Chapter 2
Intuitionistic Fuzzy IMs

Here, following [11, 16], we extend the concept of IM, introducing the concept of an Intuitionistic Fuzzy IM (IFIM) and Extended IFIM (EIFIM).

2.1 Short Remarks on Intuitionistic Fuzziness

Initially, we give some remarks on Intuitionistic Fuzzy Sets (IFSs, see, e.g., [7, 13]) and especially, of their particular case, Intuitionistic Fuzzy Pairs (IFPs; see [26]). The IFP is an object with the form $\langle a, b\rangle$, where $a, b \in [0, 1]$ and $a + b \leq 1$, that is used as an evaluation of some object or process. Its components (a and b) are interpreted as degrees of membership and non-membership, or degrees of validity and non-validity, or degree of correctness and non-correctness, etc.

Let us have two IFPs $x = \langle a, b\rangle$ and $y = \langle c, d\rangle$.

The following relations have been defined in [26]:

$$\begin{array}{lll} x < y & \text{iff} & a < c \text{ and } b > d \\ x \leq y & \text{iff} & a \leq c \text{ and } b \geq d \\ x = y & \text{iff} & a = c \text{ and } b = d \\ x \geq y & \text{iff} & a \geq c \text{ and } b \leq d \\ x > y & \text{iff} & a > c \text{ and } b < d \end{array}$$

We define analogous of operations "conjunction" and "disjunction":

$$\begin{gathered} x \& y = \langle \min(a, c), \max(b, d)\rangle \\ x \vee y = \langle \max(a, c)), \min(b, d)\rangle \\ x + y = \langle a + c - a.c, b.d\rangle \\ x.y = \langle a.c, b + d - b.d\rangle \\ x@y = \langle \frac{a+c}{2}, \frac{b+d}{2}\rangle. \end{gathered}$$

K.T. Atanassov, *Index Matrices: Towards an Augmented Matrix Calculus*,
Studies in Computational Intelligence 573, DOI 10.1007/978-3-319-10945-9_2

In [13], definitions of 138 operations "implication" and 34 operations "negation" are given. In Table 2.1 the currently existing 45 negations are given. In some of these definitions, we use the functions sg and $\overline{\text{sg}}$ that are defined by:

$$\text{sg}(x) = \begin{cases} 1 \text{ if } x > 0 \\ 0 \text{ if } x \leq 0 \end{cases}, \quad \overline{\text{sg}}(x) = \begin{cases} 0 \text{ if } x > 0 \\ 1 \text{ if } x \leq 0 \end{cases}$$

Let a set E be fixed. An Intuitionistic Fuzzy Set (IFS) A in E is an object of the following form (see, e.g., [7, 13]):

$$A = \{\langle x, \mu_A(x), \nu_A(x)\rangle | x \in E\},$$

where functions $\mu_A \colon E \to [0, 1]$ and $\nu_A \colon E \to [0, 1]$ define the degree of membership and the degree of non-membership of the element $x \in E$, respectively, and for every $x \in E$:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1.$$

2.2 IFIMs and EIFIMs

Extending Sect. 1.1, the basic definition of the IFIM is given.

Let I be a fixed set. By IFIM with index sets K and L ($K, L \subset I$), we denote the object:

$$[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1 & \ldots & l_j & \ldots & l_n \\ \hline k_1 & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1}\rangle & \ldots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j}\rangle & \ldots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n}\rangle \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_i & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1}\rangle & \ldots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle & \ldots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n}\rangle \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_m & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1}\rangle & \ldots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j}\rangle & \ldots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n}\rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$: $0 \leq \mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \leq 1$.

For brevity, we can mention the above object by $[K, L, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle\}]$, where

$$K = \{k_1, k_2, \ldots, k_m\},$$

$$L = \{l_1, l_2, \ldots, l_n\},$$

for $1 \leq i \leq m$, and $1 \leq j \leq n$:

$$\mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0, 1].$$

Table 2.1 The currently existing 45 negations

$\neg_1$	$\langle x, b, a\rangle$
$\neg_2$	$\langle x, \overline{\text{sg}}(a), \text{sg}(a)\rangle$
$\neg_3$	$\langle x, b, a.b + a^2\rangle$
$\neg_4$	$\langle x, b, 1 - b\rangle$
$\neg_5$	$\langle x, \overline{\text{sg}}(1 - b), \text{sg}(1 - b)\rangle$
$\neg_6$	$\langle x, \overline{\text{sg}}(1 - b), \text{sg}(a)\rangle$
$\neg_7$	$\langle x, \overline{\text{sg}}(1 - b), a\rangle$
$\neg_8$	$\langle x, 1 - a, a\rangle$
$\neg_9$	$\langle x, \overline{\text{sg}}(a), a\rangle$
$\neg_{10}$	$\langle x, \overline{\text{sg}}(1 - b), 1 - b\rangle$
$\neg_{11}$	$\langle x, \text{sg}(b), \overline{\text{sg}}(b)\rangle$
$\neg_{12}$	$\langle x, b.(b + a), \min(1, a.(b^2 + a + b.a))\rangle$
$\neg_{13}$	$\langle x, \text{sg}(1 - a), \overline{\text{sg}}(1 - a)\rangle$
$\neg_{14}$	$\langle x, \text{sg}(b), \overline{\text{sg}}(1 - a)\rangle$
$\neg_{15}$	$\langle x, \overline{\text{sg}}(1 - b), \overline{\text{sg}}(1 - a)\rangle$
$\neg_{16}$	$\langle x, \overline{\text{sg}}(a), \overline{\text{sg}}(1 - a)\rangle$
$\neg_{17}$	$\langle x, \overline{\text{sg}}(1 - b), \overline{\text{sg}}(b)\rangle$
$\neg_{18}$	$\langle x, b.\text{sg}(a), a.\text{sg}(b)\rangle$
$\neg_{19}$	$\langle x, b.\text{sg}(a), 0\rangle$
$\neg_{20}$	$\langle x, b, 0\rangle$
$\neg_{21}$	$\langle x, \min(1 - a, \text{sg}(a)), \min(a, \text{sg}(1 - a))\rangle$
$\neg_{22}$	$\langle x, \min(1 - a, \text{sg}(a)), 0\rangle$
$\neg_{23}$	$\langle x, 1 - a, 0\rangle$
$\neg_{24}$	$\langle x, \min(b, \text{sg}(1 - b)), \min(1 - b, \text{sg}(b))\rangle$
$\neg_{25}$	$\langle x, \min(b, \text{sg}(1 - b)), 0\rangle$
$\neg_{26}$	$\langle x, b, a.b + \overline{\text{sg}}(1 - a)\rangle$
$\neg_{27}$	$\langle x, 1 - a, a.(1 - a) + \overline{\text{sg}}(1 - a)\rangle$
$\neg_{28}$	$\langle x, b, (1 - b).b + \overline{\text{sg}}(b)\rangle$
$\neg_{29}$	$\langle x, \max(0, b.a + \overline{\text{sg}}(1 - b)), \min(1, a.(b.a + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(1 - a))\rangle$
$\neg_{30}$	$\langle x, a.b, a.(a.b + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(1 - a)\rangle$
$\neg_{31}$	$\langle x, \max(0, (1 - a).a + \overline{\text{sg}}(a)), \min(1, a.((1 - a).a + \overline{\text{sg}}(a)) + \overline{\text{sg}}(1 - a))\rangle$
$\neg_{32}$	$\langle x, (1 - a).a, a.((1 - a).a + \overline{\text{sg}}(a)) + \overline{\text{sg}}(1 - a)\rangle$
$\neg_{33}$	$\langle x, b.(1 - b) + \overline{\text{sg}}(1 - b), (1 - b).(b.(1 - b) + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(b)\rangle$
$\neg_{34}$	$\langle x, b.(1 - b), (1 - b).(b.(1 - b) + \overline{\text{sg}}(1 - b)) + \overline{\text{sg}}(b)\rangle$
$\neg_{35}$	$\langle \frac{b}{2}, \frac{1+a}{2}\rangle$
$\neg_{36}$	$\langle \frac{b}{3}, \frac{2+a}{3}\rangle$
$\neg_{37}$	$\langle \frac{2b}{3}, \frac{2a+1}{3}\rangle$
$\neg_{38}$	$\langle \frac{1-a}{3}, \frac{2+a}{3}\rangle$
$\neg_{39}$	$\langle \frac{b}{3}, \frac{3-b}{3}\rangle$
$\neg_{40}$	$\langle \frac{2-2a}{3}, \frac{1+2a}{3}\rangle$

(continued)

Table 2.1 (continued)

$\neg_{41}$	$\langle \frac{2b}{3}, \frac{3-2b}{3} \rangle$
$\neg_{42,\lambda}$	$\langle \frac{b+\lambda-1}{2\lambda}, \frac{a+\lambda}{2\lambda}$, where $\lambda \geq 1$
$\neg_{43,\gamma}$	$\langle \frac{b+\gamma}{2\gamma+1}, \frac{a+\gamma}{2\gamma+1}$, where $\gamma \geq 1$
$\neg_{44,\alpha,\beta}$	$\langle \frac{b+\alpha-1}{\alpha+\beta}, \frac{a+\beta}{\alpha+\beta}$, where $\alpha \geq 1, \beta \in [0, \alpha]$
$\neg_{45,\varepsilon,\eta}$	$\langle \min(1, \nu_A(x) + \varepsilon), \max(0, \mu_A(x) - \eta) \rangle$

Now, for above sets K and L, the EIFIM is defined by:

$$[K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$$

$$\equiv \begin{array}{c|ccccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \ldots & l_j, \langle \alpha_j^l, \beta_j^l \rangle & \ldots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \ldots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \ldots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \ldots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \ldots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \ldots & \vdots & \ldots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \ldots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \ldots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

where for every $1 \leq i \leq m, 1 \leq j \leq n$:

$$\mu_{k_i,l_j}, \nu_{k_i,l_j}, \mu_{k_i,l_j} + \nu_{k_i,l_j} \in [0, 1],$$

$$\alpha_i^k, \beta_i^k, \alpha_i^k + \beta_i^k \in [0, 1],$$

$$\alpha_j^l, \beta_j^l, \alpha_j^l + \beta_j^l \in [0, 1]$$

and here and below,

$$K^* = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | k_i \in K\} = \{\langle k_i, \alpha_i^k, \beta_i^k \rangle | 1 \leq i \leq m\},$$

$$L^* = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | l_j \in L\} = \{\langle l_j, \alpha_j^l, \beta_j^l \rangle | 1 \leq j \leq n\}.$$

Let

$$K^* \subset P^* \text{ iff } (K \subset P) \ \& \ (\forall k_i = p_i \in K)((\alpha_i^k < \alpha_i^p) \ \& \ (\beta_i^k > \beta_i^p)).$$

$$K^* \subseteq P^* \text{ iff } (K \subseteq P) \ \& \ (\forall k_i = p_i \in K)((\alpha_i^k \leq \alpha_i^p) \ \& \ (\beta_i^k \geq \beta_i^p)).$$

All operations and relations over EIFIM must be re-defined, because they have different forms from the above ones. Obviously, the hierarchical operators are not applicable now.

2.3 Standard Operations Over EIFIMs

For the EIFIMs $A = [K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$, $B = [P^*, Q^*, \{\langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle\}]$, operations that are analogous to the usual matrix operations of addition and multiplication are defined, as well as other specific ones.

Addition-(max,min)

$$A \oplus_{(\max,\min)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$T^* = K^* \cup P^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cup P\},$$
$$V^* = L^* \cup Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in L \cup Q\},$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u \in K - P \\ \alpha_r^p, & \text{if } t_u \in P - K \\ \max(\alpha_i^k, \alpha_r^p), & \text{if } t_u \in K \cap P \end{cases},$$

$$\beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w \in L - Q \\ \beta_s^q, & \text{if } t_w \in Q - L \\ \min(\beta_j^l, \beta_s^q), & \text{if } t_w \in L \cap Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \\ & \text{and } v_w = l_j \in L; \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P \\ & \text{and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \\ & \text{and } v_w = q_s \in Q; \\ \langle \max(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \quad \min(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle, & \text{and } v_w = l_j = q_s \in L \cap Q \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

Addition-(min,max)

$$A \oplus_{(\min,\max)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where T^*, V^*, α_u^t, β_w^v, have the above forms but

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - Q \\ & \text{or } t_u = k_i \in K - P \\ & \text{and } v_w = l_j \in L; \\ \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P \\ & \text{and } v_w = q_s \in Q - L \\ & \text{or } t_u = p_r \in P - K \\ & \text{and } v_w = q_s \in Q; \\ \\ \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), & \text{if } t_u = k_i = p_r \in K \cap P \\ \ \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle, & \text{and } v_w = l_j = q_s \in L \cap Q \\ \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

Termwise multiplication-(max,min)

$$A \otimes_{(\max,\min)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$\begin{aligned} T^* &= K^* \cap P^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cap P\}, \\ V^* &= L^* \cap Q^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in L \cap Q\}, \\ \alpha_u^t &= \min(\alpha_i^k, \alpha_r^p), \ \text{for } t_u = k_i = p_r \in K \cap P, \\ \beta_w^v &= \min(\beta_j^l, \beta_s^q), \ \text{for } v_w = l_j = q_s \in L \cap Q \end{aligned}$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \langle \max(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \min(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle.$$

Termwise multiplication-(min,max)

$$A \otimes_{(\min,\max)} B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where T^*, V^*, α_u^t, β_w^v, have the above forms but

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \langle \min(\mu_{k_i,l_j}, \rho_{p_r,q_s}), \max(\nu_{k_i,l_j}, \sigma_{p_r,q_s}) \rangle.$$

Multiplication-(max,min)

$$A \odot_{(\max,\min)} B = [T^*, V^*, \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$T^* = (K \cup (P - L))^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K \cup (P - L)\},$$
$$V^* = (Q \cup (L - P))^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in Q \cup (L - P)\},$$

$$\alpha_u^t = \begin{cases} \alpha_i^k, & \text{if } t_u = k_i \in K \\ \alpha_r^p, & \text{if } t_u = p_r \in P - L \end{cases},$$

$$\beta_w^v = \begin{cases} \beta_j^l, & \text{if } v_w = l_j \in L - P \\ \beta_s^q, & \text{if } t_w = q_s \in Q \end{cases},$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P - Q \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P - L - K \\ & \text{and } v_w = q_s \in Q \\ \langle \max\limits_{l_j = p_r \in L \cap P} (\min(\mu_{k_i,l_j}, \rho_{p_r,q_s})), & \text{if } t_u = k_i \in K \\ & \text{and } v_w = q_s \in Q \\ \min\limits_{l_j = p_r \in L \cap P} (\max(\nu_{k_i,l_j}, \sigma_{p_r,q_s}))\rangle, & \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

Multiplication-(min,max)

$$A \odot_{(\min,\max)} B = [T^*, V^*, \langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where T^*, V^*, α_u^t, β_w^v, have the above forms but

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } t_u = k_i \in K \\ & \text{and } v_w = l_j \in L - P - Q \\ \langle \rho_{p_r,q_s}, \sigma_{p_r,q_s} \rangle, & \text{if } t_u = p_r \in P - L - K \\ & \text{and } v_w = q_s \in Q \\ \langle \min\limits_{l_j = p_r \in L \cap P} (\max(\mu_{k_i,l_j}, \rho_{p_r,q_s})), & \text{if } t_u = k_i \in K \\ & \text{and } v_w = q_s \in Q \\ \max\limits_{l_j = p_r \in L \cap P} (\min(\nu_{k_i,l_j}, \sigma_{p_r,q_s})) \rangle, & \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases}$$

Structural subtraction

$$A \ominus B = [T^*, V^*, \{\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle\}],$$

where

$$T^* = (K - P)^* = \{\langle t_u, \alpha_u^t, \beta_u^t \rangle | t_u \in K - P\},$$
$$V^* = (L - Q)^* = \{\langle v_w, \alpha_w^v, \beta_w^v \rangle | v_w \in L - Q\},$$

for the set–theoretic subtraction operation and

$$\alpha_u^t = \alpha_i^k, \text{ for } t_u = k_i \in K - P,$$
$$\beta_w^v = \beta_j^l, \text{ for } v_w = l_j \in L - Q$$

and

$$\langle \varphi_{t_u,v_w}, \psi_{t_u,v_w} \rangle = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, \text{ for } t_u = k_i \in K - P \text{ and } v_w = l_j \in L - Q.$$

Negation of an EIFIM

$$\neg A = [T^*, V^*, \{\neg \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}],$$

where $\neg$ is one of the above intuitionistic fuzzy negations in Table 2.1, or another possible negation.

Termwise subtraction

$$A -_{\max,\min} B = A \oplus_{\max,\min} \neg B,$$

$$A -_{\min,\max} B = A \oplus_{\min,\max} \neg B.$$

Operations “reduction”, “projection” and “substitution” coincide with the respective operations defined in Chap. 1 , Sects. 1.6–1.8.

2.4 Relations Over EIFIMs

Let the two EIFIMs $A = [K^*, L^*, \{\langle a_{k,l}, b_{k,l}\rangle\}]$ and $B = [P^*, Q^*, \{\langle c_{p,q}, d_{p,q}\rangle\}]$ be given. We shall introduce the following definitions where $\subset$ and $\subseteq$ denote the relations *“strong inclusion”* and *“weak inclusion”*.

The strict relation “inclusion about dimension” is

$$A \subset_d B \text{ iff } (((K^* \subset P^*) \,\&\, (L^* \subset Q^*)) \;\vee\; ((K^* \subseteq P^*) \,\&\, (L^* \subset Q^*))$$

$$\vee((K^* \subset P^*) \,\&\, (L^* \subseteq Q^*))) \,\&\, (\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle = \langle c_{k,l}, d_{k,l}\rangle).$$

The non-strict relation “inclusion about dimension” is

$$A \subseteq_d B \text{ iff } (K^* \subseteq P^*) \,\&\, (L^* \subseteq Q^*) \,\&\, (\forall k \in K)(\forall l \in L)$$

$$(\langle a_{k,l}, b_{k,l}\rangle = \langle c_{k,l}, d_{k,l}\rangle).$$

The strict relation “inclusion about value” is

$$A \subset_v B \text{ iff } (K^* = P^*) \,\&\, (L^* = Q^*) \,\&\, (\forall k \in K)(\forall l \in L)$$

$$(\langle a_{k,l}, b_{k,l}\rangle < \langle c_{k,l}, d_{k,l}\rangle).$$

The non-strict relation “inclusion about value” is

$$A \subseteq_v B \text{ iff } (K^* = P^*) \,\&\, (L^* = Q^*) \,\&\, (\forall k \in K)(\forall l \in L)$$

$$(\langle a_{k,l}, b_{k,l}\rangle \leq \langle c_{k,l}, d_{k,l}\rangle).$$

The strict relation “inclusion” is

$$A \subset_* B \text{ iff } (((K^* \subset P^*) \,\&\, (L^* \subset Q^*)) \;\vee\; ((K^* \subseteq P^*) \,\&\, (L^* \subset Q^*))$$

$$\vee\; ((K^* \subset P^*) \,\&\, (L^* \subseteq Q^*))) \,\&\, (\forall k \in K)(\forall l \in L)(\langle a_{k,l}, b_{k,l}\rangle < \langle c_{k,l}, d_{k,l}\rangle).$$

The non-strict relation “inclusion” is

$$A \subseteq_* B \text{ iff } (K^* \subseteq P^*) \,\&\, (L^* \subseteq Q^*) \,\&\, (\forall k \in K)(\forall l \in L)$$

$$(\langle a_{k,l}, b_{k,l}\rangle \leq \langle c_{k,l}, d_{k,l}\rangle).$$

2.5 Level Operators Over EIFIMs

Let the EIFIM $A = [K^*, L^*, \{\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle\}]$ be given.

Let for $i = 1, 2, 3 : \rho_i, \sigma_i, \rho_i + \sigma_i \in [0, 1]$ be fixed numbers.

In [7,13], several level operators are defined. One of them, for a given IFS

$$X = \{\langle x, \mu_X(x), \nu_X(x)\rangle | x \in E\}$$

is

$$N_{\alpha,\beta}(X) = \{\langle x, \mu_X(x), \nu_X(x)\rangle | x \in E \ \& \ \mu_X(x) \geq \alpha \ \& \ \nu_X(x) \leq \beta\},$$

where $\alpha, \beta \in [0, 1]$ are fixed and $\alpha + \beta \leq 1$.

Here, its analogues are introduced. They are three: $N^1_{\rho_1,\sigma_1}, N^2_{\rho_2,\sigma_2}, N^3_{\rho_3,\sigma_3}$ and affect the K-, L-indices and $\langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle$-elements, respectively. The three operators can be applied over an EIFIM A either sequentially, or simultaneously. In the first case, their forms are

$$N^1_{\rho_1,\sigma_1}(A) = [N_{\rho_1,\sigma_1}(K^*), L^*, \{\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j} \rangle\}],$$

where

$$\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j} \rangle = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle$$

only for $\langle k_i, \alpha_i^k, \beta_i^k \rangle \in N_{\rho_1,\sigma_1}(K^*)$ and for each $\langle l_j, \alpha_j^l, \beta_j^l \rangle \in L^*$;

$$N^2_{\rho_2,\sigma_2}(A) = [K^*, N_{\rho_2,\sigma_2}(L^*), \{\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j} \rangle\}],$$

where

$$\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j} \rangle = \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle$$

for each $\langle k_i, \alpha_i^k, \beta_i^k \rangle \in K^*$ and only for $\langle l_j, \alpha_j^l, \beta_j^l \rangle \in N_{\rho_2,\sigma_2}(L^*)$;

$$N^3_{\rho_3,\sigma_3}(A) = [K^*, L^*, \{\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j} \rangle\}],$$

where

$$\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j} \rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } \mu_{k_i,l_j} \geq \rho_3 \ \& \ \nu_{k_i,l_j} \leq \sigma_3 \\ \langle 0, 1 \rangle, & \text{otherwise} \end{cases},$$

In the second case, their form is

$$(N^1_{\rho_1,\sigma_1}, N^2_{\rho_2,\sigma_2}, N^3_{\rho_3,\sigma_3})(A) = [N_{\rho_1,\sigma_1}(K^*), N_{\rho_2,\sigma_2}(L^*), \{\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j} \rangle\}],$$

where

$$\langle \varphi_{k_i,l_j}, \psi_{k_i,l_j} \rangle = \begin{cases} \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle, & \text{if } \langle k_i, \alpha_i^k, \beta_i^k \rangle \in N_{\rho_1,\sigma_1}(K^*) \\ & \text{and } \langle l_j, \alpha_j^l, \beta_j^l \rangle \in N_{\rho_2,\sigma_2}(L^*) \\ & \text{and } \mu_{k_i,l_j} \geq \rho_3 \ \& \ \nu_{k_i,l_j} \leq \sigma_3 \\ \langle 0, 1 \rangle, & \text{if } \langle k_i, \alpha_i^k, \beta_i^k \rangle \in N_{\rho_1,\sigma_1}(K^*) \\ & \text{and } \langle l_j, \alpha_j^l, \beta_j^l \rangle \in N_{\rho_2,\sigma_2}(L^*) \\ & \text{and } \mu_{k_i,l_j} < \rho_3 \ \vee \ \nu_{k_i,l_j} > \sigma_3 \end{cases},$$

2.6 Aggregation Operations Over EIFIMs

Let the EIFIM

$$A = \begin{array}{c|ccccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots & l_j, \langle \alpha_j^l, \beta_j^l \rangle & \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle & \dots & \langle \mu_{k_1,l_j}, \nu_{k_1,l_j} \rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_i, \langle \alpha_i^k, \beta_i^k \rangle & \langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle & \dots & \langle \mu_{k_i,l_j}, \nu_{k_i,l_j} \rangle & \dots & \langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle \\ \vdots & \vdots & \dots & \vdots & \dots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle & \dots & \langle \mu_{k_m,l_j}, \nu_{k_m,l_j} \rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle \end{array},$$

be given and let $k_0 \notin K$ and $l_0 \notin L$ be two fixed indices.

Now, we introduce the following 18 operations over it.

(max,max)-row-aggregation

$$\rho_{(\max,\max)}(A, k_0)$$

$$= \begin{array}{c|cc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots \\ \hline k_0, \langle \max\limits_{1 \le i \le m} \alpha_i^k, \min\limits_{1 \le i \le m} \beta_i^k \rangle & \langle \max\limits_{1 \le i \le m} \mu_{k_i,l_1}, \min\limits_{1 \le i \le m} \nu_{k_i,l_1} \rangle & \dots \end{array}$$

$$\begin{array}{cc} \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline \dots & \langle \max\limits_{1 \le i \le m} \mu_{k_i,l_n}, \min\limits_{1 \le i \le m} \nu_{k_i,l_n} \rangle \end{array},$$

(max,ave)-row-aggregation

$$\rho_{(\max,\max)}(A, k_0)$$

$$= \begin{array}{c|cc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots \\ \hline k_0, \langle \max\limits_{1 \le i \le m} \alpha_i^k, \min\limits_{1 \le i \le m} \beta_i^k \rangle & \langle \frac{1}{m} \sum\limits_{i=1}^{m} \mu_{k_i,l_1}, \frac{1}{m} \sum\limits_{i=1}^{m} \nu_{k_i,l_1} \rangle & \dots \end{array}$$

$$\begin{array}{c|c} \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline \dots & \langle \frac{1}{m} \sum\limits_{i=1}^{m} \mu_{k_i,l_n}, \frac{1}{m} \sum\limits_{i=1}^{m} \nu_{k_i,l_n} \rangle \end{array},$$

(max,min)-row-aggregation

$$\rho_{(\max,\max)}(A, k_0)$$

$$= \begin{array}{c|cc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots \\ \hline k_0, \langle \max\limits_{1 \le i \le m} \alpha_i^k, \min\limits_{1 \le i \le m} \beta_i^k \rangle & \langle \min\limits_{1 \le i \le m} \mu_{k_i,l_1}, \max\limits_{1 \le i \le m} \nu_{k_i,l_1} \rangle & \dots \end{array}$$

$$\begin{array}{cc} \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline \dots & \langle \min\limits_{1 \le i \le m} \mu_{k_i,l_n}, \max\limits_{1 \le i \le m} \nu_{k_i,l_n} \rangle \end{array},$$

(ave,max)-row-aggregation

$$\rho_{(\min,\max)}(A, k_0)$$

$$= \begin{array}{c|cc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots \\ \hline k_0, \langle \frac{1}{m} \sum\limits_{i=1}^{m} \alpha_i^k, \frac{1}{m} \sum\limits_{i=1}^{m} \beta_i^k \rangle & \langle \max\limits_{1 \le i \le m} \mu_{k_i,l_1}, \min\limits_{1 \le i \le m} \nu_{k_i,l_1} \rangle & \dots \end{array}$$

$$\begin{array}{cc} \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline \dots & \langle \max\limits_{1 \le i \le m} \mu_{k_i,l_n}, \min\limits_{1 \le i \le m} \nu_{k_i,l_n} \rangle \end{array},$$

(ave,ave)-row-aggregation

$$\rho_{(\max,\max)}(A, k_0)$$

$$= \begin{array}{c|cc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots \\ \hline k_0, \langle \frac{1}{m} \sum\limits_{i=1}^{m} \alpha_i^k, \frac{1}{m} \sum\limits_{i=1}^{m} \beta_i^k \rangle & \langle \frac{1}{m} \sum\limits_{i=1}^{m} \mu_{k_i,l_1}, \frac{1}{m} \sum\limits_{i=1}^{m} \nu_{k_i,l_1} \rangle & \dots \end{array}$$

$$\begin{array}{cc} \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline \dots & \langle \frac{1}{m} \sum\limits_{i=1}^{m} \mu_{k_i,l_n}, \frac{1}{m} \sum\limits_{i=1}^{m} \nu_{k_i,l_n} \rangle \end{array},$$

(ave,min)-row-aggregation

$$\rho_{(\max,\max)}(A, k_0)$$

$$= \begin{array}{c|cc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots \\ \hline k_0, \langle \frac{1}{m} \sum\limits_{i=1}^{m} \alpha_i^k, \frac{1}{m} \sum\limits_{i=1}^{m} \beta_i^k \rangle & \langle \min\limits_{1 \le i \le m} \mu_{k_i,l_1}, \max\limits_{1 \le i \le m} \nu_{k_i,l_1} \rangle & \dots \end{array}$$

$$\begin{array}{cc} \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline \dots & \langle \min\limits_{1\le i\le m} \mu_{k_i,l_n}, \max\limits_{1\le i\le m} \nu_{k_i,l_n} \rangle \end{array},$$

(min,max)-row-aggregation

$$\rho_{(\min,\max)}(A, k_0)$$

$$= \begin{array}{c|cc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots \\ \hline k_0, \langle \min\limits_{1\le i\le m} \alpha_i^k, \max\limits_{1\le i\le m} \beta_i^k \rangle & \langle \max\limits_{1\le i\le m} \mu_{k_i,l_1}, \min\limits_{1\le i\le m} \nu_{k_i,l_1} \rangle & \dots \end{array}$$

$$\begin{array}{cc} \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline \dots & \langle \max\limits_{1\le i\le m} \mu_{k_i,l_n}, \min\limits_{1\le i\le m} \nu_{k_i,l_n} \rangle \end{array},$$

(min,ave)-row-aggregation

$$\rho_{(\max,\max)}(A, k_0)$$

$$= \begin{array}{c|cc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots \\ \hline k_0, \langle \min\limits_{1\le i\le m} \alpha_i^k, \max\limits_{1\le i\le m} \beta_i^k \rangle & \langle \frac{1}{m} \sum\limits_{i=1}^{m} \mu_{k_i,l_1}, \frac{1}{m} \sum\limits_{i=1}^{m} \nu_{k_i,l_1} \rangle & \dots \end{array}$$

$$\begin{array}{cc} \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline \dots & \langle \frac{1}{m} \sum\limits_{i=1}^{m} \mu_{k_i,l_n}, \frac{1}{m} \sum\limits_{i=1}^{m} \nu_{k_i,l_n} \rangle \end{array},$$

(min,min)-row-aggregation

$$\rho_{(\max,\max)}(A, k_0)$$

$$= \begin{array}{c|cc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots \\ \hline k_0, \langle \min\limits_{1\le i\le m} \alpha_i^k, \max\limits_{1\le i\le m} \beta_i^k \rangle & \langle \min\limits_{1\le i\le m} \mu_{k_i,l_1}, \max\limits_{1\le i\le m} \nu_{k_i,l_1} \rangle & \dots \end{array}$$

$$= \begin{array}{cc} \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline \dots & \langle \min\limits_{1\le i\le m} \mu_{k_i,l_n}, \max\limits_{1\le i\le m} \nu_{k_i,l_n} \rangle \end{array},$$

(max,max)-column-aggregation

$$\sigma_{max}(A, l_0) = \begin{array}{c|c} & l_0, \langle \max\limits_{1\le i\le m} \alpha_j^l, \min\limits_{1\le i\le m} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \max\limits_{1\le j\le n} \mu_{k_1,l_j}, \min\limits_{1\le j\le n} \nu_{k_1,l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \max\limits_{1\le j\le n} \mu_{k_m,l_j}, \min\limits_{1\le j\le n} \nu_{k_m,l_j} \rangle \end{array},$$

(max,ave)-column-aggregation

$$\sigma_{max}(A,l_0)=\begin{array}{c|c} & l_0,\langle \max\limits_{1\le i\le m}\alpha^l_j,\ \min\limits_{1\le i\le m}\beta^l_j\rangle \\ \hline k_1,\langle\alpha^k_1,\beta^k_1\rangle & \langle \frac{1}{n}\sum\limits_{j=1}^{n}\mu_{k_1,l_j},\ \frac{1}{n}\sum\limits_{j=1}^{n}\nu_{k_1,l_j}\rangle \\ \vdots & \vdots \\ k_m,\langle\alpha^k_m,\beta^k_m\rangle & \langle \frac{1}{n}\sum\limits_{j=1}^{n}\mu_{k_m,l_j},\ \frac{1}{n}\sum\limits_{j=1}^{n}\nu_{k_m,l_j}\rangle \end{array},$$

(max,min)-column-aggregation

$$\sigma_{max}(A,l_0)=\begin{array}{c|c} & l_0,\langle \max\limits_{1\le i\le m}\alpha^l_j,\ \min\limits_{1\le i\le m}\beta^l_j\rangle \\ \hline k_1,\langle\alpha^k_1,\beta^k_1\rangle & \langle \min\limits_{1\le j\le n}\mu_{k_1,l_j},\ \max\limits_{1\le j\le n}\nu_{k_1,l_j}\rangle \\ \vdots & \vdots \\ k_m,\langle\alpha^k_m,\beta^k_m\rangle & \langle \min\limits_{1\le j\le n}\mu_{k_m,l_j},\ \max\limits_{1\le j\le n}\nu_{k_m,l_j}\rangle \end{array},$$

(ave,max)-column-aggregation

$$\sigma_{max}(A,l_0)=\begin{array}{c|c} & l_0,\langle \frac{1}{n}\sum\limits_{j=1}^{n}\alpha^l_j,\ \frac{1}{n}\sum\limits_{j=1}^{n}\beta^l_j\rangle \\ \hline k_1,\langle\alpha^k_1,\beta^k_1\rangle & \langle \max\limits_{1\le j\le n}\mu_{k_1,l_j},\ \min\limits_{1\le j\le n}\nu_{k_1,l_j}\rangle \\ \vdots & \vdots \\ k_m,\langle\alpha^k_m,\beta^k_m\rangle & \langle \max\limits_{1\le j\le n}\mu_{k_m,l_j},\ \min\limits_{1\le j\le n}\nu_{k_m,l_j}\rangle \end{array},$$

(ave,ave)-column-aggregation

$$\sigma_{max}(A,l_0)$$

$$=\begin{array}{c|c} & l_0,\langle \frac{1}{n}\sum\limits_{j=1}^{n}\alpha^l_j,\ \frac{1}{n}\sum\limits_{j=1}^{n}\beta^l_j\rangle \\ \hline k_1,\langle\alpha^k_1,\beta^k_1\rangle & \langle \frac{1}{n}\sum\limits_{j=1}^{n}\mu_{k_1,l_j},\ \frac{1}{n}\sum\limits_{j=1}^{n}\nu_{k_1,l_j}\rangle \\ \vdots & \vdots \\ k_m,\langle\alpha^k_m,\beta^k_m\rangle & \langle \frac{1}{n}\sum\limits_{j=1}^{n}\mu_{k_m,l_j},\ \frac{1}{n}\sum\limits_{j=1}^{n}\nu_{k_m,l_j}\rangle \end{array}.$$

(ave,min)-column-aggregation

$$\sigma_{max}(A, l_0) = \begin{array}{c|c} & l_0, \langle \frac{1}{n}\sum_{j=1}^{n} \alpha_j^l, \frac{1}{n}\sum_{j=1}^{n} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \min_{1\le j\le n} \mu_{k_1,l_j}, \max_{1\le j\le n} \nu_{k_1,l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \min_{1\le j\le n} \mu_{k_m,l_j}, \max_{1\le j\le n} \nu_{k_m,l_j} \rangle \end{array},$$

(min,max)-column-aggregation

$$\sigma_{max}(A, l_0) = \begin{array}{c|c} & l_0, \langle \min_{1\le i\le m} \alpha_j^l, \max_{1\le i\le m} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \max_{1\le j\le n} \mu_{k_1,l_j}, \min_{1\le j\le n} \nu_{k_1,l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \max_{1\le j\le n} \mu_{k_m,l_j}, \min_{1\le j\le n} \nu_{k_m,l_j} \rangle \end{array},$$

(min,ave)-column-aggregation

$$\sigma_{max}(A, l_0) = \begin{array}{c|c} & l_0, \langle \min_{1\le i\le m} \alpha_j^l, \max_{1\le i\le m} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \frac{1}{n}\sum_{j=1}^{n} \mu_{k_1,l_j}, \frac{1}{n}\sum_{j=1}^{n} \nu_{k_1,l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \frac{1}{n}\sum_{j=1}^{n} \mu_{k_m,l_j}, \frac{1}{n}\sum_{j=1}^{n} \nu_{k_m,l_j} \rangle \end{array},$$

(min,min)-column-aggregation

$$\sigma_{max}(A, l_0) = \begin{array}{c|c} & l_0, \langle \min_{1\le i\le m} \alpha_j^l, \max_{1\le i\le m} \beta_j^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & \langle \min_{1\le j\le n} \mu_{k_1,l_j}, \max_{1\le j\le n} \nu_{k_1,l_j} \rangle \\ \vdots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & \langle \min_{1\le j\le n} \mu_{k_m,l_j}, \max_{1\le j\le n} \nu_{k_m,l_j} \rangle \end{array}.$$

2.7 Extended Modal Operators Defined Over EIFIMs

Let, as above, $x = \langle a, b\rangle$ be an IFP and let $\alpha, \beta \in [0, 1]$. Some of the extended modal operators defined over x have the following forms (see [13, 26]):

$$\begin{aligned}
F_{\alpha,1-\alpha}(x) &= \langle a + \alpha.(1 - a - b), b + \beta.(1 - a - b)\rangle, \quad \text{where } \alpha + \beta \leq 1\\
G_{\alpha,\beta}(x) &= \langle \alpha.a, \beta.b\rangle\\
H_{\alpha,\beta}(x) &= \langle \alpha.a, b + \beta.(1 - a - b)\rangle\\
H^*_{\alpha,\beta}(x) &= \langle \alpha.a, b + \beta.(1 - \alpha.a - b)\rangle\\
J_{\alpha,\beta}(x) &= \langle a + \alpha.(1 - a - b), \beta.b\rangle\\
J^*_{\alpha,\beta}(x) &= \langle a + \alpha.(1 - a - \beta.b), \beta.b\rangle
\end{aligned}$$

and let the level operators have the forms:

$$\begin{aligned}
P_{\alpha,\beta}x &= \langle \max(\alpha, a), \min(\beta, b)\rangle\\
Q_{\alpha,\beta}x &= \langle \min(\alpha, a), \max(\beta, b)\rangle,
\end{aligned}$$

for $\alpha, \beta \in [0, 1]$ and $\alpha + \beta \leq 1$.

Now we define operators over EIFIMs. Let $O^1_{\alpha_1,\beta_1}, O^2_{\alpha_2,\beta_2}, O^3_{\alpha_3,\beta_3}$ be three operators and their arguments $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \beta_3$ satisfy the respective conditions, given above. The three operators affect the K-, L-indices and $\langle \mu_{k_i,l_j}, \nu_{k_i,l_j}\rangle$-elements, respectively. They can be applied over an EIFIM A sequentially, or simultaneously. In the first case, their forms are

$$(O^1_{\alpha_1,\beta_1}, \bot, \bot)(A)$$

$$= \begin{array}{c|ccc}
 & l_1, \langle \alpha^l_1, \beta^l_1\rangle & \dots & l_n, \langle \alpha^l_n, \beta^l_n\rangle \\ \hline
k_1, O^1_{\alpha_1,\beta_1}(\langle \alpha^k_1, \beta^k_1\rangle) & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1}\rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n}\rangle \\
\vdots & \vdots & \dots & \vdots \\
k_m, O^1_{\alpha_1,\beta_1}(\langle \alpha^k_m, \beta^k_m\rangle) & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1}\rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n}\rangle
\end{array},$$

$$(\bot, O^2_{\alpha_2,\beta_2}, \bot)(A)$$

$$= \begin{array}{c|ccc}
 & l_1, O^2_{\alpha_2,\beta_2}(\langle \alpha^l_1, \beta^l_1\rangle) & \dots & l_n, O^2_{\alpha_2,\beta_2}(\langle \alpha^l_n, \beta^l_n\rangle) \\ \hline
k_1, \langle \alpha^k_1, \beta^k_1\rangle & \langle \mu_{k_1,l_1}, \nu_{k_1,l_1}\rangle & \dots & \langle \mu_{k_1,l_n}, \nu_{k_1,l_n}\rangle \\
\vdots & \vdots & \dots & \vdots \\
k_m, \langle \alpha^k_m, \beta^k_m\rangle & \langle \mu_{k_m,l_1}, \nu_{k_m,l_1}\rangle & \dots & \langle \mu_{k_m,l_n}, \nu_{k_m,l_n}\rangle
\end{array},$$

$$(\bot, \bot, O^3_{\alpha_3,\beta_3})(A)$$

$$= \begin{array}{c|ccc} & l_1, \langle \alpha_1^l, \beta_1^l \rangle & \dots & l_n, \langle \alpha_n^l, \beta_n^l \rangle \\ \hline k_1, \langle \alpha_1^k, \beta_1^k \rangle & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle) \\ \vdots & \vdots & \dots & \vdots \\ k_m, \langle \alpha_m^k, \beta_m^k \rangle & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle) \end{array}.$$

In the second case, the form of the triple of operators is

$$(O^1_{\alpha_1,\beta_1}, O^2_{\alpha_2,\beta_2}, O^3_{\alpha_3,\beta_3})(A)$$

$$= \begin{array}{c|ccc} & l_1, O^2_{\alpha_2,\beta_2}(\langle \alpha_1^l, \beta_1^l \rangle) & \dots & l_n, O^2_{\alpha_2,\beta_2}(\langle \alpha_n^l, \beta_n^l \rangle) \\ \hline k_1, O^1_{\alpha_1,\beta_1}(\langle \alpha_1^k, \beta_1^k \rangle) & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_1,l_1}, \nu_{k_1,l_1} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_1,l_n}, \nu_{k_1,l_n} \rangle) \\ \vdots & \vdots & \dots & \vdots \\ k_i, O^1_{\alpha_1,\beta_1}(\langle \alpha_i^k, \beta_i^k \rangle) & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_i,l_1}, \nu_{k_i,l_1} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_i,l_n}, \nu_{k_i,l_n} \rangle) \\ \vdots & \vdots & \dots & \vdots \\ k_m, O^1_{\alpha_1,\beta_1}(\langle \alpha_m^k, \beta_m^k \rangle) & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_m,l_1}, \nu_{k_m,l_1} \rangle) & \dots & O^3_{\alpha_3,\beta_3}(\langle \mu_{k_m,l_n}, \nu_{k_m,l_n} \rangle) \end{array}.$$

2.8 An Example with Intuitionistic Fuzzy Graphs

Let $V = \{v_1, v_2, \dots, v_n\}$ be a fixed set of vertices and let each vertex x have a degree of existence $\alpha(x)$ and a degree of non-existence $\beta(x)$. Therefore, we can construct the IFS

$$V^* = \{\langle x, \alpha(x), \beta(x) \rangle | x \in V\} = \{\langle v_i, \alpha(v_i), \beta(v_i) \rangle | 1 \le i \le n\},$$

where for each $x \in V$:

$$\alpha(x), \beta(x), \alpha(x) + \beta(x) \in [0, 1].$$

Let H be a set of arcs between vertices from V. We again can juxtapose to each arc a degree of existence $\mu(x, y)$ and a degree of non-existence $\nu(x, y)$. Therefore, we can construct the new IFS

$$\begin{aligned} H^* &= \{\langle \langle x, y \rangle, \mu(x, y), \nu(x, y) \rangle | x, y \in V\} \\ &= \{\langle \langle v_i, v_j \rangle, \mu(v_i, v_j), \nu(v_i, v_j) \rangle | 1 \le i, j \le n\}, \end{aligned}$$

where for each $x, y \in V$:

$$\mu(x, y), \nu(x, y), \mu(x, y) + \nu(x, y) \in [0, 1].$$

Now, for the graph $G = (V, H)$ we can construct the Extended Intuitionistic Fuzzy Graph (EIFG) $G^* = (V^*, H^*)$. It has the following IM-representation:

$$[V^*, V^*, \{\langle \mu(v_i, v_j), \nu(v_i, v_j)\rangle\}]$$

$$= \begin{array}{l|ccc} & v_1, \langle \alpha(v_1), \beta(v_1)\rangle & \dots & v_n, \langle \alpha(v_n), \beta(v_n)\rangle \\ \hline v_1, \langle \alpha(v_1), \beta(v_1)\rangle & \langle \mu_{v_1,v_1}, \nu_{v_1,v_1}\rangle & \dots & \langle \mu_{v_1,v_n}, \nu_{v_1,v_n}\rangle \\ \vdots & \vdots & \dots & \vdots \\ v_i, \langle \alpha(v_i), \beta(v_i)\rangle & \langle \mu_{v_i,v_1}, \nu_{v_i,v_1}\rangle & \dots & \langle \mu_{v_i,v_n}, \nu_{v_i,v_n}\rangle \\ \vdots & \vdots & \dots & \vdots \\ v_n, \langle \alpha(v_n), \beta(v_n)\rangle & \langle \mu_{v_n,v_1}, \nu_{v_n,v_1}\rangle & \dots & \langle \mu_{v_n,v_n}, \nu_{v_n,v_n}\rangle \end{array},$$

where for every $1 \le i \le n$, $1 \le j \le n$: $\mu_{v_i,v_j}, \nu_{v_i,v_j} \in [0, 1]$, $\mu_{v_i,v_j} + \nu_{v_i,v_j} \in [0, 1]$, $\alpha(v_i), \beta(v_i) \in [0, 1]$, $\alpha(v_i) + \beta(v_i) \in [0, 1]$.

Let us discuss here for simplicity only the case of oriented graph. Let us denote by $x \to y$ the fact that both vertices x and y are connected by an arc and x is higher than y. Let operation $\circ \in \{+, \max, @, \min, \times\}$.

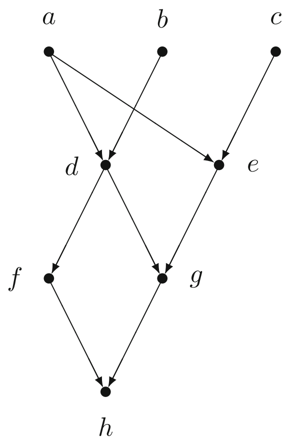
We call that the EIFG G^* is "well-top-down-(very strong, strong, middle, weak, very weak)-ordered", or shortly, "well-top-down-$\circ$-ordered", if for every two vertices v_i and v_j, such that $v_i \to v_j$, the following inequality holds:

$$\langle \alpha_i, \beta_i\rangle \circ \langle \mu_{v_i,v_j}, \nu_{v_i,v_j}\rangle \ge \langle \alpha_j, \beta_j\rangle.$$

Analogously, we call that the EIFG G^* is "well-bottom-up-(very strong, strong, middle, weak, very weak)-ordered", or shortly, "well-bottom-up-$\circ$-ordered", if for every two vertices v_i and v_j, such that $v_i \to v_j$, the following inequality holds:

$$\langle \alpha_i, \beta_i\rangle \circ \langle \mu_{v_i,v_j}, \nu_{v_i,v_j}\rangle \le \langle \alpha_j, \beta_j\rangle.$$

We illustrate the way for IM-interpretation of the EIFGs by the following example. Let us have the EIFG G^* with the form



Its H^*-component has the following form (where, obviously, the information about the IFS V^* is included in it):

$$H^* = [\{\langle a, \frac{1}{2}, \frac{1}{3}\rangle, \langle b, \frac{1}{3}, \frac{1}{3}\rangle, \langle c, \frac{1}{3}, \frac{1}{2}\rangle, \langle d, \frac{2}{3}, \frac{1}{8}\rangle, \langle e, \frac{3}{4}, \frac{1}{4}\rangle, \langle f, \frac{1}{10}, \frac{7}{8}\rangle, \langle g, \frac{2}{5}, \frac{3}{5}\rangle,$$

$$\langle h, \frac{1}{5}, \frac{1}{5}\rangle\}, \{\langle a, \frac{1}{2}, \frac{1}{3}\rangle, \langle b, \frac{1}{3}, \frac{1}{3}\rangle, \langle c, \frac{1}{3}, \frac{1}{2}\rangle, \langle d, \frac{2}{3}, \frac{1}{8}\rangle, \langle e, \frac{3}{4}, \frac{1}{4}\rangle, \langle f, \frac{1}{10}, \frac{7}{8}\rangle,$$

$$\langle g, \frac{2}{5}, \frac{3}{5}\rangle, \langle h, \frac{1}{5}, \frac{1}{5}\rangle\}, \{\mu_{x,y}, \nu_{x,y}\}].$$

Now, having in mind the discussion in Sect. 2.2 , we can modify the IM to the form

$$H^* = [\{\langle a, \frac{1}{2}, \frac{1}{3}\rangle, \langle b, \frac{1}{3}, \frac{1}{3}\rangle, \langle c, \frac{1}{3}, \frac{1}{2}\rangle, \langle d, \frac{2}{3}, \frac{1}{8}\rangle, \langle e, \frac{3}{4}, \frac{1}{4}\rangle, \langle f, \frac{1}{10}, \frac{7}{8}\rangle, \langle g, \frac{2}{5}, \frac{3}{5}\rangle\},$$

$$\{\langle d, \frac{2}{3}, \frac{1}{8}\rangle, \langle e, \frac{3}{4}, \frac{1}{4}\rangle, \langle f, \frac{1}{10}, \frac{7}{8}\rangle, \langle g, \frac{2}{5}, \frac{3}{5}\rangle, \langle h, \frac{1}{5}, \frac{1}{5}\rangle\}, \{\mu_{x,y}, \nu_{x,y}\}].$$

The form of the new IM is

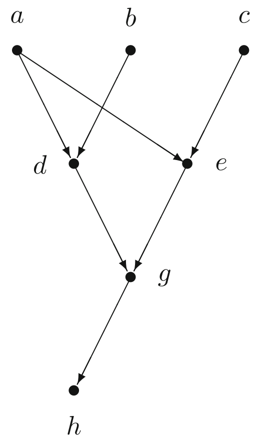
	$d, \langle\frac{2}{3}, \frac{1}{8}\rangle$	$e, \langle\frac{3}{4}, \frac{1}{4}\rangle$	$f, \langle\frac{1}{10}, \frac{7}{8}\rangle$	$g, \langle\frac{2}{5}, \frac{3}{5}\rangle$	$h, \langle\frac{1}{5}, \frac{1}{5}\rangle$
$a, \langle\frac{1}{2}, \frac{1}{3}\rangle$	$\langle\frac{3}{4}, \frac{1}{5}\rangle$	$\langle\frac{1}{2}, \frac{1}{4}\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$
$b, \langle\frac{1}{3}, \frac{1}{3}\rangle$	$\langle\frac{2}{3}, 0\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$
$c, \langle\frac{1}{3}, \frac{1}{2}\rangle$	$\langle 0, 1\rangle$	$\langle\frac{1}{5}, \frac{2}{5}\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$
$d, \langle\frac{2}{3}, \frac{1}{8}\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle\frac{1}{5}, \frac{2}{5}\rangle$	$\langle\frac{3}{4}, \frac{1}{5}\rangle$	$\langle 0, 1\rangle$
$e, \langle\frac{3}{4}, \frac{1}{4}\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle\frac{2}{3}, \frac{1}{6}\rangle$	$\langle 0, 1\rangle$
$f, \langle\frac{1}{10}, \frac{7}{8}\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle\frac{1}{3}, \frac{1}{4}\rangle$
$g, \langle\frac{2}{5}, \frac{3}{5}\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle 0, 1\rangle$	$\langle\frac{3}{5}, \frac{1}{5}\rangle$

Now, we can apply one or more of the level-operators $N^1_{\rho_1,\sigma_1}$, $N^2_{\rho_2,\sigma_2}$, $N^3_{\rho_3,\sigma_3}$ and as a result, the form of the graph will be changed. It is important to mention that in the present case (when the two index sets coincide), the first two level operators must have equal parameters and, therefore, if some vertex has to be omitted from one of both index sets, it will be omitted from the other index set, too. For example, if we apply operator $N^1_{\frac{1}{5},\frac{1}{4}}$ over G^*, we obtain

$N^1_{\frac{1}{5},\frac{1}{4}}(G^*) =$

	$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$h, \langle \frac{1}{5}, \frac{1}{5} \rangle$
$a, \langle \frac{1}{2}, \frac{1}{3} \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$b, \langle \frac{1}{3}, \frac{1}{3} \rangle$	$\langle \frac{2}{3}, 0 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$c, \langle \frac{1}{3}, \frac{1}{2} \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{1}{5}, \frac{2}{5} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle 0, 1 \rangle$
$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{2}{3}, \frac{1}{6} \rangle$	$\langle 0, 1 \rangle$
$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{5}, \frac{1}{5} \rangle$

and the new graph has the form

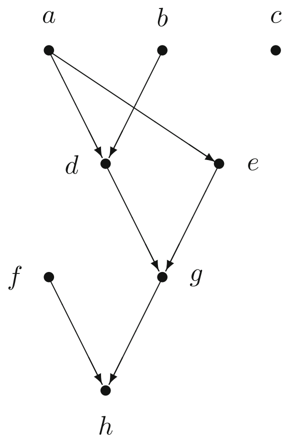


On the other hand, if we can apply, e.g., operator $N^3_{\frac{1}{4},\frac{1}{3}}$ over G^*, we obtain

$N^3_{\frac{1}{4},\frac{1}{3}}(G^*) =$

	$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$f, \langle \frac{1}{10}, \frac{7}{8} \rangle$	$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$h, \langle \frac{1}{5}, \frac{1}{5} \rangle$
$a, \langle \frac{1}{2}, \frac{1}{3} \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle \frac{1}{2}, \frac{1}{4} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$b, \langle \frac{1}{3}, \frac{1}{3} \rangle$	$\langle \frac{2}{3}, 0 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$c, \langle \frac{1}{3}, \frac{1}{2} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle 0, 1 \rangle$
$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{2}{3}, \frac{1}{6} \rangle$	$\langle 0, 1 \rangle$
$f, \langle \frac{1}{10}, \frac{7}{8} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{1}{3}, \frac{1}{4} \rangle$
$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{5}, \frac{1}{5} \rangle$

and the new graph has the form



and the IM G^* can be reduced to

	$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$h, \langle \frac{1}{5}, \frac{1}{5} \rangle$
$a, \langle \frac{1}{2}, \frac{1}{3} \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle \frac{1}{2}, \frac{1}{4} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$b, \langle \frac{1}{3}, \frac{1}{3} \rangle$	$\langle \frac{2}{3}, 0 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$c, \langle \frac{1}{3}, \frac{1}{2} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$
$d, \langle \frac{2}{3}, \frac{1}{8} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{4}, \frac{1}{5} \rangle$	$\langle 0, 1 \rangle$
$e, \langle \frac{3}{4}, \frac{1}{4} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{2}{3}, \frac{1}{6} \rangle$	$\langle 0, 1 \rangle$
$f, \langle \frac{1}{10}, \frac{7}{8} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{1}{3}, \frac{1}{4} \rangle$
$g, \langle \frac{2}{5}, \frac{3}{5} \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle 0, 1 \rangle$	$\langle \frac{3}{5}, \frac{1}{5} \rangle$

Obviously, vertex f remains in the first index set, because an arc goes out of it. On the other hand, vertex c here is an isolated one.

The present text was written in the end of 2013 and it was published in [19], when I, as an Editor-in-Chief of the journal "Notes on Intuitionistic Fuzzy Sets", received (in March 2014) the paper of Parvathi Rangasamy [45] that contains very similar ideas. My paper was published in No. 1 for 2014 and Parvathi's paper in No. 2 of the above mentioned journal.

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