Chapter 2
Homomorphisms of von Neumann Algebras

Abstract We introduce the tensor category structure of endomorphisms of infinite (type III) von Neumann factors. We review the basic concepts of conjugate homomorphisms between a pair of infinite factors, including the dimension, and discuss the generalization to homomorphisms of a factor into a von Neumann algebra with a centre.

Let N and M be two von Neumann algebras, and α, β a pair of homomorphisms : $N \to M$. (Without further mentioning, the notion "homomorphism" will include the * and unit-preserving properties $\alpha(n^*) = \alpha(n)^*$ and $\alpha(\mathbf{1}_N) = \mathbf{1}_M$.) An operator $t \in M$ such that

$$t \cdot \alpha(n) = \beta(n) \cdot t \qquad \text{for all} \quad n \in N$$

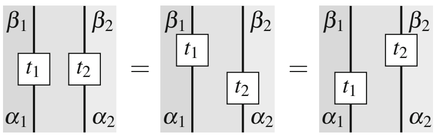
is called an **intertwiner**, writing $t : \alpha \to \beta$ or $t \in \mathrm{Hom}(\alpha, \beta)$. Clearly, if $t \in \mathrm{Hom}(\alpha, \beta)$, then $t^* \in \mathrm{Hom}(\beta, \alpha)$; $\mathrm{Hom}(\alpha, \beta)$ is a complex vector space, and $\mathrm{Hom}(\alpha, \alpha)$ is a C*-algebra.

A homomorphism $\alpha : N \to M$ is composed with a homomorphism $\beta : M \to L$, such that $\beta \circ \alpha : N \to L$.

Likewise, for any three homomorphisms $\alpha, \beta, \gamma : N \to M$ and intertwiners $t \in \mathrm{Hom}(\alpha, \beta)$ and $s \in \mathrm{Hom}(\beta, \gamma)$, the product in M gives an intertwiner $s \cdot t \in \mathrm{Hom}(\alpha, \gamma)$.

These structures turn the endomorphisms of a von Neumann algebra N into a strict tensor category $\mathrm{End}(N)$, and the homomorphisms between von Neumann algebras $N, M, \ldots$ into a strict tensor 2-category, where the concatenation of morphisms is the product of intertwiners: $s \circ t := s \cdot t$, the monoidal product of objects is the composition of endomorphisms: $\beta \times \alpha := \beta \circ \alpha$, and the monoidal product of morphisms $t_i : \alpha_i \to \beta_i$ is the product

$$t_1 \times t_2 = t_1 \cdot \alpha_1(t_2) = \beta_1(t_2) \cdot t_1 :$$



(This graphical notation, directly appealing to the underlying tensor category point of view, will render the structure of many algebraic computations more transparent.

M. Bischoff et al., *Tensor Categories and Endomorphisms of von Neumann Algebras*,
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Its basic rules are self-explaining from this example: Different shades indicate different von Neumann algebras, and we usually reserve the lightest shade for N, lines are homomorphisms, boxes and similar symbols to appear later are intertwiners, the monoidal product is horizontal juxtaposition, and the concatenation product is read from the bottom to the top. The operator adjoint is represented by up-down reflection.)

Notice that *as operators*, $t \times 1_\alpha = t$ is the same operator in a different intertwiner space, whereas $1_\alpha \times t = \alpha(t)$. To enhance readability, we shall occasionally suppress the concatenation symbol and write simply $s \circ t$ as the operator product st.

Because all intertwiner spaces $\mathrm{Hom}(\alpha, \beta)$ are linear subspaces of the target von Neumann algebra, they inherit its weak and norm topologies. In particular, $\mathrm{End}(N)$ is a C* tensor category, and the self-intertwiners $\mathrm{Hom}(\alpha, \alpha)$ form a C* algebra. Important consequences are that $t^* \circ t \equiv t^*t$ is a positive operator in $\mathrm{Hom}(\beta, \beta)$, and that $t^* \circ t = 0$ implies $t = 0$.

2.1 Endomorphisms of Infinite Factors

A von Neumann algebra N is a **factor** iff its centre $N' \cap N \equiv \mathrm{Hom}(\mathrm{id}_N, \mathrm{id}_N) = \mathbb{C} \cdot \mathbf{1}_N$. Since id_N is the monoidal unit in the tensor category, this is the same as saying that the category $\mathrm{End}(N)$ is simple.

These elementary facts can be supplemented by further structure. If $u : \alpha \to \beta$ is unitary, α and β are said to be **unitarily equivalent**. The unitary equivalence class of α is called the **sector** $[\alpha]$. An endomorphism α is **irreducible** iff $\mathrm{Hom}(\alpha, \alpha) = \mathbb{C} \cdot \mathbf{1}_N$.

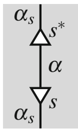
In an **infinite** ($\Leftrightarrow$ purely infinite, type III) von Neumann factor acting on a separable Hilbert space (which we shall henceforth assume throughout), every projection $e \neq 0$ can be written as $e = ss^*$ where $s^*s = 1$, and one can always choose decompositions of the unit $1 = \sum_i s_i s_i{}^*$ such that $s_i{}^* s_j = \delta_{ij}$. The algebra generated by bounded quantum mechanical observables (= the algebra $\mathscr{B}(\mathscr{H})$ of all bounded operators) does not share this property; instead, the local algebras of quantum field theory are generically infinite von Neumann factors.

Thanks to this property, one can define

(i) an inclusion relation for endomorphisms: $\beta \prec \alpha$ iff there is $s : \beta \to \alpha$ with $s^*s = 1_\beta$.

(ii) subobjects: if $e : \alpha \to \alpha$ is a projection, then there is a sub-endomorphism α_s defined by the choice of s such that $ss^* = e$, $s^*s = 1$, and putting

$$\alpha_s(\cdot) = s^*\alpha(\cdot)s :$$



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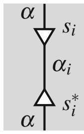
We refer to $\alpha_s \prec \alpha$ as the **range** of e. We shall sometimes write α_e instead, in order to emphasize that the unitary equivalence class of α_s does not depend on the choice of s. (Categories where subobjects exist are also called "Karoubian", thus $\mathrm{End}(N)$ is Karoubian if N is an infinite factor.)

(iii) direct sums of endomorphisms:

$$\alpha(\cdot) := \sum_i s_i \alpha_i(\cdot) s_i{}^* : \qquad \sum_i$$



is an endomorphism, $\alpha_i \prec \alpha$. Suppressing the dependence on the isometries s_i, we write sloppily $\alpha \simeq \bigoplus_i \alpha_i$. Since the choice of the isometries s_i is irrelevant for the unitary equivalence class (sector) $[\alpha]$, the direct sum should be understood as a direct sum of sectors. We emphasize this by writing also

$$[\alpha] = \bigoplus_i [\alpha_i].$$

2.2 Homomorphisms and Subfactors

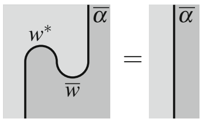
All notions of the preceding presentation can be transferred to homomorphisms $\varphi : N \to M$ where both N and M are infinite factors. Notice that intertwiners $t \in \mathrm{Hom}(\varphi_1, \varphi_2)$ are elements of M.

Admitting several factors, one obtains a 2-category, whose objects are the factors, the 1-morphisms are the homomorphisms, and the 2-morphisms are their intertwiners.

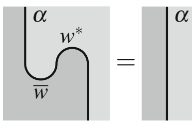
If $N \subset M$ is a **subfactor** (i.e., both N and M are factors), then the identical map $\iota : N \to M$, $n \mapsto n$, is a nontrivial homomorphism, that describes the embedding of N into M.

One can define [1, Chap. 3] a **dimension** function on the homomorphisms $N \to M$ when both N and M are infinite factors, which is additive under direct sums and multiplicative under composition. It is defined through the notion of **conjugates**: $\alpha : N \to M$ and $\overline{\alpha} : M \to N$ are said to be conjugates of each other whenever there is a pair of intertwiners $N \ni w : \mathrm{id}_N \to \overline{\alpha}\alpha$ and $M \ni \overline{w} : \mathrm{id}_M \to \alpha\overline{\alpha}$ satisfying the **conjugacy relations**

$$(w^* \times 1_{\overline{\alpha}}) \circ (1_{\overline{\alpha}} \times \overline{w}) = 1_{\overline{\alpha}} :$$



$$(1_\alpha \times w^*) \circ (\overline{w} \times 1_\alpha) = 1_\alpha :$$



(2.2.1)

Being self-intertwiners of id_N, resp. id_M, $w^* w = d \cdot \mathbf{1}_N$ and $\overline{w}^* \overline{w} = d' \cdot \mathbf{1}_M$ are positive scalars, and $w, \overline{w}$ can be normalized such that $d = d'$. The **dimension**

$\dim(\alpha) = \dim(\overline{\alpha})$ is defined to be

$$\dim(\alpha) = \dim(\overline{\alpha}) := \inf_{(w,\overline{w})} d \tag{2.2.2}$$

where the infimum is taken over all solutions $(w, \overline{w})$ of the conjugacy relations Eq. (2.2.1) with $d = d'$. A solution saturating the infimum is called **standard solution** or **standard pair**. If α and β are irreducible, every solution with $d = d'$ is standard, because $\dim\mathrm{Hom}(\mathrm{id}, \alpha\overline{\alpha}) = \dim\mathrm{Hom}(\overline{\alpha}\alpha)$=1. In the general case, standard solutions always exists, and are unique up to unitary equivalence [1, 2].

(Here is a simple explicit proof: For $[\alpha] = \bigoplus_i n_i[\alpha_i]$ and $[\overline{\alpha}] = \bigoplus_i \overline{n}_i[\overline{\alpha}_i]$ with $\alpha_i, \overline{\alpha}_i$ irreducible, one may choose standard pairs $(w_i, \overline{w}_i)$ for $\alpha_i, \overline{\alpha}_i$ and orthonormal bases $s^i_a \in \mathrm{Hom}(\alpha_i, \alpha)$, $\overline{s}^i_b \in \mathrm{Hom}(\overline{\alpha}_i, \overline{\alpha})$. Then the most general element of $\mathrm{Hom}(\mathrm{id}, \overline{\alpha}, \alpha)$ is of the form $w = \sum_i \sum_{ab} c^i_{ab}\overline{\alpha}(s^i_a)\overline{s}^i_b w_i$, and similarly $\overline{w} = \sum_i \sum_{ab} c'^i_{ab}\, \alpha(\overline{s}^i_b)s^i_a\overline{w}_i$. These solve the conjugacy relations iff the coefficient matrices satisfy $c'^i = (c^i)^{-1*}$ (in particular, the multiplicities $\overline{n}_i = n_i$ must be the same), and one has $d = \sum_i \dim(\alpha_i)\,\mathrm{Tr}(c^i)^*c^i$, $d' = \sum_i \dim(\overline{\alpha}_i)\,\mathrm{Tr}(c^i)^{-1*}(c^i)^{-1}$. The variational problem $d[c]d'[c] \stackrel{!}{=} \min$ with $d = d'$ is solved by any family of unitary matrices c^i.)

The conjugate of an endomorphism is unique up to unitary equivalence. Endomorphisms which do not have conjugates can be assigned the dimension ∞.

The dimension is always $\geq$1, and a homomorphism α is an isomorphism iff $\dim(\alpha) = 1$. In this case, α^{-1} is a conjugate of α. More generally, the dimension is the square root of the (minimal) index [3, 4]:

$$\dim(\alpha)^2 = [M : \alpha(N)].$$

In particular, for a subfactor $N \subset M$, $\dim(\iota)$ is the square root of the index $[M : N]$ [5]. In this case, $\iota\overline{\iota} \in \mathrm{End}(M)$ is called the **canonical endomorphism**, and $\overline{\iota}\iota \in \mathrm{End}(N)$ the **dual canonical endomorphism**.

Lemma 2.1 ([1]) (i) *Let* $(w_1, \overline{w}_1)$ *and* $(w_2, \overline{w}_2)$ *be standard pairs for* $(\alpha_1, \overline{\alpha}_1)$ *and for* $(\alpha_2, \overline{\alpha}_2)$*, respectively. Then*

$$w = \overline{\alpha}_1(w_2)w_1, \quad \overline{w} = \alpha_2(\overline{w}_1)\overline{w}_2$$

is a standard pair for $(\alpha_2\alpha_1, \overline{\alpha}_1\overline{\alpha}_2)$.

(ii) *Let* $(w_i, \overline{w}_i)$ *be standard pairs for* $(\alpha_i, \overline{\alpha}_i)$*, and* $[\alpha] = \bigoplus_i[\alpha_i]$, $[\overline{\alpha}] = \bigoplus_i[\overline{\alpha}_i]$. *Choose orthonormal isometries* $s_i \in \mathrm{Hom}(\alpha_i, \alpha)$ *and* $\overline{s}_i \in \mathrm{Hom}(\overline{\alpha}_i, \overline{\alpha})$. *Then*

$$w = \sum_i (\overline{s}_i \times s_i) \circ w_i, \quad \overline{w} = \sum_i (s_i \times \overline{s}_i) \circ \overline{w}_i$$

is a standard pair for $(\alpha, \overline{\alpha})$.

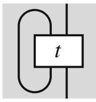
Corollary 2.2 ([1]) *The conjugate respects direct sums, and the dimension is additive and multiplicative:*

$$\dim(\alpha_2 \circ \alpha_1) = \dim(\alpha_2) \cdot \dim(\alpha_1), \quad \dim(\alpha) = \sum_i \dim(\alpha_i) \;\; \textit{if} \;\; [\alpha] = \bigoplus_i [\alpha_i].$$

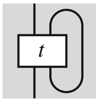
It should, of course, be emphasized that all the notions of direct sums, subobjects, conjugates and dimension respect the unitary equivalence relation.

Definition 2.3 The left and right **traces** are the faithful positive maps

$$\mathrm{LTr}_\alpha : \mathrm{Hom}(\alpha\beta, \alpha\beta') \to \mathrm{Hom}(\beta, \beta'),$$
$$t \mapsto (w^* \times 1_\beta) \circ (1_{\overline{\alpha}} \times t) \circ (w \times 1_\beta) \tag{2.2.3}$$



$$\mathrm{RTr}_\alpha : \mathrm{Hom}(\beta\alpha, \beta'\alpha) \to \mathrm{Hom}(\beta, \beta'),$$
$$t \mapsto (1_\beta \times \overline{w}^*) \circ (t \times 1_{\overline{\alpha}}) \circ (1_\beta \times \overline{w}) \tag{2.2.4}$$

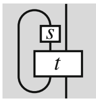


for the conjugate homomorphisms $\alpha, \overline{\alpha}$.

Proposition 2.4 ([1, Lemma 3.7]) *Let N and M be infinite factors, and let the traces* LTr_α *and* RTr_α *be defined w.r.t. a standard solution* $(w, \overline{w})$ *of the conjugacy relations for $\alpha : N \to M$ and $\overline{\alpha} : M \to N$.*

The traces do not depend on the choice of the conjugate and of the standard solution, and satisfy the trace property

$$\mathrm{LTr}_\alpha (s \times 1_{\beta'}) \circ t = \mathrm{LTr}_{\alpha'}\, t \circ (s \times 1_\beta)$$



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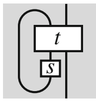

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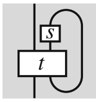
$$\mathrm{RTr}_\alpha (1_{\beta'} \times s) \circ t = \mathrm{RTr}_{\alpha'}\, t \circ (1_\beta \times s) \tag{2.2.5}$$



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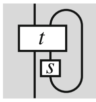

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for $s \in \mathrm{Hom}(\alpha', \alpha)$ and $t \in \mathrm{Hom}(\alpha\beta, \alpha'\beta')$ resp. $t \in \mathrm{Hom}(\beta\alpha, \beta'\alpha')$. For $\beta = \beta' = \mathrm{id}$, both traces coincide and are denoted $\mathrm{Tr}_\alpha : \mathrm{Hom}(\alpha, \alpha) \to \mathbb{C}$. In particular,

$$\mathrm{Tr}_\alpha \, 1_\alpha = \dim(\alpha). \tag{2.2.6}$$

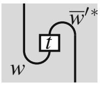
The latter property can in fact be adopted as an alternative definition for standardness, since one also has

Proposition 2.5 ([1, Lemma 3.9]) *Let N and M be infinite factors, and let the traces* LTr_α *and* RTr_α *be defined as in Definition 2.3 w.r.t.* any (*i.e., not necessarily standard*) *solution* $(w, \overline{w})$ *of the conjugacy relations for* $\alpha : N \to M$ *and* $\overline{\alpha} : M \to N$. *Then* LTr_α *and* RTr_α *coincide if and only if* $(w, \overline{w})$ *is standard.*

If $(w, \overline{w})$ is not standard, the maps LTr_α and RTr_α on $\mathrm{Hom}(\alpha, \alpha) \to \mathbb{C}$ may happen to be traces, without being equal. E.g., for reducible α every $n \in \mathrm{Hom}(\alpha, \alpha)$ gives rise to a deformation $w' := (1_{\overline{\alpha}} \times n) \circ w$, $\overline{w}' := (n^{*-1} \times 1_{\overline{\alpha}}) \circ \overline{w}$ of a standard pair $(w, \overline{w})$, which still solves the conjugacy relations. Then LTr'_α and RTr'_α defined with $(w', \overline{w}')$ are traces if and only if n^*n is central in $\mathrm{Hom}(\alpha, \alpha)$, while $(w', \overline{w}')$ is standard iff $n^*n = 1_\alpha$. One has the following characterization [1, Lemma 2.3]:

Proposition 2.6 *Let* $(w, \overline{w})$ *and* $(w', \overline{w}')$ *be solutions of the conjugacy relations for* $\alpha, \overline{\alpha}$ *and for* $\alpha', \overline{\alpha}'$, *not necessarily standard. Define* LTr_α *as in Definition* 2.3 *w.r.t. these pairs. The following are equivalent:*

(i) *For* $t \in \mathrm{Hom}(\alpha, \alpha')$ *and* $s \in \mathrm{Hom}(\alpha', \alpha)$, *one has* $\mathrm{LTr}_\alpha(st) = \mathrm{LTr}_{\alpha'}(ts)$.
(ii) *For* $t \in \mathrm{Hom}(\alpha, \alpha')$, *one has*



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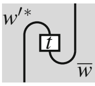
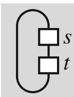


$\in \mathrm{Hom}(\overline{\alpha}, \overline{\alpha}')$.

The same is true, replacing LTr *by* RTr *in* (i), *or replacing* t *by* $s \in \mathrm{Hom}(\alpha', \alpha)$ *in* (ii).

In particular, (ii) *holds if* $(w, \overline{w})$ *and* $(w', \overline{w}')$ *are standard.*

Proof "(i) ⇒ (ii)" is the statement of [1, Lemma 2.3c], although the authors actually prove also the converse. The proof proceeds by noting that

$\mathrm{LTr}_\alpha(st) =$


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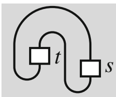
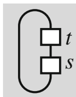


, $\mathrm{RTr}_{\alpha'}(ts) =$


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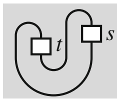

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Now, (ii) trivially implies equality of the two expressions, hence (i). Conversely, (i) implies (ii) because $(1_{\overline{\alpha}'} \times s) \circ w'$ is an arbitrary element of $\mathrm{Hom}(\mathrm{id}, \overline{\alpha}\alpha')$.

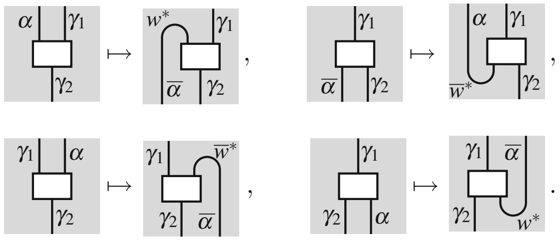
The variants of the statement follow by obvious modifications.

Finally, if $(w, \overline{w})$ and $(w', \overline{w}')$ are standard, then Proposition 2.4 implies (i), hence (ii). □

For a single infinite von Neumann factor N, $\mathrm{End}_0(N)$ is the full subcategory of $\mathrm{End}(N)$, whose objects are the endomorphisms of finite dimension. This is a "rigid" category since left and right duals exist for all objects (namely, the conjugate).

All intertwiner spaces $\mathrm{Hom}(\alpha, \beta)$ in $\mathrm{End}_0(N)$ are finite-dimensional, and $\mathrm{Hom}(\alpha, \alpha)$ are isomorphic with a direct sum of matrix algebras $\bigoplus_\lambda \mathrm{Mat}_{\mathbb{C}}(n_\lambda)$, where λ are the equivalence classes of irreducible sub-endomorphisms of α and n_λ their multiplicities in α.

Whenever α has finite dimension (and hence a conjugate $\overline{\alpha}$ exists), one can use a standard solution $(w, \overline{w})$ to define linear bijections (left and right **Frobenius conjugations**) between the spaces $\mathrm{Hom}(\gamma_2, \alpha\gamma_1)$ and $\mathrm{Hom}(\overline{\alpha}\gamma_2, \gamma_1)$, and between $\mathrm{Hom}(\gamma_2, \gamma_1\alpha)$ and $\mathrm{Hom}(\gamma_2\overline{\alpha}, \gamma_1)$,



These maps along with the ensuing equalities of the dimensions of the intertwiner spaces,

$$\dim\mathrm{Hom}(\gamma_2, \alpha\gamma_1) = \dim\mathrm{Hom}(\overline{\alpha}\gamma_2, \gamma_1),$$
$$\dim\mathrm{Hom}(\gamma_2, \gamma_1\alpha) = \dim\mathrm{Hom}(\gamma_2\overline{\alpha}, \gamma_1),$$

are usually referred to as **Frobenius reciprocities**.

2.3 Non-factorial Extensions

We want to extend our setup to N being a factor, while M is admitted to be a properly infinite von Neumann algebra with finite centre. For a related analysis, see [6, 7].

M is a direct sum of finitely many infinite factors

$$M = \bigoplus_i M_i.$$

The units of M_i are the minimal central projections e_i of M. A homomorphism $\varphi : N \to M$ can then be written as

$$\varphi(n) = \bigoplus_i \varphi_i(n).$$

Unlike the direct sum of sectors involving isometric intertwiners, cf. Sect. 2.1, this is the true direct sum of homomorphisms $\varphi_i : N \to M_i$, which is a homomorphism $N \to \bigoplus_i M_i$.

Notice that the central projections $e_i \in M$ are self-intertwiners of φ, but e_i can *not* be split as ss^* with isometries $s \in M$. Therefore, the direct sum of sectors $[\varphi_i]$ as in Sect. 2.1 is not defined.

Proposition 2.7 *If all* $\varphi_i : N \to M_i$ *have conjugates* $\overline{\varphi}_i$*, then a conjugate homomorphism* $\overline{\varphi} : M \to N$ *of* φ *can be defined as*

$$\overline{\varphi}(m) = \sum_i s_i \overline{\varphi}_i(m_i) s_i^*$$

where $m = \bigoplus_i m_i$*,* $m_i \in M_i$*, and* s_i *are isometries in* N *satisfying* $s_i^* s_j = \delta_{ij}$ *and* $\sum_i s_i s_i^* = \mathbf{1}_N$*. The dimension of* φ *is*

$$\dim(\varphi) = \big(\sum_i \dim(\varphi_i)^2\big)^{\frac{1}{2}}. \tag{2.3.1}$$

The dimension $\dim(\varphi)$ is defined by the same infimum as Eq. (2.2.2), taken over all solutions $(w, \overline{w})$ of the conjugacy relations such that $w^* w = d \cdot \mathbf{1}_N$, $\overline{w}^* \overline{w} = d \cdot \mathbf{1}_M$. Notice that it is no longer additive, as in the factor case.

Proof One easily sees that the solutions of the conjugacy relations are parameterized by

$$w = \sum_i \lambda_i \cdot s_i w_i, \qquad \overline{w} = \bigoplus_i \overline{\lambda}_i^{-1} \cdot \varphi_i(s_i) \overline{w}_i,$$

with parameters $\lambda_i \in \mathbb{C}$. Here, $(w_i, \overline{w}_i)$ are solutions for $(\varphi, \overline{\varphi})$ satisfying $w_i^* w_i = d_i \cdot \mathbf{1}_N$ and $\overline{w}_i^* \overline{w}_i = d_i \cdot \mathbf{1}_{M_i}$. Imposing $w^* w = d \cdot \mathbf{1}_N$ and $\overline{w}^* \overline{w} = d \cdot \mathbf{1}_M$ fixes the numerical coefficients by $|\lambda_i|^2 = d/d_i$ and $d^2 = \sum_i d_i^2$. This quantity is minimized if all d_i are minimal, i.e., all $(w_i, \overline{w}_i)$ are standard, and $d_i = \dim(\varphi_i)$. This completes the proof. □

Remark 2.8 For standard pairs $(w, \overline{w})$ of multiples of isometries satisfying the minimality condition, the tracial properties (Propositions 2.4–2.6) fail in general, when M (or N) is not a factor. The authors of [7] propose a different "normalization condition" (Eq. (4.3) in [7]) for solutions to the conjugacy relations, with $w^* w \in N$ and $\overline{w}^* \overline{w} \in M$ central but in general not multiples of $\mathbf{1}$. In the case of N and M both being factors, their condition amounts to the equality of the left and right traces, hence is equivalent to standardness by Proposition 2.5, but it distinguishes different normalizations otherwise. In the case at hand, it would rather fix $|\lambda_i|^2 = 1$, so that $\overline{w}^* \overline{w}$ is no longer a multiple of an isometry.

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