

# Contents

*Preface for the Instructor* xi

*Preface for the Student* xv

*Acknowledgments* xvii

## 1 Vector Spaces 1

### 1.A $\mathbf{R}^n$ and $\mathbf{C}^n$ 2

Complex Numbers 2

Lists 5

$\mathbf{F}^n$  6

Digression on Fields 10

Exercises 1.A 11

### 1.B Definition of Vector Space 12

Exercises 1.B 17

### 1.C Subspaces 18

Sums of Subspaces 20

Direct Sums 21

Exercises 1.C 24

## 2 Finite-Dimensional Vector Spaces 27

### 2.A Span and Linear Independence 28

Linear Combinations and Span 28

Linear Independence 32

Exercises 2.A 37

2.B Bases **39**

Exercises 2.B **43**

2.C Dimension **44**

Exercises 2.C **48**

**3 Linear Maps 51**

3.A The Vector Space of Linear Maps **52**

Definition and Examples of Linear Maps **52**

Algebraic Operations on  $\mathcal{L}(V, W)$  **55**

Exercises 3.A **57**

3.B Null Spaces and Ranges **59**

Null Space and Injectivity **59**

Range and Surjectivity **61**

Fundamental Theorem of Linear Maps **63**

Exercises 3.B **67**

3.C Matrices **70**

Representing a Linear Map by a Matrix **70**

Addition and Scalar Multiplication of Matrices **72**

Matrix Multiplication **74**

Exercises 3.C **78**

3.D Invertibility and Isomorphic Vector Spaces **80**

Invertible Linear Maps **80**

Isomorphic Vector Spaces **82**

Linear Maps Thought of as Matrix Multiplication **84**

Operators **86**

Exercises 3.D **88**

3.E Products and Quotients of Vector Spaces **91**

Products of Vector Spaces **91**

Products and Direct Sums **93**

Quotients of Vector Spaces **94**

Exercises 3.E **98**

**3.F Duality 101**The Dual Space and the Dual Map **101**The Null Space and Range of the Dual of a Linear Map **104**The Matrix of the Dual of a Linear Map **109**The Rank of a Matrix **111**Exercises 3.F **113****4 Polynomials 117**Complex Conjugate and Absolute Value **118**Uniqueness of Coefficients for Polynomials **120**The Division Algorithm for Polynomials **121**Zeros of Polynomials **122**Factorization of Polynomials over  $\mathbf{C}$  **123**Factorization of Polynomials over  $\mathbf{R}$  **126**Exercises 4 **129****5 Eigenvalues, Eigenvectors, and Invariant Subspaces 131****5.A Invariant Subspaces 132**Eigenvalues and Eigenvectors **133**Restriction and Quotient Operators **137**Exercises 5.A **138****5.B Eigenvectors and Upper-Triangular Matrices 143**Polynomials Applied to Operators **143**Existence of Eigenvalues **145**Upper-Triangular Matrices **146**Exercises 5.B **153****5.C Eigenspaces and Diagonal Matrices 155**Exercises 5.C **160****6 Inner Product Spaces 163****6.A Inner Products and Norms 164**Inner Products **164**Norms **168**Exercises 6.A **175**

6.B Orthonormal Bases **180**

Linear Functionals on Inner Product Spaces **187**

Exercises 6.B **189**

6.C Orthogonal Complements and Minimization Problems **193**

Orthogonal Complements **193**

Minimization Problems **198**

Exercises 6.C **201**

**7 Operators on Inner Product Spaces 203**

7.A Self-Adjoint and Normal Operators **204**

Adjoints **204**

Self-Adjoint Operators **209**

Normal Operators **212**

Exercises 7.A **214**

7.B The Spectral Theorem **217**

The Complex Spectral Theorem **217**

The Real Spectral Theorem **219**

Exercises 7.B **223**

7.C Positive Operators and Isometries **225**

Positive Operators **225**

Isometries **228**

Exercises 7.C **231**

7.D Polar Decomposition and Singular Value Decomposition **233**

Polar Decomposition **233**

Singular Value Decomposition **236**

Exercises 7.D **238**

**8 Operators on Complex Vector Spaces 241**

8.A Generalized Eigenvectors and Nilpotent Operators **242**

Null Spaces of Powers of an Operator **242**

Generalized Eigenvectors **244**

Nilpotent Operators **248**

Exercises 8.A **249**

8.B Decomposition of an Operator	<b>252</b>
Description of Operators on Complex Vector Spaces	<b>252</b>
Multiplicity of an Eigenvalue	<b>254</b>
Block Diagonal Matrices	<b>255</b>
Square Roots	<b>258</b>
Exercises 8.B	<b>259</b>

8.C Characteristic and Minimal Polynomials	<b>261</b>
The Cayley–Hamilton Theorem	<b>261</b>
The Minimal Polynomial	<b>262</b>
Exercises 8.C	<b>267</b>

8.D Jordan Form	<b>270</b>
Exercises 8.D	<b>274</b>

## **9 Operators on Real Vector Spaces 275**

9.A Complexification	<b>276</b>
Complexification of a Vector Space	<b>276</b>
Complexification of an Operator	<b>277</b>
The Minimal Polynomial of the Complexification	<b>279</b>
Eigenvalues of the Complexification	<b>280</b>
Characteristic Polynomial of the Complexification	<b>283</b>
Exercises 9.A	<b>285</b>

9.B Operators on Real Inner Product Spaces	<b>287</b>
Normal Operators on Real Inner Product Spaces	<b>287</b>
Isometries on Real Inner Product Spaces	<b>292</b>
Exercises 9.B	<b>294</b>

## **10 Trace and Determinant 295**

10.A Trace	<b>296</b>
Change of Basis	<b>296</b>
Trace: A Connection Between Operators and Matrices	<b>299</b>
Exercises 10.A	<b>304</b>

10.B Determinant **307**

Determinant of an Operator **307**

Determinant of a Matrix **309**

The Sign of the Determinant **320**

Volume **323**

Exercises 10.B **330**

*Photo Credits* **333**

*Symbol Index* **335**

*Index* **337**



<http://www.springer.com/978-3-319-11079-0>

Linear Algebra Done Right

Axler, S.

2015, XVII, 340 p. 26 illus. in color., Hardcover

ISBN: 978-3-319-11079-0