Chapter 1
Introduction

Abstract The introduction contains a brief and informal description of the concept of parabolic renormalization and of our main results.

Keyword Inou-Shishikura renormalization fixed point

Parabolic renormalization was first introduced by Shishikura in his celebrated work [SH] on the Hausdorff dimension of the boundary of the Mandelbrot set. More recently, the result of Inou and Shishikura [IS] on the convergence of parabolic renormalization became a key to the construction of quadratic Julia sets of positive measure by Buff and Chéritat [BS]. Thus, parabolic renormalization is clearly a powerful and important tool; indeed, it is one of the most important analytic tools to emerge in studying the measure and dimension of Julia sets. Yet it remains one of the more difficult and subtle chapters of modern Complex Dynamics, still imperfectly understood and in many ways mysterious.

Indeed, even the definition of parabolic renormalization is quite complicated. Skipping all of the (important) details, we attempt to summarize it below, as follows. Start with a simple parabolic germ of an analytic function at the origin of the form

$$f(z) = z + a_2 z^2 + \sum_{n\geq 3} a_n z^n, \quad \text{with } a_2 \neq 0.$$

Using a linear change of coordinates we assure without loss of generality that $a_2 = 1$. The classical Leau-Fatou flower theorem (presented with great care in e.g. [Mil1]) describes the local dynamics of f near the origin as follows. There exists a topological disk P_A, known as an attracting petal of f, such that $f(P_A) \subset P_A \cup \{0\}$, and the iterates $f^n(z)$ converge to 0 uniformly for $z \in P_A$. Moreover, every orbit of f which converges to 0 eventually lands in P_A. A repelling petal P_R of f is defined to be an attracting petal for the local branch of f^{-1} which fixes the origin. Together, P_R and P_A form a punctured neighborhood of the origin.

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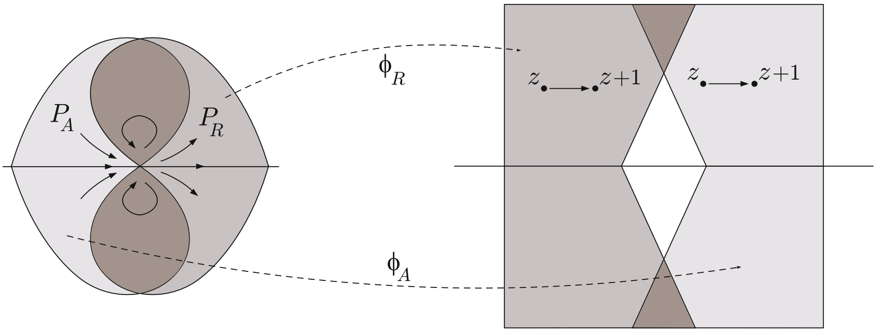

Fig. 1.1 Attracting and repelling petals of a simple parabolic germ of the form $f(z) = z + z^2 + \cdots$, and the corresponding Fatou coordinates

Inside a petal, the dynamics of f can be linearized. There exists a conformal map

$$\phi_A : P_A \to \mathbb{C},$$

which solves the Abel functional equation

$$\phi_A(f(z)) = \phi_A(z) + 1.$$

The map ϕ_A is known as an *attracting Fatou coordinate*; it is defined uniquely up to an addition of a constant. A *repelling Fatou coordinate* ϕ_R is similarly defined as a solution of the Abel equation in a repelling petal. The petals can be chosen so that the image $\phi_A(P_A)$ contains a sector $\{|\mathrm{Arg}(z - C)| < \pi/2 + \varepsilon\}$ for a sufficiently large $C > 0$ and some $\varepsilon > 0$, and similarly the image $\phi_R(P_R)$ contains a sector $\{|\mathrm{Arg}(z + C)| > \pi/2 - \varepsilon\}$ (Fig. 1.1).

By design, the composition $\phi_A \circ (\phi_R)^{-1}$ commutes with the unit translation $z \mapsto z + 1$. Using this fact, it is not difficult to see that it is defined for all z with $|\mathrm{Im} z| > M$ for a sufficiently large value of M. If we denote

$$\mathrm{ixp}(z) \equiv \exp(2\pi i z),$$

then the composition

$$\mathbf{h} \equiv \mathrm{ixp} \circ \phi_A \circ (\phi_R)^{-1} \circ \mathrm{ixp}^{-1} \tag{1.1}$$

defines a pair of analytic maps h^+, h^- defined in punctured neighborhoods of 0 and ∞ respectively. These maps have removable singularities at 0 and ∞, and $h^+(0) = 0$, $h^-(\infty) = \infty$. The multipliers of the fixed points 0 and ∞ are both nonzero.

The pair of analytic germs of h^+ at 0 and h^- at ∞ is Voronin's form of the Écalle-Voronin conformal conjugacy invariant of f [Ec, Vor]. These germs are not quite uniquely defined: a choice of additive constants in the definition of ϕ_A, ϕ_R induces a pre- and post-composition of $\mathbf{h}$ with multiplications by nonzero complex numbers. Now let us say that f is *renormalizable* if

$$h^+(z) = b_1 z + b_2 z^2 + \cdots$$

with $b_2 \neq 0$. Then there is a unique choice of nonzero constants α, β for which

$$\alpha h^+(\beta z) = z + z^2 + \cdots . \tag{1.2}$$

We call the germ (1.2) the *parabolic renormalization* of f.

The term *renormalization* has an established meaning in dynamics: it stands for a rescaled first return map. Although it is not at all obvious from the above description, parabolic renormalization can be interpreted as a limiting case of an appropriately conformally rescaled renormalization of almost parabolic germs (see e.g. [IS]).

In [IS], Inou and Shishikura demonstrated that the successive parabolic renormalizations $\mathscr{P}^n(f_0)$ of the quadratic polynomial $f_0(z) = z + z^2$ converge to an analytic map f_* defined in a neighborhood of the origin, which satisfies the fixed point equation

$$\mathscr{P}(f_*) = f_*. \tag{1.3}$$

They proved that in a suitably restricted class of maps, f_* is a globally attracting fixed point of $\mathscr{P}$.

We note that in general the Eq. (1.3) has many different solutions. Indeed, Schäfke (private communication) has recently described a nondynamical construction of fixed points of the operator $\mathscr{P}$ with an arbitrarily specified h^-. However, if our germ f extends to an analytic map with nice global covering properties, then, generically, its renormalizations will converge to f_*.

This work grew out of our efforts to provide a natural geometric description for the class of maps invariant under $\mathscr{P}$ (and, in particular, for the fixed point f_* itself), and to carry out a computer-assisted study of f_* and $\mathscr{P}$. We describe a natural class of analytic maps $\mathbf{P}_0$ which have a maximal analytic extension to a Jordan domain satisfying the invariance property

$$\mathscr{P} : \mathbf{P}_0 \to \mathbf{P}_0.$$

The covering properties of a map $f \in \mathbf{P}_0$ admit an explicit topological model, which we describe in some detail. We prove that the Inou-Shishikura fixed point f_* of $\mathscr{P}$ is contained in $\mathbf{P}_0$, and use the convergence result of [IS] to show that successive renormalizations of any map $f \in \mathbf{P}_0$ converge to f_*.

When it comes to a numerical study of the action of $\mathscr{P}$, one encounters an immediate challenge: estimating the attracting and repelling Fatou coordinates of an analytic germ in the definition (1.1) with sufficient precision. We approach their computation from a new angle, utilizing an asymptotic series for a Fatou coordinate. The existence of such an asymptotic series has at least in some cases been known from the work of Écalle [Ec] on Resurgence Theory. We give an elementary analytic proof of this fact in the general case. We then use the asymptotic series to design a computational scheme for $\mathscr{P}$, and use it to compute with high accuracy Taylor's expansion of f_*, as well as to compute the boundary of its maximal domain of analyticity. We also produce an explicit estimate of the spectral radius of the linearization $D\mathscr{P}|_{f_*}$ in a suitable Banach ball.

We have strived to make our exposition self-contained. In Chap. 2 the reader will find a detailed exposition of the local theory of simple parabolic germs. In addition to standard material, in Sect. 2.2 we present a proof of the existence of an asymptotic series of the Fatou coordinate at infinity, which plays a key role in our numerical experiments. We make a brief note of the role this series plays in Écalle's Resurgence Theory for Fatou coordinates in Sect. 2.2.1. In Sect. 2.3, after a discussion of Écalle-Voronin conformal conjugacy invariants, the parabolic renormalization operator $\mathscr{P}$ makes its first appearance. Chapter 3 discusses global properties of parabolic renormalization, starting with a detailed discussion of parabolic renormalization of the quadratic polynomial $f_0(z) = z + z^2$ in Sect. 3.2. We define the class $\mathbf{P}_0$ in Sect. 3.5. Section 3.6 contains the core of the proof of invariance of $\mathbf{P}_0$ under the action of $\mathscr{P}$. In Sect. 3.7 we use the results of Inou and Shishikura to show that parabolic renormalizations of a map in $\mathbf{P}_0$ converge to the Inou-Shishikura fixed point f_*. Chapter 4 contains a numerical study of the parabolic renormalization operator and its fixed point f_*. Here we make use of the asymptotic series for the Fatou coordinates of a parabolic germ. Some amusing examples are left for dessert in Chap. 5.

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