Chapter 2
Random Process Fundamentals

The chapter deals with the basic notions that characterize a random process in a statistical sense.

The properties of stationarity and ergodicity are also introduced, in that they play a significant role in practice.

For the sake of simplicity, the topic is explored with reference to continuous-time, continuous-valued processes.

Results obtained in this section can easily be extended to all other types of random processes by employing the δ generalized function in probability law description [1] of random process sections.

2.1 Distributions and Moments of a Random Process

Let X(t) be a random variable (R.V.) section of a random process x(ω, t) at an arbitrary time instant t ∈ T.

In general, different t points of T correspond to different X(t) (Fig. 2.1). It follows that the cumulative distribution functions (CDFs)\(^1\) of sections X(t₁), X(t₂), ..., X(tₙ) and for any set (t₁, t₂, ..., tₙ) – n = 1, 2, ...—are time functions.\(^2\)

By means of the common symbols for joint cumulative distribution function (JCDF) and probability density function (PDF) of a section X(t), we put respectively

\[
F(x, t) = P[X(t) \leq x] \quad (2.1)
\]

\[
f(x, t) = \frac{\partial F(x, t)}{\partial x} \quad (2.2)
\]

\(^1\) As known, cumulative distribution functions are also termed distribution functions.
In general we have
\[ F(x_i, x_j, \ldots, x_n; t_i, t_j, \ldots, t_n) = P[X(t_i) \leq x_i; X(t_j) \leq x_j; \ldots; X(t_n) \leq x_n] \quad (2.3) \]
\[ f(x_i, x_j, \ldots, x_n; t_i, t_j, \ldots, t_n) = \frac{\partial F(x_i, x_j, \ldots, x_n; t_i, t_j, \ldots, t_n)}{\partial x_i \cdot \partial x_j \cdot \ldots \cdot \partial x_n} \quad (2.4) \]

A real-valued random process \( x(x, t) \) is statistically determined if we know its nth-order CDFs (2.3) for any integer \( n \) and for any point set \( (t_1, t_2, \ldots, t_n) \) of \( T \). These functions are not arbitrary, but they must satisfy certain conditions.

To this end, the following theorem (Kolmogorov Consistency Theorem, 1933) is fundamental:

*a family of finite joint cumulative distribution functions statistically characterizes a random process only if they result to be symmetrical and consistent.*

An \( F(x_i, x_j, \ldots, x_n; t_i, t_j, \ldots, t_n) \) is symmetrical if it is invariant with regard to each simultaneous permutation of \( x_i \) and \( t_i \). It is consistent when, for some \( x_i \to \infty \), it tends to the distribution function of the remaining \( X(t_r), r \neq i \).

In other words, by means of a consistency property a CDF of a given order is determined from a CDF of higher order. The analyses of \( n \) random variable systems get more complex when \( n \geq 3 \), as is well known by the theory of probability.

In practice, we then use appropriate time functions which synthetically represent the main properties of random processes.
These functions have the same role as random variable moments and are commonly called process moments.

Among them, the most important are the mean value function (or just mean) $m(t)$, the autocorrelation function $B(t, t')$ and the autocovariance function $C(t, t')$.

We therefore define as mean value function $m(t)$ of a random process $x(\omega, t)$, the mean $E[X(t)]$ of the corresponding process section $X(t)$ for each $t$.

In symbols

$$m(t) = E[X(t)] = \int_{-\infty}^{+\infty} x f(x, t) \, dx$$  \hspace{1cm} (2.5)

**Example** Let $\Omega = \{\omega_1, \omega_2\}$ be a sample space and $Y$ and $Z$ two RVs on $\Omega$ such that: $Y(\omega_1) = 0$; $Y(\omega_2) = 1$; $Z(\omega_1) = 2$; $Z(\omega_2) = 3$. Besides, let these values be all associated to the same probability $p = 0.5$.

It thus follows that $E[Y] = 0.5$; $E[Z] = 0.25$.

We consider now the random process

$$x(\omega, t) \equiv Y + Z \cdot t$$  \hspace{1cm} (2.6)

The two straight lines $x(\omega_1, t) = 2t$; $x(\omega_2, t) = 1 + 3t$ (Fig. 2.2) represent the realizations of $x(\omega, t)$.

As to the mean value function $m(t)$, by recalling that the mean operator is a linear operator, we obtain from (2.6)
\[ m(t) = E[Y] + E[Z \cdot t] = E[Y] + E[Z] \cdot t = 0.5 + 2.5 \cdot t. \]

\( m(t) \) is represented by the dotted line in Fig. 2.2.

As in case of random variables, the mean value function \( m(t) \) of a random process \( x(\omega, t) \) represents a set of values around which the realizations \( x_i(t) \) of \( x(\omega, t) \) are grouped and oscillate in its neighbourhood. For any time \( t \), \( m(t) \) can then be associated to the same meaning of position index as attributed to the expected value of a R.V.

We now consider two arbitrary time instants \( t \) and \( t' \) of a parameter space \( T \) of \( x(\omega, t) \) (Figs. 2.3 and 2.4). We define as autocorrelation function \( B(t, t') \), the joint second-order moment of two sections \( X(t), X(t') \) of \( x(\omega, t) \):

\[
B(t, t') = E[X(t) X(t')] = \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x x' f(x, x'; t, t') \, dx \, dx' \tag{2.7}
\]

\( B(t, t') \) is therefore a function of the two variables \( t, t' \in T \).
In what follows, instead of section variables of \( x(\omega, t), X(t) \) and \( X(t') \), we consider, in general, the sections of a random process centred around their respective mean values, i.e. the deviation of \( X(t) \) and \( X(t') \) from their means:

\[
\tilde{X}(t) = X(t) - m(t) \quad \tilde{X}(t') = X(t') - m(t') \tag{2.8}
\]

For a process with all sections centred around their respective mean values, it clearly results that

\[
m(t) = E[\tilde{X}(t)] = 0 \tag{2.9}
\]

The autocorrelation function (2.7) of the R.V. section (2.8) is termed autocovariance function (or covariance kernel) \( C(t, t') \):

\[
C(t, t') = E[(X(t) - m(t)) \cdot (X(t') - m(t'))] \tag{2.10}
\]

If we put \( t = t' \), from formula (2.10) a new function only of \( t \) is obtained and assumes the meaning of variance function \( \text{Var}(t) \) of the process for \( t \) belonging to \( T \). In symbols we put

\[
C(t, t) = E[(X(t) - m(t))^2] = \text{Var}(t) = \sigma^2(t) \tag{2.11}
\]

Finally, starting from (2.10) and (2.11) a new function \( \rho(t, t') \) of the paired instants \( t, t' \) is introduced.

It is given by

\[
\rho(t, t') = \frac{C(t, t')}{\sqrt{\sigma^2(t) \cdot \sigma^2(t')}} \tag{2.12}
\]

Formula (2.12) defines the normed autocovariance function of a random process. It is also called autocorrelation coefficient.

**Example** With reference to the same process as in the previous example \( x(\omega, t) \equiv Y + Zt \), for the autocorrelation function (2.7) we obtain

\[
B(t, t') = E[(Y - Zt) \cdot (Y - Zt')] = E[Y^2] - E[YZ](t + t') + E[Z^2]tt'
\]

It is worth recalling that \( E[Y^2], E[YZ], E[Z^2] \) are constants, while the variables are \( t, t' \). The numerical determination of these averages is left to the reader.

It should explicitly be underlined that in time series analyses frequently the normed autocovariance function (2.12) is called autocorrelation function. In this
book we use the term autocorrelation function only for $B(t, t')$ (see expression (2.7)).

As defined,\footnote{It is known that the product mean of random variables is not influenced by the presentation order of variables on which it operates.} functions $B(t, t')$, $C(t, t')$ and $\rho(t, t')$ are symmetrical, in the sense that

\begin{align*}
B(t, t') &= B(t', t) \\
C(t, t') &= C(t', t) \\
\rho(t, t') &= \rho(t', t)
\end{align*}

The introduction of an autocovariance function allows to obtain further information for a random function than what can be inferred from the only knowledge of $m(t)$ and $\sigma^2(t)$ (see formulas (2.9) and (2.11)).

In fact autocovariance, as well as covariance, between $X(t)$ and $X(t')$ measures the tendency of the two random variable sections of the random process, to assume simultaneously higher or lower values than their respective means (namely it is a measure of how much $X(t)$ and $X(t')$ unanimously vary).

In this regard, we observe the families of realizations from two distinct random processes in Figs. 2.3 and 2.4. The two processes present the same mean value function $m_1(t) = m_2(t) = m(t)$ and variance function $\sigma_1(t) = \sigma_2(t) = \sigma(t)$.

In both Figs. 2.3 and 2.4 there are also reported two dotted curves distant $3\sigma(t)$ from the mean function. The two curves define a domain that contains almost all the realizations of the corresponding random process.

Therefore, the two moments $m(t)$ and $\sigma^2(t)$ do not account for the wide difference in time behaviour of the two random processes. One can see that the essential features of realizations in Figs. 2.3 and 2.4 are completely different for these two random processes. The realizations of the former process (see Fig. 2.3) in fact prove to have a substantially regular behaviour with $t$. In other words, they
are devoid of oscillations around the mean value function which characterize the realizations of the latter random process (see Fig. 2.4). Notwithstanding, mean values functions and variance functions of the former process coincide with the latter.

The structural difference between the two processes is, on the other hand, pointed out by the behaviour of their respective autocovariance functions, as shown below.

We denote with \( x_i(t) \) and \( x_j(t) \) two arbitrary realizations (trajectories of a random process \( x(\omega, t) \)).

With the behaviour of the process realizations in Fig. 2.3, should for instance be \( x_i(t) > x_j(t) \) in \( t \), in most cases there will be \( x_i(t') > x_j(t') \) at an arbitrary time \( t' \in T \).

Moreover, fluctuation values around the mean of \( X(t) \) are substantially kept also in \( X(t') \).

The statistical relationship between \( X(t) \) and \( X(t') \) is, thus, maintained when two arbitrary times \( t, t' \) vary in \( T \), with the same features and virtually the same intensity. For the autocovariance function it follows a weak variability (decrease) in \( T \),

\[
C(t, t') \approx \text{const.} \quad \forall \, t, t' \in T.
\]

On the other hand, for the process in Fig. 2.4, rapid trajectory oscillations around the mean value function do not make a stable statistical relationship between realizations detectable. In this case the statistical relationship between two arbitrary sections \( X(t) \) and \( X(t') \) is not maintained in sections concerning another pair of instants in \( T \). This occurs even if the above-mentioned arbitrary pair of time instants is close to that previously considered.

For the process in Fig. 2.4 it follows that its autocovariance function rapidly decreases when \( t \) increases.

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3 Fluctuations around the mean in a process Section \( X(t) \) are the quantities \( z_i(t) = x_i(t) - m(t) \), or the determinations of the R.V. “centred Section \( Z(t) \)” of the process \( Z(t) = X(t) - m(t) \).

4 The constancy of \( C(t, t') \) for pairs of time \( t, t' \), anyway placed on \( T \), is a specific form of stationarity, namely the autocovariance stationarity. The autocovariance stationarity is a particular form of weak stationarity (see the following Sect. 2.2).
For more practice, the interested reader can extend the comments about processes in Figs. 2.3 and 2.4 to those shown in Figs. 2.5 and 2.6.

### 2.2 Stationarity of a Random Process

Generally a dynamical system which tends towards equilibrium evolves into one or more states (called transient states) before reaching steady-state conditions (Fig. 2.7).

For an uncertain dynamical system, the equilibrium conditions must be connoted statistically.

In qualitative terms, an uncertain dynamical system is at statistical equilibrium when its probabilistic features do not vary during $T$. $T$ indicates, as usual, the selected observation period.

In these circumstances the probability laws recurring throughout the analysis of the phenomenon in question are called stationary.

Similarly, a random process is stationary with regard to its given moments if these latter do not depend on time $t$. This is equivalent to the invariance of the aforesaid statistics with regard to an arbitrary translation on the time axis.

Thus, different specifications can be provided for stationarity. They depend on the more or less restrictive statistical regularity conditions which can be detected for a process. A rigorous mathematical characterization of distinct stationarity forms lies beyond the scope of this book. In this regard, see for instance [1, 2].

A broad outline of strict-sense stationarity (SSS), or strict-stationarity, and some more preliminary notions on wide-sense stationarity (WSS), or weak stationarity, are given below.

A random process is strict-sense stationary if, for all sets of sections $X(t_1), X(t_2), \ldots, X(t_n)$ which can be extracted from it, and for all possible dimensions $n$ of the multivariate distribution, it results that
A strict-sense stationarity implies that if there are any, the joint moments of n-tuple RVs forming the process up to the order $\infty$, are not functions of $t$. In a $k$ finite-order stationarity, expression (2.16) is not satisfied for any $n$, but only for $n \leq k$. Thus a random process is classified as first-order stationary if we have from (2.16)

$$F(x, t) = F(x, t + \tau) = F(x).$$

(2.16')

It is classified as second-order stationary if we have

$$F(x_1, x_2; t_1, t_2) = F(x_1, x_2; t_1 + \tau, t_2 + \tau) = F(x_1, x_2, \tau)$$

(2.16'')

If Eq. 2.16 is valid for $n = k$, then it is valid for any $n < k$. In fact, the $k$-order distribution function determines all those of lower orders (see Sect. 2.1, Kolmogorov Consistency Theorem).

On the other hand, a wide-sense stationarity requires that:

(a) the process mean value function $m(t)$ is constant

$$m(t) = \text{const.} \quad \forall t \in T$$

(2.17)

(b) the autocovariance function only depends on $\tau$ and results to be finite in $\tau = 0$ ($\forall t = t'$)

$$C(t, t') = c(\tau) \quad \tau = t' - t \quad \forall t, t' \in T$$

(2.18)

$$c(0) < +\infty$$

(2.19)

Example We consider a random process $x(\omega, t)$ defined on $\Omega = \{\omega_1, \omega_2, \omega_3\}$. Realizations at an arbitrary time $t$, $X(\omega_1, t) = 2$, $X(\omega_2, t) = \sin(2t)$, $X(\omega_3, t) = \cos(2t)$ are equiprobable with $p(\omega_1) = p(\omega_2) = p(\omega_3) = 1/3$. For the mean value function we immediately obtain $m(t) = E[X(t)] = 1/3[2 + \sin(2t) + \cos(2t)]$.

$m(t)$, depending on $t$, shows evidence for non-stationarity of $x(\omega, t)$.

For a wide-sense stationary process, from the symmetry property of the autocovariance function, it follows that
\[ c(\tau) = c(-\tau) \] (2.20)

In fact, being \( C(t, t') = C(t', t) \), we obtain

\[ C(t, t + \tau) = C(t', t' - \tau) \] (2.21)

and for the supposed dependence of \( C(\cdot, \cdot) \) only on \( \tau \) (the process being wide-sense stationary), expression (2.20) follows.

In case of a discrete-time process, the time \( \tau = k \) between two arbitrary sections is called lag \( k \).

From the above-mentioned autocovariance property it follows that a wide-sense stationary process is also constant in variance.

This immediately results from (2.18) when \( t = t' \)

\[ C(t, t) = \sigma^2(t) = c(0) = \text{const.} \] (2.22)

In the hypothesis of wide-sense stationarity, the previous properties with respect to \( \tau \) are plainly valid for function \( \rho(\cdot) \) (see formula (2.12)).

In other words, we have

\[
\begin{align*}
\rho(t, t') &= r(\tau) = c(\tau)/c(0) \quad \tau = t' - t \quad \forall t, t' \in T \quad (2.23) \\
\rho(t, t) &= r(0) = \sigma^2(t)/\sigma^2(t) = 1 \quad (2.24)
\end{align*}
\]

If we now consider that any process can be made at a constant (zero) mean (see formula (2.9)), the invariance condition of \( c(\tau) \) for translation along time axis is the only one required for a wide-sense stationarity.

It is worth noting that a strict-sense stationarity implies a wide-sense stationarity if, and only if, the first two moments (2.17) and (2.18) of the process are finite.

This condition is not necessary for a strict-sense stationary process.

A wide-sense stationarity does not imply the strict-sense one necessarily. In fact, the validity of expressions (2.17) and (2.18) does not involve conditions (2.16).

Moreover, other definitions of wide-sense stationarity can be given than those established with expressions (2.17), (2.18), (2.19) (see [1], [2]). An example is then the autocorrelation stationarity. This occurs if, with a mean value function variable with \( t \), the only dependence on \( \tau \) is required for function (2.7), \( B(t, t') \): \( B(t, t') = b(\tau), \tau = t' - t, \forall t, t' \in T \).

Finally, in practice the term wide-sense stationarity is often used to mean exclusively the constancy of the mean value function \( m(t) \), or the \( m(t) \) and variance \( \sigma^2(t) \) of a process for \( t \) belonging to \( T \).

In the former case the term used is stationarity in mean. In the latter case stationarity is said to be in mean and variance.
The presence or not of mean, variance and covariance stationarity can also be detected in single realizations of a process. Such characteristics can, in other words, be detected also in a time series. The exploratory analysis of time series allows to carry out this type of study. In Fig. 2.8 there are clear examples of stationary and non-stationary (known as evolutionary) series.

Fig. 2.8 Stationary and non-stationary time series
In practice a wide-sense stationarity—in \( m(t) \) and \( \sigma^2(t) \)—of a random process is generally postulated on the basis of the phenomenon features in question.

### 2.3 Ergodicity of a Random Process

A rigorous definition of ergodicity and its forms requires some knowledge on the convergence in probability which cannot obviously be given here.

It is worth remembering here that a succession \( Y_n \) of random variables is said to converge in probability towards a R.V. \( Y \) if, given in any case \( \varepsilon > 0 \), we have

\[
\lim_{n \to \infty} P[|Y_n - Y| \geq \varepsilon] = 0.
\]

As to probability convergence see for instance [1, 3].

In order to introduce an elementary notion of ergodicity we consider Fig. 2.9a which shows \( n \) realizations \( x_i(t) \) of a random process \( x(\omega, t) \).

Let \( X(t^*) \) be a section of the process at time \( t^* \). We consider, moreover, the arithmetic mean value

\[
\bar{m}(t^*) = \frac{1}{n} \sum_{i=1}^{n} x_i(t^*) \quad t^* \in T
\]  

(2.25)

Formula (2.25) provides the mean value estimation of \( E[X(t^*)] \) from a sample of \( n \) realizations \( x_i(t^*) \), \( i = 1, 2, \ldots, n \), of \( X(t) \) in \( t^* \) (see Fig. 2.9a).

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**Fig. 2.9**  
(a) Realizations of a random process \( x(\omega, t) \);  
(b) single realization of \( x(\omega, t) \)
On the other hand, Fig. 2.9b shows only one arbitrary process realization (trajectory) denoted with \(x_k(t)\). For \(x_k(t)\) we have identified \(m\) time \(t_i\) and \(m\) determinations \(x_k(t_i)\).

Thus, we can calculate a further mean \(m_x\)

\[
m_x = \frac{\sum_{i=1}^{m} x_k(t_i)}{m} \quad t_i \in T
\]

\(m_x\) is called time mean related to an arbitrary process realization \(x(t)\).

In case of a stationary process, when \(n\) increases indefinitely, the mean \(\bar{m}(t^*)\) tends to the mean value of process (2.5) with \(m(t) = \text{const.}\).

In fact, \(\bar{m}(t^*)\) is the correct estimation of the mean (2.5).

A stationary process is said to be ergodic in mean if the time mean (2.26) of each random process realization tends to the mean value (2.25) when in (2.25) and in (2.26) \(n \rightarrow \infty\) and \(m \rightarrow \infty\) respectively.

Strictly speaking, the definition of ergodicity as given here should be provided for “nearly all” the realizations of the process. For some (rare) realizations, as a matter of fact, there could not find any equality between the relative \(m_x\) and the mean value of the process (however high be \(n\) and \(m\)). But the probability for these realizations to occur is null.

Ergodicity can also be defined for other process statistics, e.g. for autocovariance.

Thus, generally, in case of a continuous-time and continuous-valued process, ergodicity can be expressed as follows

\[
E\{\varphi[X(t)]\} = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \varphi[x(t)] \, dt
\]

where \(\varphi(\cdot)\) is an arbitrary real function. \(X(t)\) is, as usual, a R.V. section of the process \(x(\omega, t)\) and \(x(t)\) is one of its realizations.

The second member of the previous expression indicates the time-mean value of \(\varphi[X(t)]\).

In brief, with regard to a moment \(\beta\) (mean, variance, autocorrelation, etc.) ergodicity implies that:

1. the process is stationary with regard to \(\beta\) (stationarity in mean, in variance, etc.);
2. time means of \(\beta\) are equal for nearly all process realizations, so that we can talk of a time-mean value of \(\beta\);
3. time-mean value of \(\beta\) and the corresponding moment of the random process are equal with probability 1.

In short, we can say that ergodicity with regard to a random process consists in the fact that every single realization of the process is a ‘plenipotentiary representative’ of the whole family of realizations. Thus, a long-term realization can
In this direction, Fig. 2.8 clearly illustrates typical realizations of ergodic process in mean, variance and autocovariance in the left column graphs. The identification of necessary and sufficient conditions for ergodicity in random processes seems to be very complex. For instance, the ergodicity condition with respect to the mean function is the following (Slutsky’s Theorem, 1937):

\[ \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} r(j) = 0. \]

substitute, in terms of information provided, a set of realizations of the same duration.

In this direction, Fig. 2.8 clearly illustrates typical realizations of ergodic process in mean, variance and autocovariance in the left column graphs.

The identification of necessary and sufficient conditions for ergodicity in random processes seems to be very complex. For instance, the ergodicity condition with respect to the mean function is the following (Slutsky’s Theorem, 1937):

an necessary and sufficient condition for a stationary process to be ergodic with regard to the mean value \( m(t) \) is that it results \( \lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^{n} r(j) = 0. \)
It should be remembered that $r(s)$ is the autocorrelation coefficient given by formula (2.12) (see also formula (2.23)).

The graph of $r(\tau)$ is called correlogram. The correlogram reveals a lot of information on evolutionary characteristics of a random process (see Eq. (2.34) and worked example at end of this Section).

Figure 2.10 shows two correlogram examples sampled for a stationary (a) and alternating (b) time series.

Figure 2.11 shows the tendency of a correlogram sampled for a non-stationary process realization (with trend).

It must explicitly be said that ergodicity does not always ensue from a process stationarity (Fig. 2.12). In general, ergodicity (which will later be considered as
related to mean and autocovariance) is generally assumed on the basis of the nature of the problem under study.

In brief ergodicity, like stationarity, is generally postulated on the basis of the characteristics of the phenomenon observed.

### 2.4 Numerical Evaluation of Random Process Characteristics

In general, random process characteristics can be estimated from a sample of process realizations. We have \( n \) realizations. We consider a set of times \( t_1, t_2, \ldots, t_m \) (wherever possible, equispaced) called reference points, and the sections \( X(t_1), X(t_2), \ldots, X(t_m) \). From every \( X(t_j) \) \( n \) values are obtained for each \( t_j \) (\( j = 1, 2, \ldots, m \))

\[
X(t_1) = \{x_1(t_1), x_2(t_1), \ldots, x_n(t_1)\}
\]
\[
X(t_2) = \{x_1(t_2), x_2(t_2), \ldots, x_n(t_2)\}
\]
\[
\vdots
\]
\[
X(t_m) = \{x_1(t_m), x_2(t_m), \ldots, x_n(t_m)\}
\]

On the whole, \( n \times m \) data are obtained and \( n \) time series \( x_i(t) \) are also surveyed (see Table 2.1).

Data in Table 2.1 allow to obtain the estimations of:

- mean value function, represented section by section with:

\[
\bar{m}(t_k) = \frac{1}{n} \sum_{i=1}^{n} x_i(t_k)
\]

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**Table 2.1** Random process sampling

<table>
<thead>
<tr>
<th>( x_i(t) )</th>
<th>( t_j )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( t_1 )</td>
<td>( t_2 )</td>
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<tr>
<td>( x_1(t) )</td>
<td>( x_1(t_1) )</td>
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<td>( x_n(t) )</td>
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<td>( x_n(t) )</td>
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</tr>
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variance, given by

\[ \sigma^2(t_k) = \frac{\sum_{i=1}^{n} [x_i(t_k) - \bar{m}(t_k)]^2}{n - 1} \]  \hspace{1cm} (2.29)

• covariance function with

\[ C(t_k, t_l) = \frac{\sum_{i=1}^{n} [x_i(t_k) - \bar{m}(t_k)][x_i(t_l) - \bar{m}(t_l)]}{n - 1} \]  \hspace{1cm} (2.30)

Thus for a sample time set \((t_1, \ldots, t_m)\) the behaviours of the functions \(\bar{m}(t), \sigma^2(t)\) and the surface \(C(t, t')\) can then be evaluated at points.

From (2.29) and (2.30) and by recalling formula (2.12) we can also estimate the standard autocovariance surface \(\tilde{\rho}(t, t')\).

The analytical relations approximating \(\bar{m}(t), \sigma^2(t), C(t, t')\) and \(\tilde{\rho}(t, t')\) can be obtained by means of numerical analysis method. It goes without saying that the higher the size \(n\) of the realization sample and the size \(m\) of the reference point set, the more reliable the estimations are.

A significant number allows, moreover, to get direct information on the probabilistic features of a random process.

For this purpose we consider, for instance, Fig. 2.13.

An estimation \(F(x, t')\) of \(F(x, t)\) in \(t'\) is immediately provided by

\[ F(x, t') = \frac{n}{N} \]  \hspace{1cm} (2.31)

where \(n\) is the number of realizations, among all \(N\) of the sample, such that the ordinates in \(t'\) do not overcome \(x\).
The same method can be used to evaluate, e.g. the joint distribution function
\[ F(x_0, x_{00}; t_0, t_{00}) \]. It follows that
\[ F(x_0, x_{00}; t_0, t_{00}) = \frac{n}{N} \]
where \( n \) is, in this case, the number of realizations which in \( t_0 \) and \( t_{00} \) do not
overcome \( x_0 \) and \( x_{00} \) respectively (see Fig. 2.14).

If the process is ergodic, given \( N \) observations \( x(t_1), x(t_2), \ldots x(t_N) \) on a single
realization (see Fig. 2.15), for large \( N \) we assume:
- as mean estimation \( m_x = \frac{1}{N} \sum_{i=1}^{N} x(t_i) \) and as variance estimation expression
  (2.33) with \( k = 0 \);
- as autocovariance estimation

\[ c(k) = \frac{1}{N} \sum_{i=1}^{N-k} [x(t_i) - m_x][x(t_{i+k}) - m_x] \quad k = 1, 2, \ldots m \text{ where } m < N \] (2.33)
For the correlogram we then have, by means of formula (2.33),

\[
r(k) = c(k)/c(0) = \frac{\sum_{i=1}^{N-k} [x(t_i) - m_x] \cdot [x(t_{i+k}) - m_x]}{\sum_{i=1}^{N} [x(t_i) - m_x]^2}
\]

(2.34)

The estimation of distribution functions \( \hat{F}(x) \) is easily obtained by means of such expressions as

\[
\hat{F}(x) = \frac{n}{N}
\]

(2.35)

where \( n \) is the number of realizations, among all \( N \) of the sample \( (x(t_1), x(t_2), \ldots, x(t_N)) \), such that their values do not overcome \( x \).

\( \Delta t_i, i = 1, 2, \ldots, n \), denote the intervals of \( T \) in all the points whose process realization appears to be equal or lower than value \( x \).

On the general principles about the estimation of parameters of random processes from sample realizations see for instance [2].

**Example** In this example we use a dataset and the statistical analysis employed in [4] for a random process different from that we are examining here.

We record the speeds (in m/s) \( v_i(s_j) \) of \( i = 1, 2, \ldots, 12 \) vehicles in \( S_j, j = 0, 1, \ldots, 6 \) road cross-sections of abscissa \( s_j \) along a road segment.

Road cross-sections are mutually 0.4 km apart.

For each of the 12 vehicles we calculate the difference between the spot speeds recorded in road cross-sections \( S_{j+1} \) and \( S_j \) and attribute it to abscissa \( s_j \)

\[
x_i(s_j) = v_i(s_{j+1}) - v_i(s_j)
\]

(2.36)

\( x_i(s_j) \) is a point at \( s = s_j \) of a realization \( i \) of the process \( x(\omega, s) \) defined by (2.36).

Here the arbitrary abscissa \( s_j \) has obviously the role of “generalized time” \( t_j \).

Therefore we set \( x(\omega, s) = x(\omega, t) \) and \( x_i(s_j) = x_i(t_j) \).

Following the scheme in Table 2.1 the random process \( x(\omega, t) \) is sampled in Table 2.2.

With formula (2.28), by means of dataset in Table 2.2 we estimate the function \( \bar{m}(t) \) (Table 2.3).

With data in Table 2.2 expression (2.30) allows to estimate the autocorrelation (variance and autocovariance) matrix of the process \( x(\omega, t) \) under examination in Table 2.4.

In Table 2.4 the terms below the main diagonal are omitted, the matrix being symmetrical. On the main diagonal there are variance estimations (Table 2.4).

By extracting the square root, we obtain the mean square deviation \( \sigma(t) \) in function of time (see Tables 2.5 and 2.6).
Table 2.2 Random process sampling

<table>
<thead>
<tr>
<th>Realizations</th>
<th>t</th>
<th>0</th>
<th>0.4</th>
<th>0.8</th>
<th>1.2</th>
<th>1.6</th>
<th>2.0</th>
<th>2.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.64</td>
<td>0.74</td>
<td>0.62</td>
<td>0.59</td>
<td>0.35</td>
<td>−0.09</td>
<td>−0.39</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>0.54</td>
<td>0.37</td>
<td>0.06</td>
<td>−0.32</td>
<td>−0.60</td>
<td>−0.69</td>
<td>−0.676</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.34</td>
<td>0.50</td>
<td>0.37</td>
<td>0.26</td>
<td>−0.52</td>
<td>−0.72</td>
<td>0.42</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.23</td>
<td>0.26</td>
<td>0.35</td>
<td>0.55</td>
<td>0.69</td>
<td>0.75</td>
<td>0.80</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.12</td>
<td>0.20</td>
<td>0.24</td>
<td>0.18</td>
<td>−0.20</td>
<td>−0.42</td>
<td>−0.46</td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>−0.16</td>
<td>−0.12</td>
<td>−0.15</td>
<td>0.05</td>
<td>0.29</td>
<td>0.43</td>
<td>0.63</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>−0.22</td>
<td>−0.29</td>
<td>−0.38</td>
<td>−0.24</td>
<td>−0.06</td>
<td>0.07</td>
<td>−0.16</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>−0.26</td>
<td>−0.69</td>
<td>−0.70</td>
<td>−0.61</td>
<td>−0.43</td>
<td>−0.22</td>
<td>0.29</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>−0.50</td>
<td>−0.60</td>
<td>−0.68</td>
<td>−0.62</td>
<td>−0.68</td>
<td>−0.56</td>
<td>−0.54</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>−0.30</td>
<td>0.13</td>
<td>0.75</td>
<td>0.84</td>
<td>0.78</td>
<td>0.73</td>
<td>0.71</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>−0.69</td>
<td>−0.40</td>
<td>0.08</td>
<td>0.16</td>
<td>0.12</td>
<td>0.18</td>
<td>0.33</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>0.18</td>
<td>−0.79</td>
<td>−0.56</td>
<td>−0.39</td>
<td>−0.42</td>
<td>−0.58</td>
<td>−0.53</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.3 Mean m(t) values

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>0.4</th>
<th>0.8</th>
<th>1.2</th>
<th>1.6</th>
<th>2.0</th>
<th>2.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>m(t)</td>
<td>−0.007</td>
<td>−0.057</td>
<td>0.000</td>
<td>0.037</td>
<td>−0.057</td>
<td>−0.093</td>
<td>0.036</td>
</tr>
</tbody>
</table>

Table 2.4 Autocorrelation matrix values

<table>
<thead>
<tr>
<th>t'</th>
<th>t</th>
<th>0</th>
<th>0.4</th>
<th>0.8</th>
<th>1.2</th>
<th>1.6</th>
<th>2.0</th>
<th>2.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.163</td>
<td>0.138</td>
<td>0.080</td>
<td>0.046</td>
<td>−0.011</td>
<td>−0.064</td>
<td>−0.065</td>
<td></td>
</tr>
<tr>
<td>0.4</td>
<td>0.138</td>
<td>0.239</td>
<td>0.202</td>
<td>0.162</td>
<td>0.083</td>
<td>0.023</td>
<td>0.025</td>
<td></td>
</tr>
<tr>
<td>0.8</td>
<td>0.080</td>
<td>0.202</td>
<td>0.236</td>
<td>0.215</td>
<td>0.153</td>
<td>0.099</td>
<td>0.090</td>
<td></td>
</tr>
<tr>
<td>1.2</td>
<td>0.046</td>
<td>0.162</td>
<td>0.215</td>
<td>0.221</td>
<td>0.191</td>
<td>0.149</td>
<td>0.132</td>
<td></td>
</tr>
<tr>
<td>1.6</td>
<td>−0.011</td>
<td>0.083</td>
<td>0.153</td>
<td>0.191</td>
<td>0.241</td>
<td>0.235</td>
<td>0.171</td>
<td></td>
</tr>
<tr>
<td>2.0</td>
<td>−0.064</td>
<td>0.023</td>
<td>0.098</td>
<td>0.149</td>
<td>0.235</td>
<td>0.269</td>
<td>0.211</td>
<td></td>
</tr>
<tr>
<td>2.4</td>
<td>−0.065</td>
<td>0.025</td>
<td>0.090</td>
<td>0.132</td>
<td>0.171</td>
<td>0.211</td>
<td>0.288</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.5 Variance σ²(t) values

<table>
<thead>
<tr>
<th>t</th>
<th>0</th>
<th>0.4</th>
<th>0.8</th>
<th>1.2</th>
<th>1.6</th>
<th>2.0</th>
<th>2.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>σ²(t)</td>
<td>0.163</td>
<td>0.239</td>
<td>0.236</td>
<td>0.221</td>
<td>0.241</td>
<td>0.269</td>
<td>0.288</td>
</tr>
</tbody>
</table>
If we divide the values in Table 2.4 by the corresponding products of mean square deviations, we can draw the table of estimated values of the standard correlation function $q(t; t')$ (2.15) in Table 2.7.

We hypothesize now that $x(\omega, t)$ be stationary. An average estimation $E[X(t)] = \text{const.}$ is obviously obtained from

$$m = \frac{m(0) + m(0.4) + m(0.8) + \cdots + m(2.4)}{7} \approx -0.02 \text{ km/h}$$

Similarly for $\sigma^2(t)$

$$\sigma^2(t) = \frac{\sigma^2(0) + \sigma^2(0.4) + \sigma^2(0.8) + \cdots + \sigma^2(2.4)}{7} \approx 0.236 \text{ (km/h)}^2$$

$$\sigma(t) = \sqrt{0.236} \approx 0.486 \text{ km/h}$$

If we divide the values in Table 2.4 by the corresponding products of mean square deviations, we can draw the table of estimated values of the standard correlation function $\rho(t, t')$ (2.15) in Table 2.7.

We hypothesize now that $x(\omega, t)$ be stationary. An average estimation $E[X(t)] = \text{const.}$ is obviously obtained from

$$m = \frac{m(0) + m(0.4) + m(0.8) + \cdots + m(2.4)}{7} \approx -0.02 \text{ km/h}$$

Similarly for $\sigma^2(t)$

$$\sigma^2(t) = \frac{\sigma^2(0) + \sigma^2(0.4) + \sigma^2(0.8) + \cdots + \sigma^2(2.4)}{7} \approx 0.236 \text{ (km/h)}^2$$

$$\sigma(t) = \sqrt{0.236} \approx 0.486 \text{ km/h}$$

Finally, we estimate the standard correlation function. For a stationary process the correlation function (and therefore the standard correlation function) only depends on $\tau = t' - t$; consequently, for constant $\tau$ the correlation function must be constant. In Table 2.7 the main diagonal ($\tau = 0$) and parallels to it ($\tau = 0.4; \tau = 0.8; \tau = 1, 2, \ldots$) correspond to a constant $\tau$. The values of the function $\bar{r}(\tau)$ can be calculated by working out the mean of correlation function determinations along the parallels to the main diagonal (Table 2.8). The graph of the function $\bar{r}(\tau)$ (i.e. the correlogram) is represented in Fig. 2.16.

Finally, it should be noted that the characterization of random processes requires the study of other important topics before being used for practical

### Table 2.6 Mean square deviation $\sigma(t)$ values

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>0.4</th>
<th>0.8</th>
<th>1.2</th>
<th>1.6</th>
<th>2.0</th>
<th>2.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bar{\sigma}(t)$</td>
<td>0.404</td>
<td>0.489</td>
<td>0.486</td>
<td>0.470</td>
<td>0.491</td>
<td>0.519</td>
<td>0.537</td>
</tr>
</tbody>
</table>

### Table 2.7 Standard autocorrelation values

<table>
<thead>
<tr>
<th>$t'$</th>
<th>0</th>
<th>0.4</th>
<th>0.8</th>
<th>1.2</th>
<th>1.6</th>
<th>2.0</th>
<th>2.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t$</td>
<td>0</td>
<td>0.4</td>
<td>0.8</td>
<td>1.2</td>
<td>1.6</td>
<td>2.0</td>
<td>2.4</td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td>0.700</td>
<td>0.405</td>
<td>0.241</td>
<td>-0.053</td>
<td>-0.306</td>
<td>-0.299</td>
</tr>
<tr>
<td>0.4</td>
<td>0.700</td>
<td>1</td>
<td>0.856</td>
<td>0.707</td>
<td>0.345</td>
<td>0.090</td>
<td>0.095</td>
</tr>
<tr>
<td>0.8</td>
<td>0.405</td>
<td>0.856</td>
<td>1</td>
<td>0.943</td>
<td>0.643</td>
<td>0.390</td>
<td>0.344</td>
</tr>
<tr>
<td>1.2</td>
<td>0.241</td>
<td>0.707</td>
<td>0.943</td>
<td>1</td>
<td>0.829</td>
<td>0.612</td>
<td>0.524</td>
</tr>
<tr>
<td>1.6</td>
<td>-0.053</td>
<td>0.345</td>
<td>0.643</td>
<td>0.829</td>
<td>1</td>
<td>0.923</td>
<td>0.650</td>
</tr>
<tr>
<td>2.0</td>
<td>-0.306</td>
<td>0.090</td>
<td>0.390</td>
<td>0.612</td>
<td>0.923</td>
<td>1</td>
<td>0.760</td>
</tr>
<tr>
<td>2.4</td>
<td>-0.299</td>
<td>0.095</td>
<td>0.344</td>
<td>0.524</td>
<td>0.650</td>
<td>0.760</td>
<td>1</td>
</tr>
</tbody>
</table>
applications. Among these, for instance, there are complex-valued random processes, the canonical development of random variables and their spectral representations. Like the differential calculus of random processes, these subjects lie beyond the scope of this book. In this regard, see e.g. [1, 5, 6].

**References**


<table>
<thead>
<tr>
<th>$\tau$</th>
<th>0</th>
<th>0.4</th>
<th>0.8</th>
<th>1.2</th>
<th>1.6</th>
<th>2.0</th>
<th>2.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\tilde{r}(\tau)$</td>
<td>1.00</td>
<td>0.84</td>
<td>0.60</td>
<td>0.38</td>
<td>0.13</td>
<td>−0.10</td>
<td>−0.30</td>
</tr>
</tbody>
</table>

**Fig. 2.16** Correlogram for values in Table 2.8
Traffic and Random Processes
An Introduction
Mauro, R.
2015, IX, 119 p. 51 illus., Hardcover
ISBN: 978-3-319-09323-9