Finite or Infinite Number of Solutions of Polynomial Congruences in Two Positive Integer Variables

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1 Basic Definitions, Formulation of the Problem, and Statement of Result

Assume that x, y are two variables taking positive integral values.

Also assume that $f(x), g(y)$ are polynomials with nonnegative and integral coefficients and thus taking nonnegative integer function values for positive integer values of the variables.

Now let $h = \deg(f(x))$ and $k = \deg(g(y))$ be the degrees of these polynomials. The *polynomial system of congruences* is the system

$$\begin{aligned} g(y) &\equiv 0 \mod x \\ f(x) &\equiv 0 \mod y\ . \end{aligned} \tag{1}$$

Immediately the question arises whether there can be an infinity of (positive integral distinct) solutions (x, y) for this system. To make this question meaningful we want to assume that $f(0) > 0$ and $g(0) > 0$ hold. We can almost completely answer this question under this assumption.

Although this problem may be difficult to decide (or at least difficult to prove) for certain specific systems, we can provide an easy answer in terms of the pair of degrees (h, k) of the polynomials as given above. In the following we only consider the case of nonconstant polynomials as it is clear that in case one of the polynomials above, $f(x)$ say, is a positive integer constant, then there are only finitely many

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B. Heim et al. (eds.), *Automorphic Forms*, Springer Proceedings
in Mathematics & Statistics 115, DOI 10.1007/978-3-319-11352-4_2

positive y dividing that constant and then there are only finitely values $g(y)$ and only finitely many values x. Let $\mathcal{L}_{fin}$ be the set of pairs $(h, k) \in \mathbb{N} \times \mathbb{N}$ such that each pair of positive polynomials $f(x), g(y)$ with the said degrees and with $f(0), g(0) > 0$ has only finitely many positive integer solutions. Thus we have

$$\mathcal{L}_{inf} = \{(h, k) : \exists f(x), g(y);\ \ h = \deg f(x),\ k = \deg g(y) \text{ and} \tag{2}$$

$$\text{the system (1) has infinitely many positive solutions.}\ \}$$

and its complement is

$$\mathcal{L}_{fin} = \mathbb{N} \times \mathbb{N} \backslash \mathcal{L}_{inf}.$$

We shall prove that

Theorem 1.

$$\mathcal{L}_{fin} = \{(1, 1), (1, 2), (1, 3), (2, 1), (3, 1)\}\ \ .$$

2 The Examples

Example 1. Let f_n be the sequence of Fibonacci numbers. Then it is well known that for $n \geq 3$

$$f_{n+2} f_{n-2} - f_n^2 = (-1)^{n-1}.$$

By shifting indices up by two we see that for odd n any pair $(x, y) = (f_n, f_{n+2})$ satisfies the system

$$y^2 + 1 \equiv 0 \mod x \quad \text{and} \qquad x^2 + 1 \mod y \equiv 0 \mod y. \tag{3}$$

Thus (3) has infinitely many positive solutions.

This example shows that $(2, 2) \in \mathcal{L}_{inf}$.

We can now generalize this example. Assume that $k, h > 0$ are fixed integers and that a_0, a_1, b_0, b_1 are four positive integers satisfying the two equations

$$a_0 a_1 = b_1^h + 1\ ,\ \ b_0 b_1 = a_0^k + 1\ . \tag{4}$$

We easily see that b_1 divides $a_1^k + 1$. Indeed

$$(a_0 a_1)^k = (b_1^h + 1)^k = b_1^h \cdot X_0 + 1$$

for some positive integer X_0 coming out of the binomial expansion of $(b_1^h + 1)^k$. Thus

$$a_0^k a_1^k + a_0^k = b_1^h \cdot X_0 + a_0^k + 1 = b_1^h \cdot X_0 + b_1 b_0 = b_1(b_1^{h-1} X_0 + b_0).$$

Assume now that (a priori rational) numbers a_2, b_2 are defined by

$$b_1 b_2 = a_1^k + 1 \, , a_1 a_2 = b_2^h + 1 \, . \tag{5}$$

First note that from the above divisibility $b_1 \mid (a_1^k + 1)$ we get that b_2 is an integer. Then we have from the second equation in the system (4) that

$$(b_1 b_2)^h = (a_1^k + 1)^h = a_1^k \cdot X_1 + 1$$

for some positive integer X_1 that comes out of the binomial expansion of $(a_1^k + 1)^h$. Thus from the first equation in the system (4)

$$b_1^h b_2^h + b_1^h = a_1^k X_1 + b_1^h + 1 = a_1^k X_1 + a_0 a_1 = a_1 \cdot Y_1$$

for some positive integer Y_1.

This implies that a_1 divides the product $b_1^h(b_2^h + 1)$. But from the first part of (4) we get $\gcd(a_1, b_1) = 1$. Thus a_1 divides $b_2^h + 1$. Therefore also a_2 is an integer.

We also may verify the divisibility property b_2 divides $a_2^k + 1$ for this step. From the second equation of (5) we get

$$(a_1 a_2)^k = (b_2^h + 1)^k = b_2^h \cdot Z_1 + 1$$

for some positive integer Z_1 coming from the corresponding binomial expansion. Now

$$a_1^k a_2^k + a_1^k = b_2^h Z_1 + a_1^k + 1 = b_2 \cdot (b_2^{h-1} Z_1 + b_1)$$

and hence b_2 divides $a_1^k(a_2^k + 1)$. As again $\gcd(a_1, b_2) = 1$ this shows that b_2 divides $a_2^k + 1$.

We now assume in general that we have given integers $a_0, a_1, \ldots, a_n$ and $b_0, b_1, \ldots, b_n$ satisfying the equations

$$a_i a_{i-1} = b_i^h + 1 \, , \;\; b_j b_{j+1} = a_j^k + 1 \, , \tag{6}$$

for $i = 0, 1, \ldots, n$ and for $j = 0, 1, \ldots, n-1$ and the divisibility relation b_n divides $a_n^k + 1$.

Then let b_{n+1}, a_{n+1} be defined for $n \in \mathbb{Z}$ by

$$b_n b_{n+1} = a_n^k + 1 \, , \; a_n a_{n+1} = b_{n+1}^h + 1 \, . \tag{7}$$

Then we can show that also a_{n+1}, b_{n+1} are actually integers.

First note that from the above divisibility that b_{n+1} is an integer.

Then we have from the second equation in the system (7) that

$$(b_n b_{n+1})^h = (a_n^k + 1)^h = a_n^k \cdot X_n + 1$$

for some positive integer X_n that comes out of the binomial expansion of $(a_n^k + 1)^h$. Thus from the first equation in the system (7)

$$b_n^h b_{n+1}^h + b_n^h = a_n^k X_n + b_n^h + 1 = a_n^k X_n + a_{n-1} a_n = a_n \cdot Y_n$$

for some positive integer Y_n. Thus a_n divides the product $b_n^h (b_{n+1}^h + 1)$. Now as $\gcd(a_n, b_n) = 1$ by the first equation in (7) we obtain that a_n divides $b_{n+1}^h + 1$. This then shows via the second equation of (7) that also a_{n+1} is an integer.

We also may verify the divisibility property b_{n+1} divides $a_{n+1}^k + 1$ for this step. From the second equation of (7) we get

$$(a_n a_{n+1})^k = (b_{n+1}^h + 1)^k = b_{n+1}^h \cdot Z_n + 1$$

for some positive integer Z_n coming from the corresponding binomial expansion. Now

$$a_n^k a_{n+1}^k + a_n^k = b_{n+1}^h Z_n + a_n^k + 1 = b_{n+1} \cdot (b_{n+1}^{h-1} Z_n + b_n)$$

and hence b_{n+1} divides $a_n^k (a_{n+1}^k + 1)$. As again $\gcd(a_n, b_{n+1}) = 1$ this shows that b_{n+1} divides $a_{n+1}^k + 1$.

This completes the verification of the induction step.

We have proved that all the numbers a_n, b_n are positive integers for $n \geq 2$ once the four numbers a_0, a_1, b_0, b_1 are positive integers satisfying (4).

It remains to show that for all $h, k > 0$ there exists at least one such example. But this is trivial; we may take $a_0 = 1, a_1 = 2$ and $b_0 = 2, b_1 = 1$.

We may also continue the process downward obtaining $a_{-1}, b_{-1}, a_{-2}, b_{-2}$, etc., thus obtaining two 2−way sequences a_n, b_n with $n \in \mathbb{Z}$.

This starts from the condition that a_0 divides $b_0^h + 1$. Details are exactly the same as above and may be omitted.

Pompe [3] observed the special case $h = k$ of this example. He introduced the sequences A003818, A003819, and A 003820 in the online encyclopedia of integer sequences which are the same as ours for $h = k$.

Floor v Lamoen [3] has obtained (and he characterized) in A002310 the sequence a_n of the special case $k = 4, h = 1$.

3 Existence Results

3.1 Existence Part of Theorem 1

We note that the construction in the previous section provides positive polynomials with strictly positive constant coefficients. Thus if we take a pair $(f(x) = x^h + 1, g(y) = y^k + 1)$ of such polynomials with degree pair (h, k), we need only to inquire whether the existing sequences $a_n,\ b_n$ for the initial values $a_0 = 1, b_0 = 2$, $a_1 = 2, b_1 = 1$ are infinite or not.

For the existence result of infinitely many solutions it is enough to show that for all pairs $(h, k) \neq (1, 1), (1, 2), (2, 1), (1, 3), (3, 1)$ the sequence a_n is eventually strictly increasing.

We first consider the case $h, k \geq 2$. We first check that $a_1 = 2, b_2 = 2^k + 1, a_2 = \frac{(2^k+1)^h+1}{2}$. Thus it holds that $a_1 < b_2 < a_2$. Now make the inductive assumption $a_{n-1} < b_n < a_n$. Then we compute

$$\frac{b_{n+1}}{a_n} = \frac{a_n^k + 1}{a_n b_n} > \frac{a_n^k}{a_n b_n} = \frac{a_n^{k-1}}{b_n} \geq \frac{a_n}{b_n} > 1\,. \tag{8}$$

Similarly by using this inequality just proved we get

$$\frac{a_{n+1}}{b_{n+1}} = \frac{b_{n+1}^h + 1}{a_n b_{n+1}} > \frac{b_{n+1}^h}{a_n b_{n+1}} = \frac{b_{n+1}^{h-1}}{a_n} \geq \frac{b_{n+1}}{a_n} > 1\,. \tag{9}$$

Of course in the first case we have used $k \geq 2$ and in the second case $h \geq 2$.

Now consider the case when one of the numbers h, k is 1. Without loss of generality we may assume that $k \geq h = 1$.

In this case we use as inductive assumption $a_{n-1}^2 < b_n < a_n^2$. This is true for $n = 2$ and $k \geq 4$ as $a_1 = 2, b_2 = 2^k + 1, a_2 = 2^{k-1} + 1$ holds. For the induction step

$$\frac{b_{n+1}}{a_n^2} = \frac{a_n^k + 1}{a_n^2 b_n} > \frac{a_n^k}{a_n^2 b_n} = \frac{a_n^{k-2}}{b_n} \geq \frac{a_n^2}{b_n} > 1\,. \tag{10}$$

Using the inequality just proved we also get

$$\frac{a_{n+1}^2}{b_{n+1}} = \frac{(b_{n+1} + 1)^2}{a_n^2 b_{n+1}} > \frac{b_{n+1}^2}{a_n^2 b_{n+1}} = \frac{b_{n+1}}{a_n^2} > 1\,. \tag{11}$$

Of course in the first case we have used $k \geq 4$.

In either case we have that a_n is a strictly increasing and thus infinite sequence of positive integers. Thus in all cases except for
$(h, k) = (1, 1), (1, 2), (1, 3), (2, 1), (3, 1)$ there exist polynomials with an infinity of solutions for the system (1).

We note that the above sequences for the cases $(h, k) = (1, 1), (1, 2), (2, 1), (1, 3), (3, 1)$ become periodic and thus provide only a finite number of solutions.

3.2 A More General Existence Result in the Biquadratic Case

We describe now another more general existence result (without proof) for the biquadratic case. Let $s, t \geq 1$ be integers with $s + t \geq 2$ to avoid trivialities.

Then the system

$$x^2 + st - 1 \equiv 0 \mod y \tag{12}$$

$$y^2 + t \equiv 0 \mod x \tag{13}$$

has infinitely many solutions in the form of a pair of two-way double recursion sequences with initial values

$$(x_0, y_0) = (t + 1, t);\ (x_1, y_1) = (s^2 t + 1, st)$$

and with recursion

$$x_{n+1} = \theta \cdot x_n - x_{n-1},\ y_{n+1} = \theta \cdot y_n - y_{n-1},$$

where $\theta = s^2 t - 2st + t + 2 = t(s - 1)^2 + 2$.

4 Finiteness Results

4.1 The Linear Case

In this section we use matrix notation.

It was shown by the author in his University of Malaya (KL) lecture notes [1] that for positive integers a_{12}, a_{21}, b_1, b_2 the system

$$a_{12} x_2 + b_1 = d_1 x_1 \tag{14}$$

$$a_{21} x_1 + b_2 = d_2 x_2 \tag{15}$$

for positive integers d_1, d_2 has only finitely many positive solutions. Some estimates for the number of solutions were also given.

Here we sketch the argument of [1]. We first can cross multiply with d_2, d_1, respectively. This leads to the system

$$(d_1 d_2 - a_{12}a_{21}) \cdot x_2 = a_{21}b_1 + d_1 b_2 \tag{16}$$

$$(d_1 d_2 - a_{12}a_{21}) \cdot x_1 = a_{12}b_2 + d_2 b_1 \ . \tag{17}$$

Put $\Delta = d_1 d_2 - a_{12}a_{21}$. If $\Delta \leq 0$, then there are no positive solutions, so we may assume that Δ is a positive integer. We shorten the notation to

$$\Delta \cdot x_2 = a_{21}b_1 + d_1 b_2 \tag{18}$$

$$\Delta \cdot x_1 = a_{12}b_2 + d_2 b_1 \tag{19}$$

We now derive a tight bound on the size of x_1 and x_2.

Proposition 1. *For any positive solution of (14) and (15) we have*

$$x_1 \leq a_{12}b_2 + b_1(1 + a_{12}a_{21}) \ . \tag{20}$$

$$x_2 \leq a_{21}b_1 + b_2(1 + a_{12}a_{21}) \ . \tag{21}$$

By symmetry we only need to prove (20). Consider the quantity $x_1 - a_{12}b_2$. If it is not positive, then $x_1 \leq a_{12}b_2$ and (20) holds with strict inequality.
Hence we may assume that $x_1 - a_{12}b_2 > 0$ is a positive integer. Then we get

$$x_1 - a_{12}b_2 \leq d_1(x_1 - a_{12}b_2) \text{ and using (14)} \tag{22}$$

$$= a_{12}x_2 + b_1 - a_{12}d_1 b_2 \tag{23}$$

$$\leq a_{12}\Delta x_2 + b_1 - a_{12}d_1 b_2 \text{ and using (18)} \tag{24}$$

$$= a_{12}a_{21}b_1 + a_{12}b_2 d_1 + b_1 - a_{12}d_1 b_2 \tag{25}$$

$$= b_1(1 + a_{12}a_{21}). \tag{26}$$

This shows (20). The inequality (21) follows by symmetry.

We note that the case of equality occurs in (20) iff both $d_1 = 1$ and $\Delta = 1$. Hence if $x_1 = a_{12}b_2 + b_1(1 + a_{12}a_{21})$, then by (18) we get $x_2 = a_{21}b_1 + b_2$. This then is the solution of (14) and (15) with maximal x_1. Similarly for $d_2 = 1$ and $\Delta = 1$ there is the solution $x_2 = a_{21}b_1 + b_2(1 + a_{12}a_{21})$ and $x_1 = a_{12}b_2 + b_1$ with maximal x_2. We call these two solutions the *two maximal* solutions.

We now assume more generally that only $\Delta = 1$ holds. Then d_1, d_2 are a pair of complementary divisors of $1 + a_{12}a_{21}$. For any such pair of divisors by (18) and (19) we obtain the solutions

$$x_1 = a_{12}b_2 + d_2 b_1, \ x_2 = a_{21}b_1 + d_1 b_2 \ . \tag{27}$$

For an integer m let $\tau(m)$ be the number of positive divisors of m. We see that there are $\tau(1 + a_{12}a_{21})$ distinct solutions to (14) and (15). We refer to these solutions as the *large* $\Delta = 1$ *solutions.*

The statement given is also valid if one of b_1, b_2 (but not both!) is zero.

From that a certain classification of the solutions can be obtained. Details are omitted.

4.2 The Semiquadratic Case

Here we show that for nonnegative integers a, b, c, d, e with $c > 0, e > 0$ and $a > 0, d > 0$ the system

$$ax^2 + bx + c \equiv 0 \mod y \quad \text{and} \quad dy + e \equiv 0 \mod x$$

has only finitely many (positive integral) solutions.

Clearly there exist positive integers h, k such that

$$ky = ax^2 + bx + c \tag{28}$$

$$hx = dy + e \tag{29}$$

Since x divides $k(dy + e) = dky + ke = adx^2 + bdx + cd + ke$, there exists a positive integer m with $mx = cd + ke$.

Now consider the positive integral expression

$$(mx - cd)hxy = e(ax^2 + bx + c)(dy + e) =$$

$$ehkxy = adex^2y + ae^2x^2 + bdexy + be^2x + cdey + ce^2. \tag{30}$$

Note first that from

$$(mh - ade)x^2y = cdhxy + bdexy + ae^2x^2 + be^2x + cdey + ce^2 > 0$$

it follows that $mh > ade$.

The above expression (30) also gives

$$(mh - ade)x^2y - (cdh + bde)xy - ae^2x^2 - be^2x - cdey = ce^2\ .$$

Since $x \geq 1$ we get

$$(mh - ade)x^2y - d(ch + be)xy - e^2(a + b)x^2 - cdexy \leq ce^2\ ,$$

and hence

$$x \cdot ((mh - ade)xy - d(ch + be)y - e^2(a + b)x - cdey) \leq ce^2\ . \tag{31}$$

If there exists an unbounded (in x strictly increasing) sequence of solutions $x_i \mapsto +\infty, y_i \mapsto +\infty$, then clearly y_i also becomes unbounded and we must have that for such solutions $(x, y) = (x_i, y_i)$ eventually (for all but finitely many values of i)

$$(mh - ade)xy - d(ch + be + ce)y - e^2(a + b)x \le 0\ . \tag{32}$$

But then introducing the nonnegative rational bounded constants

$$A = \frac{d(ch + be + ce)}{mh - ade} \tag{33}$$

$$B = \frac{e^2(a + b)}{mh - ade} > 0 \tag{34}$$

it follows that

$$xy - Ax - By + AB = (x - B)(y - A) \le AB.$$

This contradicts the fact that (x_i, y_i) are an unbounded sequence.

Thus the set of positive integer solutions is bounded and hence it is finite. We thus have established the following result.

Proposition 2. *For nonnegative integers $a, b, c, d, e \ge 0$ with $a > 0, d > 0,\ c > 0, e > 0$ the system*

$$ax^2 + bx + c \equiv 0 \mod y \quad \textit{and} \quad dy + e \equiv 0 \mod x$$

has only finitely many positive integral solutions (x, y).

4.3 The Semicubic Case

In this part we show that for positive integers $a, b, c, d > 0$ and for integers $e, f \ge 0$ and positive integral $x,\ y$ the system

$$ay + b \equiv 0 \mod x \tag{35}$$

$$cx^3 + ex^2 + fx + d \equiv 0 \mod y \tag{36}$$

has only finitely many positive integer solution pairs (x, y).

We first need some notation. Whenever (x, y) is a (positive) solution of (35) and (36), then we may define positive integers h, k (depending on x, y) by

$$hx = ay + b \tag{37}$$

$$ky = cx^3 + ex^2 + fx + d\ . \tag{38}$$

There are four further positive integral quantities (all depending on x, y) called *divisants* and generically denoted by Δ which are associated to this situation. In the following lemma we derive some of these quantities and analyze their respective relationships.

Lemma 1. *There exist positive integers Δ_m, Δ_n such that*

$$\Delta_m \cdot x = ad + bk\ ; \tag{39}$$

$$\Delta_n \cdot y = b^3c + b^2eh + bfh^2 + dh^3\ .\ . \tag{40}$$

For the proof of (39) multiply (37) by k and use (38) to get

$$hkx = (ay + b)k = acx^3 + aex^2 + afx + ad + bk,$$

and after subtracting we see that

$$x(hk - acx^2 - aex - af) = ad + bk,$$

so that we may choose

$$\Delta_m = hk - acx^2 - aex - af. \tag{41}$$

For the proof of (40) multiply (38) by h^3 and use (37) to get

$$\begin{aligned} kh^3y &= h^3(cx^3 + ex^2 + fx + d) \\ &= c(ay + b)^3 + eh(ay + b)^2 + fh^2(ay + b) + dh^3 \\ &= y(a^3cy^2 + 3a^2bcy + 3ab^2c + a^2ehy + 2abeh + afh^2) + b^3c + b^2eh + bfh^2 + dh^3. \end{aligned}$$

After subtracting the bracket we get

$$(kh^3 - a^3cy^2 - 3a^2bcy - 3ab^2c - a^2ehy - 2abeh - afh^2)y = b^3c + b^2eh + bfh^2 + dh^3,$$

so that we may choose

$$\Delta_n = h^3k - a^3yc^2 - 3a^2bcy - 3ab^2c - a^2ehy - 2abeh - afh^2\ . \tag{42}$$

This completes the proof of Lemma 1.

Lemma 2. *There exists a positive integer Δ_s such that*

$$\Delta_s \cdot h = \Delta_n + ab^2c. \tag{43}$$

Use the number (42) and add ab^2c and then simplify with (38):

$$\Delta_n + ab^2c = h(h^2k - a^2ey - 2abe - afh) - ac(a^2y^2 - b^2) - 3abc(ay + b)$$

$$= h(h^2k - a^2ey - 2abe - afh) - achx(ay - b) - 3abchx$$
$$= h(h^2k - a^2ey - 2abe - afh - a^2cxy - 2abcx)\ ;$$

so that we may choose

$$\Delta_s = h^2k - a^2ey - 2abe - afh - a^2cxy - 2abcx\ . \tag{44}$$

This proves Lemma 2.

Lemma 3. *The two product equations involving Δ_s give*

$$\Delta_s x = b\Delta_m + abf + adh\ , \tag{45}$$
$$\Delta_s y = b^2e + bfh + dh^2 + b^2cx\ . \tag{46}$$

First we prove (46) by multiplying (43) by h and using (40)

$$\begin{aligned}\Delta_s hy &= \Delta_n y + ab^2cy \\ &= b^3c + b^2eh + bfh^2 + dh^3 + ab^2cy \\ &= +b^2eh + bfh^2 + dh^3 + b^2c(b + ay) \\ &= +b^2eh + bfh^2 + dh^3 + b^2chx \\ &= h(b^2e + bfh + dh^2 + b^2cx)\end{aligned}$$

so by cancelling h we get (46).

We now use (46) and multiply it by x to obtain (45):

$$\begin{aligned}\Delta_s xy &= b^2ex + bfhx + dh^2x + b^2cx^2 \\ &= b^2ex + b^2cx^2 + bd(ay + b) + ch(ay + b) \\ &= b\Delta_m y + abfy + adhy\ .\end{aligned}$$

Cancelling y we get (45). This proves Lemma 3.

Lemma 4. *The divisants Δ_m, Δ_n also satisfy the equations*

$$\Delta_m y = bcx^2 + bex + dh + bf\ , \tag{47}$$
$$\Delta_n x = b\Delta_s + ab^2e + abfh + adh^2\ . \tag{48}$$

Multiply (39) by y and simplify using (37)

$$\begin{aligned}\Delta_m xy &= ady + bky \\ &= ady + bcx^3 + bex^2 + bfx + bd\end{aligned}$$

$$= bcx^3 + bex^2 + bfx + d(ay + b)$$
$$= x(bcx^2 + bex + bf + dh) .$$

Now cancelling x gives (47).

Similarly multiply (40) by x and simplify using (37) and (46) to get

$$\begin{aligned}\Delta_n xy &= b^3cx + b^2ehx + bfh^2x + dh^3x \\ &= b^3cx + b^3e + b^2fh + bdh^2 + y(ab^2e + abfh + adh^2) \\ &= b\Delta_s y + y(ab^2e + abfh + adh^2)\end{aligned}$$

and cancel y to get (48). This proves Lemma 4.

There is a formula for the product $h \cdot \Delta_m$:

$$h \cdot \Delta_m = \Delta_s + abe + abcx . \tag{49}$$

For the proof we simplify $h\Delta_m - \Delta_s$ as follows. First we use the formula for Δ_m in (41) and then multiply it by h and then in the third line substituting (44) we get

$$\begin{aligned}h\Delta_m - \Delta_s &= h^2k - acx(hx) - ae(hx) - afh - \Delta_s \\ &= h^2k - acx(ay + b) - ae(ay + b) - afh - \Delta_s \\ &= abcx + abe\end{aligned}$$

From the above we obtain an equation involving several of the divisants.

Proposition 3. *The following equation is valid:*

$$(\Delta_s - abc)(x - 1) + (h - b)(\Delta_m - ad) = ab(c + d + e + f) . \tag{50}$$

The proof can be obtained from the above relations (49) and (45):

$$\begin{aligned}(\Delta_s - abc)(x - 1) + (h - b)(\Delta_m - ad) &= \Delta_s x - abcx - \Delta_s + abc + h\Delta_m - b\Delta_m - adh + abd \\ &= b\Delta_m + abf + adh - abcx - \Delta_s + abc \\ &\quad + \Delta_s + abe + abcx - b\Delta_m - adh + abd \\ &= abf + abc + abe + abd .\end{aligned}$$

In order to apply the proposition we need to introduce one more parameter.

Lemma 5. *There exists a positive integer t such that*

$$k \cdot t = a^3cd^3 + a^2d^2e\Delta_m + adf\Delta_m^2 + d\Delta_m^3 . \tag{51}$$

For the proof we use

$$\begin{aligned} ky\Delta_m^3 &= \Delta_m^3(cx^3 + ex^2 + fx + d) \\ &= c(\Delta_m x)^3 + e\Delta_m(\Delta_m x)^2 + f\Delta_m^2(\Delta_m x) + d\Delta_m^3 \\ &= c(ad + bk)^3 + e\Delta_m(ad + bk)^2 + f\Delta_m^2(ad + bk) + d\Delta_m^3 \\ &= k(b^3ck^2 + 3ab^2cdk + 3a^2bcd^2 + b^2ek\Delta_m + 2abde\Delta_m + bf\Delta_m^2) \\ &\quad + a^3cd^3 + a^2d^2e\Delta_m + adf\Delta_m^2 + d\Delta_m^3 . \end{aligned}$$

Now bringing the terms with k to the left-hand side we get

$$\begin{aligned} &k(y\Delta_m^3 - b^3ck^2 - 3ab^2cdk - 3a^2bcd^2 - b^2ek\Delta_m - 2abde\Delta_m - bf\Delta_m^2) \\ &= a^3cd^3 + a^2d^2e\Delta_m + adf\Delta_m^2 + d\Delta_m^3 . \end{aligned}$$

Thus we may take t to be

$$t = y\Delta_m^3 - b^3ck^2 - 3ab^2cdk - 3a^2bcd^2 - b^2ek\Delta_m - 2abde\Delta_m - bf\Delta_m^2 . \tag{52}$$

First we show that

Lemma 6. *The integer d divides the integer t. Thus we may introduce a further divisant as $\Delta_t = \dfrac{t}{d}$.*

Into the expression (52) we substitute the formula (47) to get

$$\begin{aligned} t = \Delta_m^2(bcx^2 + bex + dh + bf) - b^3ck^2 - b^2ek\Delta_m - bf\Delta_m^2 \\ - 2abde\Delta_m - 3ab^2cdk - 3a^2bcd^2 \end{aligned}$$

where the third term in the bracket and the last three terms are already divisible by d. Thus there exists an integer z such that using (39) in the third line below

$$\begin{aligned} t &= d \cdot z + \Delta_m^2(bcx^2 + bex + bf) - b^3ck^2 - b^2ek\Delta_m - bf\Delta_m^2 \\ &= d \cdot z + bc(\Delta_m x)^2 + be\Delta_m(\Delta_m x) + bf\Delta_m^2 - b^3ck^2 - b^2ek\Delta_m - bf\Delta_m^2 \\ &= d \cdot z + bc(ad + bk)^2 + be\Delta_m(ad + bk) - b^3ck^2 - b^2ek\Delta_m \\ &= d \cdot z + d(a^2bcd + 2ab^2k \cdot c + abe\Delta_m) \end{aligned}$$

which proves the lemma.

Since z in the above proof is $z = h\Delta_m^2 - 2abe\Delta_m - 3ab^2ck - 3a^2bcd$ we obtain another expression for t as

$$t = d\cdot(h\Delta_m^2 - 2abe\Delta_m - 3abcdk - 3a^2bcd + a^2bcd + 2ab^2kc + abe\Delta_m)\,. \tag{53}$$

$$\Delta_t = h\Delta_m^2 - 2abe\Delta_m - 3abcdk - 3a^2bcd + a^2bcd + 2ab^2kc + abe\Delta_m\,. \tag{54}$$

Lemma 7. *For the positive integer t and the divisants Δ_m, Δ_s, the following relation holds:*

$$d\cdot\Delta_m\cdot\Delta_s = t + a^2bcd^2. \tag{55}$$

For the proof of Lemma 7 we use the expression (52) and substitute (47) where we can cancel the term $bf\Delta_m^2$; and then in the third line we cancel further expressions. This gives

$$\begin{aligned} t &= \Delta_m^2x^2bc + be\Delta_m^2x + d\Delta_m^2h - b^3ck^2 - 3ab^2cdk - 3a^2bcd^2 - b^2ek\Delta_m - 2abde\Delta_m \\ &= bc(\Delta_m x)^2 + be\Delta_m(\Delta_m x) + d\Delta_m(\Delta_s + abe + abcx) - b^2ek\Delta_m - 2abde\Delta_m \\ &\quad -b^3ck^2 - 3ab^2cdk - 3a^2bcd^2 \\ &= bc(ad+bk)^2 + be\Delta_m(ad+bk) + d\Delta_m\Delta_s + abcd\Delta_m x - b^2ek\Delta_m - abde\Delta_m \\ &\quad -b^3ck^2 - 3ab^2cdk - 3a^2bcd^2 \\ &= a^2bcd^2 + 2ab^2cdk + d\Delta_m\Delta_s + abcd(ad+bk) - 3ab^2cdk - 3a^2bcd^2 \\ &= +d\Delta_m\Delta_s - a^2bcd^2 \ . \ \text{This proves (55)}. \end{aligned}$$

Next we give the product formula for $h\cdot t$ in terms of Δ_s and Δ_m :

$$h\cdot t = d\Delta_s^2 + abde\Delta_s + ab^2cd\Delta_m + a^2b^2cdf \ . \tag{56}$$

For the proof of (56) we combine (55) with (49) and (45):

$$\begin{aligned} ht &= dh\Delta_m\Delta_s - a^2bcd^2h \\ &= d\Delta_s(\Delta_s + abe + abcx) - a^2bcd^2h \\ &= d\Delta_s^2 + abde\Delta_s + abcd\Delta_s x - a^2bcd^2h \\ &= d\Delta_s^2 + abde\Delta_s + ab^2cd\Delta_m + a^2b^2cdf \ . \ \text{This proves (56)}. \end{aligned}$$

Now we come to the crucial formula which shows that the product $\Delta_n\cdot t$ is a cubic polynomial function of Δ_s :

$$\Delta_n\cdot t = d\cdot\Delta_s^3 + abde\cdot\Delta_s^2 + a^2b^2cdf\cdot\Delta_s + a^3b^3c^2d^2 \ . \tag{57}$$

For the proof multiply (56) by Δ_s and use (55) again:

$$\begin{aligned} ht\Delta_s &= d\Delta_s^3 + abde\Delta_s^2 + ab^2cd\Delta_s\Delta_m + a^2b^2cdf\Delta_s \\ &= d\Delta_s^3 + abde\Delta_s^2 + a^2b^2cdf\Delta_s + ab^2ct + ab^2c(a^2bcd^2)\ . \end{aligned}$$

Hence by (43)

$$\Delta_n \cdot t = h \cdot \Delta_s \cdot t - ab^2ct = d\Delta_s^3 + abde\Delta_s^2 + a^2b^2cdf\Delta_s + a^3b^3c^2d^2\ , \quad (58)$$

which proves (57). Thus the product of the two divisants Δ_n, Δ_t is

$$\Delta_n\Delta_t = \Delta_s^3 + abe\Delta_s^2 + a^2b^2cf\Delta_s + a^3b^3c^2d\ \ . \quad (59)$$

We remark that in a different interpretation of these results we can make the following integral substitution:

$$x_1 = \Delta_s,\ y_1 = \Delta_n,\ h_1 = h,\ k_1 = \frac{t}{d} = \Delta_t, \quad (60)$$

$$s_1 = a^2b^3c^2x,\ n_1 = a^3b^3c^2y,\ m_1 = ab^2c\Delta_m,\ t_1 = a^3b^9c^3dk\ \ . \quad (61)$$

Then with the constants

$$e_1 = abe, f_1 = a^2b^2cf, d_1 = a^3b^3c^2d, b_1 = ab^2c, c_1 = a_1 = 1 \quad (62)$$

and with $\Delta_{t,1} = b^6ck$ the system $(x_1, y_1, h_1, k_1, m_1, n_1, s_1, \Delta_{t,1})$ also satisfies all the above equations.

In particular the *new* equation corresponding to (38) then is

$$k_1y_1 = x_1^3 + e_1x_1^2 + f_1x_1 + d_1$$

which after multiplication with the constant factor d gives us exactly (57).

From the formulas given above we can now derive the following main theorem.

Theorem 2. *For any integers $a, b, c, d > 0$ and for integers $e, f \geq 0$ the number of solutions of the system (35) and (36) is finite.*

We prove this by contradiction. Assume that (37) and (38) have infinitely many solutions. For each such solution (x_i, y_i) we may derive all the other quantities $h_i, k_i, \Delta_{m,i}, \Delta_{n,i}, \Delta_{s,i}, t_i$ as given above.

Among these solutions we will find a sequence of pairs (x_i, y_i) such that either the first or the second coordinates of this solution sequence is an unbounded sequence of integers $x_i \to \infty$ or $y_i \to \infty$. The equations (37) and (38) then show that both sequences x_i, y_i must be unbounded. Then the associated sequence k_i is unbounded because of (39), and the associated sequence h_i is unbounded because of (40). The associated sequence $\Delta_{m,i}$ is unbounded because of (51). The associated sequence $\Delta_{n,i}$ is unbounded because of (43). Finally the associated sequence $\Delta_{s,i}$ is unbounded because of (57). But then there exists a positive integer i_0 such that for all $i > i_0$ all bracket terms in (50), that is, all of $\Delta_{s,i} - def, x_i - 1, h_i - d, \Delta_{m,i} - ef$, are strictly positive. Thus the left-hand side of (50) becomes positive and as large as we may like for sufficiently large i. But this implies a contradiction as the right-hand side of (50) is a constant. Thus an infinite sequence of solutions cannot exist. This proves the theorem.

We remark that in [2] for a large class of positive polynomials $(p(x), q(x))$ of degrees $(\deg(p), \deg(q)) \neq (1,1), (1,2), (2,1), (1,3), (3,1)$ the system

$$x \mid p(y),\ \ y \mid q(x)$$

was shown to have infinitely many positive integral solutions.

5 Open Problem and Comment

We have recently shown [4] that for any positive integers $b, d > 0$ and for $a = c = 1$ the biquadratic system

$$ax^2 + ex + b \equiv 0 \mod y \text{ and } cy^2 + fy + d \equiv 0 \mod x$$

for $e, f \geq 0$ and for $b = d$ always has infinitely many positive solutions. The general biquadratic case is still open.

We may ask whether there may exist any such polynomial congruence with $(\deg(p(x)), \deg(q(x))) \neq (1,1), (1,2), (1,3), (2,1), (3,1)$ which has only finitely many solutions.

Acknowledgements I should like to thank the referee for useful comments on a previous version of this text. My particular gratitude goes to Prof Bernhard Heim for the opportunity to contribute to the wonderful Oman conference of GUTech in 2012, for his untiring commitment before, during and after this conference, and for his patience with my efforts during the preparation of this paper.

References

1. T. Bier, *Non Standard Topics in Number Theory: Division Systems*. Lecture Notes (University of Malaysia, KL, Malaysia, 2004)
2. T. Bier, O. Dira, Construction of integer sequences (submitted)
3. Online Encyclopedia of Integer Sequences. Sequences A 002310, A 002320 (contributed by C. Kimberling) and A003818 ff (contributed by W Pompe) in the website www.oeis.org
4. T. Bier, Classifications of solutions of certain biquadratic division systems (submitted)

http://www.springer.com/978-3-319-11351-7

Automorphic Forms
Research in Number Theory from Oman
Heim, B.; Al-Baali, M.; Ibukiyama, T.; Rupp, F. (Eds.)
2014, XIV, 242 p. 9 illus., 3 illus. in color., Hardcover
ISBN: 978-3-319-11351-7