Regular Algebraic Surfaces, Ramification Structures and Projective Planes

N. Barker, N. Boston, N. Peyerimhoff and A. Vdovina

Abstract Regular algebraic surfaces isogenous to a higher product of curves can be obtained from finite groups with ramification structures. We find unmixed ramification structures for finite groups constructed as p-quotients of particular infinite groups with special presentation related to finite projective planes.

2000 Mathematics Subject Classification: 14L30 · 20F32 · 51E24

1 Introduction

An algebraic surface is *isogenous to a higher product* (of curves) if it admits a finite unramified covering which is isomorphic to a product of curves $C_1 \times C_2$ of genera $g(C_i) \geq 2$. It was shown in [10] that every such surface S has a unique *minimal* realisation $S \cong (C_1 \times C_2)/G$, where G is a finite group acting freely on $C_1 \times C_2$ and C_1 and C_2 have the smallest possible genera. Moreover, G respects the product structure by either acting diagonally on each factor (unmixed case) or there are elements in G interchanging the factors (mixed case). Surfaces isogenous to a higher product are always minimal and of general type. Particularly interesting examples

N. Barker (✉) · A. Vdovina
Department for Pure Mathematics and Mathematical Statistics,
University of Cambridge, CB3 OWB, Cambridge, UK
e-mail: nb443@cam.ac.uk

A. Vdovina
e-mail: alina.vdovina@ncl.ac.uk

N. Boston
Department of Mathematics,
University of Wisconsin, 303 Van Vleck Hall, 480 Lincoln Drive,
Madison, WI 53706, USA
e-mail: boston@math.wisc.edu

N. Peyerimhoff
Department of Mathematical Sciences, Science Laboratories,
Durham University, South Road, DH1 3LE, Durham, UK
e-mail: norbert.peyerimhoff@durham.ac.uk

I. Bauer et al. (eds.), *Beauville Surfaces and Groups*, Springer Proceedings
in Mathematics & Statistics 123, DOI 10.1007/978-3-319-13862-6_2

are *Beauville surfaces*, i.e., algebraic surfaces isogeneous to a higher product which are rigid (i.e., do not admit nontrivial deformations).

The *irregularity* $q(S)$ of a surface S is the difference between its geometric and its algebraic genus, and agrees with the Hodge number $h^{1,0}(S)$. Surfaces with vanishing irregularity are called *regular*. Since $q(S) = g(C_1/G) + g(C_2/G)$ (see [24, Proposition 2.2]), we have $C_i/G \cong \mathbb{P}^1$ for both curves in the minimal realisation of a regular surface.

Every surface S isogenous to a higher product gives rise to a finite group G via its minimal realisation. This process can be reversed. Starting with a finite group G, the existence of a so called *ramification structure* can be used to construct a regular surface of the form $(C_1 \times C_2)/G$. We will discuss ramification structures and the construction of the associated surfaces in Sect. 2. Bauer, Catanese and Grunewald [5] used this group theoretical description to classify all regular surfaces S isogenous to a product of curves with vanishing geometric genus $p_g(S) = h^{2,0}(S)$. The process in [5] was aided by the reduction of the search of ramification structures to groups of order less than 2000, for which the MAGMA library of small groups could then be used. They saw this classification as the solution in a very special case to the open problem posed by Mumford: "Can a computer classify all surfaces of general type with $p_g = 0$?"

The infinite group in [15, Example 6.3] given by the presentation

$$G_0 := \langle x_0, \ldots, x_6 \mid x_i x_{i+1} x_{i+3} \ (i \in \mathbb{Z}_7) \rangle \tag{1}$$

was used in [1, 2] to construct finite 2-groups with special unmixed and mixed ramification structures, giving rise to unmixed and mixed Beauville surfaces. These finite 2-groups were the maximal 2-quotients of 2-class k of both the group G_0 and its index two subgroup H_0 generated by x_0 and x_1. We like to mention that very little is known about *mixed* Beauville structures and it is generally assumed that they are very rare. Results in [13, 14] show that the symmetric groups S_n and all almost simple groups with sporadic derived groups cannot have mixed structures. The examples in [2] are the first known *infinite* family of 2-groups with mixed Beauville structures. It is immediate from the definition that a p-group can only admit a mixed structure if $p = 2$. The only other known construction of groups admitting mixed Beauville structures was given in [4]. But this general construction is very different in nature and does not provide examples of 2-*groups* admitting mixed Beauville structures. In this paper we restrict our considerations, however, to the *unmixed case*.

The above group G_0 belongs to a family called *groups with special presentation*. These groups were introduced by Howie [15] and are related to projective planes over finite fields (see Sect. 3 for more details). It was proved in [12] that all groups with special presentation are just infinite (i.e., they are infinite groups all of whose non-trivial normal subgroups have finite index). A natural question arose: *Do any other groups with special presentations give rise to finite groups with particular ramification structures?*

In this article we consider finite index subgroups of the groups listed in [11, Example 3.3], an index 3 subgroup of the following group with special presentation from [15, Example 6.4], the group

$$G := \langle x_0, \ldots, x_{12} \mid x_i^3, x_i x_{i+1} x_{i+4} \ (i \in \mathbb{Z}_{13}) \rangle \tag{2}$$

and the group given in [18, Example 2] constructed from a polyhedral presentation (a generalization of the triangle presentations defined in [8]). We use the computer program MAGMA to search for unmixed ramification structures in maximal p-quotients of p-class k of the above mentioned groups for various primes p. These ramification structures give then rise to particular regular surfaces isogenous to a higher product. Moreover, these results lead to natural conjectures about infinite families of p-groups admitting unmixed ramification structures.

Let us present an example of our results. The subgroup H of G defined in (2), and generated by x_0, x_1, x_2 has index 3. Let $H_{3,k}$ be the maximal 3-quotient of 3-class k. For simplicity, let us denote the elements in $H_{3,k}$ corresponding to x_0, x_1, x_2, again, by x_0, x_1, x_2. Let $y_0 = x_0 x_1^2 x_2^2$, $y_1 = x_0^2 x_1 x_2^2$ and $y_2 = x_1 x_2^{-1} x_2^{x_0}$. Then we have the following result.

Theorem 1.1 *For $k = 2, \ldots, 60$, the groups $H_{3,k}$ are of order 3^{a_k} and admit unmixed ramification structures (T_1, T_2) of type $([3,3,3,3^{d_k}], [3^{b_k},3^{b_k},3^{b_k},3^{b_k}])$, where $T_1 = (x_0, x_1, x_2, (x_0x_1x_2)^{-1})$, $T_2 = (y_0, y_1, y_2, (y_0y_1y_2)^{-1})$, $b_k = 1+[\log_3 \frac{3k}{4}]$, $d_k = 1+[\log_3 k]$, and*

$$a_k = \begin{cases} 8j & \text{if } k = 3j, \\ 8j+3 & \text{if } k = 3j+1, \\ 8j+6 & \text{if } k = 3j+2. \end{cases}$$

Here $[x]$ denotes the largest integer $\leq x$.

This result motivates the following conjecture.

Conjecture 1.2 *Let H be the index 3 subgroup of the group (2), generated by x_0, x_1, x_2. Then, for all $k \geq 2$, the maximal 3-quotients $H_{3,k}$ of 3-class k are of order 3^{a_k} and admit unmixed ramification structures (T_1, T_2) of type $([3,3,3,3^{d_k}], [3^{b_k},3^{b_k},3^{b_k},3^{b_k}])$, where a_k, b_k, d_k and T_1 and T_2 are explicitly given in Theorem 1.1.*

A promising approach to prove this conjecture is to employ the matrix representation of (2) in Appendix 2: "Representation for the Group G". (Such a matrix representation was key in [2] to prove that infinitely many 2-quotients of the group (1) admit mixed Beauville structures.)

The article is organised as follows. Section 2 presents fundamental facts about ramification structures and algebraic surfaces. Our results on ramification structures are presented in Sect. 3 below. Appendix 1: "Expanders Associated to the Group G_0" is concerned with the derivation of an explicit matrix representation of the group (2). Finally, Appendix 1: "Expanders Associated to the Group G_0" provides a brief survey about explicit recent expander constructions related to the group (1).

2 Ramification Structures and Associated Surfaces

2.1 Group Theoretical Structures

Following [5] closely, we give the definition of an (unmixed) ramification structure of a finite group G.

An r-tuple $T = [g_1, \ldots, g_r]$ of non-trivial elements of G is called a *spherical system of generators*, if $g_1, \ldots, g_r$ generate G and $g_1 g_2 \cdots g_r = 1$. The r-tuple $[m_1, \ldots, m_r]$ of non-decreasing orders of the elements g_i is called the *type* of the spherical system T of generators, i.e., $2 \leq m_1 \leq m_2 \leq \cdots \leq m_r$ and there is a permutation $\tau \in Sym(r)$ such that $m_i = \operatorname{ord}(g_{\tau(i)})$. Let

$$\Sigma(T) := \bigcup_{g \in G} \bigcup_{j=0}^{\infty} \bigcup_{i=1}^{r} \{g g_i^j g^{-1}\}$$

be the union of all conjugates of the cyclic subgroups generated by the elements g_i of the spherical system. Two spherical systems of generators $T_1 = [g_1, \ldots, g_r]$ and $T_2 = [g'_1, \ldots, g'_s]$ are called *disjoint* if $\Sigma(T_1) \cap \Sigma(T_2) = \{1\}$. An unmixed ramification structure is defined as follows.

Definition 2.1 *(Unmixed ramification structures, see* [5, Definition 1.1]) Let $A_1 = [m_1, \ldots, m_r]$ and $A_2 = [n_1, \ldots, n_s]$ be tuples of natural numbers with $2 \leq m_1 \leq \cdots \leq m_r$ and $2 \leq n_1 \leq \cdots \leq n_s$. An *unmixed ramification structure of type* (A_1, A_2) for a finite group G is a pair (T_1, T_2) of disjoint spherical systems of generators such that T_1 has type A_1 and T_2 has type A_2.

The disjointness of the pair (T_1, T_2) of an unmixed ramification structure guarantees that G acts freely on the product $C_1 \times C_2$ of associated algebraic curves (see Sect. 2.2 and the references therein). In this article we will only consider unmixed ramification structures and their associated surfaces. For examples of the mixed case see, e.g., [1, 4, 5].

Recall that *unmixed Beauville structures* are unmixed ramification structures with two spherical systems (T_1, T_2) of length 3, i.e., $r = s = 3$. They are of particular interest, since they give rise to the Beauville surfaces mentioned in the Introduction.

2.2 From Ramification Structures to Algebraic Surfaces

In this section we explain how to construct an algebraic surface $S = (C_{T_1} \times C_{T_2})/G$ from a given finite group G with an unmixed ramification structure (T_1, T_2).

Let G be a finite group and $T = [g_1, \ldots, g_r]$ be a spherical system of generators with $m_i = \operatorname{ord}(g_{\tau(i)})$. For $1 \leq i \leq r$, let $P_1, \ldots, P_l \in \mathbb{P}^1$ be a sequence of points ordered counterclockwise around a base point P_0 and $\gamma_i \in \pi(\mathbb{P}^1 - \{P_1, \ldots, P_r\}, P_0)$ be represented by a simple counterclockwise loop around P_i, such that $\gamma_1\gamma_2\dots\gamma_r = 1$. By Riemann's existence theorem, we obtain a surjective homomorphism

$$\Phi : \pi(\mathbb{P}^1 - \{P_1, \dots, P_r\}, P_0) \to G$$

with $\Phi(\gamma_i) = g_i$ and a Galois covering $\lambda : C_T \to \mathbb{P}^1$ with ramification indices equal to the orders of the elements $g_1, \dots, g_r$. These data induce a well defined action of G on the curve C_T, and by the Riemann-Hurwitz formula, we have

$$g(C_T) = 1 + \frac{|G|}{2}\left(r - 2 - \sum_{l=1}^{r} \frac{1}{m_l}\right). \tag{3}$$

Now, we assume that G admits an unmixed ramification structure (T_1, T_2). This leads to a diagonal action of G on the product $C_{T_1} \times C_{T_2}$, and the disjointness of the two spherical systems of generators ensures that G acts freely on the product of curves. The associated algebraic surface S is the quotient $(C_{T_1} \times C_{T_2})/G$. By the Theorem of Zeuthen-Segre, we have for the topological Euler number

$$e(S) = 4\frac{(g(C_{T_1}) - 1)(g(C_{T_2}) - 1)}{|G|},$$

as well as the relations (see [10, Theorem 3.4]),

$$\chi(S) = \frac{e(S)}{4} = \frac{K_S^2}{8},$$

where K_S^2 is the self intersection number of the canonical divisor and $\chi(S) = 1 + p_g(S) - q(S)$ is the holomorphic Euler-Poincaré characteristic of S. Assume that (T_1, T_2) is of the type (A_1, A_2) with $A_1 = [m_1, \dots, m_r]$ and $A_2 = [n_1, \dots, n_s]$. Then the above relations imply for the associated surface S that

$$\chi(S) = \frac{|G|}{4}\left(r - 2 - \sum_{l=1}^{r} \frac{1}{m_l}\right)\left(s - 2 - \sum_{l=1}^{s} \frac{1}{n_l}\right).$$

3 Groups with Special Presentations

As mentioned in [11], small cancellation groups are generalizations of surface groups and satisfy many of the nice properties of those groups. It was proved in [11] (with a small list of exceptions) that almost all groups with a presentation satisfying the small cancellation conditions $C(3)$ and $T(6)$ contain a free subgroup of rank 2.

Further, [15] proved that most $C(3)$, $T(6)$ groups G (namely, the ones which do *not* have special presentations) are SQ-universal. (A group G is called SQ-universal if every countable group can be embedded in a quotient group of G.) In that article, a group presentation was called *special* if every relator has length 3 and the star graph is isomorphic to the incidence graph of a finite projective plane. (See [21, p. 61] for a textbook reference on the *star graph* of a presentation. In short, the star graph of a presentation $\langle \mathbf{x} \mid \mathbf{r} \rangle$ is defined as follows (see [15]): its vertex set are the elements in $\mathbf{x} \cup \mathbf{x}^{-1}$ and there is an edge from x to y if there is a cyclically reduced word beginning in x and ending in y, which is a cyclic permutation of a relator or its inverse.) Moreover, it was asked (see [15, Question 6.11]) whether any or all of the groups with special presentations are SQ-universal. It was proved in [12] that groups with special presentation are just infinite (i.e., all non-trivial normal subgroups have finite index) and, therefore, cannot be SQ-universal. (Note that special presentations in the sense of [15] are (3, 3)-special in the sense of [12].)

Howie [15] also set up an example machine (see Theorem 3.1 below) to create infinitely many groups with special presentations. More precisely, he constructed a special presentation with star graph isomorphic to the incidence graph of the projective plane over every finite field $\mathbb{F}_q$, where q is a prime power. Until then, only seven examples of special presentations were known (see [11, Example 3.3]), and each of them has a star graph isomorphic to the Heawood graph (i.e., the incidence graph of the 7-point projective plane over $\mathbb{F}_2$).

Given a finite field $K = \mathbb{F}_q$ (for q a prime power), a *positive* presentation with star graph isomorphic to this incidence graph of the Desarguesian projective plane over K is formed.

The construction takes a cubic extension of K, namely $F = \mathbb{F}_{q^3}$, and identifies the cyclic group $C_m = F^\times / K^\times$ with the points of the projective pane $\mathcal{P}$ over K, where $m = q^2 + q + 1$.

The group C_m acts on $\mathcal{P}$ via multiplication in F, and this action is regular on both the points and lines of $\mathcal{P}$ i.e. C_m is a Singer group, see [16]. The lines of $\mathcal{P}$ can be identified with the subset σL of C_m, where σ ranges over C_m and L is a fixed line or perfect difference set i.e. a set of residues $a_1, \ldots, a_{q+1} \mod m$ such that every non-zero residue modulo $m = q^2 + q + 1$ can be expressed uniquely in the form $a_i - a_j$.

Theorem 3.1 ([15, Theorem 6.2]) *Let q be a prime power and $m = q^2 + q + 1$. Then there exists a subset l of $q + 1$ elements of $\mathbb{Z}_m$ such that*

$$\langle x_0, \ldots, x_{m-1} \mid x_i x_{i+\lambda} x_{i+\lambda+q\lambda} \ (i \in \mathbb{Z}_m, \lambda \in l) \rangle$$

is a special presentation whose star graph is isomorphic to the incidence graph of the projective plane over $GF(q)$.

Let us now present results on ramification structures of finite groups obtained from particular groups G with special presentations. These finite groups are generated via the lower, exponent p-central series, i.e.,

$$G = P_0(G) \geq \cdots \geq P_{i-1}(G) \geq P_i(G) \geq \ldots,$$

where $P_i(G) = [P_{i-1}(G), G]P_{i-1}(G)^p$ for $i \geq 1$. The finite groups $G_{p,k}$ under considerations are then the maximal p-quotients of p-class k, denoted by $G_{p,k}$ and given by $G_{p,k} = G/P_k(G)$.

The results discussed below are obtained via the computer program MAGMA (see [6]). Note that the algorithm `pQuotient` constructs, for a given group G, a consistent power-conjugate presentation for $G_{p,k}$.

3.1 Ramification Structures for the Group in [15, Example 6.3]

The group G_0 in (1) with seven generators $x_0, \ldots, x_6$ appeared as Example 6.3 in [15]. (G_0 is constructed using Theorem 3.1 with $q = 2$, $m = 7$ and $l = \{1, 2, 4\}$.) The subgroup H_0 generated by x_0, x_1 has index two. In [1, Theorems 4.1 and 4.2], we presented unmixed ramification structures for the 2-groups $(H_0)_{2,k}$ for $3 \leq k \leq 64$ (for k not a power of 2), as well as mixed ramification structures for the 2-groups $(G_0)_{2,k}$ for $3 \leq k \leq 10$ (again, for k not a power of 2). Since the involved spherical systems of generators consist of three elements, these ramification structures are actually Beauville structures and lead to new examples of Beauville surfaces.

Moreover, [2] presents a rigorous proof that an *infinite family* of 2-quotients of G_0 admits mixed Beauville structures. [2] uses a faithful matrix representation of G_0 by infinite upper triangular matrices (created in an analogous way as described in Appendix 1: "Expanders Associated to the Group G_0"), and the quotients are obtained via truncations at upper diagonals. MAGMA calculations show that the first 100 of these quotients are isomorphic to the quotients $(G_0)_{2,k}$. It is natural to conjecture that all of these quotients of G_0 constructed in these two different ways are pairwise isomorphic (see [22, Conjecture 1]).

The group G_0 appears also in [9, Sect. 4] as the group A.1. (The articles [8, 9] are concerned with simply transitive group actions on the vertices of $\widetilde{A}_2$-buildings.) The index two subgroup H_0 was also used in [17, 22] to construct families of expander graphs of vertex degrees 4 and 3. These expander graph constructions are briefly described in Appendix 1: "Expanders Associated to the Group G_0".

3.2 Ramification Structures for the Groups in [11, Example 3.3]

There, a list of seven special group presentations $G_i = \langle \mathbf{x} \mid \mathbf{r}_i \rangle$, $1 \leq i \leq 7$, with

$$r_1 = \{ab^{-1}d, a^{-1}dc, d^{-1}ea, b^2 f, ceg, cgf, efg\},$$
$$r_2 = \{a^{-1}df, b^{-1}ed, c^{-1}fe, a^2 g, bdg, bec, cgf\},$$
$$r_3 = \{abc, ade, afg, cge, bef, bdg, dfc\},$$
$$r_4 = \{a^2 b, acd, bde, bfc, ceg, dgf, efg\},$$

$$r_5 = \{a^2b, acd, bde, bfc, ceg, dg^2, ef^2\},$$
$$r_6 = \{a^2b, acb, bef, bge, d^2f, c^2g, egf\},$$
$$r_7 = \{abc, adb, acd, bef, cge, dfg, egf\},$$

were given ($\mathbf{x} = \{a, b, c, d, e, f, g\}$). The star graphs of all seven presentations are isomorphic to the incidence graph of the 7-point projective plane. See Fig. 1 for the star graph of the group G_1.

Our group G_0 in (1) coincides with their group G_3, which was discussed in Sect. 3.1. It is stated in [11] that the only isomorphism between abelianised groups G_i^{ab} is between G_4^{ab} and G_6^{ab}. However, if one looks at the commutator subgroup C_4 and C_6 of the groups G_4 and G_6, then $C_4^{ab} \cong \mathbb{Z}/4\mathbb{Z}$ and $C_6^{ab} \cong \mathbb{Z}/2\mathbb{Z}$. Therefore, G_4 can not be isomorphic to G_6.

We use the computer program MAGMA to search in the maximal p-quotients of maximal class k for $1 \le k \le 10$ of certain finite index subgroups of the groups G_i for unmixed ramification structures.

Special Presentation G_1

There is a subgroup H_1 of index 4 in G_1 generated by b. Thus, as $H_1 \cong \mathbb{Z}$ all maximal p-quotients of p-class k of H_1 are cyclic groups of order p^k. Therefore, there will be no unmixed ramification structures coming from the groups $(H_1)_{p,k}$.

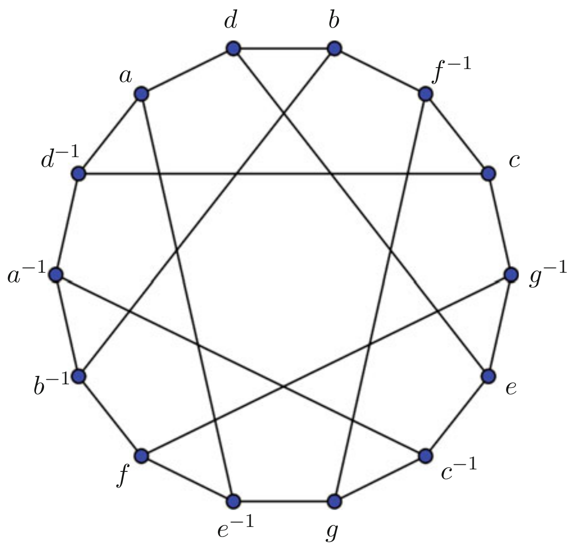

Fig. 1 Star graph of $G_1 = \langle \mathbf{x} \mid \mathbf{r}_1 \rangle$

Special Presentation G_2 and G_4

There is a subgroup H_2 of index 16 in G_2 (the commutator subgroup of G_2) generated by $h_{2,0} = bd^{-1}a^{-1}bc^{-1}$, $h_{2,1} = abd^{-1}abd^{-1}bc^{-1}$, $h_{2,2} = (bc^{-1})^2$, $h_{2,3} = adc^{-1}$ and $h_{2,4} = db^{-1}a^{-1}\, db^{-1}a^{-1}bc^{-1}$. The maximal 7-quotients of 7-class k, written as $(H_2)_{7,k}$, gives rise to a disjoint pair of spherical generators of length 3 given by the tuples,

$$T_{2,1} = [h_{2,0}, h_{2,1}, h_{2,1}^{-1}h_{2,0}^{-1}] \text{ and } T_{2,2} = [h_{2,0}h_{2,1}^2, h_{2,0}h_{2,1}^3, h_{2,1}^{-3}h_{2,0}^{-1}h_{2,2}^{-2}h_{2,0}^{-1}],$$

for $1 \le k \le 10$. To simplify notation, we denoted the images of $h_{2,0}$ and $h_{2,1}$ in $(H_2)_{7,k}$, again, by $h_{2,0}$ and $h_{2,1}$. Thus, the groups $(H_2)_{7,k}$ have unmixed ramification structures.

The group G_4 of [11, Example 3.3] coincides with the group C.1. in [9, Sect. 5] (via the identification $a_0 = a, a_1 = f, a_2 = c, a_3 = d, a_4 = e, a_5 = g, a_6 = b$). We find a subgroup H_4 of index 48 in G_4 generated by $h_{4,0} = da^{-1}bc^{-1}$, $h_{4,1} = bc^{-1}bc^{-1}ea^{-1}bf^{-1}$, $h_{4,2} = cb^{-1}ae^{-1}cf^{-1}$, $h_{4,3} = (ea^{-1}bf^{-1})^2$, $h_{4,4} = cb^{-1}cb^{-1}ea^{-1}bf^{-1}$ and $h_{4,5} = ea^{-1}bc^{-1}ea^{-1}bc^{-1}ea^{-1}bf^{-1}$. For $1 \le k \le 10$, the maximal 7-quotients of 7-class k, $(H_4)_{7,k}$, gives rise to a disjoint pair of spherical generators of length 3 given by the tuples,

$$T_{4,1} = [h_{4,0}, h_{4,1}, h_{4,1}^{-1}h_{4,0}^{-1}] \text{ and } T_{4,2} = [h_{4,0}h_{4,1}^2, h_{4,0}h_{4,1}^3, h_{4,1}^{-3}h_{4,0}^{-1}h_{4,2}^{-2}h_{4,0}^{-1}].$$

The groups H_2 and H_4 have the same maximal 7-quotients of maximal 7-class k for $1 \le k \le 10$. However, the abelianizations of the infinite groups are $H_2^{ab} \cong \mathbb{Z}_7 \times \mathbb{Z}_{21}$ and $H_4^{ab} \cong \mathbb{Z}_7 \times \mathbb{Z}_7$. The following theorem summarizes the above observations.

Theorem 3.2 *For $r = 2,4$, $k = 1,\ldots,10$, the groups $(H_r)_{7,k}$ are of order 7^a and admit unmixed ramification structures $(T_{r,1}, T_{r,2})$ of type $([7^b,7^b,7^b], [7^b,7^b,7^b])$ for*

$$a = \begin{cases} 2k & \text{if } k = 1, 4, 5, 8, 9, \\ 2k - 1 & \text{if } k = 2, 3, 6, 7, 10. \end{cases} \quad \text{and } b = \begin{cases} 2 & \text{if } 1 \le k \le 4, \\ 3 & \text{if } 5 \le k \le 8, \\ 5 & \text{if } 9 \le k \le 10. \end{cases}$$

This result is strong evidence that the following conjecture is true.

Conjecture 3.3 *For $r = 2,4$ and all $k \in \mathbb{N}$, the maximal 7-quotients $(H_r)_{7,k}$ of 7-class k admit unmixed ramification structures with disjoint spherical systems $(T_{r,1}, T_{r,2})$ introduced above.*

The unmixed ramification structures given for the groups $(H_2)_{7,k}$, $(H_4)_{7,k}$ above give rise to unmixed Beauville surfaces $S = (C_{T_1} \times C_{T_2})/(H_n)_{7,k}$ for $n = 2, 4$. For example, the order of the group $(H_2)_{7,1}$ and $(H_4)_{7,1}$ is 7^2. Therefore, the genera of the curves C_{T_i} is (see (3))

$$g(C_{T_1}) = g(C_{T_2}) = 1 + 2 \times 7 = 15,$$

and the holomorphic Euler-Poincaré characteristic of S is

$$\chi(S) = \frac{(g(C_{T_1}) - 1)(g(C_{T_2}) - 1)}{|G|} = 4.$$

Remark 3.4 (see [10, Beauville's Example 3.22]) The groups $(H_2)_{7,1}$, $(H_4)_{7,1}$ are isomorphic to the group $(\mathbb{Z}/7\mathbb{Z})^2$ and the two curves $C_{T_1} = C_{T_2}$ are given by the Fermat curve $x^7 + y^7 + z^7 = 0$ of degree 7. The group $(\mathbb{Z}/7\mathbb{Z})^2$ acts on $C_{T_1} \times C_{T_2}$ by the following rule

$$(\alpha, \beta) \cdot ([x : y : z], [u : v : w]) = ([\xi^{\alpha} x : \xi^{\beta} y : z], [\xi^{\alpha+2\beta} u : \xi^{\alpha+3\beta} v : w]),$$

where $\xi = e^{\frac{2\pi i}{7}}$ and $\alpha, \beta \in \mathbb{Z}/7\mathbb{Z}$. We identify $h_{n,0} \mapsto \alpha$ and $h_{n,1} \mapsto \beta$ for $n = 2, 4$.

Special Presentation G_5

The group G_5 coincides with the group A.2 in [9, Sect. 5]. We find a subgroup H_5 of index 3, generated by $h_{5,0} = ba^{-1}$, $h_{5,1} = ca^{-1}$, $h_{5,2} = da^{-1}$, $h_{5,3} = ea^{-1}$, $h_{5,4} = fa^{-1}$ and $h_{5,5} = ga^{-1}$ which have the same maximal 2-quotients of 2-class k as the group G_0 in (1) for $1 \leq k \leq 10$. However, the abelianization of this group is $H_5^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_{14}$ which is not isomorphic $G_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$.

In addition, we have a subgroup F_5 in H_5 of index 2, which appears to have the same maximal 2-quotients of 2-class k as H_0 (the subgroup of G_0 generated by x_0, x_1) for $1 \leq k \leq 10$. The abelianization of this group is $F_5^{ab} \cong \mathbb{Z}_4 \times \mathbb{Z}_{28}$ which is not isomorphic $G_0^{ab} \cong \mathbb{Z}_4 \times \mathbb{Z}_{14}$.

These results lead naturally to the following conjecture.

Conjecture 3.5 *Let H_5 be the index 3 subgroup of G_5 introduced above and let G_0 be the group given in (1). Let F_5 be the above mentioned index 2 subgroup of H_5, and H_0 the index 2 subgroup of G_0 generated by x_0, x_1. Even though H_5 and G_0 are not isomorphic, all corresponding maximal 2-quotients of H_5 and G_0 agree. Moreover, the same curious fact holds true for their subgroups F_5 and H_0.*

Special Presentation G_6 and G_7

The group G_6 coincides with the group B.2 in [9, Sect. 5]. We have the group specified by relations $\mathbf{r}_6 = \{a^2b, acb, bef, bge, d^2f, c^2g, egf\}$ on 7 generators but can be rewritten to a group generated by $\{x = e, y = f\}$ with relations

$$\mathbf{r}_6' = \{y^{-1}x^{-1}y^2x^{-2}y^{-3}x^{-1}, x^3yxyx^{-2}y^2\}$$

(see [3, Sect. 2.7]). The group G_7 coincides with the group B.1 in [9, Sect. 5].

We see that both groups have a subgroup H_6, H_7 of index 24 in G_6, G_7, respectively, which gives rise to maximal 3-quotients of 3-class k for $1 \leq k \leq 10$. However, the 3-groups are too large to successfully search for unmixed ramification structures. The abelianization of both groups is $H_6^{ab} \cong H_7^{ab} \cong \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_3$.

3.3 Ramification Structures for the Group in [15, Example 6.4]

The group G in (2) with 13 generators $x_0, \ldots, x_{12}$ appeared as Example 6.4 in [15]. (G is constructed using Theorem 3.1 with $q = 3$, $m = 13$ and $l = \{0, 1, 3, 9\}$.) The subgroup H generated by x_0, x_1, x_2 has index 3. Again, the group G can also be found in [9, Sect. 4] as the group 1.1 (via the identification $a_i = x_{2i}$, where the indices are taken modulo 13).

The existence of ramification structures for $k = 2, \ldots, 60$ for the finite groups $H_{3,k}$ was already formulated in the Introduction (see Theorem 1.1) and was obtained by using MAGMA. See Conjecture 1.2 in the Introduction for the associated conjecture.

The ramification structures of $H_{3,k}$ in Theorem 1.1 give rise to algebraic surfaces $S = (C_{T_1} \times C_{T_2})/H_{3,k}$. For example, the order of the group $H_{3,2}$ is $a_2 = 3^6$. Therefore, the genera of the curves C_{T_i} is (see (3))

$$g(C_{T_1}) = g(C_{T_2}) = 1 + 3^5 = 244,$$

and the holomorphic Euler-Poincaré characteristic of S is

$$\chi(S) = \frac{(g(C_{T_1}) - 1)(g(C_{T_2}) - 1)}{|G|} = 81.$$

3.4 Ramification Structures for the Groups of Theorem 3.1 with $q \geq 4$

The construction given by Theorem 3.1 is for any q a prime power. For $q = 4$ the group below is given.

Example 3.6 ([15, Example 6.5]) We have that $q^2 + q + 1 = 21$ and so $\mathbb{F}_{q^3}^\times/\mathbb{F}_q^\times$ is identified with $\mathbb{Z}_{21}$. The group $\widehat{G}$ is given by the presentation,

$$\widehat{G} := \langle x_0, \ldots, x_{20} | x_i x_{i+7} x_{i+14}, x_i x_{i+14} x_{i+7}, x_i x_{i+3} x_{i+15} \text{ for } i \in \mathbb{Z}_{21} \rangle. \quad (4)$$

The abelianization of this group is $\widehat{G}^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$. The group has maximal 2-quotients of 2-class k for $1 \leq k \leq 10$. However, it is extremely difficult to search for ramification structures of the maximal p-quotients of p-class k for $q \geq 4$. The finite groups are too large and have too many conjugacy classes, which leads to a computational expensive search.

3.5 Ramification Structures for the Group in [18, Example 2]

In [18], a new construction of groups presentations based on finite projective planes was introduced, generalizing the triangle presentations of [8, 9]. For the reader's convenience, we explain this briefly. The construction is based on the following general definition.

Definition 3.7 (*see* [18]) Let $\mathcal{P}_1, \ldots, \mathcal{P}_n$ be n disjoint finite projective planes of order q. Let P_i and L_i be the sets of points and lines respectively in $\mathcal{P}_i$. Let $P = \cup P_i$, $L = \cup L_i$, $P_i \cap P_j = \emptyset$ for $i \neq j$ and let λ be a bijection $\lambda : P \to L$.

A set $\mathcal{K}$ of k-tuples $(x_1, \ldots, x_k)$ will be called a *polyhedral presentation* over P compatible with λ if

(1) given $x_1, x_2 \in P$ then $(x_1, \ldots, x_k) \in \mathcal{K}$ for some $x_3, \ldots, x_k$ if and only if x_2 and $\lambda(x_1)$ are incident;
(2) $(x_1, \ldots, x_k) \in \mathcal{K}$ implies that $(x_2, \ldots, x_k, x_1) \in \mathcal{K}$;
(3) given $x_1, x_2 \in P$, then $(x_1, \ldots, x_k) \in \mathcal{K}$ for at most one $x_3 \in P$.

We call λ a *basic bijection*.

A polyhedral presentation $\mathcal{K}$ gives rise to a group presentation $G_{\mathcal{K}}$ in the following way: the generators of $G_{\mathcal{K}}$ are given by $\cup P_i$ and the relations are the k-tuples of $\mathcal{K}$, each written as a product.

Example 3.8 The *triangle presentations* listed in [9] can be seen as special cases of polyhedral presentations for $n = 1$, $k = 3$ and $q = 2, 3$.

We now discuss the case $n = 1$, $q = 2$. We enumerate the points of the projective plane by $1, 2, \ldots, 6$. The following array illustrates a basic projection λ:

$$\begin{array}{l} 0 : 1\ 4\ 2 \\ 1 : 3\ 2\ 5 \\ 2 : 4\ 3\ 6 \\ 3 : 0\ 4\ 5 \\ 4 : 1\ 5\ 6 \\ 5 : 0\ 2\ 6 \\ 6 : 0\ 1\ 3. \end{array}$$

Here, every point k represents a row and is followed by the points contained in the associated line $\lambda(k)$. For example, the line $\lambda(3)$ consists of the points 0, 4, 5.

A triangle presentation $\mathcal{T}$ for the group A.1 in [9] is given by

$$(0, 1, 3), (1, 2, 4), (2, 3, 5), (3, 4, 6), (4, 5, 0), (5, 6, 1), (6, 0, 2),$$

and all the cyclic permutations, i.e. for $(0, 1, 3) \in \mathcal{T}$ we also have $(1, 3, 0)$, $(3, 0, 1) \in \mathcal{T}$. The associated group presentation $G_{\mathcal{T}}$ agrees with the presentation of G_0 in (1).

Example 3.9 [18, Example 2] The projective plane $\mathcal{P}$ of order 4 can be partitioned by three projective planes of order two (see [7]). We denote points of the subplane $\mathcal{P}_i$ for $i = 1, 2, 3$ by numbers from $7i - 6$ to $7i$. Note that lines in $\mathcal{P}$ consist of five points, while the lines in $\mathcal{P}_i$ consist of three points. A basic projection λ for $\mathcal{P}$ is given below. Note that each subplane $\mathcal{P}_i$ has its own basic projection, denoted by λ_i, satisfying $\lambda_i(k) \subset \lambda(k)$. In the array below, the row associated to the point k lists first the three points in the associated line via the basic bijection in the subplane, followed up by the two remaining points in $\lambda(k)$.

$$\begin{array}{rl} 4: & 5\ 6\ 7\quad 12\ 18 \\ 7: & 1\ 2\ 5\quad 8\ 21 \\ 2: & 3\ 4\ 5\quad 14\ 16 \\ 5: & 1\ 3\ 6\quad 10\ 19 \\ 1: & 2\ 4\ 6\quad 9\ 15 \\ 3: & 1\ 4\ 7\quad 11\ 17 \\ 6: & 2\ 3\ 7\quad 13\ 20 \end{array} \qquad \begin{array}{rl} 9: & 12\ 13\ 14\quad 1\ 15 \\ 11: & 8\ 9\ 12\quad 3\ 17 \\ 14: & 10\ 11\ 12\quad 2\ 16 \\ 12: & 8\ 10\ 13\quad 4\ 18 \\ 10: & 9\ 11\ 13\quad 5\ 19 \\ 13: & 8\ 11\ 13\quad 6\ 20 \\ 8: & 9\ 10\ 14\quad 7\ 21 \end{array}$$

$$\begin{array}{rl} 18: & 19\ 20\ 21\quad 4\ 12 \\ 21: & 15\ 16\ 19\quad 7\ 8 \\ 16: & 17\ 18\ 19\quad 2\ 14 \\ 19: & 15\ 17\ 20\quad 5\ 10 \\ 15: & 16\ 18\ 20\quad 1\ 9 \\ 17: & 15\ 18\ 21\quad 3\ 11 \\ 20: & 16\ 17\ 21\quad 6\ 13 \end{array}$$

The above basic projections give rise to the following polyhedral presentation $\mathcal{K}$ for a projective plane of order 4, induced by polyhedral presentations of projective planes of order 2

$$\begin{gathered} (1, 9, 15), (1, 15, 9), (2, 14, 16), (2, 16, 14), (3, 11, 17), (3, 17, 11), (4, 12, 18), \\ (4, 18, 12), (5, 10, 19), (5, 19, 10), (6, 13, 20), (6, 20, 13), (7, 8, 21), (7, 21, 8), \\ (1, 2, 3), (1, 4, 5), (1, 6, 7), (3, 4, 6), (3, 7, 5), (2, 5, 6), (2, 4, 7), (8, 9, 12), (8, 10, 13), \\ (8, 14, 11), (9, 14, 10), (9, 13, 11), (12, 13, 14), (10, 11, 12), (15, 16, 17), (15, 18, 19), \\ (17, 18, 20), (17, 21, 19), (16, 19, 20), (16, 18, 21), \end{gathered}$$

and all their cyclic permutations.

All relators in the group presentation $G_{\mathcal{K}}$ given by $\mathcal{K}$ are of length 3 and the star graph is isomorphic to the incidence graph of a finite projective plane of order 4. This means, by [15], that the group given by this presentation $G_{\mathcal{K}}$ is a *special presentation*.

It also can be seen that this group acts on a Euclidean building where the vertex links are incidence graphs of projective planes of order 4, see [18].

There are remarkable connections between the group $G_{\mathcal{K}}$ and the group G_0 discussed in Sect. 3.1. Firstly, we can present the group $G_{\mathcal{K}}$ in an alternative way with different generators:

$$G_{\mathcal{K}} = \langle w_0, \ldots, w_6, y_0, \ldots, y_6, z_0, \ldots, z_6 \mid w_i w_{i+1} w_{i+3}, y_i y_{i+1} y_{i+3}, z_i z_{i+1} z_{i+3}, \\ w_i^{-1} y_{6(1+i)} z_i^{-1}, w_i^{-1} z_i^{-1} y_{6(1+i)} (i \in \mathbb{Z}_7) \rangle, \tag{5}$$

where each of the three subsets of generators has very similar relators like those appearing for the group G_0 in (1), with only two more series of relators added representing connections between the generators of different subsets.

Secondly, the maximal 2-quotients of 2-class k of the group $G_{\mathcal{K}}$ are isomorphic to the maximal 2-quotients of 2-class k for the group G_0 (given by the presentation in (1)) for $1 \le k \le 20$. However, the groups $G_{\mathcal{K}}$ and G_0 are not isomorphic, as they have different abelianized groups $G_{\mathcal{K}}^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_6 \times \mathbb{Z}_6$, while $G_0^{ab} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_6$. This gives rise to the following conjecture.

Conjecture 3.10 *Let $G_{\mathcal{K}}$ be the group introduced above (e.g., by the presentation (5)), and G_0 be the group given in (1). Even though $G_{\mathcal{K}}$ and G_0 are not isomorphic, all corresponding maximal 2-quotients of these two groups agree.*

Remark 3.11 If we replace the relators $w_i^{-1} y_{6(1+i)} z_i^{-1}$, $w_i^{-1} z_i^{-1} y_{6(1+i)}$ in (5) by the relators $x_i y_i z_i$, $x_i z_i y_i$ we obtain a group G' with the following presentation

$$G' = \langle w_0, \ldots, w_6, y_0, \ldots, y_6, z_0, \ldots, z_6 \mid w_i w_{i+1} w_{i+3}, y_i y_{i+1} y_{i+3}, z_i z_{i+1} z_{i+3}, \\ w_i y_i z_i, w_i z_i y_i (i \in \mathbb{Z}_7) \rangle.$$

This group G' is isomorphic to the group $\widehat{G}$ given by the presentation (4) under the identifications

$$\begin{array}{lll}
x_0 \mapsto z_0^{-1}, & x_7 \mapsto w_0^{-1}, & x_{14} \mapsto y_0^{-1}, \\
x_1 \mapsto w_1^{-1}, & x_8 \mapsto y_1^{-1}, & x_{15} \mapsto z_1^{-1}, \\
x_2 \mapsto y_2^{-1}, & x_9 \mapsto z_2^{-1}, & x_{16} \mapsto w_2^{-1}, \\
x_3 \mapsto z_3^{-1}, & x_{10} \mapsto w_3^{-1}, & x_{17} \mapsto y_3^{-1}, \\
x_4 \mapsto w_4^{-1}, & x_{11} \mapsto y_4^{-1}, & x_{18} \mapsto z_4^{-1}, \\
x_5 \mapsto y_5^{-1}, & x_{12} \mapsto z_5^{-1}, & x_{19} \mapsto w_5^{-1}, \\
x_6 \mapsto z_6^{-1}, & x_{13} \mapsto w_6^{-1}, & x_{20} \mapsto y_6^{-1}.
\end{array}$$

Acknowledgments We thank Donald Cartwright for the representations and method given in Appendix 2 and helpful correspondences. The first author also wishes to thank Uzi Vishne for useful correspondences.

Appendix 1: Expanders Associated to the Group G_0

Expander graphs are defined with the help of the edge expansion rate.

Definition A. 1 Let $\mathcal{G} = (V, E)$ be a combinatorial graph with vertex set V and edge set E. Then the *edge expansion rate* $h(\mathcal{G})$ is defined as

$$h(\mathcal{G}) = \inf_{\text{finite} A \subset V} \frac{|\partial A|}{\min(|A|, |V \backslash A|)},$$

where $\partial A \subset E$ is the set of all edges connecting a vertex of A with a vertex of $V \backslash A$.

Expanders are infinite families of finite graphs which are both sparse and highly connected. They are not only theoretically important but have also applications in computer science for, e.g., robust network designs.

Definition A. 2 A sequence $\mathcal{G}_n = (V_n, E_n)$ of connected finite graphs with $|V_n| \to \infty$ is called a *family of expanders* if there exists $k \geq 2$ and $\epsilon > 0$ such that

(a) all graphs $\mathcal{G}_n$ are k-regular,
(b) $h(\mathcal{G}_n) \geq \epsilon$ for all n.

It was observed in [22] that the subgroup H_0 of G_0 generated by x_0, x_1 has index 2, and that both groups H_0 and G_0 are just infinite and have Kazhdan property (T). Property (T) implies that, for a fixed choice of generators, the Cayley graphs of all quotients by finite index normal subgroups have a uniform positive lower bound for their edge expansion rate (see [19, Proposition 3.3.1]). A presentation of the subgroup H_0 is given by

$$H_0 = \langle x_0, x_1 \mid r_1, r_2, r_3 \rangle,$$

where

$$\begin{aligned} r_1 &= (x_1 x_0)^3 x_1^{-3} x_0^{-3}, \\ r_2 &= x_1 x_0^{-1} x_1^{-1} x_0^{-3} x_1^2 x_0^{-1} x_1 x_0 x_1, \\ r_3 &= x_1^3 x_0^{-1} x_1 x_0 x_1 x_0^2 x_1^2 x_0 x_1 x_0. \end{aligned}$$

We have the following Cayley graph expanders obtained from finite groups with just two generators and four relations.

Theorem A. 3 (cf. [22, Theorem 1]) *The groups*

$$H_k = \langle x_0, x_1 \mid r_1, r_2, r_3, [x_1, \underbrace{x_0, \ldots, x_0}_{k}] \rangle$$

are finite with $|H_k| \to \infty$*, and the associated Cayley graphs with respect to the generators* x_0, x_1 *define an infinite family of expanders of vertex degree 4.*

Using the faithful matrix representation of H_0 by infinite upper triangular matrices and their truncations at the kth upper diagonal as mentioned in Sect. 3.1, we obtain another family $\widetilde{H}_k$ of finite *nilpotent* groups whose associated Cayley graphs $\mathcal{G}_k$ with respect to the generators x_0, x_1 are another family of expander graphs which form a tower of coverings

$$\cdots \mathcal{G}_k \to \mathcal{G}_{k-1} \to \cdots \to \mathcal{G}_1 \to \mathcal{G}_0,$$

whose covering indices are powers of 2 (for more details, see [22]). It was conjectured in [22, Conjecture 2] that the covering indices follow the pattern 4, 8, 4, 8, 8, 4, 8, 8, 4, 8, 8, ... See Fig. 2 for the graph $\mathcal{G}_2$. We use the notation $z_1 = [x_0, x_1]$, $z_2 = x_0^2$, $z_3 = x_1^2$ and $z_{ij} = z_i z_j$ and $z_{ijk} = z_i z_j z_k$. The elements expressed by z_i lie in the centre of $\widetilde{H}_2$. The same graph was illustrated in [22, Fig. 4], but the illustration given here is more symmetric. Solid edges from vertices with label i to vertices with label $i+1 \pmod 4$ represent right multiplication by x_0, while dashed edges from vertices with label i to vertices with label $i+1 \pmod 4$ represent right multiplication by x_1. Note that the solid 4-cycles as well as the dashed 4-cycles in $\mathcal{G}_3$ are consequences of $x_0^4 = x_1^4 = 1$ in $\widetilde{H}_2$.

Another construction of 3-regular expanders was given in [17]. Starting from the same groups $\widetilde{H}_k$, we now consider the associated Cayley graphs X_k with respect to

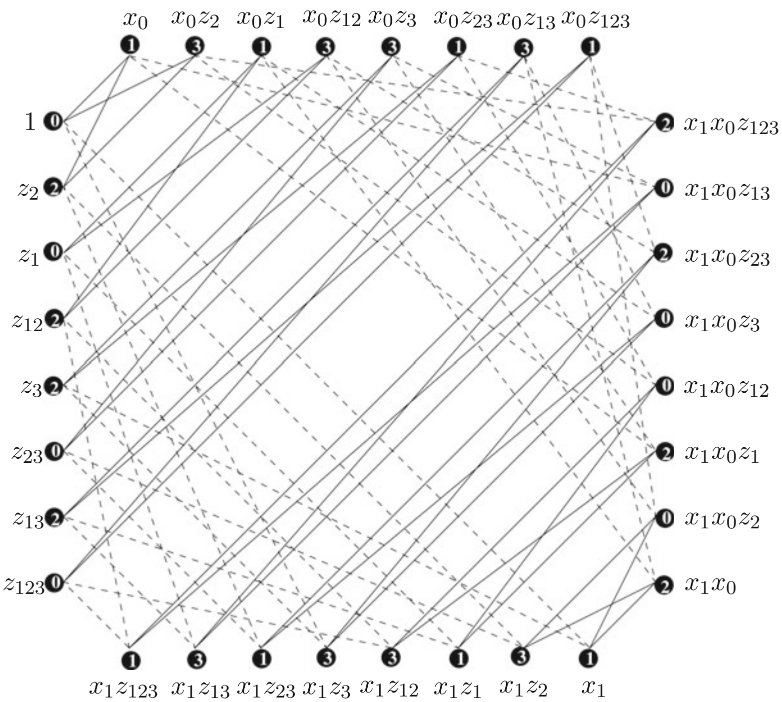

Fig. 2 The graph $\mathcal{G}_2$

the generators x_0, x_1, x_3 where $x_3 = x_1^{-1}x_0^{-1}$. The graphs X_k are 6-regular and $\mathcal{G}_k$ is a subgraph of X_k with the same number of vertices. Property (T) guarantees that the graphs X_k are also a family of expanders. One can check that X_k forms a tessellation of a closed Riemann surface by triangles and by 2^l-gons (with l only depending on k) and that every edge of X_k belongs to precisely one triangle of X_k. Now we apply a $\Delta - Y$ transformation to the graphs X_k to obtain new graphs T_k. The $\Delta - Y$ transformation removes the edges of every triangle in the original graph X_k, adds a new vertex in its centre, and connects this new central vertex with new edges to the 3 original vertices of the triangle. It turns out that the vertex set of the new graph T_k is twice as large as the vertex set of old graph X_k, and that T_k is 3-regular. Moreover, there is an explicit connection between the eigenfunctions of the adjacency matrix of X_k and the eigenfunctions of the adjacency matrix of T_k (see [17, Theorem 2.1]). The spectral characterisation of expander graphs then implies that the new family T_k of 3-regular graphs is, again, a family of expanders.

For yet another expander graphs construction from the group G_0 see [20, 23].

Appendix 2: Representation for the Group G

We include a representation for the group G (given by (2)) in $\mathrm{GL}(9, \mathbb{F}_3[1/Y])$, which may be useful in the future (as the matrix representations for the group G_0 with presentation (1) were useful for several works [8, 20, 22]). The representation is due to Donald Cartwright and the algebra program REDUCE. Recall that the group G coincides with the group 1.1 in [9], where we relate the generators by $a_i = x_{2i}$ for $i = 0, \ldots, 12$, with indices taken modulo 13. We set

$$x_0 : \begin{pmatrix} 1&1&1&0&2&2&0&1&1\\ 0&1&2&0&0&1&0&0&2\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&1&1&1&0&2&2\\ 0&0&0&0&1&2&0&0&1\\ 0&0&0&0&0&1&0&0&0\\ 0&0&0&0&0&0&1&1&1\\ 0&0&0&0&0&0&0&1&2\\ 0&0&0&0&0&0&0&0&1 \end{pmatrix} + \frac{1}{Y}\begin{pmatrix} 0&2&2&0&1&1&0&2&2\\ 0&0&1&0&0&2&0&0&1\\ 0&0&0&0&0&0&0&0&0\\ 0&1&1&0&2&2&0&1&1\\ 0&0&2&0&0&1&0&0&2\\ 0&0&0&0&0&0&0&0&0\\ 0&2&2&0&1&1&0&2&2\\ 0&0&1&0&0&2&0&0&1\\ 0&0&0&0&0&0&0&0&0 \end{pmatrix}$$

and

$$\tau : \begin{pmatrix} 1&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&1&0&0&0\\ 0&0&0&1&0&1&0&0&0\\ 0&0&0&0&0&0&2&2&0\\ 0&0&0&0&0&0&0&1&2\\ 0&0&0&0&0&0&2&0&1 \end{pmatrix},$$

where the other generators $x_1, \ldots, x_{12}$ are formed via conjugation of x_0 by τ, i.e. $x_i = \tau^i x_0 \tau^{-i}$ for $i = 1, \ldots, 12$.

The idea in creating this representation is to write $\mathbb{F}_{27} = \mathbb{F}_3(\theta)$, where θ is a primitive element on $\mathbb{F}_{27}$ satisfying $\theta^3 = \theta + 1$, and to use the basis $\{\theta^i \sigma^j | i, j = 0, 1, 2\}$ for the divison algebra $\mathcal{A}$ over $\mathbb{F}_{27}(Y)$ for an indeterminate Y (in the order $1, \theta, \theta^2, \sigma, \theta\sigma, \ldots, \theta^2\sigma^2$). Here σ is assumed to satisfy $\sigma^3 = Y - 1$ (which implies $(1+\sigma)^{-1} = (1/Y)(1 - \sigma + \sigma^2)$) and $\sigma\theta\sigma^{-1} = \theta^3$. The generators of $\mathcal{T}_{\mathcal{K}}$, where $\mathcal{K}$ is a triangle presentation from [8, 9], are the $a_u = u^{-1}(1+\sigma)u$, where $u \in \mathbb{F}_{27}^\times / \mathbb{F}_3^\times$. Since $\mathbb{F}_{27}^\times = \mathbb{F}_3^\times \cdot \{1 = \theta^{13}, \theta, \ldots, \theta^{12}\}$, we choose $\alpha_k = \theta^{-k}(1+\sigma)\theta^k$ as in [9, p. 178]. The α_k's act on $\mathcal{A}$ by conjugation. A straightforward calculation yields

$$\alpha_k \theta^i \sigma^j \alpha_k^{-1} = \theta^i \sigma^j \frac{1}{Y} + \left(\theta^{3i+2k} - \theta^{i+2\cdot 3^j k}\right) \sigma^{j+1} \frac{1}{Y} + \left(\theta^{i+8\cdot 3^j k} - \theta^{3i+2k+2\cdot 3^{j+1} k}\right) \sigma^{j+2} \frac{1}{Y} + \theta^{3i+2k+8\cdot 3^{j+1} k} \sigma^j \frac{Y-1}{Y}.$$

Expressing the conjugation by α_k with respect to the above basis of $\mathcal{A}$ then gives rise to a representation as a 9×9 matrix over the field $\mathbb{F}_3(1/Y)$. We conclude from [9] that the matrices associated to the α_k satisfy the relations of our generators x_k. Note, finally, that the above matrix for τ represents the conjugation by θ in $\mathcal{A}$, i.e., $z \mapsto \theta^{-1} z \theta$.

References

1. N. Barker, N. Boston, N. Peyerimhoff, A. Vdovina, New examples of Beauville surfaces. Monatsh. Math. **166**(3–4), 319–327 (2012)
2. N. Barker, N. Boston, N. Peyerimhoff A. Vdovina, An infinite family of 2-groups with mixed Beauville structures. Int. Math. Res. Not. (2014). doi:10.1093/imrn/rnu045
3. S. Barre, Polyédres de rang deux, Thesis ENS Lyon, December (1996). http://web.univ-ubs.fr/lmam/barre/these1.pdf
4. I.C. Bauer, F. Catanese F. Grunewald, Beauville surfaces without real structures, in *Geometric Methods in Algebra and Number Theory*. Progress in Mathematics, vol. 235 (Birkhäuser, Boston, 2005)
5. I.C. Bauer, F. Catanese, F. Grunewald, The classification of surfaces with $p_g = q = 0$ isogenous to a product of curves. Pure Appl. Math. Q. **4**(2), 547–586 (2008)

6. W. Bosma, J. Cannon, C. Playoust, The Magma algebra system. I. The user language. J. Symb. Comput. **24**(3–4), 235–265 (1997)
7. R.H. Bruck, Quadratic extensions of cyclic planes, in *Proceedings Symposium in Applied Mathematics*, pp. 15–44 (1960)
8. D.I. Cartwright, A.M. Mantero, T. Steger, A. Zappa, Groups acting simply transitively on the vertices of a building of type $\widetilde{A}_2$, I. Geom. Dedicata **47**(2), 143–166 (1993)
9. D.I. Cartwright, A.M. Mantero, T. Steger, A. Zappa, Groups acting simply transitively on the vertices of a building of type $\widetilde{A}_2$, II. Geom. Dedicata **47**(2), 167–223 (1993)
10. F. Catanese, Fibred surfaces, varieties isogenous to a product and related moduli spaces. Am. J. Math. **122**(1), 1–44 (2000)
11. M. Edjvet, J. Howie, Star graphs, projective planes and free subgroups in small cancellation groups. Proc. Lond. Math. Soc. **57**(2), 301–328 (1988)
12. M. Edjvet, A. Vdovina, On the SQ-universality of groups with special presentations. J. Group Theory **13**(6), 923–931 (2010)
13. B. Fairbairn, Some exceptional Beauville structures. J. Group Theory **15**, 631–639 (2012)
14. Y. Fuertes, G. González-Diez, On Beauville structures on the groups S_n and A_n. Math. Z. **264**, 959–968 (2010)
15. J. Howie, On the SQ-universality of $T(6)$-groups. Forum Math. **1**(3), 251–272 (1989)
16. D.R. Hughes, F.C. Piper, *Projective Planes* (Springer, New York, 1973)
17. I. Ivrissimtzis, N. Peyerimhoff, A. Vdovina, Trivalent expandes and hyperbolic surfaces (2012) arXiv:1202.2304
18. S. Immervol, A. Vdovina, Partitions of projective planes and construction of polyhedra, Max-Planck-Institut fur Mathematik, Bonn, Preprint Series 23 (2001)
19. A. Lubotzky, *Discrete Groups, Expanding Graphs and Invariant Measures* (Birkhäuser, Basel, 2010)
20. A. Lubotzky, B. Samuels, U. Vishne, Explicit construction of Ramanujan complexes of type $\widetilde{A}_d$. Eur. J. Comb. **26**(6), 965–993 (2005)
21. R.C. Lyndon, P.E. Schupp, *Combinatorial Group Theory, Classics in Mathematics*. Reprint of the 1977 edition (Springer, Berlin, 2001)
22. N. Peyerimhoff, A. Vdovina, Cayley graph expanders and groups of finite width. J. Pure Appl. Algebra **215**(11), 2780–2788 (2011)
23. A. Sarveniazi, Explicit construction of a Ramanujan $(n_1, n_2, \ldots, n_{d-1})$-regular hypergraph. Duke Math. J. **139**(1), 141–171 (2007)
24. F. Serrano, Isotrivial fibred surfaces. Ann. Mat. Pura Appl. **171**(4), 63–81 (1996)

http://www.springer.com/978-3-319-13861-9

Beauville Surfaces and Groups
Bauer, I.; Garion, S.; Vdovina, A. (Eds.)
2015, IX, 183 p. 23 illus., 3 illus. in color., Hardcover
ISBN: 978-3-319-13861-9