Chapter 2
Topological Spaces and Continuity

Starting from metric spaces as they are familiar from elementary calculus, one observes that many properties of metric spaces like the notions of continuity and convergence do not depend on the detailed information about the metric: instead, only the coarser knowledge of the set of open subsets is needed. This motivates the definition of a topological space as a set together with a collection of subsets which are declared to be the *open subsets*. The precise definition requires crucial properties of open subsets as they are valid for metric spaces. Having managed this axiomatization of "openness" it is fairly easy to transfer the notions of continuity and neighbourhoods to general topological spaces.

Already for metric spaces and now for general topological spaces there are several notions of connectedness which we shall discuss in some detail. New for topological spaces is the need to specify and require separation properties: unlike for a metric spaces we can not necessarily separate different points by open subsets anymore. We will discuss some of these new phenomena in this chapter.

2.1 Metric Spaces

Before defining general topological spaces we recall some basic definitions and results on metric spaces as they should be familiar from undergraduate courses. We start recalling the main definition of a metric space:

Definition 2.1.1 (*Metric space*) A metric space is a pair (M, d) of a (non-empty) set M together with a map

$$d\colon M \times M \longrightarrow [0, \infty), \tag{2.1.1}$$

called the metric, such that

(i) $d(x, y) = 0$ iff $x = y$,
(ii) $d(x, y) = d(y, x)$,
(iii) $d(x, y) \leq d(x, z) + d(z, y)$

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for all $x, y, z \in M$.

If one relaxes the first condition to $d(x, y) = 0$ if $x = y$ then d is sometimes called a *pseudo-metric*. The third condition is the triangle inequality. The geometric idea behind this definition is that $d(x, y)$ is a measure for the *distance* between the points x and y.

There are some easy and well-known examples of metric spaces; the verification of the defining properties is a simple exercise.

Example 2.1.2 (Metric spaces)

(i) The real numbers $\mathbb{R}$ with the usual absolute value $|\cdot|$ give a metric $d(x, y) = |x - y|$. Analogously, $\mathbb{C}$ becomes a metric space, too.

(ii) The previous example can be generalized in various ways. One important way is a real or complex normed vector space (V, $\|\cdot\|$). Then $d(x, y) = \|x - y\|$defines a metric on V. Here we mention the following particular cases:

- $\mathbb{R}^n$ or $\mathbb{C}^n$ with the Euclidean metric

$$d(x, y) = \sqrt{\sum_{k=1}^{n} |x_k - y_k|^2}. \tag{2.1.2}$$

- $\mathbb{R}^n$ or $\mathbb{C}^n$ with the metric coming from the *p-norm* with $p \in [1, \infty)$,

$$d_p(x, y) = \sqrt[p]{\sum_{k=1}^{n} |x_k - y_k|^p}. \tag{2.1.3}$$

- $\mathbb{R}^n$ or $\mathbb{C}^n$ with the metric coming from the supremum norm

$$d_\infty(x, y) = \|x - y\|_\infty = \max_{k=1}^{n} |x_k - y_k|. \tag{2.1.4}$$

(iii) Consider the space $\mathbb{R}[[x]]$ of real *formal power series* $a = \sum_{n=0}^{\infty} a_n x^n$ with coefficients $a_n \in \mathbb{R}$ and a variable x, where we do not care about convergence at all. By termwise operations $\mathbb{R}[[x]]$ is a real vector space. We define now the *order* of a to be

$$o(a) = \begin{cases} \infty & \text{if } a = 0 \\ \min\{n \mid a_n \neq 0\} & \text{if } a \neq 0. \end{cases} \tag{2.1.5}$$

Then the definition

$$d(a, b) = 2^{-o(a-b)} \tag{2.1.6}$$

gives a metric on $\mathbb{R}[[x]]$ where we set $2^{-\infty} = 0$ as usual, see also Exercise 2.7.2 for more details on this metric space.

(iv) If M is any set we define the *discrete metric* by

$$d(p, q) = \begin{cases} 0 & \text{if } p = q \\ 1 & \text{if } p \neq q. \end{cases} \tag{2.1.7}$$

An easy verification shows that this is indeed a metric space with $d(p, q) \leq 1$ for all points $p, q \in M$.

(v) If (M, d_M) is a metric space and $N \subseteq M$ a subset then $d_N = d\big|_{N \times N}$ gives a metric on N, called the *induced metric* or *subspace metric*. This way, many interesting geometric objects like the spheres

$$\mathbb{S}^n = \left\{ x \in \mathbb{R}^{n+1} \;\middle|\; \|x\| = 1 \right\} \tag{2.1.8}$$

or the tori

$$\mathbb{T}^n = \left\{ z \in \mathbb{C}^n \;\middle|\; z = (z_1, \ldots, z_n) \quad \text{with} \quad |z_1| = \cdots = |z_n| = 1 \right\} \tag{2.1.9}$$

inherit a metric structure by considering them as subsets of a suitably chosen Euclidean space.

(vi) If (M, d) is a metric space then also

$$d'(p, q) = \frac{d(p, q)}{1 + d(p, q)} \tag{2.1.10}$$

is a metric on M. Now all points have distance $d'(p, q) < 1$. To validate the triangle inequality for d' we first note that the function $f \colon \xi \mapsto \frac{\xi}{1+\xi}$ is monotonically increasing on $\mathbb{R}_0^+$ and we have $f(\alpha + \beta) \leq f(\alpha) + f(\beta)$ for all $\alpha, \beta \geq 0$.

We see that there is a whole world of metric spaces to be explored. As a first step we define the open subsets of a metric space in a similar way open subsets of $\mathbb{R}$ are defined in elementary calculus.

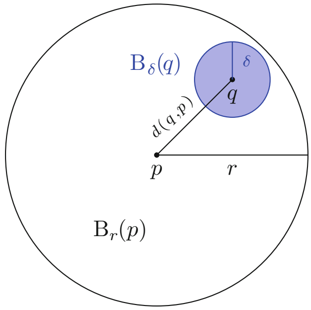
Definition 2.1.3 (*Open subsets in metric space*) Let (M, d) be a metric space.

(i) The open ball $\mathrm{B}_r(p)$ around $p \in M$ of radius $r > 0$ is defined by

$$\mathrm{B}_r(p) = \{q \in M \mid d(p, q) < r\}. \tag{2.1.11}$$

(ii) A subset $\mathcal{O} \subseteq M$ is called open if for all $p \in \mathcal{O}$ one finds a radius $r > 0$ (depending on p) such that $\mathrm{B}_r(p) \subseteq \mathcal{O}$.

Fig. 2.1 Open metric balls are open



Of course, calling a ball open does not make it an open subset in the sense of the definition. Here we have to prove something. We collect this and some further properties of open subsets in the following proposition:

Proposition 2.1.4 (Open subsets in metric space) *Let* (M, d) *be a metric space.*

(i) *All open balls* $\mathrm{B}_r(p)$ *with* $p \in M$ *and* $r > 0$ *are open.*
(ii) *The empty set* $\emptyset$ *and* M *are open.*
(iii) *If* $\{\mathcal{O}_i\}_{i \in I}$ *is an arbitrary collection of open subsets of* M *then* $\bigcup_{i \in I} \mathcal{O}_i$ *is open, too.*
(iv) *If* $\mathcal{O}_1, \ldots, \mathcal{O}_n$ *are finitely many open subsets of* M *then* $\mathcal{O}_1 \cap \cdots \cap \mathcal{O}_n$ *is open, too.*

Proof Let $q \in \mathrm{B}_r(p)$ be given. Then $r - d(p, q) > 0$ and thus we find a $\delta > 0$ with $\delta < r - d(p, q)$, see Fig. 2.1. Now consider $q' \in \mathrm{B}_\delta(q)$. We claim that $q' \in \mathrm{B}_r(p)$. Indeed, by the triangle inequality we have

$$d(p, q') \leq d(p, q) + d(q, q') < d(p, q) + \delta < r.$$

Thus $\mathrm{B}_\delta(q) \subseteq \mathrm{B}_r(p)$, showing that $\mathrm{B}_r(p)$ is open. The second part is clear. For the third, let $p \in \bigcup_{i \in I} \mathcal{O}_i$ then there is at least one $i_0 \in I$ with $p \in \mathcal{O}_{i_0}$. Since $\mathcal{O}_{i_0}$ is open, we find $r > 0$ with $\mathrm{B}_r(p) \subseteq \mathcal{O}_{i_0}$. But then

$$\mathrm{B}_r(p) \subseteq \mathcal{O}_{i_0} \subseteq \bigcup_{i \in I} \mathcal{O}_i$$

shows that the union is open. For the last part, let $p \in \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_n$ if the intersection is non-empty. Then we have $r_1, \ldots, r_n > 0$ with $\mathrm{B}_{r_1}(p) \subseteq \mathcal{O}_1, \ldots, \mathrm{B}_{r_n}(p) \subseteq \mathcal{O}_n$ since all of them are open. Taking the minimum $r = \min\{r_1, \ldots, r_n\}$ gives an open ball $\mathrm{B}_r(p)$ contained in all $\mathrm{B}_{r_1}(p), \ldots, \mathrm{B}_{r_n}(p)$. Thus $\mathrm{B}_r(p) \subseteq \mathcal{O}_1 \cap \cdots \cap \mathcal{O}_n$ follows, showing the last part. $\square$

Analogously to the open balls we define the *closed balls* to be

$$\mathrm{B}_r(p)^{\mathrm{cl}} = \{q \in M \mid d(p, q) \leq r\}, \tag{2.1.12}$$

where again $p \in M$ and $r > 0$. Then we have the trivial inclusion

$$\mathrm{B}_r(p) \subseteq \mathrm{B}_r(p)^{\mathrm{cl}} \tag{2.1.13}$$

for all $p \in M$ and all $r > 0$. Moreover, a subset $A \subseteq M$ is called *closed* if its complement $M \setminus A$ is open. We get the following characterization of closed subsets:

Proposition 2.1.5 *Let (M, d) be a metric space.*

(i) All closed balls are closed.
(ii) The empty set $\emptyset$ and M are closed.
(iii) If $\{A_i\}_{i \in I}$ is an arbitrary collection of closed subsets of M then $\bigcap_{i \in I} A_i$ is closed, too.
(iv) If $A_1, \ldots, A_n$ are finitely many closed subsets then their union $A_1 \cup \cdots \cup A_n$ is closed, too.

Proof Let $\mathrm{B}_r(p)^{\mathrm{cl}}$ be given and $q \in M \setminus \mathrm{B}_r(p)^{\mathrm{cl}}$. Then $d(p, q) > r$ and hence we find a $\delta > 0$ with $r + \delta < d(p, q)$. Now let $q' \in \mathrm{B}_\delta(q)$. Then by the triangle inequality

$$d(p, q) \leq d(p, q') + d(q', q) < d(p, q') + \delta,$$

and thus $d(p, q') > d(p, q) - \delta > r$ showing that $q' \in M \setminus \mathrm{B}_r(p)^{\mathrm{cl}}$. This gives the first part. The remaining parts are obtained from the corresponding statements on open subsets in Proposition 2.1.4 by passing to the complements. ☐

In a next step we recall the $\epsilon\delta$-definition of continuity and translate it into a statement involving only the open subsets. This is quite remarkable as the open subsets carry much less information than the metric:

Example 2.1.6 Let (M, d) be a metric space and define the metric d' as in Example 2.1.2, (vi). Then $\mathcal{O} \subseteq M$ is open with respect to d iff it is open with respect to d'. Indeed, the function $f(\xi) = \frac{\xi}{1+\xi}$ for $\xi \in [0, \infty)$ has the inverse function $f^{-1}(\eta) = \frac{\eta}{1-\eta}$. From this it follows that $d(p, q) < r$ iff $d'(p, q) < r'$ where $r' = f(r)$. This shows that the open balls for the two metrics coincide after rescaling the radii with the maps f and f^{-1}, respectively.

Definition 2.1.7 ($\epsilon\delta$*-Continuity*) Let $f\colon (M, d_M) \longrightarrow (N, d_N)$ be a map between metric spaces.

(i) The map f is called continuous at $p \in M$ if for all $\varepsilon > 0$ there is a $\delta > 0$ with

$$d_N(f(p), f(q)) < \varepsilon \quad \text{for} \quad d_M(p, q) < \delta. \tag{2.1.14}$$

(ii) The map f is called continuous if f is continuous at all points $p \in M$.

Proposition 2.1.8 *Let* $f : (M, d_M) \longrightarrow (N, d_N)$ *be a map between metric spaces.*

(i) The map f *is continuous at* $p \in M$ *iff for every open subset* $U \subseteq N$ *with* $f(p) \in U$ *the preimage* $f^{-1}(U)$ *contains an open subset* $\mathcal{O} \subseteq f^{-1}(U)$ *with* $p \in \mathcal{O}$.
(ii) The map f *is continuous iff the preimage* $f^{-1}(U)$ *of every open subset* U *of* N *is open in* M.

Proof Let f be continuous at p and let $U \subseteq N$ be open with $f(p) \in U$. Then there exists an open ball $\mathrm{B}_\varepsilon(f(p)) \subseteq U$ for some $\varepsilon > 0$. By assumption, we find a $\delta > 0$ for which (2.1.14) applies. Thus for $q \in \mathrm{B}_\delta(p)$ we have $f(q) \in \mathrm{B}_\varepsilon(f(p)) \subseteq U$ showing that $\mathrm{B}_\delta(p) \subseteq f^{-1}(U)$. Conversely, assume f fulfills the condition of (i) and let $\varepsilon > 0$ be given. Then $f^{-1}(\mathrm{B}_\varepsilon(f(p)))$ contains an open subset $\mathcal{O}$ which contains p. But then there is a $\delta > 0$ with $\mathrm{B}_\delta(p) \subseteq \mathcal{O} \subseteq f^{-1}(\mathrm{B}_\varepsilon(f(p)))$. This means that for $q \in M$ with $d_M(p, q) < \delta$ we have $f(q) \in \mathrm{B}_\varepsilon(f(p))$ which gives $d_N(f(p), f(q)) < \varepsilon$. Thus the first part is shown. For the second assume first that f is continuous and $\mathcal{O} \subseteq N$ is open. We can now apply the first part to every $p \in f^{-1}(\mathcal{O})$, to show that there is an open ball around p which is entirely contained in $f^{-1}(\mathcal{O})$. Thus $f^{-1}(\mathcal{O})$ is open. Conversely, assume $f^{-1}(\mathcal{O})$ is open for every open $\mathcal{O}$. Again, we can apply the first part to every $p \in M$ and every open ball $\mathrm{B}_\varepsilon(f(p))$ and get the openness of $f^{-1}(\mathrm{B}_\varepsilon(f(p)))$. Thus there is an open ball $\mathrm{B}_\delta(p) \subseteq f^{-1}(\mathrm{B}_\varepsilon(f(p)))$ which gives the continuity at p. □

Passing again to complements we get the following alternative characterization using closed subsets:

Corollary 2.1.9 *Let* $f : (M, d_M) \longrightarrow (N, d_N)$ *be a map between metric spaces. Then* f *is continuous iff the preimage of every closed subset of* N *is closed in* M.

In the formulation of the continuity at a given point the subsets which contain an open ball $\mathrm{B}_r(p)$ play a particular role. This justifies the following definition:

Definition 2.1.10 (*Neighbourhood*) Let (M, d) be a metric space and let $p \in M$. Then a subset $U \subseteq M$ is called neighbourhood of p if $p \in U$ and U contains an open ball $\mathrm{B}_r(p) \subseteq U$. The set of all neighbourhoods of p is denoted by $\mathfrak{U}(p)$.

Note that a neighbourhood U of p needs not to be open itself, though it always contains an open neighbourhood. The following proposition lists a few elementary properties of neighbourhoods:

Proposition 2.1.11 *Let* (M, d) *be a metric space and let* $p \in M$.

(i) If U *is a neighbourhood of* p *and* $U \subseteq U'$ *then* U' *is a neighbourhood of* p*, too.*
(ii) If $U_1, \ldots, U_n$ *are neighbourhoods of* p *then* $U_1 \cap \cdots \cap U_n$ *is a neighbourhood of* p*, too.*
(iii) Any neighbourhood of p *contains* p.

(iv) If U is a neighbourhood of p then there exists a neighbourhood $V \subseteq U$ of p such that V is a neighbourhood of all $q \in V$.

Proof The first part is clear since with $\mathrm{B}_r(p) \subseteq U$ we have also $\mathrm{B}_r(p) \subseteq U'$. The second part is obtained by taking again the minimum of the relevant radii. The third is clear. For the fourth part we note that an open subset, like an open ball, is a neighbourhood of all its points. This was precisely the content the definition of an open set in Definition 2.1.3, (ii). Then the fourth part follows by taking $V = \mathrm{B}_r(p) \subseteq U$ for a suitable radius $r > 0$. □

Using the notion of neighbourhoods we can rephrase the statement of Proposition 2.1.8, (i), as follows:

Corollary 2.1.12 *A map $f : (M, d_M) \longrightarrow (N, d_N)$ between metric spaces is continuous at p if the preimage of every neighbourhood of $f(p)$ is a neighbourhood of p.*

Again we see that the notion of a neighbourhood refers to the open subsets of M only: the detailed information about the metric is also lost here, the neighbourhood systems of the metric spaces (M, d) and (M, d') with d' as in Example 2.1.2, (vi), are the same even though the metrics are very different.

The next concept is also transferred easily from elementary calculus to metric spaces: convergence of sequences and the notion of completeness.

Definition 2.1.13 (*Convergence and completeness*) Let (M, d) be a metric space and let $(p_n)_{n\in\mathbb{N}}$ be a sequence in M.

(i) The sequence $(p_n)_{n\in\mathbb{N}}$ converges to $p \in M$ if for every $\varepsilon > 0$ one finds an $N \in \mathbb{N}$ such that for all $n \geq N$ one has

$$d(p, p_n) < \varepsilon. \tag{2.1.15}$$

(ii) The sequence $(p_n)_{n\in\mathbb{N}}$ is called a Cauchy sequence if for all $\varepsilon > 0$ one finds an $N \in \mathbb{N}$ such that for all $n, m \geq N$ one has

$$d(p_n, p_m) < \varepsilon. \tag{2.1.16}$$

(iii) The metric space (M, d) is called complete if every Cauchy sequence converges.

The third part is reasonable as every convergent sequence is clearly a Cauchy sequence by the triangle inequality. Again we can formulate convergence in form of open subsets and neighbourhoods alone. The notion of Cauchy sequences and completeness requires some additional structure which we shall not consider at the moment.

Proposition 2.1.14 *Let (M, d) be a metric space and let $(p_n)_{n\in\mathbb{N}}$ be a sequence in M. Then $(p_n)_{n\in\mathbb{N}}$ converges to $p \in M$ iff for every neighbourhood U of p there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $p_n \in U$.*

Proof The equivalence of the two statements is now easy to see, we leave this as a little task for the reader. □

As a conclusion of this discussion we arrive at the point of view that many features of metric spaces actually do not depend on the metric but only on the system of open subsets. This suggest to consider an axiomatized version of “open subsets” from the beginning and study this theory of topological spaces instead of the theory of metric spaces. It will turn out that this has several benefits:

(i) There will be important examples of “spaces” which can not be treated as metric spaces but as topological spaces only: among many other examples the notion of a differentiable manifold in differential geometry does not refer to a metric space from the beginning but to a topological space (even though it turns out a posteriori that they do carry a compatible metric structure). In functional analysis many important topological vector spaces are known to be “non-metrizable” like e.g. the space of test functions $\mathscr{C}_0^\infty(\mathbb{R})$. A precise formulation and a proof of this statement will require some more advanced technology which we will learn in the sequel, see Exercise 7.4.8.
(ii) Many concepts and proofs simplify drastically after taking the point of view of topological spaces compared to the usage of metric spaces, even though they also apply for metric spaces.
(iii) Topology will be an ideal playground to practice “axiomatization” of a mathematical concept, a technology which will be useful at many other places as well.

On the other hand one should not forget that the theory of metric spaces provides a finer structure and thus more specific features which will not be captured by topological spaces, the notion of Cauchy sequences and completeness is one first example. There is an appropriate axiomatization of this as well in the theory of uniform spaces, but for the time being we shall not touch this. Another important aspect of metric spaces will be the theory of coarse spaces or asymptotic geometry, certainly asking for a separate course.

2.2 Topological Spaces

We come now to the definition of a general topological space. There are at least two approaches: Historically, Hausdorff axiomatized the properties of the neighbourhoods of a point as found in [10, Chap. XII, §1]. Shortly later, Alexandroff axiomatized the properties of the system of open subsets, which is now the usual approach. In any case, both definitions turn out to be equivalent. For a given set M we denote the power set of M by 2^M.

The main idea is now very simple: we want to axiomatize the behaviour of open subsets in a metric space as found in Proposition 2.1.4 without explicit reference to the metric itself:

Definition 2.2.1 (*Topological space*) Let M be a set. Then a subset $\mathcal{M} \subseteq 2^M$ of the power set of M is called a topology if the following properties are fulfilled:

(i) The empty set $\emptyset$ and M are in $\mathcal{M}$.
(ii) If $\{\mathcal{O}_i\}_{i \in I}$ with $\mathcal{O}_i \in \mathcal{M}$ is an arbitrary collection of elements in $\mathcal{M}$ then also $\bigcup_{i \in I} \mathcal{O}_i \in \mathcal{M}$.
(iii) If $\mathcal{O}_1, \ldots, \mathcal{O}_n \in \mathcal{M}$ are finitely many elements in $\mathcal{M}$ then also $\mathcal{O}_1 \cap \cdots \cap \mathcal{O}_n \in \mathcal{M}$.

A set M together with a topology $\mathcal{M} \subseteq 2^M$ is called a topological space $(M, \mathcal{M})$ and the sets $\mathcal{O} \in \mathcal{M}$ are called the open subsets of $(M, \mathcal{M})$.

Other commonly used notations for topologies on M are τ_M or T_M, sometimes also just τ if the reference to M is clear.

Example 2.2.2 (Topologies) Let M be a set.

(i) The power set $\mathcal{M}_{\text{discrete}} = 2^M$ is a topology on M, called the *discrete* or *finest topology* of M.
(ii) Taking $\mathcal{M}_{\text{indiscrete}} = \{\emptyset, M\}$ gives a topology, called the *indiscrete* or *trivial* or *coarsest topology* of M.
(iii) If d is a metric for M then taking $\mathcal{M}$ to be the open subsets in the sense of Definition 2.1.3 gives a topology on M, called the *metric topology* of the metric space (M, d). From e.g. Example 2.1.6 and Exercise 2.7.1 we see that very different metrics can yield the same topology. Note also that the metric topology of the discrete metric on a set M from Example 2.1.2, (iv), is the discrete topology on M.
(iv) Consider the collection $\mathcal{M}_{\text{cofinite}} \subseteq 2^M$ of all those subsets which have *finite* complements and $\emptyset$. This yields the *cofinite topology* on M.

Complementary to the definition of open subsets we have the closed subsets:

Definition 2.2.3 (*Closed subset*) A subset $A \subseteq M$ of a topological space $(M, \mathcal{M})$ is called closed if $M \setminus A$ is open.

As in Proposition 2.1.5 we concluded that $\emptyset$ and M are both closed and finite unions as well as arbitrary intersections of closed subsets are again closed. Thus we can equivalently characterize a topological space by means of its closed subsets instead of the open ones.

The next observation will be important for constructing topologies:

Proposition 2.2.4 *If $\{\mathcal{M}_i\}_{i \in I}$ are topologies on M then also $\mathcal{M} = \bigcap_{i \in I} \mathcal{M}_i$ is a topology on M.*

Proof Note that here we take the intersection of subsets of the power set 2^M. Let $\mathcal{M} = \bigcap_{i \in I} \mathcal{M}_i \subseteq 2^M$. Since $\emptyset, M \in \mathcal{M}_i$ for all $i \in I$ we also have $\emptyset, M \in \mathcal{M}$. Now let $\{\mathcal{O}_j\}_{j \in J}$ be a collection of subsets in $\mathcal{M}$, i.e. $\mathcal{O}_j \in \mathcal{M}$. Then $\mathcal{O}_j \in \mathcal{M}_i$ for all $i \in I$ and hence also $\bigcup_{j \in J} \mathcal{O}_j \in \mathcal{M}_i$ for all $i \in I$, since $\mathcal{M}_i$ is a topology. But this means $\bigcup_{j \in J} \mathcal{O}_j \in \mathcal{M}$, too. Analogously, we get for $\mathcal{O}_1, \ldots, \mathcal{O}_n \in \mathcal{M}$ that $\mathcal{O}_1 \cap \cdots \cap \mathcal{O}_n \in \mathcal{M}$. $\square$

Given two topologies $\mathcal{M}_1, \mathcal{M}_2$ on M we want to compare them. Since $\mathcal{M}_1, \mathcal{M}_2 \in 2^M$ are both subsets of a common set, the power set, there is an obvious way of how to do that:

Definition 2.2.5 (*Finer and coarser*) Let $\mathcal{M}_1, \mathcal{M}_2 \subseteq 2^M$ be topologies on M. Then $\mathcal{M}_1$ is called finer than $\mathcal{M}_2$ if $\mathcal{M}_2 \subseteq \mathcal{M}_1$. In this case $\mathcal{M}_2$ is called coarser than $\mathcal{M}_1$.

This explains the notation in Example 2.2.2, (i) and (ii): the discrete topology is indeed the finest possible topology on M we can have while the indiscrete topology is the coarsest one. Note that we use "finer" in the sense of "finer or equal" but not in the sense of "strictly finer".

Often we have a space M together with a collection $\mathcal{S} \subseteq 2^M$ of subsets which we would like to be the open subsets of a topology. However, $\mathcal{S}$ may fail to be a topology directly. Thus one is looking for a topology containing $\mathcal{S}$ in a economical way: it should be as coarse as possible. Clearly the finest topology $\mathcal{M}_{\text{finest}} = 2^M$ will contain $\mathcal{S}$ for trivial reasons so this other extreme case is uninteresting.

Proposition 2.2.6 *Let $\mathcal{S} \subseteq 2^M$ be a subset of the power set of M containing $\emptyset$ and M.*

(i) There exists a unique topology $\mathcal{M}(\mathcal{S})$ which is coarser than every other topology containing $\mathcal{S}$.
(ii) This topology $\mathcal{M}(\mathcal{S})$ can be obtained by the following two-step procedure: first we take all finite intersections of subsets in $\mathcal{S}$ and afterwards we take arbitrary unions of the resulting subsets.

Proof Since $\mathcal{M}_{\text{discrete}} = 2^M$ is a topology containing $\mathcal{S}$ there is at least one topology containing $\mathcal{S}$. Now we take the intersection of *all* these topologies containing $\mathcal{S}$. By Proposition 2.2.4 this is again a topology, which still contains $\mathcal{S}$. Apparently, it is contained in any other topology containing $\mathcal{S}$ by the very construction. Thus it is also the unique one with this property, proving the first part. For the second part, let $\tilde{\mathcal{M}}(\mathcal{S})$ be the collection of subsets we get by first taking finite intersections and arbitrary unions afterwards. Clearly, $\tilde{\mathcal{M}}(\mathcal{S}) \subseteq \mathcal{M}(\mathcal{S})$ since a topology is stable under taking finite intersections and arbitrary unions. We have to show that they are equal. If $\mathcal{O}_i \in \tilde{\mathcal{M}}(\mathcal{S})$ for $i \in I$ with some index set I then each $\mathcal{O}_i = \bigcup_{j \in J_i} \mathcal{O}_{ij}$ and each $\mathcal{O}_{ij} = S_{ij1} \cap \cdots \cap S_{ijn}$ with $S_{ij1}, \ldots, S_{ijn} \in \mathcal{S}$ and n depending on i and j. Note that by repeating $\mathcal{O}_{ij}$'s we can assume that the index set $J = J_i$ is actually the same for all i. But then

$$\mathcal{O} = \bigcup_{i \in I} \mathcal{O}_i = \bigcup_{i \in I, j \in J} \mathcal{O}_{ij} = \bigcup_{i \in I, j \in J} S_{ij1} \cap \cdots \cap S_{ijn}$$

is again a union of the desired form. Hence $\mathcal{O} \in \tilde{\mathcal{M}}(\mathcal{S})$. For a finite I we get

$$\bigcap_{i \in I} \mathcal{O}_i = \bigcap_{i \in I} \bigcup_{j \in J} \mathcal{O}_{ij} = \bigcup_{j \in J} \bigcap_{i \in I} S_{ij1} \cap \cdots \cap S_{ijn}.$$

But the intersection is always finite for a fixed $j \in J$. Hence also $\mathcal{O} \in \tilde{\mathcal{M}}(\mathcal{S})$ proving that $\tilde{\mathcal{M}}(\mathcal{S})$ is a topology. By the first part it has to coincide with $\mathcal{M}(\mathcal{S})$. □

Of course the condition $\emptyset, M \in \mathcal{S}$ is easy to achieve by augmenting $\mathcal{S}$ if needed: we included this more by convenience. The construction in Proposition 2.2.4 allows to generate topologies very easily:

Example 2.2.7 The metric topology on a metric space (M, d) is the topology generated by the open balls, i.e. the coarsest topology containing all open balls. Indeed, an open subset $\mathcal{O} \subseteq M$ in the metric topology can be written as

$$\mathcal{O} = \bigcup_{p \in \mathcal{O}} \mathrm{B}_{r(p)}(p) \tag{2.2.1}$$

with suitable radii $r(p) > 0$ such that $\mathrm{B}_{r(p)}(p) \subseteq \mathcal{O}$. Thus here we do not even need to take finite intersections first. By Proposition 2.1.4 this is a topology, coarser than the one constructed in the second part of Proposition 2.2.6. By the first part of Proposition 2.2.6 it coincides with the coarsest one containing the open balls.

The subset $\mathcal{S}$ generating a topology is often very useful and justifies its own terminology:

Definition 2.2.8 (*Basis and Subbasis*) Let $(M, \mathcal{M})$ be a topological space and let $\mathcal{S}, \mathcal{B} \subseteq \mathcal{M}$ be subsets (containing already $\emptyset$ and M).

(i) The set $\mathcal{B}$ is called a basis of $\mathcal{M}$ if every open subset is a union of subsets from $\mathcal{B}$.
(ii) The set $\mathcal{S}$ is called a subbasis of $\mathcal{M}$ if the collection of all finite intersections of sets from $\mathcal{S}$ forms a basis of $\mathcal{M}$.

Clearly, a basis is also a subbasis and $\mathcal{S}$ is a subbasis if $\mathcal{M}(\mathcal{S}) = \mathcal{M}$. Thus, in view of Proposition 2.2.6, a subbasis seems to be the more important concept. For a metric space, Example 2.2.7 shows that the open balls form a basis and not just a subbasis: we do not need to take finite intersections of open balls, they are already obtained by taking suitable unions of smaller balls.

The last general construction we shall discuss here is given by the subspace topology. If $N \subseteq M$ is a subset of a topological space $(M, \mathcal{M})$ then we define

$$\mathcal{M}\big|_N = \left\{ \mathcal{O} \cap N \;\middle|\; \mathcal{O} \in \mathcal{M} \right\} \subseteq 2^N, \tag{2.2.2}$$

i.e. all subsets of N which are obtained by intersecting N with an open subset of M, see also Fig. 2.2.

Proposition 2.2.9 (Subspace topology) *Let $(M, \mathcal{M})$ be a topological space and $N \subseteq M$ a subset. Then $\mathcal{M}\big|_N$ is a topology for N.*

The verification is straightforward. Nevertheless, some caution is necessary: an open subset $U \subseteq N$ with respect to $\mathcal{M}\big|_N$ can also be considered as subset of M but then it fails to be open with respect to $\mathcal{M}$ in general. The same holds for closed subsets of $(N, \mathcal{M}\big|_N)$ viewed as subsets of $(M, \mathcal{M})$, see also Exercise 2.7.3.

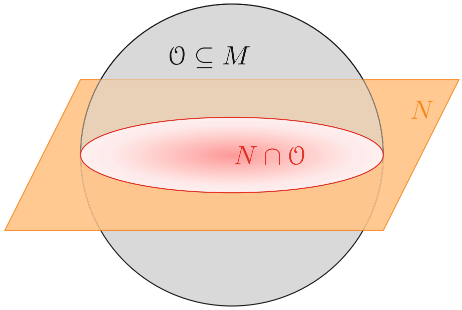


Fig. 2.2 An open set in the subspace topology

2.3 Neighbourhoods, Interiors, and Closures

As in the metric case we define now the neighbourhood system of a point in a topological space:

Definition 2.3.1 (*Neighbourhood*) Let $(M, \mathcal{M})$ be a topological space and $p \in M$.

(i) A subset $U \subseteq M$ is called neighbourhood of p if there exists an open subset $\mathcal{O} \subseteq M$ with $p \in \mathcal{O} \subseteq U$.
(ii) The collection of all neighbourhoods of p is called the neighbourhood system (or neighbourhood filter) of p, denoted by $\mathfrak{U}(p)$.

The properties of neighbourhoods in the metric case as discussed in Proposition 2.1.11 carry over to the general situation of a topological space:

Proposition 2.3.2 *Let $(M, \mathcal{M})$ be a topological space and $p \in M$.*

(i) A subset $\mathcal{O} \subseteq M$ is a neighbourhood of all of its points iff $\mathcal{O}$ is open.
(ii) For $U \in \mathfrak{U}(p)$ and $U \subseteq U'$ we have $U' \in \mathfrak{U}(p)$.
(iii) For $U_1, \ldots, U_n \in \mathfrak{U}(p)$ we have $U_1 \cap \cdots \cap U_n \in \mathfrak{U}(p)$.
(iv) For $U \in \mathfrak{U}(p)$ we have $p \in U$.
(v) For $U \in \mathfrak{U}(p)$ there exists a $V \in \mathfrak{U}(p)$ with $V \subseteq U$ and $V \in \mathfrak{U}(q)$ for all $q \in V$.

Proof Suppose $\mathcal{O}$ is open, then clearly $\mathcal{O}$ is a neighbourhood for all $p \in \mathcal{O}$. Thus assume $\mathcal{O} \in \mathfrak{U}(p)$ for all $p \in \mathcal{O}$. Then we get an open $\mathcal{O}_p \subseteq M$ for every $p \in \mathcal{O}$ with $p \in \mathcal{O}_p \subseteq \mathcal{O}$. It follows that $\mathcal{O} = \bigcup_{p \in \mathcal{O}} \mathcal{O}_p$ is open, proving the first part. The parts (ii)–(v) are now analogous to the statements of Proposition 2.1.11. $\square$

Remark 2.3.3 Conversely, given a non-empty system of subsets $\mathfrak{U}(p)$ for every $p \in M$ satisfying the properties (ii)–(v), which we then call a *system of neighbourhoods*, we can reconstruct a unique topology $\mathcal{M}$ on M such that the $\mathfrak{U}(p)$ are the neighbourhood systems with respect to $\mathcal{M}$: One defines $\mathcal{O} \subseteq M$ to be open if it is a neighbourhood of all of its points. This was Hausdorff's original approach in [10, Sect. Chap. VII, §1], see also Exercise 2.7.4.

Definition 2.3.4 (*Neighbourhood basis*) Let $(M, \mathcal{M})$ be a topological space and $p \in M$. Then a subset $\mathfrak{B}(p) \subseteq \mathfrak{U}(p)$ of neighbourhoods of $p \in M$ is called a neighbourhood basis of p if for every $U \in \mathfrak{U}(p)$ there is a $B \in \mathfrak{B}(p)$ with $B \subseteq U$.

Example 2.3.5 (Neighbourhood bases) Let M be a set.

(i) For a metric d on M and $p \in M$ the open balls $\mathrm{B}_{r_n}(p)$ with $r_n > 0$ being a zero sequence constitute a *countable* neighbourhood basis for the metric topology.
(ii) For the discrete topology $\mathcal{M}_{\text{discrete}}$ on M the set $\{\{p\}\} = \mathfrak{B}(p)$ is a *finite* neighbourhood basis since $\{p\}$ is open.
(iii) Let M be uncountable and consider the cofinite topology $\mathcal{M}_{\text{cofinite}}$. Then a neighbourhood basis of $p \in M$ is necessarily uncountable. While this seems to be a rather artificial example there are (more non-trivial) examples of function spaces in functional analysis which have topologies such that neighbourhood bases are very large, i.e. uncountable.

To capture this phenomenon of having very different sizes of neighbourhood bases, one introduces the following two countability axioms:

Definition 2.3.6 (*First and second countability*) Let $(M, \mathcal{M})$ be a topological space.

(i) The space M is called first countable (at $p \in M$) if every point (the point p) has a countable neighbourhood basis.
(ii) The space M is called second countable if $\mathcal{M}$ has a countable basis.

Clearly second countable implies first countable, the converse needs not to be true. The second countability plays an important role in e.g. differential geometry where it guarantees that manifolds do not become “too big”.

Example 2.3.7 The metric topology of $\mathbb{R}^n$ is second countable as it is sufficient to consider only those open balls $\mathrm{B}_r(x)$ where $r > 0$ is rational and $x \in \mathbb{Q}^n$, see Exercise 2.7.6.

Given an arbitrary subset $A \subseteq M$ of a topological space $(M, \mathcal{M})$ one can construct certain other subsets using the topology. The following definitions are motivated by the geometric intuition in $\mathbb{R}^n$:

Definition 2.3.8 (*Interior, closure and boundary*) Let $(M, \mathcal{M})$ be a topological space and $A \subseteq M$.

(i) A point $p \in M$ is called inner point of A if $A \in \mathfrak{U}(p)$.
(ii) The interior A° of A is the set of all inner points of A.
(iii) A point $p \in M$ is called boundary point of A if for every neighbourhood $U \in \mathfrak{U}(p)$ we have $A \cap U \neq \emptyset \neq (M \setminus A) \cap U$.
(iv) The boundary ∂A of A is the set of all boundary points of A.
(v) The closure A^{cl} of A is the set of all points $p \in M$ such that all $U \in \mathfrak{U}(p)$ satisfy $U \cap A \neq \emptyset$.

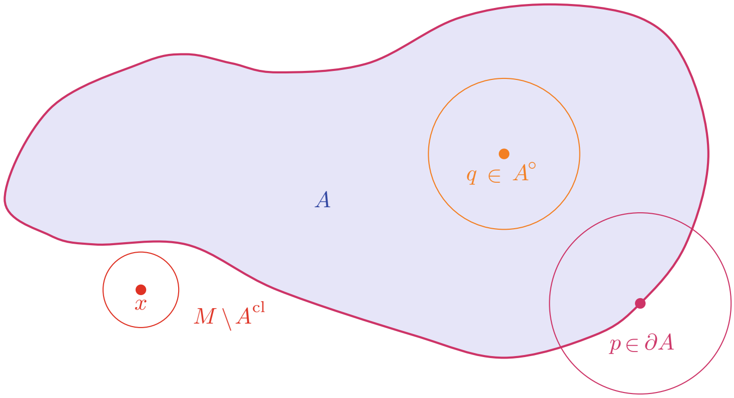


Fig. 2.3 The open interior, the boundary and the closure of a subset A: one has $q \in A^\circ$, $p \in \partial A$, and $x \in M \setminus A^{\mathrm{cl}}$

A commonly used alternative notation is $\overline{A}$ for the closure A^{cl} and $\mathrm{int}(A)$ for the interior A°. The geometric intuition behind these definitions can easily be visualized, see Fig. 2.3. We get the following alternative characterizations of the interior, the closure, and the boundary.

Proposition 2.3.9 *Let $(M, \mathcal{M})$ be a topological space and $A \subseteq M$.*

(i) The interior A° of A is the largest open subset inside A.
(ii) The closure A^{cl} of A is the smallest closed subset containing A.
(iii) The boundary ∂A of A is closed and $\partial A = A^{\mathrm{cl}} \setminus A^\circ$.

Proof First we show that A° is open at all. For $p \in A^\circ$ we know $A \in \mathfrak{U}(p)$ and hence there is an open subset $U \subseteq A$ with $p \in U$. But then for all $q \in U$ we have $A \in \mathfrak{U}(q)$ as well, showing $q \in A^\circ$, too. Hence $U \subseteq A^\circ$ follows, i.e. $A^\circ \in \mathfrak{U}(p)$. Since this holds for all $p \in A^\circ$, we have an open subset A° by Proposition 2.3.2, (i). Conversely, if $\mathcal{O} \subseteq A$ is an open subset then $p \in \mathcal{O} \subseteq A$ is an interior point of A, showing $\mathcal{O} \subseteq A^\circ$. Thus the first statement follows. For the second part, let $p \notin A^{\mathrm{cl}}$ then there exists a $U \in \mathfrak{U}(p)$ with $U \cap A = \emptyset$. Without restriction, we can assume U to be open. But then $U \in \mathfrak{U}(q)$ for all $q \in U$ showing that also $q \notin A^{\mathrm{cl}}$. Hence $U \subseteq M \setminus A^{\mathrm{cl}}$ implies that for every point $p \in M \setminus A^{\mathrm{cl}}$ a whole open neighbourhood of p is in $M \setminus A^{\mathrm{cl}}$. Thus $M \setminus A^{\mathrm{cl}}$ is open and A^{cl} is closed. Clearly $A \subseteq A^{\mathrm{cl}}$. Now let $\tilde{A} = \bigcap_{A \subseteq B,\, B \text{ closed}} B$ be the intersection of all closed subsets containing A. Clearly $\tilde{A}$ is the smallest closed subset containing A and thus $\tilde{A} \subseteq A^{\mathrm{cl}}$. Passing to complements gives

$$M \setminus \tilde{A} = \bigcup_{\substack{U \subseteq M \setminus A \\ U \text{ is open}}} U,$$

and hence for $p \in M \setminus \tilde{A}$ one has an open subset $U \subseteq M \setminus A$ with $p \in U$. But then $p \in U$ with $U \cap A = \emptyset$ shows $p \notin A^{\mathrm{cl}}$. Thus $A^{\mathrm{cl}} \cap (M \setminus \tilde{A}) = \emptyset$ or $A^{\mathrm{cl}} \subseteq \tilde{A}$ which proves (ii). For the last part we first note that $\partial A \subseteq A^{\mathrm{cl}}$. In fact, the definition shows

$$\partial A = A^{\mathrm{cl}} \cap (M \setminus A)^{\mathrm{cl}},$$

and hence ∂A is closed by (ii). Now let $p \in A^{\mathrm{cl}} \setminus \partial A$ then for all $U \in \mathfrak{U}(p)$ we have $U \cap A \neq \emptyset$ but there is a $U \in \mathfrak{U}(p)$ with $U \cap (M \setminus A) = \emptyset$. For this U we have $U \subseteq A$. This shows that every point $p \in A^{\mathrm{cl}} \setminus \partial A$ is inner, i.e. $A^{\mathrm{cl}} \setminus \partial A \subseteq A^{\circ}$. Conversely, for an interior point $p \in A^{\circ}$ we clearly have $p \notin \partial A$ but $p \in A^{\mathrm{cl}}$, showing the last part. □

By taking the closure, a subset is enlarged. Taking the open interior enlarges the complement, in both cases by the missing boundary points. The following two extreme cases will be of particular interest:

Definition 2.3.10 (*Dense and nowhere dense*) Let $(M, \mathcal{M})$ be a topological space.

(i) A subset $A \subseteq M$ is called dense if $A^{\mathrm{cl}} = M$.
(ii) A subset $A \subseteq M$ is called nowhere dense if $(A^{\mathrm{cl}})^{\circ} = \emptyset$.

Note that nowhere dense is strictly stronger than not dense, examples will be discussed in Exercise 2.7.24, see also Exercise 2.7.12.

We collect now some useful properties of closures, open interiors and boundaries and their behaviour with respect to unions, intersections, and complements.

Proposition 2.3.11 *Let $(M, \mathcal{M})$ be a topological space and let $A, B \subseteq M$ be subsets.*

(i) One has $\emptyset^{\circ} = \emptyset = \emptyset^{\mathrm{cl}}$ and $M^{\circ} = M = M^{\mathrm{cl}}$ as well as $\partial\emptyset = \emptyset = \partial M$.
(ii) One has $A^{\circ} \subseteq A \subseteq A^{\mathrm{cl}}$ and

$$(A^{\circ})^{\circ} = A^{\circ}, \quad \partial(\partial A) \subseteq \partial A, \quad \textit{and} \quad (A^{\mathrm{cl}})^{\mathrm{cl}} = A^{\mathrm{cl}}. \tag{2.3.1}$$

(iii) For $A \subseteq B$ one has $A^{\circ} \subseteq B^{\circ}$ and $A^{\mathrm{cl}} \subseteq B^{\mathrm{cl}}$.
(iv) One has

$$A^{\circ} \cup B^{\circ} \subseteq (A \cup B)^{\circ}, \quad \partial(A \cup B) \subseteq \partial A \cup \partial B, \quad \textit{and} \quad A^{\mathrm{cl}} \cup B^{\mathrm{cl}} = (A \cup B)^{\mathrm{cl}}. \tag{2.3.2}$$

(v) One has

$$A^{\circ} \cap B^{\circ} = (A \cap B)^{\circ} \quad \textit{and} \quad (A \cap B)^{\mathrm{cl}} \subseteq A^{\mathrm{cl}} \cap B^{\mathrm{cl}}. \tag{2.3.3}$$

(vi) One has

$$(M \setminus A)^{\circ} = M \setminus A^{\mathrm{cl}}, \quad \partial(M \setminus A) = \partial A, \quad \textit{and} \quad (M \setminus A)^{\mathrm{cl}} = M \setminus A^{\circ}. \tag{2.3.4}$$

Proof The first part is clear, either directly using the definitions or the characterizations from Proposition 2.3.9. For the second, an inner point of A is in particular a point in A and a point in A is in the closure since every neighbourhood $U \in \mathfrak{U}(p)$ intersects A at least in $\{p\}$ itself. Thus $A^\circ \subseteq A \subseteq A^{\mathrm{cl}}$ follows. Since A° is the largest open set inside A, we can apply this to A° and get that $(A^\circ)^\circ$ is the largest open set inside A°. Since A° is already open, we have $(A^\circ)^\circ = A^\circ$. Analogously, we can argue for the closure to get $(A^{\mathrm{cl}})^{\mathrm{cl}} = A^{\mathrm{cl}}$. Since a boundary is always closed, we have from Proposition 2.3.9, (iii), the relation $\partial(\partial A) = (\partial A)^{\mathrm{cl}} \setminus (\partial A)^\circ = \partial A \setminus (\partial A)^\circ \subseteq \partial A$, completing the second part. The third part is again clear from Proposition 2.3.9, (i) and (ii). For the fourth part we have $A^\circ \subseteq A \subseteq A \cup B$ and also $B^\circ \subseteq A \cup B$. Thus $A^\circ \cup B^\circ \subseteq A \cup B$ is an open subset of $A \cup B$ and hence contained in the largest such open subset, i.e. in $(A \cup B)^\circ$. Next, $A \subseteq A \cup B$ implies $A^{\mathrm{cl}} \subseteq (A \cup B)^{\mathrm{cl}}$ and analogously $B^{\mathrm{cl}} \subseteq (A \cup B)^{\mathrm{cl}}$ showing $A^{\mathrm{cl}} \cup B^{\mathrm{cl}} \subseteq (A \cup B)^{\mathrm{cl}}$. Hence $A^{\mathrm{cl}} \cup B^{\mathrm{cl}}$ is a closed subset containing $A \cup B$. Since the smallest such closed subset is $(A \cup B)^{\mathrm{cl}}$ we get $(A \cup B)^{\mathrm{cl}} \subseteq A^{\mathrm{cl}} \cup B^{\mathrm{cl}}$. Thus we have equality $A^{\mathrm{cl}} \cup B^{\mathrm{cl}} = (A \cup B)^{\mathrm{cl}}$. Finally, we get

$$\begin{aligned} \partial(A \cup B) &= (A \cup B)^{\mathrm{cl}} \setminus (A \cup B)^\circ \\ &\subseteq (A^{\mathrm{cl}} \cup B^{\mathrm{cl}}) \setminus (A^\circ \cup B^\circ) \\ &\subseteq (A^{\mathrm{cl}} \setminus A^\circ) \cup (B^{\mathrm{cl}} \setminus B^\circ) \\ &= \partial A \cup \partial B, \end{aligned}$$

since $A^\circ \subseteq A^{\mathrm{cl}}$ and $B^\circ \subseteq B^{\mathrm{cl}}$ together with the relations we already obtained for $(A \cup B)^{\mathrm{cl}}$ and $(A \cup B)^\circ$. This completes the fourth part. For the fifth we argue dually: $A \cap B \subseteq A, B$ shows $(A \cap B)^\circ \subseteq A^\circ \cap B^\circ$. However $A^\circ \subseteq A$ and $B^\circ \subseteq B$ gives $A^\circ \cap B^\circ \subseteq A \cap B$ and hence $A^\circ \cap B^\circ$ is an open subset inside $A \cap B$, the largest with this property is $(A \cap B)^\circ$. Hence they coincide. Next $A \cap B \subseteq A, B$ gives $(A \cap B)^{\mathrm{cl}} \subseteq A^{\mathrm{cl}} \cap B^{\mathrm{cl}}$ at once, showing the fifth part. For the last part, we first notice that the definition of a boundary point of A is symmetric in A and $M \setminus A$. Thus $\partial(M \setminus A) = \partial A$ follows immediately. Now suppose $p \in M \setminus A^\circ$. This is equivalent to $p \notin A^\circ$ and thus equivalent to $A \notin \mathfrak{U}(p)$. But A is not a neighbourhood of p iff for all neighbourhoods $U \in \mathfrak{U}(p)$ of p we have $U \cap (M \setminus A) \neq \emptyset$. This means that $p \in (M \setminus A)^{\mathrm{cl}}$ showing the equality $(M \setminus A)^{\mathrm{cl}} = M \setminus A^\circ$. Using this we get $M \setminus (M \setminus A)^\circ = (M \setminus (M \setminus A))^{\mathrm{cl}} = A^{\mathrm{cl}}$ and thus $M \setminus A^{\mathrm{cl}} = (M \setminus A)^\circ$. □

In Exercise 2.7.9 one can find examples of subsets where all the above inclusions are shown to be proper: thus they cannot be improved in general.

2.4 Continuous Maps

We come now to the central definition of continuity of maps. Motivated by the considerations in Proposition 2.1.8 one states the following definition:

Definition 2.4.1 (*Continuity*) Let $f\colon (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be a map between topological spaces.

(i) The map f is called continuous at $p \in M$ if for every neighbourhood $U \in \mathfrak{U}(f(p))$ of $f(p)$ also $f^{-1}(U)$ is a neighbourhood of p.
(ii) The map f is called continuous if the preimage of every open subset of N is open in M.
(iii) The set of continuous maps will be denoted by

$$\mathscr{C}(M, N) = \left\{ f\colon M \longrightarrow N \;\middle|\; f \text{ is continuous} \right\}, \tag{2.4.1}$$

and we set $\mathscr{C}(M) = \mathscr{C}(M, \mathbb{C})$ for the complex-valued continuous functions on M.

We know by Proposition 2.1.8 that this reproduces the $\epsilon\delta$-continuity for metric spaces. Moreover, also in this more general situation the two notions are consistent in the following sense:

Proposition 2.4.2 *Let $f\colon (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be a map between topological spaces. Then the following statements are equivalent:*

(i) The map f is continuous at every point.
(ii) The map f is continuous.
(iii) The subset $f^{-1}(A) \subseteq M$ is closed for every closed $A \subseteq N$.
(iv) The subset $f^{-1}(\mathcal{O})$ is open for every $\mathcal{O} \in \mathcal{S}$ in a subbasis $\mathcal{S}$ of N.

Proof The equivalence of (ii) and (iii) is clear by taking complements. Assume (i), and let $\mathcal{O} \subseteq N$ be open and let $p \in f^{-1}(\mathcal{O})$. Then $f(p) \in \mathcal{O}$ shows $\mathcal{O} \in \mathfrak{U}(f(p))$ and thus $f^{-1}(\mathcal{O}) \in \mathfrak{U}(p)$. Since this holds for all $p \in f^{-1}(\mathcal{O})$, we have $f^{-1}(\mathcal{O})$ open. This gives (i) $\implies$ (ii). Conversely, suppose (ii), and let $U \in \mathfrak{U}(f(p))$. Then there is an open $\mathcal{O} \subseteq U$ with $f(p) \in \mathcal{O}$ and hence $p \in f^{-1}(\mathcal{O}) \in \mathfrak{U}(p)$, since $f^{-1}(\mathcal{O})$ is open by continuity. But then $f^{-1}(\mathcal{O}) \subseteq f^{-1}(U)$ shows $f^{-1}(U) \in \mathfrak{U}(p)$ giving (ii) $\implies$ (i) Finally, the compatibility of $\bigcap$ and $\bigcup$ with preimages shows the equivalence of (ii) and (iv). □

In particular, the fourth part is often very convenient for checking continuity as we can use a rather small and easy subbasis instead of the typically huge and complicated topology.

To show the efficiency of this definition of continuity we first prove the following statements on compositions of maps:

Proposition 2.4.3 *Let $f\colon (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ and $g\colon (N, \mathcal{N}) \longrightarrow (K, \mathcal{K})$ be maps between topological spaces.*

(i) If f is continuous at $p \in M$ and g is continuous at $f(p) \in N$ then $g \circ f$ is continuous at p.
(ii) If f and g are continuous then $g \circ f$ is continuous.

Proof Both statements rely on the simple fact that the preimage maps

$$f^{-1}\colon 2^N \longrightarrow 2^M \quad \text{and} \quad g^{-1}\colon 2^K \longrightarrow 2^N$$

satisfy

$$(g \circ f)^{-1} = f^{-1} \circ g^{-1}.$$

Then the preimages of neighbourhoods are mapped to neighbourhoods and the preimages of open subsets are mapped again to open subsets. □

Remark 2.4.4 The very same argument is used in measure theory to show that the composition of measurable maps is again measurable.

While the preimages of open or closed subsets behave nicely under continuous maps, the images will show no particularly simple behaviour in general. The image of the continuous map

$$f\colon \mathbb{R} \ni x \;\mapsto\; \frac{1}{1+x^2} \in \mathbb{R} \tag{2.4.2}$$

of the open (or closed) subset $\mathbb{R}$ is the half-open interval $(0, 1]$. Thus the following definitions provide additional features of maps:

Definition 2.4.5 (*Open and closed maps*) Let $f\colon (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be a map between topological spaces.

(i) The map f is called open if $f(\mathcal{O}) \subseteq N$ is open for all open $\mathcal{O} \subseteq M$.
(ii) The map f is called closed if $f(A) \subseteq N$ is closed for every closed subset $A \subseteq M$.

Suppose that a point $q \in N$ yields a closed subset $\{q\} \subseteq N$ then a constant map $f\colon M \ni p \mapsto q \in N$ is always closed since it simply maps every subset of M to a closed subset. It is also continuous but typically not open unless the single point $\{q\}$ is also an open subset of N. Consider the projection $\mathrm{pr}_1 : \mathbb{R}^2 \longrightarrow \mathbb{R}$ onto the first component. This is an open map but not a closed map, see Fig. 2.4. Thus the notions of continuous, open, and closed maps are rather independent, see also Exercise 2.7.18.

Finally, we introduce the notion of "isomorphism" between topological spaces. Even though isomorphism would be a conceptually more appropriate name, we stick to the traditional notion:

Definition 2.4.6 (*Homeomorphism*) Let $f\colon (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be a map between topological spaces.

(i) The map f is called a homeomorphism if f is bijective, continuous, and if f^{-1} is continuous.
(ii) If there is a homeomorphism $f\colon (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ then the spaces $(M, \mathcal{M})$ and $(N, \mathcal{N})$ are called homeomorphic.

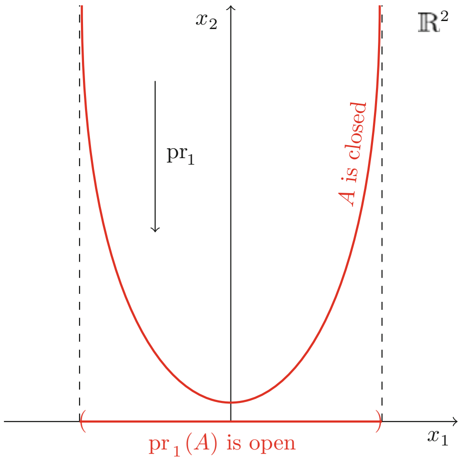


Fig. 2.4 The image of a closed curve in $\mathbb{R}^2$ under the continuous projection onto the first coordinates may be open

(iii) The map f is called an embedding if f is injective and if

$$f\colon (M, \mathcal{M}) \longrightarrow \Big(f(M), \mathcal{N}\big|_{f(M)}\Big) \tag{2.4.3}$$

is a homeomorphism.

In Exercise 2.7.15 we have an example that the inverse of a continuous bijection needs not to be continuous at all. Hence the requirement of the continuity of f^{-1} in the definition is *not* superfluous. Moreover, being homeomorphic is clearly an equivalence relation: the definition is symmetric in M and N and the composition of homeomorphisms is again a homeomorphism. Finally, we note that an injective continuous map is in general *not* an embedding, see again Exercise 2.7.15.

Example 2.4.7 Let $(M, \mathcal{M})$ be a topological space and let $N \subseteq M$ be a subset. Endow N with the subspace topology $\mathcal{N} = \mathcal{M}\big|_N$. Then the canonical inclusion map

$$\iota\colon N \longrightarrow M, \tag{2.4.4}$$

which identifies the points of N as (particular) points of M, is continuous and even an embedding. Indeed, for $\mathcal{O} \subseteq M$ open the preimage of $\mathcal{O}$ is $\iota^{-1}(\mathcal{O}) = N \cap \mathcal{O}$ which is open in $\mathcal{N}$ by the very definition of the subspace topology. Moreover, ι is clearly a bijection onto its image. Finally, the homeomorphism property is clear by the very definition. This example is the prototype of an embedding and motivates the name.

The next proposition gives some useful equivalent characterizations of homeomorphisms:

Proposition 2.4.8 *Let* $f\colon (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ *be a map between topological spaces. Then the following statements are equivalent:*

(i) The map f is a homeomorphism.
(ii) The map f is continuous, bijective, and open.
(iii) The map f is continuous, bijective, and closed.
(iv) The map f is continuous and there exists a continuous map $g\colon N \longrightarrow M$ with $g \circ f = \mathsf{id}_M$ and $f \circ g = \mathsf{id}_N$.

Proof The equivalence of (i) and (iv) is clear by taking $g = f^{-1}$. Suppose (i) and let $\mathcal{O} \subseteq M$ be open. Since f^{-1} is continuous, $(f^{-1})^{-1}(\mathcal{O}) \subseteq N$ is open. But this is just $f(\mathcal{O})$, showing that f is open. Note that here we have two different meanings of "-1": we take the preimage of $\mathcal{O}$ under the inverse map. Analogously, one shows (i) $\implies$ (iii). Now assume (ii) and let $\mathcal{O} \subseteq M$ be open. Then $f(\mathcal{O}) \subseteq N$ is open as f is an open map. But again $f(\mathcal{O})$ is the preimage of $\mathcal{O}$ under the inverse map of f^{-1}, showing that f^{-1} is continuous. Hence (ii) $\implies$ (i) follows. Finally, (iii) $\implies$ (i) is again analogous. □

Let us conclude this section with some more conceptual aspects. Recall that a *category* $\mathfrak{C}$ consists of a class of *objects* $\mathsf{Obj}(\mathfrak{C})$ and a set $\mathsf{Morph}(a, b)$ for any two objects $a, b \in \mathsf{Obj}(\mathfrak{C})$, the *morphisms* from a to b, such that one has a *composition*

$$\circ\colon \mathsf{Morph}(b, c) \times \mathsf{Morph}(a, b) \longrightarrow \mathsf{Morph}(a, c) \tag{2.4.5}$$

and a *unit morphism* $\mathsf{id}_a \in \mathsf{Morph}(a, a)$ such that the composition of morphisms is associative whenever it is defined and id_a serves as unit for the composition whenever it can be composed. More background information on the theory of categories can e.g. be found in [24]. It is common and useful to depict the morphisms as arrows between the objects which can be composed whenever the tail and the head match. There are many examples of categories in mathematics: in general one can say that whenever one introduces a new type of structure on certain sets one should immediately ask for the structure preserving maps. Together this should yield a category.

Example 2.4.9 Without verifying the properties of a category, we just list some well-known examples:

(i) The category Set of sets with maps as arrows between them.
(ii) The category Group of groups with group homomorphisms as arrows between them.
(iii) The category Ring of unital rings with unital ring homomorphisms as arrows between them.
(iv) The category of complex vector spaces $\mathsf{Vect}_{\mathbb{C}}$ with linear maps as arrows between them.
(v) The trivial category $\mathbb{1}$ with one object 1 and one arrow id_1.

In a category $\mathfrak{C}$ one calls two objects $a, b \in \mathsf{Obj}(\mathfrak{C})$ *isomorphic* if there are morphisms $f \in \mathsf{Morph}(a, b)$ and $g \in \mathsf{Morph}(b, a)$ with $f \circ g = \mathsf{id}_b$ and $g \circ f = \mathsf{id}_a$. Clearly, in all the above examples this gives then the correct notion of isomorphisms.

The conclusion of this section can now be rephrased as follows: we have found a category of topological spaces with continuous maps as morphisms between them:

Proposition 2.4.10 *The topological spaces form a category* top *with respect to the continuous maps as morphisms between them. The isomorphisms in* top *are precisely the homeomorphisms.*

Proof The main point is that the composition of continuous maps is again continuous and that the identity map $\mathsf{id}_M : M \longrightarrow M$ is continuous, too. The associativity is always fulfilled for compositions of maps. Finally, the homeomorphisms are the isomorphisms thanks to Proposition 2.4.8, (iv). □

2.5 Connectedness

In this short section we discuss some further easy properties of topological spaces: connectedness and path-connectedness. The motivation of the definition of a connected topological space originates form the following observation:

Lemma 2.5.1 *Consider the closed interval* $M = [0, 1]$ *with its usual topology. Suppose we have two open subsets* $\mathcal{O}_1, \mathcal{O}_2 \subseteq [0, 1]$ *with* $\mathcal{O}_1 \cup \mathcal{O}_2 = [0, 1]$ *and* $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$*. Then necessarily* $\mathcal{O}_1$ *and* $\mathcal{O}_2$ *are just* $[0, 1]$ *and* $\emptyset$*.*

Proof Suppose we have two such open subsets $\mathcal{O}_1, \mathcal{O}_2$ in $[0, 1]$, both non-empty. Without restriction we find $x \in \mathcal{O}_1$ and $y \in \mathcal{O}_2$ such that $0 < x < y < 1$. Indeed, the openness of $\mathcal{O}_1$ and $\mathcal{O}_2$ allows to find more points than just the boundary points 0 and 1 inside $\mathcal{O}_1$ and $\mathcal{O}_2$. Now consider all those numbers $\xi \in [0, 1]$ with

$$[x, \xi] \subseteq \mathcal{O}_1$$

and define z to be their supremum. Since $[0, 1]$ is closed, $z \in [0, 1]$. Moreover, since $0 < x$ we get $0 < z$ and since $y < 1$ we also get $z < 1$, again by the openness of $\mathcal{O}_1$ and $\mathcal{O}_2$. Suppose $z \in \mathcal{O}_1$ then also the open interval $(z - \varepsilon, z + \varepsilon) \subseteq \mathcal{O}_1$ is in $\mathcal{O}_1$ for some small enough $\varepsilon > 0$ since $\mathcal{O}_1$ is open. But in this case $[x, z + \frac{\varepsilon}{2}] \subseteq \mathcal{O}_1$ contradicting the supremum property of z. Thus $z \in \mathcal{O}_2$ as $\mathcal{O}_1 \cup \mathcal{O}_2$ is the whole interval. But then again $(z - \varepsilon, z + \varepsilon) \subseteq \mathcal{O}_2$ by openness of $\mathcal{O}_2$ for some small $\varepsilon > 0$. Hence $z - \frac{\varepsilon}{2} \in \mathcal{O}_2$ can not be in $\mathcal{O}_1$, contradicting again the supremum property of z. This is the final contradiction yielding the proof. □

Definition 2.5.2 (*Connectedness*) Let $(M, \mathcal{M})$ be a topological space. Then M is called connected if there are no two open, disjoint subsets $\mathcal{O}_1, \mathcal{O}_2 \subseteq M$ with $\mathcal{O}_1 \cup \mathcal{O}_2 = M$ beside M and $\emptyset$. A subset $A \subseteq M$ is called connected if $(A, \mathcal{M}\big|_A)$ is connected.

Corollary 2.5.3 *The unit interval* $[0, 1]$ *is connected.*

With an analogous argument one shows that all other types of intervals in $\mathbb{R}$ like $[a, b]$, $(a, b]$, $[a, b)$, and (a, b) for $-\infty \leq a \leq b \leq \infty$ are connected, too. In fact, these are the only subsets of $\mathbb{R}$ which are connected:

Proposition 2.5.4 *Let $A \subseteq \mathbb{R}$ and $a, b \in A$. If A is connected, then $[a, b] \subseteq A$.*

Proof Suppose $z \in [a, b]$ does not belong to A. Then $(-\infty, z)$ and (z, ∞) are both open subsets of $\mathbb{R}$ and hence $\mathcal{O}_1 = A \cap (-\infty, z)$ as well as $\mathcal{O}_2 = A \cap (z, \infty)$ are open in the subspace topology of A. By assumption $\mathcal{O}_1 \cup \mathcal{O}_2 = A$, $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$, and $a \in \mathcal{O}_1$ while $b \in \mathcal{O}_2$ as z is different from a and b. This contradicts the connectedness of A. □

Connectedness behaves well under continuous maps:

Proposition 2.5.5 *Let $f: (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be a continuous map between topological spaces. If M is connected then $f(M)$ is connected, too.*

Proof Suppose $f(M)$ is not connected and let $\mathcal{O}_1, \mathcal{O}_2 \subseteq f(M)$ be open and disjoint with $f(M) = \mathcal{O}_1 \cup \mathcal{O}_1$ but $\mathcal{O}_1, \mathcal{O}_2$ both be non-empty. Then we find $U_1, U_2 \subseteq N$ open with $\mathcal{O}_1 = f(M) \cap U_1$ and $\mathcal{O}_2 = f(M) \cap U_2$. Moreover, $f^{-1}(U_k) = f^{-1}(U_k \cap f(M)) = f^{-1}(\mathcal{O}_k)$ for $k = 1, 2$ shows that $f^{-1}(\mathcal{O}_1)$ and $f^{-1}(\mathcal{O}_2)$ are open in M, both non-empty, and still disjoint with $f^{-1}(\mathcal{O}_1) \cup f^{-1}(\mathcal{O}_2) = M$. But this contradicts the connectedness of M. □

This simple fact together with the result of Proposition 2.5.4 can be seen as the topological "reason" for the intermediate value theorem in calculus: The continuous image of an interval is again an interval.

Corollary 2.5.6 (Intermediate value theorem) *Let $f: (M, \mathcal{M}) \longrightarrow \mathbb{R}$ be a continuous function on a connected topological space. If $a, b \in f(M)$ then also $[a, b] \subseteq f(M)$.*

Connectedness can also be understood by the idea that we can join any two points by a continuous path. This motivates the following definition:

Definition 2.5.7 (*Path-Connectedness*) Let $(M, \mathcal{M})$ be a topological space.

(i) A path in M is a continuous map $\gamma: [0, 1] \longrightarrow M$.
(ii) The space M is called path-connected if for any $p, q \in M$ one finds a path γ with

$$\gamma(0) = p \quad \text{and} \quad \gamma(1) = q. \tag{2.5.1}$$

Since we can reparametrize the "time" variable t of a path, there is no real restriction in requiring the domain of definition to be $[0, 1]$ instead of $[a, b] \subseteq \mathbb{R}$ for some $a < b$. We have now the following statement:

Proposition 2.5.8 *Let $(M, \mathcal{M})$ be a path-connected topological space. Then M is connected, too.*

Proof Suppose M is not connected and let $\mathcal{O}_1, \mathcal{O}_2 \subseteq M$ be open, disjoint, $M = \mathcal{O}_1 \cup \mathcal{O}_2$ with both $\mathcal{O}_1, \mathcal{O}_2$ being non-empty. Then let $p \in \mathcal{O}_1$ and $q \in \mathcal{O}_2$ and join them by a continuous path $\gamma: [0, 1] \longrightarrow M$, i.e. $\gamma(0) = p$ and $\gamma(1) = q$. Then $\gamma^{-1}(\mathcal{O}_1)$, $\gamma^{-1}(\mathcal{O}_2)$ are open, disjoint, both non-empty as $0 \in \gamma^{-1}(\mathcal{O}_1)$, $1 \in \gamma^{-1}(\mathcal{O}_2)$ and $[0, 1] = \gamma^{-1}(\mathcal{O}_1) \cup \gamma^{-1}(\mathcal{O}_2)$ since $\mathcal{O}_1 \cup \mathcal{O}_2 = M$. This contradicts the connectedness of $[0, 1]$. □

In general, the reverse implication is not true: there are connected spaces which are not path-connected, see Exercise 2.7.23. Moreover, since the compositions of continuous maps are continuous, it is trivial to see that the image of a path-connected topological space under a continuous map is again path-connected.

If $(M, \mathcal{M})$ is not (path-)connected we can still ask for the largest subset containing a given point $p \in M$ which is (path-)connected. These subsets are characterized in the following Proposition:

Proposition 2.5.9 *Let $(M, \mathcal{M})$ be a topological space.*

(i) If $\{C_i\}_{i \in I}$ is a family of (path-)connected subsets of M such that $\bigcap_{i \in I} C_i \neq \emptyset$ then $\bigcup_{i \in I} C_i$ is again (path-)connected.
(ii) If $A \subseteq B \subseteq A^{\mathrm{cl}} \subseteq M$ and A is a connected subset then B is connected as well. In particular, A^{cl} is connected.
(iii) The union $\mathfrak{C}(p)$ of all connected subsets of M which contain p is connected and closed.
(iv) The union $\Pi(p)$ of all path-connected subsets of M which contain p is path-connected and

$$\Pi(p) \subseteq \mathfrak{C}(p). \tag{2.5.2}$$

Proof Let $C = \bigcup_{i \in I} C_i$. Moreover, let $\mathcal{O}_1, \mathcal{O}_2 \subseteq M$ be open subsets with $(\mathcal{O}_1 \cap C) \cap (\mathcal{O}_2 \cap C) = \emptyset$ and $C \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$. Since $C_i \subseteq C$ we have $\mathcal{O}_1 \cup \mathcal{O}_2 \supseteq C_i$ for all $i \in I$. For a fixed $i_0 \in I$ we have $\mathcal{O}_1 \cap C_{i_0}$ or $\mathcal{O}_2 \cap C_{i_o}$ empty by the connectedness of C_{i_0}. Without restriction, we can assume $\mathcal{O}_2 \cap C_{i_0} = \emptyset$ and thus $C_{i_0} \subseteq \mathcal{O}_1 \cap C_{i_0}$. But then the non-empty set $\bigcap_{j \in I} C_j$ is also contained in $\mathcal{O}_1 \cap C_{i_0}$ and hence $C_j \cap \mathcal{O}_1 \neq \emptyset$ for all $j \in I$. By the connectedness of all the C_j we conclude that $C_j \subseteq \mathcal{O}_1 \cap C_j$ for all $j \in I$ and thus $C \subseteq \mathcal{O}_1 \cap C$ proving that C is connected. The path-connected case is easier by joining two points $p, q \in C$ with $p \in C_i$ and $q \in C_j$ first to a common point $x \in \bigcap_{i \in I} C_i$ and reparametrizing the joined paths afterwards. Note that this gives indeed a continuous path again, see also Exercise 2.7.21. For the second part, let $A \subseteq B \subseteq A^{\mathrm{cl}}$ be given, with A being connected. Suppose B is not connected. Then we find two open subsets $\mathcal{O}_1, \mathcal{O}_2 \subseteq M$ with $B \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$, $(B \cap \mathcal{O}_1) \cap (B \cap \mathcal{O}_2) = \emptyset$ and $B \cap \mathcal{O}_1 \neq \emptyset \neq B \cap \mathcal{O}_2$. Then also $A \subseteq \mathcal{O}_1 \cup \mathcal{O}_2$ and $(A \cap \mathcal{O}_1) \cap (A \cap \mathcal{O}_2) = \emptyset$. Since $B \subseteq A^{\mathrm{cl}}$ we have for every $p \in B$ and every open subset $\mathcal{O}$ with $p \in \mathcal{O}$, a non-trivial intersection $A \cap \mathcal{O} \neq \emptyset$. Choosing $b_1 \in B \cap \mathcal{O}_1$ and $b_2 \in B \cap \mathcal{O}_2$ shows $A \cap \mathcal{O}_1 \neq \emptyset \neq A \cap \mathcal{O}_2$, a contradiction to the connectedness of A. The second part implies that for every connected $C \subseteq M$ also C^{cl} is connected. Thus we conclude that $\mathfrak{C}(p) = \mathfrak{C}(p)^{\mathrm{cl}}$ is again connected. For the last part we consider all path-connected subsets containing p. Their intersection is non-empty as it still contains p. Hence we can apply the first part. Since a path-connected subset is also connected by Proposition 2.5.8, the conclusion follows. □

In general, the closure of a path-connected subset is no longer path-connected but only connected, see Exercise 2.7.23. The subsets $\mathfrak{C}(p)$ and $\Pi(p)$ deserve particular attention:

Definition 2.5.10 (*Connected components*) Let $(M, \mathcal{M})$ be a topological space and $p \in M$.

(i) The subset $\mathfrak{C}(p)$ is called the connected component of p.
(ii) The subset $\Pi(p)$ is called the path-connected component of p.

It is now fairly easy to see that $q \in \mathfrak{C}(p)$ holds iff $p \in \mathfrak{C}(q)$. Moreover, if $q \in \mathfrak{C}(p)$ and $x \in \mathfrak{C}(q)$ then $x \in \mathfrak{C}(p)$. Thus we get an equivalence relation of *being in the same connected component* of M, see Exercise 2.7.22. The same holds for the path-connected components.

The connectedness is a global feature of a topological space, however, many properties relying on connectedness can also hold true if the connectedness is satisfied only locally. This yields the definition of locally (path-)connected spaces. One version to formulate the other extreme case is the notion of totally disconnected spaces, see also e.g. [32, Part I, Sect. 4] for many further notions of (dis-)connectedness.

Definition 2.5.11 (*Local connectedness and total disconnectedness*) Let $(M, \mathcal{M})$ be a topological space.

(i) If for every point $p \in M$ every neighbourhood $U \in \mathfrak{U}(p)$ contains a (path-) connected neighbourhood of p then $(M, \mathcal{M})$ is called locally (path-) connected.
(ii) If $\mathfrak{C}(p) = \{p\}$ for all $p \in M$ then $(M, \mathcal{M})$ is called totally disconnected.

Some illustrating examples of these extreme cases are discussed in Exercise 2.7.23 and Exercise 2.7.24.

Proposition 2.5.12 *Let $(M, \mathcal{M})$ be a topological space.*

(i) If M is locally connected then the connected component $\mathfrak{C}(p)$ of $p \in M$ is open.
(ii) The space M is locally connected iff the connected open subsets form a basis of the topology.
(iii) If M is locally path-connected then for all $p \in M$ we have

$$\mathfrak{C}(p) = \Pi(p). \tag{2.5.3}$$

(iv) Suppose M is locally path-connected. Then M is connected iff M is path-connected.

Proof Let $p, q \in M$ with $q \in \mathfrak{C}(p)$ be given. Then for a connected neighbourhood U of q we have $U \cap \mathfrak{C}(p) \neq \emptyset$ since q belongs to this intersection. By Proposition 2.5.9, (i), we have that $U \cup \mathfrak{C}(p)$ is still connected, hence $U \subseteq \mathfrak{C}(p)$ follows since $\mathfrak{C}(p)$ is the largest connected subset containing p. Thus $\mathfrak{C}(p)$ is a neighbourhood of q and thus open by Proposition 2.3.2, (i), proving the first part. If M is locally connected, then the open connected neighbourhoods of a point form a basis of neighbourhoods of that point. Hence they also form a basis of the topology. The converse is true by the same line of argument. Now let M be locally path-connected and hence locally connected. An analogous argument as for the first part shows that $\Pi(p)$ is open for every $p \in M$. From Proposition 2.5.9, (iv), we have $\Pi(p) \subseteq \mathfrak{C}(p)$. Suppose

$q \in \mathfrak{C}(p) \setminus \Pi(p)$. Then $\Pi(q) \subseteq \mathfrak{C}(p) \setminus \Pi(p)$ since if $\Pi(q) \cap \Pi(p) \neq \emptyset$ we would have already $\Pi(q) = \Pi(p)$. This shows

$$\mathfrak{C}(p) \setminus \Pi(p) = \bigcup_{q \in \mathfrak{C}(p) \setminus \Pi(p)} \Pi(q),$$

and hence $\mathfrak{C}(p) \setminus \Pi(p)$ is open. This gives a non-trivial decomposition of $\mathfrak{C}(p)$ into the disjoint open subsets $\Pi(p)$ and $\mathfrak{C}(p) \setminus \Pi(p)$. Since $\mathfrak{C}(p)$ is connected, one of them has to be empty. Since $p \in \Pi(p)$, we have $\mathfrak{C}(p) \setminus \Pi(p) = \emptyset$, which is the third part. The fourth is then a trivial consequence. $\square$

Example 2.5.13 An open subset $\mathcal{O} \subseteq \mathbb{R}^n$ is locally path-connected since clearly every open ball $\mathrm{B}_r(x) \subseteq \mathcal{O}$ is path-connected: the straight line $[0, 1] \ni t \mapsto (1-t)x + ty \in \mathrm{B}_r(x)$ for $y \in \mathrm{B}_r(x)$ connects x and y. Thus the notions of connectedness and path-connectedness for open subsets in $\mathbb{R}^n$ coincide.

2.6 Separation Properties

In a totally disconnected space it follows that all points constitute closed subsets $\{p\} \subseteq M$. Of course, there are many other topological spaces with this property without being totally disconnected as e.g. $\mathbb{R}$ with its standard topology. The following separation properties or "axioms" collect such features of how points in a topological space can be separated from each other.

Definition 2.6.1 (*Separation properties*) Let $(M, \mathcal{M})$ be a topological space.

(i) The space M is called a T_0-space if for each two different points $p \neq q$ in M we find an open subset which contains only one of them.
(ii) The space M is called a T_1-space if for each two different points $p \neq q$ in M we find open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ with $p \in \mathcal{O}_1$ and $q \in \mathcal{O}_2$ but $p \notin \mathcal{O}_2$ and $q \notin \mathcal{O}_1$.
(iii) The space M is called a T_2-space or a Hausdorff space if for each two different points $p \neq q$ in M we find disjoint open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ with $p \in \mathcal{O}_1$ and $q \in \mathcal{O}_2$.
(iv) The space M is called a T_3-space if for every closed subset $A \subseteq M$ and every $p \in M \setminus A$ there are disjoint open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ with $A \subseteq \mathcal{O}_1$ and $p \in \mathcal{O}_2$.
(v) The space M is called a T_4-space if for two disjoint closed subsets $A_1, A_2 \subseteq M$ there are disjoint open subsets $\mathcal{O}_1$ and $\mathcal{O}_2$ with $A_1 \subseteq \mathcal{O}_1$ and $A_2 \subseteq \mathcal{O}_2$.

These are only the most common and important separation properties, many more like $T_{2\frac{1}{2}}$ etc. can be found in e.g. [27, Sect. 6A] or [32, Part I, Sect. 2]. The heuristic meaning of these properties can easily be visualized, see Fig. 2.5.

Example 2.6.2 Consider $M = \mathbb{R}$ with the following non-standard topology: let $\mathcal{O} \subseteq \mathbb{R}$ be open if $\mathcal{O} = (-\infty, a)$ for some $a \in \mathbb{R}$ or $\mathcal{O} = \mathbb{R}, \emptyset$. This indeed defines

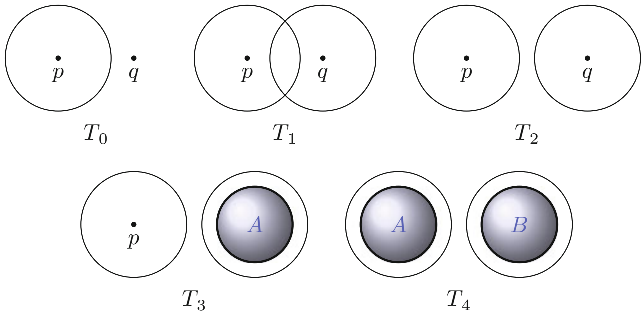


Fig. 2.5 The separation properties

a topology on $\mathbb{R}$. For two points $x, y \in \mathbb{R}$ with $x \neq y$ we have without restriction $x < y$. Then there is an open neighbourhood $\mathcal{O} = (-\infty, x + \varepsilon)$ of x with $y \notin \mathcal{O}$ by taking $\varepsilon > 0$ small enough. However, every open subset containing y also contains x. This shows that T_0 in general does not imply T_1. Of course, T_1 implies T_0.

Example 2.6.3 For the cofinite topology, the space $\mathbb{R}$ is a T_1-space: indeed for $x \neq y$, the subsets $\mathcal{O}_1 = \mathbb{R} \setminus \{y\}$ and $\mathcal{O}_2 = \mathbb{R} \setminus \{x\}$ will do the job. However, it is not a T_2-space as all non-empty open subsets overlap non-trivially. Hence T_1 does not imply T_2 but of course T_2 implies T_1. Since the only closed subsets are the finite ones, it is also easy to see that the cofinite topology does neither fulfill T_3 nor T_4.

Since single points need not to be closed, neither T_3 nor T_4 implies T_1 or T_2. Many examples and counterexamples can be found in [32].

We collect now some useful reformulations and simple implications between combinations of the separation axioms.

Proposition 2.6.4 *A topological space $(M, \mathcal{M})$ is a T_1-space iff every point $p \in M$ gives a closed subset $\{p\} \subseteq M$.*

Proof The T_1-property means that for $p \in M$ the complement of p is a neighbourhood of any point in the complement. This is equivalent to $M \setminus \{p\}$ is open and thus $\{p\}$ is closed. □

Proposition 2.6.5 *A topological space $(M, \mathcal{M})$ is a T_3-space iff for every $p \in M$ and every open $\mathcal{O} \in \mathfrak{U}(p)$ one finds an open $U \in \mathfrak{U}(p)$ with*

$$p \in U \subseteq U^{\mathrm{cl}} \subseteq \mathcal{O}. \tag{2.6.1}$$

This means that there is a neighbourhood basis of closed subsets for each point $p \in M$.

Proof Let $\mathcal{O} \in \mathfrak{U}(p)$ be open then $M \setminus \mathcal{O}$ is closed and $p \notin M \setminus \mathcal{O}$. Hence there are open subsets $V, U \subseteq M$ with $V \cap U = \emptyset$ and $p \in U$ as well as $M \setminus \mathcal{O} \subseteq V$, see Fig. 2.6. But then $M \setminus V$ is closed and

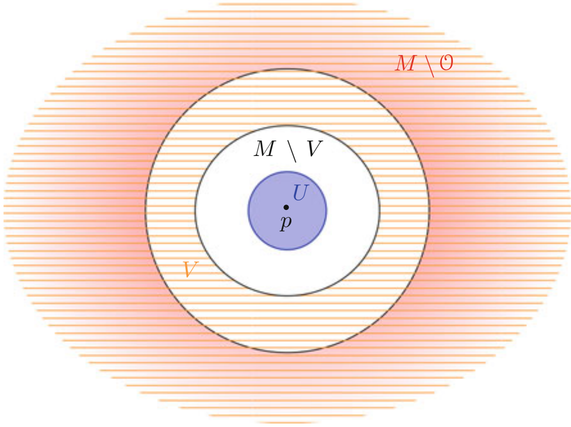


Fig. 2.6 The separating open subsets V and U in a T_3-space

$$U \subseteq M \setminus V \subseteq \mathcal{O}.$$

Since U^{cl} is the smallest closed subset containing U we have $U^{\mathrm{cl}} \subseteq M \setminus V \subseteq \mathcal{O}$ and hence we have found U with (2.6.1). Conversely, assume (2.6.1) and let $A \subseteq M$ be closed and $p \notin A$. Then $p \in M \setminus A$ and $M \setminus A$ is open. Hence we can choose an open $U \subseteq U^{\mathrm{cl}} \subseteq M \setminus A$ with $p \in U$. Clearly $U^{\mathrm{cl}} \cap A = \emptyset$ and thus $A \subseteq M \setminus U^{\mathrm{cl}} = \mathcal{O}$. Then U and $\mathcal{O}$ will separate $\{p\}$ and A as required for T_3. □

With an analogous argument one shows that the T_4 property can be formulated equivalently as follows:

Proposition 2.6.6 *A topological space* $(M, \mathcal{M})$ *is a* T_4*-space iff for every closed subset* $A \subseteq M$ *and every open* $\mathcal{O} \subseteq M$ *with* $A \subseteq \mathcal{O}$ *we have an open* $U \subseteq M$ *with*

$$A \subseteq U \subseteq U^{\mathrm{cl}} \subseteq \mathcal{O}. \tag{2.6.2}$$

Since the axioms T_3 and T_4 are quite unrelated to T_1 and T_2 it seems reasonable to require both: separation of points and separation of closed subsets. This motivates the following definition:

Definition 2.6.7 (*Regular and normal spaces*) Let $(M, \mathcal{M})$ be a topological space.

(i) The space $(M, \mathcal{M})$ is called regular if it is T_1 and T_3.
(ii) The space $(M, \mathcal{M})$ is called normal if it is T_1 and T_4.

Proposition 2.6.8 *A regular space is Hausdorff and a normal space is regular.*

Proof Since by T_1 all points $\{p\} \subseteq M$ are closed, T_3 separates a point p from the closed subset $\{q\}$ for $p \neq q$ by disjoint open subsets. Thus T_2 follows. Again by T_1 a point is closed and hence T_4 implies T_3. □

Thanks to this proposition it will be mainly the combination of T_2 with some of the remaining separation properties which will be of most importance. The normal spaces will enjoy several other nice properties, similar to metric spaces. Indeed, metric spaces are normal:

Proposition 2.6.9 *A metric space is normal and hence T_1, T_2, T_3, T_4.*

Proof Obviously, a single point $\{p\}$ is closed as $M \setminus \{p\} = \bigcup_{q \in M \setminus \{p\}} \mathrm{B}_{r_q}(q)$ with $0 < r_q < d(q, p)$ is open. Now let $A, B \subseteq M$ be closed with $A \cap B = \emptyset$. For $p \in A$ we have a radius $r_p > 0$ with $\mathrm{B}_{r_p}(p) \cap B = \emptyset$ since $p \in M \setminus B$ and $M \setminus B$ is open. Analogously, for $q \in B$ we find $r_q > 0$ with $\mathrm{B}_{r_q}(q) \cap A = \emptyset$. Define the open subsets

$$U = \bigcup_{p \in A} \mathrm{B}_{r_p/2}(p) \quad \text{and} \quad V = \bigcup_{q \in A} \mathrm{B}_{r_q/2}(q).$$

Then $A \subseteq U$ and $B \subseteq V$ is clear. Moreover, since $r_p < d(p, q)$ for all $q \in B$ and $r_q < d(q, p)$ for all $p \in A$ we see that $\mathrm{B}_{r_p/2}(p) \cap \mathrm{B}_{r_q/2}(q) = \emptyset$, by the triangle inequality, see also Fig. 2.7. But then $U \cap V = \emptyset$ follows which gives T_4. $\square$

We will come back to the separation axioms at several instances. In particular, the existence of sufficiently non-trivial continuous functions relies heavily on the separation properties. As a last application of the Hausdorff property we mention the following two statements which turn out to be very useful at many places:

Proposition 2.6.10 *Let $f, g\colon (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ be continuous maps between topological spaces and assume that $(N, \mathcal{N})$ is Hausdorff.*

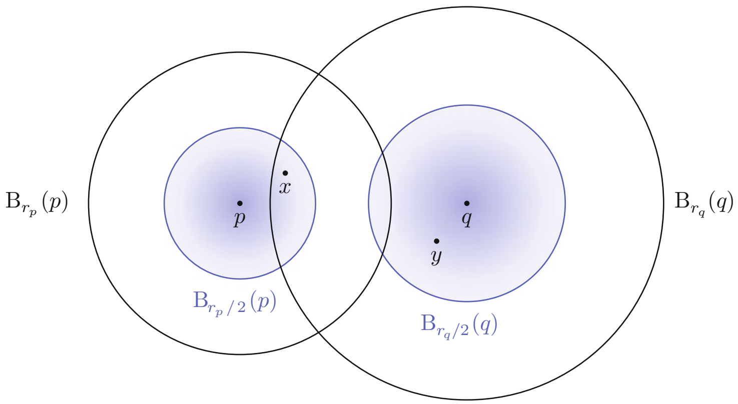


Fig. 2.7 A metric space is T_4

(i) The coincidence set $\{p \in M \mid f(p) = g(p)\} \subseteq M$ *is closed.*
(ii) If $U \subseteq M$ *is dense then* $f\big|_U = g\big|_U$ *implies* $f = g$.

Proof Let $q \in M$ be a point with $f(q) \neq g(q)$. Then the Hausdorff property implies that we find open subsets $\mathcal{O}_1, \mathcal{O}_2 \subseteq N$ with $f(q) \in \mathcal{O}_1$ and $g(q) \in \mathcal{O}_2$ but $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. By continuity, $f^{-1}(\mathcal{O}_1)$ and $g^{-1}(\mathcal{O}_2)$ are open and $q \in f^{-1}(\mathcal{O}_1) \cap g^{-1}(\mathcal{O}_2)$. If q' is another point in this intersection then $f(q') \in \mathcal{O}_1$ and $g(q') \in \mathcal{O}_2$ yielding $f(q') \neq g(q')$ as $\mathcal{O}_1 \cap \mathcal{O}_2 = \emptyset$. This shows the first part by taking complements. The second part is now easy as the set $U \subseteq M$ is in the *closed* coincidence set and thus also U^{cl} is in the closed coincidence set. But $U^{\mathrm{cl}} = M$ is already everything. □

This feature and many other pleasant properties of Hausdorff spaces justify to consider them with special care:

Definition 2.6.11 (*Hausdorff spaces*) The subcategory of top consisting of all Hausdorff topological spaces will be denoted by Top.

As the final remark we note that most of the separation properties behave nicely when passing to subspaces:

Proposition 2.6.12 *Let* $(M, \mathcal{M})$ *be a* T_k*-space with* $k \in \{0, 1, 2, 3\}$ *and let* $N \subseteq M$ *be endowed with the subspace topology. Then* $(N, \mathcal{M}\big|_N)$ *is* T_k*, too.*

Proof The proof is similar for all cases, we illustrate it for T_3: thus let $p \in N$ and $A \subseteq N$ be closed with $p \notin A$. We know that there is a closed $B \subseteq M$ with $A = N \cap B$ in this case, see Exercise 2.7.3, (ii). Since $p \in N$ we conclude that $p \notin B$, too. Hence we can separate $\{p\}$ and $B \in M$ by $U_1, U_2 \in \mathcal{M}$, i.e. $p \in U_1$, $B \subseteq U_2$, and $U_1 \cap U_2 = \emptyset$. But then $\mathcal{O}_1 = N \cap U_1$, $\mathcal{O}_2 = N \cap U_2$ are open in N and separate $\{p\}$ from A as wanted. □

2.7 Exercises

Exercise 2.7.1 (Cartesian product of metric spaces) Let (M_1, d_1) and (M_2, d_2) be metric spaces. Consider their Cartesian product $M = M_1 \times M_2$ with

$$d((x_1, x_2), (y_1, y_2)) = d_1(x_1, y_1) + d_2(x_2, y_2), \tag{2.7.1}$$

$$d'((x_1, x_2), (y_1, y_2)) = \sqrt{d_1(x_1, y_1)^2 + d_2(x_2, y_2)^2}, \tag{2.7.2}$$

and

$$d''((x_1, x_2), (y_1, y_2)) = \max\{d_1(x_1, y_1), d_2(x_2, y_2)\}, \tag{2.7.3}$$

where $(x_1, x_2), (y_1, y_2) \in M_1 \times M_2$.

(i) Show that d, d', and d'' yield metrics on M.
(ii) Show that the open subsets with respect to all these three metrics d, d', and d'' coincide.
(iii) Show that the canonical projections

$$M_1 \xleftarrow{\mathrm{pr}_1} M_1 \times M_2 \xrightarrow{\mathrm{pr}_2} M_2 \tag{2.7.4}$$

are continuous.
(iv) Generalize these results to finite Cartesian products with more than two factors.

For a countable Cartesian product one can still construct a metric: let (M_n, d_n) be metric spaces for $n \in \mathbb{N}$. Then the Cartesian product $M = \prod_{n=1}^{\infty} M_n$ is the space of all sequences $x = (x_n)_{n \in \mathbb{N}}$, where the n-th term x_n is in M_n. One defines

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)}. \tag{2.7.5}$$

(v) Show that d is a well-defined metric on M and verify that the projection pr_n onto the n-th component is continuous for all $n \in \mathbb{N}$.
Hint: Use Example 2.1.2, (vi).

Exercise 2.7.2 (Formal power series) Consider the real formal power series $\mathbb{R}[[\lambda]]$ in a formal parameter λ. Let $o\colon \mathbb{R}[[\lambda]] \longrightarrow \mathbb{N}_0 \cup \{+\infty\}$ be the *order* of the power series as in Example 2.1.2, (iii), with the corresponding metric $d(a, b) = 2^{-o(a-b)}$ for $a, b \in \mathbb{R}[[\lambda]]$ where as usual we set $2^{-\infty} = 0$.

(i) Show that d is a metric for $\mathbb{R}[[\lambda]]$ satisfying the stronger version of the triangle inequality

$$d(a, b) \le \max\{d(a, c), d(c, b)\} \tag{2.7.6}$$

for all $a, b, c \in \mathbb{R}[[\lambda]]$. A metric with this additional property is also called an *ultrametric*.
(ii) Endow $\mathbb{R}[[\lambda]] \times \mathbb{R}[[\lambda]]$ with one of the (equivalent) product metrics from Exercise 2.7.1 and show that the addition of formal power series as well as the multiplication defined by the Cauchy product

$$ab = \left(\sum_{n=0}^{\infty} \lambda^n a_n\right)\left(\sum_{m=0}^{\infty} \lambda^m b_m\right) = \sum_{k=0}^{\infty} \lambda^k \sum_{n+m=k} a_n b_m \tag{2.7.7}$$

are continuous.

(iii) Rephrase the condition for a Cauchy sequence in terms of the order and show that $\mathbb{R}[[\lambda]]$ is complete.
(iv) Show that the subspace topology of $\mathbb{R}$ induced from $\mathbb{R} \subseteq \mathbb{R}[[\lambda]]$ is the discrete topology. Show also that the topology of $\mathbb{R}[[\lambda]]$ is not the discrete one.
(v) Show that the polynomials $\mathbb{R}[\lambda] \subseteq \mathbb{R}[[\lambda]]$ are dense. More precisely, show that

$$\lim_{N \longrightarrow \infty} \sum_{n=0}^{N} \lambda^n a_n = \sum_{n=0}^{\infty} \lambda^n a_n \tag{2.7.8}$$

for every formal power series.

Exercise 2.7.3 (Subspace topology) Let $(M, \mathcal{M})$ be a topological space and let $N \subseteq M$ be a subset.

(i) Show that N is open in M iff every open subset $U \subseteq N$ with respect to the subspace topology $\mathcal{M}\big|_N$ is also open in M.
(ii) Show that $B \subseteq N$ is closed with respect to the subspace topology $\mathcal{M}\big|_N$ iff there is a closed subset $A \subseteq M$ with $B = A \cap N$.
(iii) Formulate and prove an analogous statement to (i) for closed subsets.

Exercise 2.7.4 (Neighbourhoods determine the topology) Let M be a set. For every point $p \in M$ consider a non-empty system of subsets $\mathfrak{U}(p)$ of M, such that the properties (ii), (iii), (iv), and (v) of Proposition 2.3.2 are satisfied.

(i) Define $\mathcal{O} \subseteq M$ to be open, if $\mathcal{O} \in \mathfrak{U}(p)$ for all $p \in \mathcal{O}$. Show that this defines a topology $\mathcal{M}$ on M.
(ii) Determine the neighbourhoods $\tilde{\mathfrak{U}}(p)$ of $p \in M$ for this topology $\mathcal{M}$ and show $\tilde{\mathfrak{U}}(p) = \mathfrak{U}(p)$ for all points $p \in M$.

This shows that the characterization of topological spaces via neighbourhood systems is equivalent to the characterization via topologies.

Exercise 2.7.5 (Finer and coarser topologies) Consider $\mathbb{R}$ with the discrete, the indiscrete, the cofinite, the topology from Example 2.6.2, and the standard (metric) topology. Order them with respect to being finer.

Exercise 2.7.6 (Rational balls) Consider $\mathbb{R}^n$ with its usual Euclidean metric and the corresponding topology. Show that every open subset can be obtained as union of open balls of the form $\mathrm{B}_r(p)$ with $r \in \mathbb{Q}^+$ and $p \in \mathbb{Q}^n$.

Exercise 2.7.7 (Countability and subspaces) Let $(M, \mathcal{M})$ be a topological space and let $N \subseteq M$ be a subset being endowed with the subspace topology.

(i) Show that if $(M, \mathcal{M})$ is first countable at every point of N, then $(N, \mathcal{M}\big|_N)$ is first countable as well.
(ii) Show that if $(M, \mathcal{M})$ is second countable, then $(N, \mathcal{M}\big|_N)$ is second countable, too.

Exercise 2.7.8 (Closed and open subsets) Let $(M, \mathcal{M})$ be a topological space and let $A \subseteq M$ be a subset.

(i) Show that A is closed iff $A = A^{\mathrm{cl}}$ iff $\partial A \subseteq A$.
(ii) Show that A is open iff $A = A^{\circ}$.

Exercise 2.7.9 (Closures, open interiors, and boundaries) Find and describe examples of topological spaces $(M, \mathcal{M})$ and subsets $A, B \subseteq M$ for the following statements:

(i) The boundary of the boundary of a subset can but needs not to be empty.
(ii) Let $A \subseteq B$. Show that the following three situations are possible: a strict inclusion $\partial A \subseteq \partial B$, a strict inclusion $\partial B \subseteq \partial A$, a trivial intersection $\partial A \cap \partial B = \emptyset$ with both boundaries being non-empty.
(iii) The open interior of a union $A \cup B$ can be strictly larger than the union of the open interiors $A^{\circ} \cup B^{\circ}$.
(iv) The open interior of a boundary can be non-empty.
(v) The intersection of the boundaries $\partial A \cap \partial B$ can be strictly contained in the boundary of the intersection $A \cap B$.
(vi) The boundary of the intersection $A \cap B$ of two subsets can be strictly contained in the intersection of the boundaries of A and B.

Exercise 2.7.10 (Closure in the subspace topology) Let $(M, \mathcal{M})$ be a topological space and let $N \subseteq M$ be a subset endowed with the subspace topology $\mathcal{M}\big|_N$. Furthermore, let $A \subseteq N$ and denote the closure of A with respect to $\mathcal{M}\big|_N$ by $\overline{A}$. Show that $\overline{A} = A^{\mathrm{cl}} \cap N$.

Exercise 2.7.11 (Dense subsets) Let $(M, \mathcal{M})$ be a topological space and let $A \subseteq M$. Show that A is dense iff $A \cap \mathcal{O} \neq \emptyset$ for all non-empty open $\mathcal{O} \in \mathcal{M}$.

Exercise 2.7.12 (Nowhere dense subsets) Let $(M, \mathcal{M})$ be a topological space and let $A \subseteq M$ be a subset. Show that A is nowhere dense iff $(M \setminus A^{\mathrm{cl}})^{\mathrm{cl}} = M$ iff $M \setminus A^{\mathrm{cl}}$ is a dense open subset.

Exercise 2.7.13 (Density of $\mathbb{Q}^n$ in $\mathbb{R}^n$) Show that the set of rational points $\mathbb{Q}^n$ is dense in $\mathbb{R}^n$.

Exercise 2.7.14 (Topologies and continuous maps) Consider the sets $M_1 = \{1, 2\}$ und $M_2 = \{1, 2, 3\}$.

(i) Determine all topologies on M_1 and M_2.
(ii) Order the topologies on M_2 with respect to being finer. Can all of them be compared?
(iii) Determine all continuous maps $f\colon M_1 \longrightarrow M_2$ for all combinations for the topologies on M_1 and M_2, respectively.

Exercise 2.7.15 (Continuous maps and homeomorphisms) Consider the unit circle $\mathbb{S}^1 \subseteq \mathbb{C}$ and the half-open interval $[0, 2\pi)$, both endowed with their usual subspace topologies.

(i) Show that the map

$$f: [0, 2\pi) \ni t \ \mapsto \ e^{it} \in \mathbb{S}^1 \tag{2.7.9}$$

is continuous.
(ii) Show that f is bijective.
(iii) Show that the inverse map $f^{-1}: \mathbb{S}^1 \longrightarrow [0, 2\pi)$ is not continuous.

Viewing f as a map from $[0, 2\pi)$ into $\mathbb{C}$ this gives also an example of a injective continuous map which is not an embedding.

Exercise 2.7.16 (Finer and coarser topologies and continuity) Let $f: (M, \mathcal{M}_1) \longrightarrow (N, \mathcal{N}_1)$ be a continuous map between topological spaces.

(i) Discuss whether or not f stays continuous if the topology $\mathcal{M}_1$ on the domain M is replaced by a finer (or coarser) topology $\mathcal{M}_2$, respectively.
(ii) Discuss whether or not f stays continuous if the topology $\mathcal{N}_1$ on the target N is replaced by a finer (or coarser) topology $\mathcal{N}_2$, respectively.
(iii) Show that the identity map $\mathsf{id}: (M, \mathcal{M}_1) \longrightarrow (M, \mathcal{M}_2)$ is continuous iff $\mathcal{M}_1$ is finer than $\mathcal{M}_2$.
(iv) Show that the discrete topology on M is the unique topology $\mathcal{M}$ such that every map $f: (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ into a topological space is continuous. Formulate and prove the analogous statement for the indiscrete topology.

Exercise 2.7.17 (Open map via a basis) Sometimes it is useful to characterize the openness of a map in terms of a basis of the topology: Show that a map $f: (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ between topological spaces is open iff for a basis $\mathcal{B} \subseteq \mathcal{M}$ and all subsets $\mathcal{O} \in \mathcal{B}$ one has $f(\mathcal{O}) \in \mathcal{N}$.

Exercise 2.7.18 (Open, closed, and continuous maps)

(i) Find an example of a closed but discontinuous map $f: \mathbb{R} \longrightarrow \mathbb{R}$. Can you arrange it such that it is not open?
(ii) Find an example of an open but discontinuous map. The easiest way might be to use the discrete topology.
(iii) Consider finally the inclusion $\mathbb{R} \longrightarrow \mathbb{R}^2$ and the projection $\mathrm{pr}_1: \mathbb{R}^2 \longrightarrow \mathbb{R}$ and discuss whether they are open, closed, or continuous.

Exercise 2.7.19 (The algebra $\mathscr{C}_b(M)$) Let $(M, \mathcal{M})$ be a topological space and consider the set $\mathscr{C}(M)$ of $\mathbb{C}$-valued continuous functions on M.

(i) Show that $\mathscr{C}(M)$ is a vector space with respect to the pointwise addition and multiplication with a scalar in $\mathbb{C}$. Show that the pointwise complex conjugation yields an involution on $\mathscr{C}(M)$.
(ii) Show that $\mathscr{C}(M)$ becomes a commutative associative algebra with unit with respect to the pointwise product.
(iii) Show that the maximum and the minimum of real-valued continuous functions as well as the absolute value of a continuous function are again continuous.

(iv) Consider now the $\mathbb{C}$-valued bounded continuous functions

$$\mathscr{C}_b(M) = \left\{ f \in \mathscr{C}(M) \;\middle|\; \sup_{p \in M} |f(p)| < \infty \right\} \tag{2.7.10}$$

on M. Show that they form a subalgebra of $\mathscr{C}(M)$ with unit which is closed under max, min, $|\cdot|$, and under complex conjugation.

(v) Define the supremum norm

$$\|f\|_\infty = \sup_{p \in M} |f(p)| \tag{2.7.11}$$

for $f \in \mathscr{C}_b(M)$. Show that $\|\cdot\|$ is a norm on $\mathscr{C}_b(M)$.

(vi) Show that $\|fg\|_\infty \leq \|f\|_\infty \|g\|_\infty$ for all $f, g \in \mathscr{C}_b(M)$ and find analogous estimates, equalities, or counter-examples for the pointwise maximum, minimum, the absolute value, and the complex conjugation instead of the product.

(vii) Show that $\mathscr{C}_b(M)$ is a complete normed space , i.e. a Banach space, with respect to the supremum norm. Which well-known theorem from elementary calculus is contained in this statement?

Together with the previous property, the completeness makes $\mathscr{C}_b(M)$ a *Banach algebra*, see also Definition 6.2.1.

Exercise 2.7.20 (Continuity is a local property) Consider a map $f\colon (M, \mathcal{M}) \longrightarrow (N, \mathcal{N})$ between topological spaces. Show that the following statements are equivalent:

(i) The map f is continuous.

(ii) For all subsets $A \subseteq M$ the maps $f\big|_A\colon (A, \mathcal{M}\big|_A) \longrightarrow (N, \mathcal{N})$ are continuous.

(iii) For all open covers $\{\mathcal{O}_i\}_{i \in I}$, i.e. $\mathcal{O}_i \in \mathcal{M}$ and $M = \bigcup_{i \in I} \mathcal{O}_i$, the restrictions $f\big|_{\mathcal{O}_i}\colon (\mathcal{O}_i, \mathcal{M}\big|_{\mathcal{O}_i}) \longrightarrow (N, \mathcal{N})$ are continuous for all $i \in I$.

(iv) There exists an open cover $\{\mathcal{O}_i\}_{i \in I}$ of M such that the restrictions $f\big|_{\mathcal{O}_i}\colon (\mathcal{O}_i, \mathcal{M}\big|_{\mathcal{O}_i}) \longrightarrow (N, \mathcal{N})$ are continuous for all $i \in I$.

Exercise 2.7.21 (Gluing of paths) Let $(M, \mathcal{M})$ be a topological space. Moreover, let $\gamma_1\colon [a, b] \longrightarrow M$ and $\gamma_2\colon [b, c] \longrightarrow M$ be continuous paths in M where $a < b < c$. Show that

$$\gamma(t) = \begin{cases} \gamma_1(t) & t \in [a, b] \\ \gamma_2(t) & t \in (b, c] \end{cases} \tag{2.7.12}$$

yields a continuous path $\gamma\colon [a, c] \longrightarrow M$ provided $\gamma_1(b) = \gamma_2(b)$.

Hint: Suppose the converse and let $U \subseteq M$ be a neighbourhood of $\gamma(b) = \gamma_1(b) = \gamma_2(b)$ such that $\gamma^{-1}(V)$ is not a neighbourhood of $b \in [a, c]$. Consider then $\gamma_1^{-1}(V)$ and $\gamma_2^{-1}(V)$.

Exercise 2.7.22 (Connected components) Let $(M, \mathcal{M})$ be a topological space.

(i) Show that $p \sim q$ if $q \in \mathfrak{C}(p)$ defines an equivalence relation on M. Hence M decomposes into mutually disjoint connected components.
Hint: Use Proposition 2.5.9, (iii), to obtain the characterization that $\mathfrak{C}(p)$ is the largest connected subset of M which contains p.

(ii) Show the analogous result for the path-connected components of M.

Exercise 2.7.23 (The topologist's sine curve) Let

$$S = \left\{(x, \sin(1/x)) \;\middle|\; x \in (0, 1]\right\} \subseteq \mathbb{R}^2 \tag{2.7.13}$$

be the graph of the function $x \mapsto \sin(1/x)$ defined on the interval $(0, 1]$, endowed with the subspace topology of $\mathbb{R}^2$.

(i) Sketch the graph S as well as its closure S^{cl} in $\mathbb{R}^2$. Which points are added when passing to the closure?

(ii) Show that S is connected and even path-connected.

(iii) Show that S^{cl} is connected but not path-connected.

(iv) Show that S^{cl} is not locally (path-) connected.

Exercise 2.7.24 (Cantor set I) Consider the following subsets of the closed unit interval $A_0 = [0, 1]$: in every new step one removes the open inner third of each piece, i.e. $A_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$, then $A_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]$ etc., see Fig. 2.8. The Cantor set is then the infinite intersection

$$C = \bigcap_{n=0}^{\infty} A_n \subseteq [0, 1]. \tag{2.7.14}$$

Show the following properties of C with respect to the subspace topology of $[0, 1]$:

(i) The Cantor set C is uncountable.
Hint: Show that C is the set of those real numbers x in $[0, 1]$ which can be written as

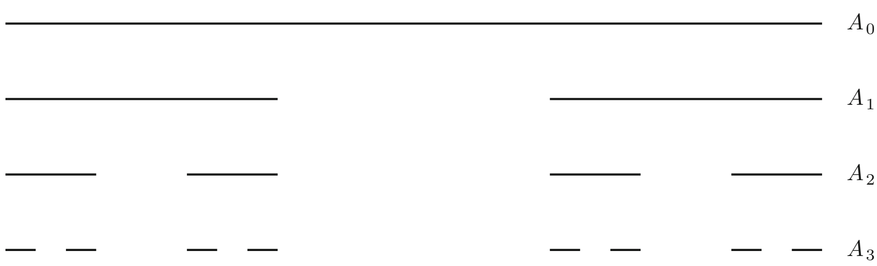


Fig. 2.8 The first iterations for the Cantor set

$$x = \sum_{n=1}^{\infty} \frac{a_n}{3^n} \tag{2.7.15}$$

with a_n either 0 or 2.

(ii) The Cantor set C is closed.

(iii) The Cantor set C is nowhere dense.

(iv) The Cantor set C is totally disconnected.

(v) The Cantor set is not discrete.

(vi) For every $\epsilon > 0$ there is an $n \in \mathbb{N}$ and open intervals $I_1, \ldots, I_n$ with total length $|I_1| + \cdots + |I_n| < \epsilon$ such that

$$C \subseteq I_1 \cup \cdots \cup I_n. \tag{2.7.16}$$

With other words, the Cantor set C has Lebesgue measure 0.

Exercise 2.7.25 (Separation properties I) Consider a set M with at least two elements endowed with the indiscrete topology. Which of the separation properties T_0, T_1, T_2, T_3, or T_4 are fulfilled? For which implications between the separation properties does this example provide counterexamples?

Exercise 2.7.26 (Separation properties II) Consider the set $M = \{1, 2, 3, 4\}$ with the topology $\mathcal{M} = \{\emptyset, \{1\}, \{1, 2\}, \{1, 3\}, \{1, 2, 3\}, M\}$.

(i) Show that $\mathcal{M}$ is indeed a topology.

(ii) Determine all closed subsets of $(M, \mathcal{M})$.

(iii) Which separation properties does $(M, \mathcal{M})$ fulfill?

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