observable $R$. The dispersion product denoted by $F_J$ is increasing with d. For small $d$ the split over J is large while for large deformation the values of $F_J$ for large $J$, are more or less the same. Here, as well as in the case of $(\alpha, \pi)$ coordinates, the coordinate dispersion is increasing with $d$, while the conjugate momentum dispersion is decreasing when $d$ increases.

Comparing the results for $(\alpha, \pi)$ and $(N, P)$ /or $(N, R)$ coordinates we notice that the interplay of quantal and classical feature depends on the pair of conjugate coordinates under study.

Before the symmetries were restored the system behaves classically which is reflected by that the uncertainty relations achieve their minima, irrespective of the chosen pair of conjugate coordinate. Moreover, the expectation value for angular momentum square has a continuous value [RCGD82]:

$$\langle \Psi | \hat{J}^2 | \Psi \rangle = 6|d|^2. \tag{2.4.3}$$

Symmetry projection leads to a J (or NJ)-dependence for the uncertainty relations which is large for small deformation. Increasing $|d|$, the system tends to recover the classical behavior.

## 2.5 The Baker-Campbell-Hausdorff Formula for the SU(2) Algebra

Along this book we shall use, in several places, the coherent state for the SU(2) algebra. In order to handle it in a comfortable manner it is necessary to use a factorized form which will be derived in what follows. Therefore, here we try to factorize an operator of the form.

$$\hat{O} = e^{\hat{A}+\hat{B}}. \tag{2.5.1}$$

In some special cases such an operator can be written as a product of two exponential factors.

$$\hat{O} = e^{\hat{A}}e^{\hat{B}}. \tag{2.5.2}$$

For example this factorization holds if the operators $A$ and $B$ commute with each other. The factorization is still simple if the commutator of the two operators is a constant. In this case the following equation holds:

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}e^{\hat{B}}e^{-\frac{1}{2}\left[\hat{A},\hat{B}\right]}. \tag{2.5.3}$$

An example of an exponential operator which can be factorized in this manner is the Glauber function or equivalently *the coherent state for the Weyl group*:

$$|d\rangle = e^{db^{\dagger}-d^{*}b}|0\rangle, \tag{2.5.4}$$

where $b$ and $b^{\dagger}$ are boson operators, while $d$ a complex parameter and $|0\rangle$ is the vacuum state for the boson operator $b$. In this case one can verify that:

$$e^{db^{\dagger}-d^{*}b}|0\rangle = e^{db^{\dagger}}e^{-d^{*}b}e^{-\frac{|d|^2}{2}}|0\rangle. \tag{2.5.5}$$

This is a direct consequence of Eq. (2.5.3), but can be also obtained by direct calculation. Indeed, if the factorization mentioned is valid then the coherent state can be written in the form:

$$|d\rangle = Ce^{db^{\dagger}}|0\rangle. \tag{2.5.6}$$

Remembering the fact that the coherent state is normalized to unity, we have:

$$\langle d|d\rangle = |C|^2\langle 0|e^{d^*b}e^{db^\dagger}|0\rangle = |C|^2 e^{|d|^2} = 1. \tag{2.5.7}$$

It results that modulo a phase factor the coherent state can be written as:

$$|d\rangle = e^{-\frac{1}{2}|d|^2}e^{db^\dagger}|0\rangle. \tag{2.5.8}$$

Thus, up to a factor depending on the annihilation operator $b$, which acting on $|0\rangle$, it reproduces it, the exponential operator has the expression from Eq. (2.5.8).

Another case which will be considered here is that where the operators $\hat{A}$ and $\hat{B}$ are generators for the $SU(2)$ algebra. This case is often met in Nuclear Physics both in microscopic and phenomenological models. Indeed, we remember that the $BCS$ function for the case where the space of correlated states is restricted to a single $j$ state is written as an exponential of a sum of the quasispin generators $S_+$ and $S_-$. Working with this function is a difficult task since the associated power series must be rewritten in a normal order. Thereby it is much simpler to use from the beginning a factorized form, two of the factors being just the exponential operators of $S_+$ and $S_-$, respectively [KIR67].

Here we shall treat the case of a rotation around the axis OY, $e^{-i\theta_2 J_y}$, which will be expressed as product of three exponential operators corresponding to the rotation generators: $J_+$, $J_-$ and $J_z$. Rotation around OY is a particular case, $s = 1$, of the more general rotation $e^{-is\theta_2 J_y}$. It is convenient to write this operator in the form:

$$e^{-i\theta_2 s J_y} = e^{F_1 J_-}e^{F_2 J_z}e^{F_3 J_+} \equiv B(s), \tag{2.5.9}$$

where $F_1$, $F_2$, $F_3$ are functions of $s$ which are to be determined such that the following restrictions be satisfied:

$$F_1(0) = F_2(0) = F_3(0) = 0. \tag{2.5.10}$$

Notations for the raising and lowering operators are the standard ones:

$$J_+ = J_x + iJ_y, \quad J_- = J_x - iJ_y. \tag{2.5.11}$$

Their commutation relations are those characterizing the $SU(2)$ algebra:

$$\left[J_+, J_-\right] = 2J_z, \quad \left[J_\pm, J_z\right] = \mp J_\pm. \tag{2.5.12}$$

If we perform the first derivative of Eq. (12.1.34) with respect to $s$ we obtain:

$$\dot{B} = \dot{F}_1\ J_- B + e^{F_1 J_-}\ \dot{F}_2\ J_z e^{F_2 J_z}e^{F_3 J_+} + e^{F_1 J_-}e^{F_2 J_z}\ \dot{F}_3\ J_+ e^{F_3 J_+}. \tag{2.5.13}$$

Now let us bring the non-exponential operators on the first position. To this aim one uses the identity

$$e^A B e^{-A} = B + \sum_{n=1} \frac{1}{n!} \underbrace{[A, [A, \ldots [A, B] \ldots]]}_{\text{n times}}. \tag{2.5.14}$$

In the present case the mentioned identity becomes:

$$e^{F_1 J_-} J_z e^{-F_1 J_-} = J_z + F_1 \left[J_-, J_z\right] = J_z + F_1 J_-. \tag{2.5.15}$$

Hence:

$$e^{F_1 J_-} J_z = (J_z + F_1 J_-) e^{F_1 J_-}. \tag{2.5.16}$$

$$\begin{aligned}
e^{F_2 J_z} J_+ e^{-F_2 J_z} &= J_+ + F_2 \left[J_z, J_+\right] + \frac{1}{2!} F_2^2 \left[J_z, \left[J_z, J_+\right]\right] + \cdots \\
&= J_+ + F_2 J_+ + \frac{1}{2!} F_2^2 J_+ + \cdots = J_+ e^{F_2} \\
e^{F_2 J_z} J_+ &= J_+ e^{F_2} e^{F_2 J_z} \\
e^{F_1 J_-} J_+ e^{-F_1 J_-} &= J_+ + F_1 \left[J_-, J_+\right] + \frac{1}{2!} F_1^2 \left[J_-, \left[J_-, J_+\right]\right] + \cdots \\
&= J_+ - 2F_1 J_z - F_1^2 J_-.
\end{aligned} \tag{2.5.17}$$

Therefore:

$$e^{F_1 J_-} e^{F_2 J_z} J_+ e^{F_3 J_+} = (J_+ - 2F_1 J_z - F_1^2 J_-) e^{F_2} B. \tag{2.5.18}$$

Finally, the derivative of B becomes:

$$\begin{aligned}
\overset{\bullet}{B} &= \left[\overset{\bullet}{F_1} J_- + (J_z + F_1 J_-) \overset{\bullet}{F_2} + (J_+ - 2F_1 J_z - F_1^2 J_-) e^{F_2} \overset{\bullet}{F_3}\right] B, \\
\frac{\theta_2}{2}(J_- - J_+) B &= \left[(\overset{\bullet}{F_1} + F_1 \overset{\bullet}{F_2} - \overset{\bullet}{F_3} F_1^2 e^{F_2}) J_- + (\overset{\bullet}{F_2} - 2F_1 e^{F_2} \overset{\bullet}{F_3}) J_z + J_+ e^{F_2} \overset{\bullet}{F_3}\right] B.
\end{aligned} \tag{2.5.19}$$

The generators $J_+$, $J_-$ and $J_z$ are linear independent operators, which result in having equal coefficients for the mentioned operators which show up in the l.h.s. and r.h.s. respectively.

$$\begin{aligned}
\overset{\bullet}{F_1} + F_1 \overset{\bullet}{F_2} - F_1^2 \overset{\bullet}{F_3} e^{F_2} &= \frac{\theta_2}{2} \Rightarrow \overset{\bullet}{F_1} = \frac{\theta_2}{2}(1 + F_1^2), \\
\overset{\bullet}{F_2} - 2F_1 \overset{\bullet}{F_3} e^{F_2} &= 0 \Rightarrow \overset{\bullet}{F_2} = -\theta_2 F_1, \\
\overset{\bullet}{F_3} e^{F_2} &= -\frac{\theta_2}{2} \Rightarrow \overset{\bullet}{F_3} = -\frac{\theta_2}{2} e^{-F_2}.
\end{aligned} \tag{2.5.20}$$

The solutions of these equations, which obey the initial conditions, are:

$$F_1 = tg\frac{\theta_2}{2}s,$$
$$F_2 = 2\ln\cos\frac{\theta_2}{2}s,$$
$$F_3 = -F_1 = -tg\frac{\theta_2}{2}s. \tag{2.5.21}$$

For the rotation operator of interest the value $s = 1$ corresponds:

$$e^{-i\theta_2 J_y} = e^{(tg\frac{\theta_2}{2})J_-}e^{(2\ln\cos\frac{\theta_2}{2})J_z}e^{-(tg\frac{\theta_2}{2})J_+}. \tag{2.5.22}$$

Let us act with this equation on an eigenstate for $J^2$ and $J_z$ of maximum projection and take into account the fact that $J_+|JJ\rangle = 0$. It results:

$$e^{-i\theta_2 J_y}|JJ\rangle = e^{(tg\frac{\theta_2}{2})J_-}\cos^{2J}\frac{\theta_2}{2}|JJ\rangle. \tag{2.5.23}$$

The same procedure could be applied to the operator

$$e^{zJ_- - z^* J_+}, \quad z = |z|e^{i\varphi}. \tag{2.5.24}$$

In this case the equations for the factor $F_1$, $F_2$, $F_3$ are:

$$\overset{\bullet}{F_1} = F_1^2 z^* + z,$$
$$\overset{\bullet}{F_2} = -2F_1 z^*,$$
$$\overset{\bullet}{F_3} = -e^{-F_2}z^*. \tag{2.5.25}$$

Integrating these equations and then put $s = 1$, one obtains:

$$F_1 = e^{i\varphi}tg|z|,$$
$$F_2 = -ln(1 + tg^2|z|),$$
$$\dot{F}_3 = -e^{-i\varphi}tg|z|. \tag{2.5.26}$$

Using the notations

$$\zeta = e^{i\varphi}tg|z|, \quad \frac{\theta}{2} = |z|, \tag{2.5.27}$$

the following factorization is obtained:

$$e^{zJ_- - z^* J_+} = e^{\zeta J_-} e^{-ln(1+|\zeta|^2)J_z} e^{-\zeta^* J_+}. \tag{2.5.28}$$

If we want that the product to have as first factor the exponential operator corresponding to the raising operator $J_+$, one could repeat the previous procedure with the result:

$$e^{zJ_- - z^* J_+} = e^{-\zeta^* J_+} e^{-ln(1+|\zeta|^2)J_z} e^{\zeta J_-}. \tag{2.5.29}$$

Acting with the operators involved in Eq. (2.5.28) on the state of maximum weight $|JJ\rangle$ and taking into account that $J_+|JJ\rangle = 0,\ J_z|JJ\rangle = J|JJ\rangle$, one arrives at the following expression:

$$|J, z\rangle = e^{zJ_- - z^* J_+}|JJ\rangle = \frac{1}{(1+|\zeta|^2)^J} e^{\zeta J_-}|JJ\rangle. \tag{2.5.30}$$

This function is *the coherent state for the group SU*(2). Such property results from the following considerations:

(1) The function $|JJ\rangle$ is extremal for the set of eigenstates of $J^2$ and $J_z$. Indeed, all states of this set can be obtained by successively acting on the extremal state with the lowering operator:

$$|JM\rangle = \begin{pmatrix} 2J \\ J-M \end{pmatrix}^{-1/2} \frac{1}{(J-M)!} (J_-)^{J-M} |JJ\rangle. \tag{2.5.31}$$

(2) The stability subgroup $G$ for this extremal is the group of rotations around the OZ axis.
(3) The exponential operator $e^{zJ_- - z^* J_+}$ is an element of the quotient group $SU(2)/G$.
(4) As shown before, the function $|J, z\rangle$ is obtained by acting with an element of the quotient group on the extremal state.

This function is often used in nuclear physics to study semiclassically the rotational degrees of freedom.

We notice that the stereographic projection of a sphere $S^2$ of a diameter equal to unity, $d = 1$, on a plane placed at the distance $d$ from the sphere center, achieves an one to one correspondence of the sphere points of coordinates $(1/2, \theta/2, \varphi)$ and the points $\zeta$ of the complex plane. This correspondence is illustrated in Fig. 2.6. The mentioned sphere is known as the Bloch sphere. Adding to the complex plane the point from infinity, the complex plane can be compacted, according to the theorem of Alexandrov. In this manner the correspondence evidenced above becomes a homeomorphism of two compact manifold.

The coherent states corresponding to two different points $z$ are not orthogonal except for those corresponding to the two sphere poles. Indeed, one can prove that

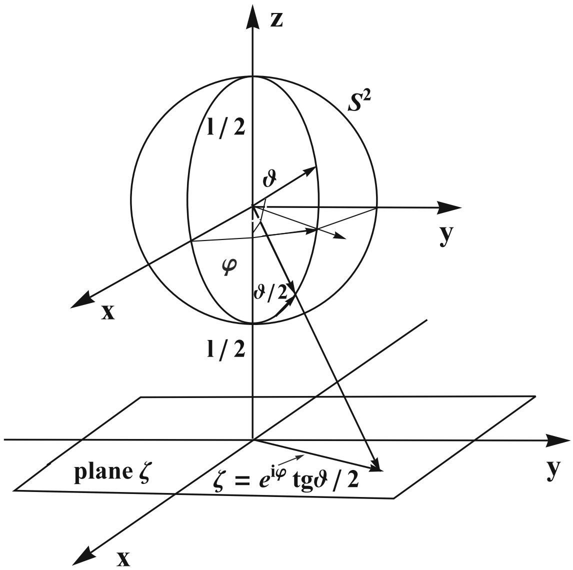


**Fig. 2.6** The correspondence between the points of the sphere $S^2$ and those of the complex plane ($\zeta$), achieved by the stereographic projection

the scalar product of two coherent states corresponding to two different points of the complex plane is:

$$|\langle J, z'|J, z\rangle|^2 = \left[\frac{1 + \mathbf{n}(\Omega') \cdot \mathbf{n}(\Omega)}{2}\right]^{2J} = cos^{4J}\frac{\Theta}{2}, \tag{2.5.32}$$

where $\mathbf{n}(\Omega)$ is the unity vector representing the point on the Bloch sphere of coordinate $(1/2, \theta/2, \varphi)$, while $\Theta$ is the angle of the two directions.

Resolution of unity for the spin coherent states is expressed by the relation:

$$\frac{2J + 1}{4\pi}\int d\Omega|J, z\rangle\langle J, z| = \sum_M |JM\rangle\langle JM| = 1. \tag{2.5.33}$$

An extremely useful property of the coherent states is that the matrix element of any operator can be obtained by calculating the derivative of one of the generating function:

$$\langle J, z|e^{\alpha_+ J_+}e^{\alpha_0 J_0}e^{\alpha_- J_-}|J, z\rangle = \frac{1}{(1 + |\zeta|^2)^{2J}}\left[e^{\frac{\alpha_0}{2}} + (\alpha_- + \zeta)(\alpha_+ + \zeta^*)e^{-\frac{\alpha_0}{2}}\right]^{2J},$$

$$\langle J, z|e^{\alpha_- J_-}e^{\alpha_0 J_0}e^{\alpha_+ J_+}|J, z\rangle = \frac{1}{(1 + |\zeta|^2)^{2J}}\left[e^{\frac{\alpha_0}{2}}(1 + \alpha_+\zeta)(1 + \alpha_-\zeta^*) + |\zeta|^2 e^{-\frac{\alpha_0}{2}}\right]^{2J}. \tag{2.5.34}$$

In order to derive these expressions some preliminaries are needed. Thus, the first relation can be obtained by a straightforward calculation and using the results:

$$e^{\alpha_- J_-}|J, z\rangle = \frac{1}{(1+|\zeta|^2)^J}\Bigg[|J, J\rangle + (\alpha_- + \zeta)\binom{2J}{1}^{1/2}|J, J-1\rangle + \cdots + (\alpha_- + \zeta)^k\binom{2J}{k}^{1/2}|J, J-k\rangle + (\alpha_- + \zeta)^{2J}|J, -J\rangle\Bigg], \quad (2.5.35)$$

$$e^{\alpha_0 J_0}e^{\alpha_- J_-}|J, z\rangle = \frac{1}{(1+|\zeta|^2)^J}\Bigg[e^{\alpha_0 J}|J, J\rangle + (\alpha_- + \zeta)\binom{2J}{1}^{1/2} \times e^{\alpha_0(J-1)}|J, J-1\rangle + \cdots + (\alpha_- + \zeta)^{2J}e^{-\alpha_0 J}|J, -J\rangle\Bigg], \quad (2.5.36)$$

$$\langle J, z|e^{\alpha_+ J_+} = \frac{1}{(1+|\zeta|^2)^J}\Bigg[\langle J, J| + (\alpha_+ + \zeta^*)\binom{2J}{1}^{1/2} \times \langle J, J-1| + \cdots + (\alpha_+ + \zeta^*)^{2J}\binom{2J}{2J}^{1/2}\langle J, -J|\Bigg]. \quad (2.5.37)$$

Concerning the second generating function, the intermediary steps lead to the result:

$$|J, z\rangle = \frac{1}{(1+|\zeta|^2)^J} \times \Bigg[|J, J\rangle + z\binom{2J}{1}^{1/2}|J, J-1\rangle + \cdots + z^k\binom{2J}{k}^{1/2}|J.J-k\rangle + \cdots + z^{2J}|J, -J\rangle\Bigg], \quad (2.5.38)$$

$$
\begin{aligned}
e^{\alpha_+ J_+}|J, z\rangle = \frac{1}{(1+|\zeta|^2)^J}
&\times \left[ |J, J\rangle (1+\alpha_+\zeta)^{2J} + |J, J-1\rangle (1+\alpha_+\zeta)^{2J-1}\zeta \binom{2J}{1}^{1/2} \right. \\
&\left. + |J, J-2\rangle (1+\alpha_+\zeta)^{2J-2}\zeta^2 \binom{2J}{2}^{1/2} + \cdots + |J, -J\rangle \zeta^{2J} \right],
\end{aligned}
\tag{2.5.39}
$$

$$
\begin{aligned}
e^{\alpha_0 J_0} e^{\alpha_+ J_+}|J, z\rangle = \frac{1}{(1+|\zeta|^2)^J}
&\times \left[ |J, J\rangle e^{\alpha_0 J} (1+\alpha_+\zeta)^{2J} + |J, J-1\rangle (1+\alpha_+\zeta)^{2J-1}\zeta \binom{2J}{1}^{1/2} e^{\alpha_0 (J-1)} \right. \\
&+ |J, J-2\rangle (1+\alpha_+\zeta)^{2J-2}\zeta^2 \binom{2J}{2}^{1/2} e^{\alpha_0 (J-2)} \\
&\left. + \cdots + |J, -J\rangle \zeta^{2J} e^{-\alpha_0 J} \right],
\end{aligned}
\tag{2.5.40}
$$

$$
\begin{aligned}
\langle J, z| e^{\alpha_- J_-} = \frac{1}{(1+|\zeta|^2)^J}
&\times \left[ \langle J, J| (1+\alpha_-\zeta^*)^{2J} + \langle J, J-1| (1+\alpha_-\zeta^*)^{2J-1}\zeta^* \binom{2J}{1}^{1/2} \right. \\
&\left. + \langle J, J-2| (1+\alpha_-\zeta^*)^{2J-2}(\zeta^*)^2 \binom{2J}{2}^{1/2} + \cdots + \langle J, -J| (\zeta^*)^{2J} \right].
\end{aligned}
\tag{2.5.41}
$$

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