Chapter 2
Fundamentals of Nonlinear Systems

In this chapter, we review some fundamental concepts and properties of nonlinear control systems that will be referred to in the subsequent chapters. In Sects. 2.1 and 2.2, we summarize the stability and robust stability concepts and the fundamental Lyapunov's stability theory. In Sect. 2.3, we establish some lemmas for the analysis of adaptive control systems. In Sect. 2.4, we introduce the concept of input-to-state stability of nonlinear control systems. In Sects. 2.5 and 2.6, we introduce the changing supply function technique and its applications to two classes of stabilzation approaches. In Sect. 2.7, we present the small gain method in the context of input-to-state stability. The notes and references are given in Sect. 2.8.

The materials in Sects. 2.1 and 2.2 are standard in nonlinear control literature and hence all proofs of results in these sections are omitted. Sections 2.3–2.7 do contain some new ingredients and detailed proofs are provided for those results which are considered non-standard.

2.1 Stability Concepts

In this section, we study the stability concepts for the general non-autonomous system described by (1.1) while viewing the autonomous systems as a special case of (1.1). To guarantee the existence of the unique solution $x(t)$ to the system (1.1) satisfying an initial condition (see Theorems 11.1 and 11.2 in the Appendix), it is assumed throughout the chapter that the function $f(x, t)$ in (1.1) is piecewise continuous in t and locally Lipschitz in x for all $t \geq t_0 \geq 0$ and all $x \in \mathbb{R}^n$. A constant vector $x_e \in \mathbb{R}^n$ is said to be an *equilibrium point* of the system (1.1) if

$$f(x_e, t) = 0, \ \forall t \geq t_0 \geq 0.$$

If a nonzero vector x_e is an equilibrium point of (1.1), then one can always introduce a new state variable $\hat{x} = x - x_e$ and define a new system $\dot{\hat{x}} = f(\hat{x} + x_e, t)$ which has $\hat{x} = 0$ as its equilibrium point. Thus, without loss of generality, one can assume

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that the origin of $\mathbb{R}^n$, i.e., $x = 0$, is an equilibrium point of the system (1.1). With respect to the equilibrium point at the origin, various stability concepts are defined below.

Definition 2.1 The equilibrium point $x = 0$ of the system (1.1) is

(i) *Lyapunov stable (or stable)* at t_0 if for any $R > 0$, there exists $r(R, t_0) > 0$ such that $\|x(t)\| < R$ for all $t \geq t_0$, and all $\|x(t_0)\| < r(R, t_0)$.
(ii) *unstable* at t_0, if it is not stable at t_0.
(iii) *asymptotically stable (AS)* at t_0 if it is stable at t_0, and there exists $\delta(t_0) > 0$ such that $\|x(t)\| \to 0$ as $t \to \infty$ for all $\|x(t_0)\| < \delta(t_0)$.
(iv) *globally asymptotically stable (GAS)* at t_0 if it is stable at t_0 and $\|x(t)\| \to 0$ as $t \to \infty$ for all $x(t_0) \in \mathbb{R}^n$.

Definition 2.2 The equilibrium point $x = 0$ of the system (1.1) is

(i) *uniformly stable (US)* if for any $R > 0$, there exists $r(R) > 0$, independent of t_0, such that $\|x(t)\| < R$ for all $t \geq t_0$, and all $\|x(t_0)\| < r(R)$.
(ii) *uniformly asymptotically stable (UAS)* if it is uniformly stable, and there exists $\delta > 0$, independent of t_0, such that, for all $\|x(t_0)\| < \delta$, $\|x(t)\| \to 0$ as $t \to \infty$ uniformly in t_0, i.e., for any $\epsilon > 0$, there exists $T > 0$, independent of t_0, such that, for all $\|x(t_0)\| < \delta$, $\|x(t)\| < \epsilon$ whenever $t > t_0 + T$.
(iii) *uniformly globally asymptotically stable (UGAS)* if it is uniformly stable, and for any $\epsilon > 0$, and any $\delta > 0$, there exists $T > 0$, independent of t_0, such that, for all $\|x(t_0)\| < \delta$, $\|x(t)\| < \epsilon$ whenever $t > t_0 + T$.

In Definition 2.2, US, UAS, UGAS can be equivalently stated in terms of class $\mathcal{K}$, class $\mathcal{K}_\infty$ and class $\mathcal{KL}$ functions described as follows.

Definition 2.3 A continuous function $\gamma : [0, a) \mapsto [0, \infty)$ is said to belong to *class* $\mathcal{K}$ if it is strictly increasing and satisfies $\gamma(0) = 0$, and is said to belong to *class* $\mathcal{K}_\infty$ if, additionally, $a = \infty$ and $\lim_{r\to\infty} \gamma(r) = \infty$.

Definition 2.4 A continuous function $\beta : [0, a) \times [0, \infty) \mapsto [0, \infty)$ is said to belong to *class* $\mathcal{KL}$ if, for each fixed s, the function $\beta(\cdot, s)$ is a class $\mathcal{K}$ function defined on $[0, a)$, and, for each fixed r, the function $\beta(r, \cdot) : [0, \infty) \mapsto [0, \infty)$ is decreasing and $\lim_{s\to\infty} \beta(r, s) = 0$.

Definition 2.5 The equilibrium point $x = 0$ of the system (1.1) is

(i) *US* if there exist a class $\mathcal{K}$ function γ and $\delta > 0$, independent of t_0, such that $\|x(t)\| \leq \gamma(\|x(t_0)\|)$ for all $t \geq t_0$, and all $\|x(t_0)\| < \delta$.
(ii) *UAS* if there exist a class $\mathcal{KL}$ function β and $\delta > 0$, independent of t_0, such that $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$ for all $t \geq t_0$, and all $\|x(t_0)\| < \delta$.
(iii) *UGAS* if there exists a class $\mathcal{KL}$ function β, independent of t_0, such that $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$ for all $t \geq t_0$, and all $x(t_0) \in \mathbb{R}^n$.
(iv) *exponentially stable (ES)* or *globally exponentially stable (GES)* if it is UAS or UGAS with

$$\beta(r, s) = kre^{-\lambda s}, \; k \geq 1, \lambda > 0.$$

For an autonomous system of the form (1.2), if $x(t)$ is the solution of (1.2) satisfying the initial condition $x(t_0) = x_0$, then $\hat{x}(t) = x(t + t_0)$ is the solution of (1.2) satisfying the initial condition $\hat{x}(0) = x_0$. Therefore, one can always assume $t_0 = 0$ for an autonomous system. Moreover, for an autonomous system, if the equilibrium point is stable (or AS, GAS) at t_0, it is also US (or UAS, UGAS).

A typical non-autonomous system whose equilibrium point is GAS but not UGAS is given as follows.

Example 2.1 Consider a first order time-varying system

$$\dot{x} = -\frac{x}{1+t}, \quad x \in \mathbb{R}. \tag{2.1}$$

It can be verified that, for any initial state $x(t_0)$ with any initial time $t_0 \geq 0$, the solution of (2.1) is

$$x(t) = x(t_0)\frac{1+t_0}{1+t}, \ \forall t \geq t_0.$$

Observe that the equilibrium point $x = 0$ is US and GAS. But, for given $\epsilon > 0$ and $\delta > 0$, in order to make $\|x(t)\| < \epsilon$ for all $\|x(t_0)\| < \delta$, t must be greater than $T = \delta(1 + t_0)/\epsilon - 1$. Since this T cannot be made independent of t_0, the equilibrium point is not UGAS.

There is another method to draw the above conclusion. Consider a class $\mathcal{KL}$ function

$$\beta(r, s) = r\frac{1+t_0}{1+t_0+s}$$

satisfying

$$\|x(t)\| = \beta\left(\|x(t_0)\|, t - t_0\right),$$

thus, the equilibrium point $x = 0$ is GAS. But the function β depends on t_0, and it is impossible to find another class $\mathcal{KL}$ function $\bar{\beta}$, independent of t_0, such that $\beta(r, s) \leq \bar{\beta}(r, s)$, which concludes that the equilibrium point is not UGAS. In fact, if this is not the case, we have

$$r\frac{1+t_0}{1+t_0+s} \leq \bar{\beta}(r, s),$$

and there exist a real number s^* satisfying $\bar{\beta}(r, s^*) \leq r/2$. Thus, for any $t_0 \geq 0$,

$$r\frac{1+t_0}{1+t_0+s^*} \leq \frac{r}{2},$$

i.e.,

$$t_0 \le s^* - 1,$$

which contradicts that t_0 is an arbitrary nonnegative real number.

For a simple system such as (2.1), one may test the stability of its equilibrium point from the analytical expression of its solution. However, it is usually impossible to obtain the analytical solution to a complicated nonlinear system. Therefore, one may have to turn to other indirect methods to test a system's stability. Lyapunov's stability theory is one of the effective methods. Let us first introduce the Lyapunov's stability theory for the autonomous system (1.2). Suppose the function $f(x)$ is continuously differentiable with x in a neighborhood of the origin of $\mathbb{R}^n$. Define the Jacobian matrix of $f(x)$ at the origin as

$$F = \frac{\partial f}{\partial x}(0).$$

Then we have the following result.

Theorem 2.1 (Lyapunov's Linearization Theorem) *Consider the system (1.2). The equilibrium point $x = 0$ is AS if all the eigenvalues of the matrix F have negative real parts; and is unstable if at least one eigenvalue of the matrix F has positive real part.*

Remark 2.1 Theorem 2.1 cannot handle the case in which none of the eigenvalues of the matrix F has positive real part, but at least one of them has zero real part, and it does not tell whether the asymptotic stability is global or local.

Example 2.2 Consider a nonlinear system

$$\dot{x} = -\sin x + x^2. \tag{2.2}$$

The Jocobian matrix (scalar) is $F = -1$. So, the system is locally asymptotically stable. On the other hand, we have $\dot{x} = -\sin x + x^2 > 0$ if $x \ge 0.88$. Therefore, $x(t) > x(0) \ge 0.88,\ \forall t > 0$, if $x(0) \ge 0.88$. It implies that the system is not globally asymptotically stable. The state trajectories of the system (2.2) are illustrated in Fig. 2.1 with different initial state values. It shows that the state trajectory converges to the equilibrium point if the initial state is $x(0) = 0.80$ or $x(0) = 0.87$, but it diverges if the initial state is $x(0) = 0.88$.

On the other hand, the following Lyapunov's direct theorem can handle the two cases mentioned in Remark 2.1.

Theorem 2.2 (Lyapunov's Direct Theorem) *Consider the system (1.1). If there exists a continuously differentiable function $V : \mathbb{R}^n \times [t_0, \infty) \mapsto \mathbb{R}^+$ such that, for some class $\mathcal{K}$ functions $\bar{\alpha}$ and $\underline{\alpha}$, defined on $[0, \delta)$ for some $\delta > 0$,*

$$\underline{\alpha}(\|x\|) \le V(x,t) \le \bar{\alpha}(\|x\|) \tag{2.3}$$

$$\dot{V}(x,t) := \frac{\partial V(x,t)}{\partial t} + \frac{\partial V(x,t)}{\partial x} f(x,t) \le 0,\ \forall \|x\| < \delta,\ \forall t \ge t_0, \tag{2.4}$$

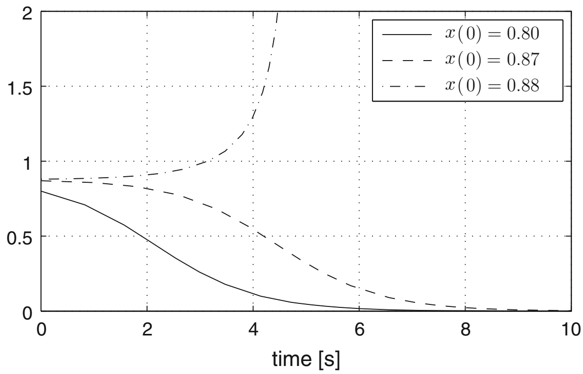

Fig. 2.1 Profile of state trajectories of the system in Example 2.2

then the equilibrium point $x = 0$ *is US. If, (2.4) is replaced by*

$$\dot{V}(x,t) \leq -\alpha(\|x\|), \ \forall \|x\| < \delta, \ \forall t \geq t_0, \tag{2.5}$$

where α *is a class* $\mathcal{K}$ *function defined on* $[0, \delta)$*, then the equilibrium point* $x = 0$ *is UAS. Moreover, if* $\delta = \infty$*, and* $\bar{\alpha}$ *and* $\underline{\alpha}$ *are class* $\mathcal{K}_\infty$ *functions, then the equilibrium point* $x = 0$ *is UGAS.*

A continuously differentiable function $V : \mathbb{R}^n \times [t_0, \infty) \mapsto \mathbb{R}^+$ satisfying (2.3) and (2.4) is called a Lyapunov function for (1.1).

Example 2.3 Consider the system (2.2) again. Let $V(x) = x^2$ whose derivative along the system trajectory satisfies

$$\dot{V}(x) = 2x(-\sin x + x^2) = -2x^2 + a(x)$$

for $a(x) = 2x(x - \sin x + x^2)$. There exists a constant δ such that $|a(x)/x^2| = 2|1 - \sin x/x + x| < 1$ for $|x| < \delta$. As a result, $|a(x)| < x^2$ for $|x| < \delta$, and hence $\dot{V}(x) \leq -x^2$. So, the system is locally asymptotically stable.

2.2 Robust Stability

In practice, a nonlinear system inevitably contains certain types of uncertainties such as external disturbances and parameter perturbations. To describe these uncertainties, we consider an uncertain nonlinear system of the following form:

$$\dot{x} = f(x, d(t)) \tag{2.6}$$

where d: $[t_0, \infty) \mapsto \mathbb{D} \subset \mathbb{R}^l$ with $\mathbb{D}$ a non-empty set represents external unpredictable disturbance and/or internal parameter variation. We assume that the function $f(x, d(t))$ is piecewise continuous in d and locally Lipschitz in x, and the function $d(t)$ is piecewise continuous in t. The system (2.6) is called a *non-autonomous/uncertain* system if d is time-varying. Taking into account the uncertainty $d(t)$, the control system (1.4)–(1.6) can be written as

$$\dot{x} = f(x, u, d(t)) \tag{2.7}$$

$$y = h(x, u, d(t)) \tag{2.8}$$

$$y_\mathrm{m} = h_\mathrm{m}(x, u, d(t)). \tag{2.9}$$

The closed-loop system composed of (2.7)–(2.9) with $y_\mathrm{m} = h_\mathrm{m}(x, d(t))$ and the controller (1.11) is

$$\dot{x}_\mathrm{c} = f_\mathrm{c}(x_\mathrm{c}, d(t)) \tag{2.10}$$

where

$$x_\mathrm{c} = \begin{bmatrix} x \\ v \end{bmatrix}, \quad f_\mathrm{c}(x_\mathrm{c}, d(t)) = \begin{bmatrix} f(x, \kappa_1(v, h_\mathrm{m}(x, d(t))), d(t)) \\ \kappa_2(v, h_\mathrm{m}(x, d(t))) \end{bmatrix}.$$

Example 2.4 In Example 1.1, if we replace the known external excitation $\cos t$ by an unknown external disturbance $d(t)$, then the system (1.3) can be expressed in the form (2.6) with

$$f(x, d(t)) = \begin{bmatrix} x_2 \\ -x_1 - x_1^3 + d(t) \end{bmatrix}.$$

For an uncertain system of the form (2.6), a constant vector $x_e \in \mathbb{R}^n$, independent of the signal $d(t)$, is said to be an *equilibrium point* of (2.6) if

$$f(x_\mathrm{e}, d(t)) = 0, \quad \forall t \geq t_0.$$

As explained for the equilibrium point of (1.1), without loss of generality, one only needs to consider the equilibrium point of the system (2.6) at the origin of $\mathbb{R}^n$, i.e., $x = 0$.

For each fixed $d(t)$, the uncertain system (2.6) reduces to the system (1.1). Therefore, various stability concepts described in Definition 2.5 for the system (1.1) also apply to the uncertain system (2.6). In this case, the functions β, γ, and the numbers δ, k, λ in Definition 2.5 may depend on specific $d(t)$. However, for some scenarios, one may find the functions β, γ, and the numbers δ, k, λ in Definition 2.5 which are independent of any $d(t) \in \mathbb{D}$. To distinguish these two scenarios, we further introduce the following *robust* stability concepts.

Definition 2.6 The equilibrium point $x = 0$ of the system (2.6) is *robustly uniformly asymptotically stable (RUAS)* or *robustly uniformly globally asymptotically stable*

(RUGAS) if it is UAS or UGAS with β and δ in Definition 2.5 independent of $d(t) \in \mathbb{D}$, or is *robustly exponentially stable (RES)* or *robustly globally exponentially stable (RGES)* if it is ES or GES with k and λ in Definition 2.5 independent of $d(t) \in \mathbb{D}$.

An uncertain system whose equilibrium point at the origin is UGAS but not RUGAS is given as follows.

Example 2.5 Consider the following system

$$\dot{x} = -\frac{1}{1+d^2}x, \quad x \in \mathbb{R}, \quad d \in \mathbb{R}. \tag{2.11}$$

It can be verified that, for any initial state $x(t_0)$ with any initial time $t_0 \geq 0$, the solution to (2.11) is

$$x(t) = x(t_0)\exp\left(\frac{-1}{1+d^2}(t-t_0)\right), \quad \forall t \geq t_0.$$

Observe that

$$\|x(t)\| = \beta(\|x(t_0)\|, t-t_0)$$

with

$$\beta(r,s) = r\exp\left(\frac{-1}{1+d^2}s\right).$$

Thus, the equilibrium point $x = 0$ is UGAS. But, the function β depends on d, and it is impossible to find another class $\mathcal{KL}$ function $\bar{\beta}$, independent of d, such that $\beta(r,s) \leq \bar{\beta}(r,s)$, which concludes that the equilibrium point is not RUGAS. In fact, if this is not the case, we have

$$r\exp\left(\frac{-1}{1+d^2}s\right) \leq \bar{\beta}(r,s),$$

and there exist a real number s^* satisfying $\bar{\beta}(r,s^*) \leq r/2$. Thus, for any $d \in \mathbb{R}$,

$$r\exp\left(\frac{-1}{1+d^2}s^*\right) \leq \frac{r}{2},$$

i.e.,

$$d^2 \leq \frac{s^*}{\ln 2} - 1,$$

which contradicts that d is an arbitrary real number.

Remark 2.2 It can be seen that if d is within a compact set, e.g., $d \in [-1, 1]$, then the origin of (2.11) is RUGAS, since $\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0)$ holds for $\beta(r, s) = re^{-0.5s}$, independent of d. In general, if the range of $d(t)$ is a compact set, then UAS, UGAS, and GES of the equilibrium point $x = 0$ of the system (2.6) imply RUAS, RUGAS, and RGES of the same equilibrium point, respectively.

The Lyapunov's stability theory, i.e., Theorems 2.1 and 2.2 can be generalized to the uncertain system (2.6). For example, the counterpart of Theorem 2.2 is stated as follows.

Theorem 2.3 (Lyapunov's Direct Theorem) *Consider the system (2.6). If there exists a continuously differentiable function* $V : \mathbb{R}^n \times [t_0, \infty) \mapsto \mathbb{R}^+$ *such that, for some class* $\mathcal{K}$ *functions* $\bar{\alpha}$ *and* $\underline{\alpha}$, *defined on* $[0, \delta)$ *for some* $\delta > 0$,

$$\underline{\alpha}(\|x\|) \leq V(x, t) \leq \bar{\alpha}(\|x\|) \tag{2.12}$$

$$\dot{V}(x, t) := \frac{\partial V(x, t)}{\partial t} + \frac{\partial V(x, t)}{\partial x} f(x, d(t)) \leq 0, \ \forall \|x\| < \delta, \ \forall t \geq t_0, \tag{2.13}$$

for all $d \in \mathbb{D}$, *then the equilibrium point* $x = 0$ *is US. If, (2.13) is replaced by*

$$\dot{V}(x, t) \leq -\alpha(\|x\|), \ \forall \|x\| < \delta, \ \forall t \geq t_0, \tag{2.14}$$

where α *is a class* $\mathcal{K}$ *function defined on* $[0, \delta)$, *then the equilibrium point* $x = 0$ *is RUAS. Moreover, if* $\delta = \infty$, *and* $\bar{\alpha}$ *and* $\underline{\alpha}$ *are class* $\mathcal{K}_\infty$ *functions, then the equilibrium point* $x = 0$ *is RUGAS.*

Theorem 2.4 *Suppose the conditions (2.12) and (2.14) in Theorem 2.3 are satisfied with*

$$\underline{\alpha}(\|x\|) = k_1 \|x\|^c, \ \bar{\alpha}(\|x\|) = k_2 \|x\|^c, \ \alpha(\|x\|) = k_3 \|x\|^c,$$

for some positive constants k_1, k_2, k_3, *and* c. *Then the equilibrium point* $x = 0$ *is RES. Moreover, if* $\delta = \infty$, *it is RGES.*

2.3 Tools for Adaptive Control

In this section, we introduce some tools for adaptive control, including Barbalat's Lemma, LaSalle-Yoshizawa Theorem, persistent excitation criteria, and a parameter convergence lemma.

Lemma 2.1 (Barbalat's Lemma) *Let* $\alpha : [t_0, \infty) \mapsto \mathbb{R}$ *be a continuously differentiable scalar function. If* $\alpha(t)$ *has a finite limit as* $t \to \infty$, *and* $\dot{\alpha}(t)$ *is uniformly continuous over* $[t_0, \infty)$, *then*

$$\lim_{t \to \infty} \dot{\alpha}(t) = 0.$$

As an application of Barbalat's Lemma, we can obtain the following result.

Theorem 2.5 (LaSalle-Yoshizawa Theorem) *Consider the system (2.6) where $f(x, d(t))$ is locally Lipschitz in x uniformly in t. If there exists a continuously differentiable function $V(x, t) : \mathbb{R}^n \times [t_0, \infty) \mapsto \mathbb{R}^+$ such that*

$$\begin{aligned} W_1(x) &\leq V(x,t) \leq W_2(x) \\ \dot{V}(x,t) &\leq -\alpha(x) \leq 0, \ \forall x \in \mathbb{R}^n, \quad \forall t \geq t_0, \end{aligned} \tag{2.15}$$

where $W_1(x)$ and $W_2(x)$ are continuous positive definite and radially unbounded[1] *functions and $\alpha(x)$ is a continuous positive semidefinite function, then the state is bounded and satisfies*

$$\lim_{t\to\infty} \alpha(x(t)) = 0.$$

Moreover, if $\alpha(x)$ is positive definite, then the equilibrium point $x = 0$ is UGAS.

Remark 2.3 By stating that $f(x, d(t))$ is locally Lipschitz in x *uniformly in t*, it means that, for any $x^* \in \mathbb{R}^n$,

$$\|f(x, d(t)) - f(y, d(t)\| \leq L\|x - y\| \tag{2.16}$$

is satisfied for all $x, y \in \{x \in \mathbb{R}^n \mid \|x - x^*\| \leq r\}$ for some $r > 0$ and for all $t \geq t_0$. The Lipchitz constant L depends on x^*, but is independent of t. If $f(x, d(t))$ is locally Lipschitz in x, and $\mathbb{D}$ is a compact set, then, clearly, $f(x, d(t))$ is locally Lipschitz in x *uniformly in t*.

Remark 2.4 Theorem 2.5 holds with $W_1(x)$ and $W_2(x)$ replaced by two class $\mathcal{K}_\infty$ functions $\underline{\alpha}(\|x\|)$ and $\bar{\alpha}(\|x\|)$, respectively.

Example 2.6 Consider a second order nonlinear system

$$\begin{aligned} \dot{x}_1 &= -x_1 k(x_1) + x_2\omega(t) \\ \dot{x}_2 &= -x_1\omega(t) \end{aligned}$$

where $x = [x_1, x_2]^\top \in \mathbb{R}^2$ is the state and $\omega(t)$ is a bounded continuous function. The function k is assumed to be a continuously differentiable and strictly positive function, i.e., $k(x_1) > k_o > 0, \ \forall x_1 \in \mathbb{R}$. The asymptotic property of the system is analyzed as follows. Consider a lower bounded function $V(x) = \|x\|^2$. Along the trajectory of the system, the derivative of $V(x)$ satisfies

[1] A function $f(x)$ is called radially unbounded if $f(x) \to \infty$ as $\|x\| \to \infty$.

$$\begin{aligned}\dot{V}(x) &= 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1\left(-x_1k(x_1) + x_2\omega\right) + 2x_2(-x_1\omega)\\ &= -2x_1^2k(x_1) \leq 0.\end{aligned} \tag{2.17}$$

Applying Theorem 2.5 and Remark 2.3 gives $\lim_{t\to\infty} x_1(t) = 0$.

The same conclusion can also be drawn by using Barbalat's Lemma. The inequality (2.17) implies that $V(x(t)) \leq V(x(0))$, $\forall t \geq 0$, and hence that $x(t)$ is bounded. Let $\alpha(t) = V(x(t))$. Its second derivative is $\ddot{\alpha}(t) = -[4x_1k(x_1) + 2x_1^2k'(x_1)][-x_1k(x_1) + x_2\omega]$ which is bounded since x and ω are bounded. Hence, $\dot{\alpha}(t)$ is uniformly continuous in t. By Lemma 2.1, $\lim_{t\to\infty} \dot{\alpha}(t) = 0$ and hence $\lim_{t\to\infty} x_1(t) = 0$.

The following material is concerned with the so-called persistent exciting property of a signal, which is widely used in the parameter convergence analysis in adaptive control.

Definition 2.7 A bounded piecewise continuous function $f : [0, \infty) \mapsto \mathbb{R}^n$ is said to be persistent exciting (PE) if there exist positive constants ϵ, t_0, T_0 such that, for any unit row vector c of dimension n,

$$\frac{1}{T_0}\int_t^{t+T_0} |cf(s)|ds \geq \epsilon, \quad \forall t \geq t_0. \tag{2.18}$$

Lemma 2.2 *A bounded piecewise continuous function* $f : [0, \infty) \mapsto \mathbb{R}^n$ *is PE if and only if there exist positive constants* ϵ, t_0, T_0 *such that*

$$\frac{1}{T_0}\int_t^{t+T_0} f(s)f^{\mathsf{T}}(s)ds \geq \epsilon^2 I, \quad \forall t \geq t_0. \tag{2.19}$$

Proof "Only if": By Jensen's inequality, i.e.,

$$(b-a)\int_a^b [g(s)]^2\, ds \geq \left(\int_a^b g(s)ds\right)^2,$$

for any integrable real-valued function g, we have

$$c\left[\frac{1}{T_0}\int_t^{t+T_0} f(s)f^{\mathsf{T}}(s)ds\right]c^{\mathsf{T}} = \frac{1}{T_0}\int_t^{t+T_0} [cf(s)]^2 ds \geq \left(\frac{1}{T_0}\int_t^{t+T_0} |cf(s)|ds\right)^2$$

for any unit row vector c of dimension n. As f is PE, one has (2.18), and hence

$$c\left[\frac{1}{T_0}\int_t^{t+T_0} f(s)f^{\mathsf{T}}(s)ds\right]c^{\mathsf{T}} \geq \epsilon^2 = c(\epsilon^2 I)c^{\mathsf{T}},$$

which implies (2.19).

"If": From (2.19), one has

$$\frac{1}{T_0}\int_t^{t+T_0} cf(s)f^{\mathsf{T}}(s)c^{\mathsf{T}}ds \geq \epsilon^2,$$

or

$$\int_t^{t+T_0} [cf(s)]^2\, ds \geq T_0\epsilon^2$$

for any unit row vector c of dimension n. Since the function f is bounded, so is $cf(s)$, i.e.,

$$|cf(s)| \leq R,\ \forall s \geq 0$$

for a constant R. Let $R_1 = \epsilon/\sqrt{2}$, $S_1 = \{s \mid |cf(s)| \geq R_1, t \leq s \leq t + T_0\}$ and $S_2 = \{s \mid |cf(s)| < R_1, t \leq s \leq t + T_0\}$. Then

$$S_1 \cup S_2 = [t, t + T_0],\ \ S_1 \cap S_2 = \emptyset.$$

Moreover, since $|cf(s)|$ is bounded and piecewise continuous, both S_1 and S_2 are Lebesgue measurable. Denote the length of a Lebesgue measurable set $S \subset [t, t+T_0]$ by $|S|$.[2] Then $0 \leq |S_i| \leq T_0$, $i = 1, 2$, and $|S_1 \cup S_2| = T_0$. Moreover,

$$\begin{aligned} T_0\epsilon^2 &\leq \int_t^{t+T_0} [cf(s)]^2\, ds = \int_{S_1} [cf(s)]^2\, ds + \int_{S_2} [cf(s)]^2\, ds \\ &\leq \int_{S_1} [cf(s)]^2\, ds + R_1^2|S_2|. \end{aligned}$$

The above inequality implies

[2] That is, $|S|$ is the Lebesgue measure on S [1].

$$|S_1|R^2 \geq \int_{S_1} [cf(s)]^2\, ds \geq T_0\epsilon^2 - R_1^2|S_2| = (T_0 - |S_2|/2)\epsilon^2 \geq T_0\epsilon^2/2 > 0$$

and

$$|S_1| \geq T_0\epsilon^2/(2R^2) > 0.$$

Next, we have

$$\frac{1}{T_0}\int_t^{t+T_0} |cf(s)|ds \geq \frac{1}{T_0}\int_{S_1} |cf(s)|ds \geq \frac{|S_1|R_1}{T_0} \geq \frac{\epsilon^3}{2\sqrt{2}R^2}$$

which is (2.18) with ϵ replaced by another constant $\epsilon^3/(2\sqrt{2}R^2)$. From the definition, f is PE. The proof is thus completed. □

Example 2.7

1. Let $f(t)$ be a nonzero constant function for all $t \geq 0$. Then $f(t)$ is PE.
2. Let $f(t) = \sin \omega t$ with $\omega > 0$. Let $T_0 = 2\pi/\omega$. Then

$$\frac{1}{T_0}\int_t^{t+T_0} |\sin \omega s|ds = \frac{2}{\pi}.$$

   Thus $f(t)$ is PE.
3. The function $f(t) = [\sin \omega t, \cos \omega t]^\top$ with $\omega > 0$ is PE while $f(t) = [\sin \omega t, \sin \omega t]^\top$ is not.

Next, we will show another criterion for the PE condition.

Lemma 2.3 *If a function* $f : [0, \infty) \mapsto \mathbb{R}^n$ *has spectral lines at frequencies* $\omega_1, \cdots, \omega_n$, *that is,*

$$\lim_{\delta\to\infty} \frac{1}{\delta}\int_t^{t+\delta} f(s)e^{-j\omega_i s}ds = \hat{f}(\omega_i) \neq 0, \quad i = 1, \cdots, n$$

uniformly in t. *Furthermore,* $\hat{f}(\omega_i)$, $i = 1, \cdots, n$, *are linearly independent in* $\mathbb{C}^n$. *Then,* $f(t)$ *is PE.*

Proof Define the matrix

$$F(t, \delta) = \frac{1}{\delta}\int_t^{t+\delta} \begin{bmatrix} e^{-j\omega_1 s} \\ \vdots \\ e^{-j\omega_n s} \end{bmatrix} f^\top(s)ds$$

and the matrix

$$F_0 = \begin{bmatrix} \hat{f}^\top(\omega_1) \\ \vdots \\ \hat{f}^\top(\omega_n) \end{bmatrix}.$$

The matrix F_0 is the limit of $F(t, \delta)$ as $\delta \to \infty$, uniformly in t. As F_0 is nonsingular by hypothesis, there exists a sufficiently large T_0, such that, for $\delta \geq T_0$, $F(t, \delta)$ is invertible and

$$\|F^{-1}(t, \delta)\| \leq 2\|F_0^{-1}\|, \ \forall t \geq 0.$$

Now, for any unit row vector $c \in \mathbb{R}^n$ and any $\omega \in \mathbb{R}$, we have

$$\begin{aligned} \frac{1}{\delta} \int_t^{t+\delta} [cf(s)]^2 ds &= \frac{1}{\delta} \int_t^{t+\delta} |cf(s)e^{-j\omega s}|^2 ds \\ &\geq \left| \frac{1}{\delta} \int_t^{t+\delta} cf(s)e^{-j\omega s} ds \right|^2 \end{aligned}$$

(by Jensen's inequality, see the proof of Lemma 2.2). For $\omega = \omega_1, \cdots, \omega_n$, one has

$$\begin{aligned} \frac{1}{\delta} \int_t^{t+\delta} [cf(s)]^2 ds &\geq \frac{1}{n} \sum_{i=1}^{n} \left| \frac{1}{\delta} \int_t^{t+\delta} cf(s)e^{-j\omega_i s} ds \right|^2 \\ &= \frac{1}{n} \|F(t, \delta)c^\top\|^2 \geq \frac{1}{n} \|F^{-1}(t, \delta)\|^{-2} \end{aligned}$$

and, for $\delta \geq T_0$,

$$\frac{1}{\delta} \int_t^{t+\delta} [cf(s)]^2 ds \geq \frac{1}{4n} \|F_0^{-1}\|^{-2}.$$

As a result,

$$c \left[\frac{1}{T_0} \int_t^{t+T_0} f(s) f^\top(s) ds - \epsilon^2 I \right] c^\top \geq 0, \ \epsilon = \|F_0^{-1}\|^{-1}/(2\sqrt{n})$$

for any unit row vector c of dimension n, which implies (2.19). The proof is thus completed. □

Example 2.8 Let

$$\tau(t) = \sum_{k=1}^{\ell} A_k \cos(\omega_k t + \phi_k)$$

for some $\ell > 0$ where A_k's are strictly positive real numbers and ω_k's are distinct strictly positive real numbers. Let

$$f(t) = [\tau(t), \dot{\tau}(t), \cdots, d^{(n-1)}\tau(t)/dt^{(n-1)}]^{\mathsf{T}}.$$

Then $f(t)$ is PE if $n \leq 2\ell$.

In fact, $\tau(t)$ and $f(t)$ can be rewritten as

$$\tau(t) = \sum_{k=1}^{\ell} A_k[e^{j(\omega_k t+\phi_k)} + e^{-j(\omega_k t+\phi_k)}]/2$$

and, respectively,

$$f(t) = \begin{bmatrix} \sum_{k=1}^{\ell} A_k[e^{j(\omega_k t+\phi_k)} + e^{-j(\omega_k t+\phi_k)}]/2 \\ \sum_{k=1}^{\ell} A_k[j\omega_k e^{j(\omega_k t+\phi_k)} + (-j\omega_k)e^{-j(\omega_k t+\phi_k)}]/2 \\ \vdots \\ \sum_{k=1}^{\ell} A_k[(j\omega_k)^{n-1} e^{j(\omega_k t+\phi_k)} + (-j\omega_k)^{n-1} e^{-j(\omega_k t+\phi_k)}]/2 \end{bmatrix}.$$

When $n \leq 2\ell$, we can pick n distinct frequencies $\hat{\omega}_i$ and the corresponding $\hat{\phi}_i$ and $\hat{A}_i$ as follows:

$$(\hat{\omega}_i, \hat{\phi}_i, \hat{A}_i) \in \{(\omega_1, \phi_1, A_1), (-\omega_1, -\phi_1, A_1), \cdots, (\omega_\ell, \phi_\ell, A_\ell), (-\omega_\ell, -\phi_\ell, A_\ell)\}, \quad i = 1, \cdots, n.$$

It is easy to check that

$$\hat{f}(\hat{\omega}_i) = \lim_{\delta \to \infty} \frac{1}{\delta} \int_t^{t+\delta} f(s) e^{-j\hat{\omega}_i s} ds = \begin{bmatrix} 1 \\ j\hat{\omega}_i \\ \vdots \\ (j\hat{\omega}_i)^{n-1} \end{bmatrix} \hat{A}_i e^{j\hat{\phi}_i}/2, \; i = 1, \cdots, n.$$

Since the frequencies $\hat{\omega}_i$, $i = 1, \cdots, n$, are distinct, the vectors $\hat{f}(\hat{\omega}_i), i = 1, \cdots, n$, are linearly independent in $\mathbb{C}^n$. Then, $f(t)$ is PE.

The PE property is useful in signal convergence analysis as illustrated in the following lemma. This lemma will be used in Chap. 5 for studying adaptive control.

Lemma 2.4 *Consider a continuously differentiable function* $g : [0, \infty) \mapsto \mathbb{R}^n$ *and a bounded piecewise continuous function* $f : [0, \infty) \mapsto \mathbb{R}^n$*, which satisfy*

$$\lim_{t \to \infty} g^{\mathsf{T}}(t) f(t) = 0. \tag{2.20}$$

Then,

$$\lim_{t \to \infty} g(t) = 0 \tag{2.21}$$

holds under the following two conditions:

(i) $\lim_{t \to \infty} \dot{g}(t) = 0$*;*
(ii) $f(t)$ *is PE.*

Proof Suppose (2.21) is not true. Then there exist a time sequence $s_1 < s_2 < \cdots$ satisfying $s_i \to \infty$ as $i \to \infty$ and a number $\delta_1 > 0$, such that $\|g(s_i)\| > \delta_1$. Under (2.20) and the condition (i), for any $\delta_2 > 0$ and $\delta_3 > 0$, there exists a time t_1, such that,

$$|\dot{g}_k(t)| \leq \delta_2, \quad \forall t \geq t_1, \; k = 1, \cdots, n,$$

and

$$|g^{\mathsf{T}}(t) f(t)| \leq \delta_3, \quad \forall t \geq t_1.$$

Also, under the condition (ii),

$$\int_t^{t+T_0} |cf(s)| ds \geq \epsilon_1 T_o, \quad \forall t \geq t_1 \tag{2.22}$$

for some constants T_0 and ϵ_1, independent of δ_3.

As a result, one has

$$|g_k(t+s) - g_k(t)| \leq \int_t^{t+s} |\dot{g}_k(x)| dx \leq \delta_2 T_0, \forall 0 \leq s \leq T_0, \; \forall t > t_1, \; k = 1, \cdots, n.$$

Let $\bar{f}$ be some real number such that $\|f(t)\| < \bar{f}, \forall t \geq 0$. Then, for any $s_i > t_1$,

$$\begin{aligned} \int_{s_i}^{s_i+T_0} |g^{\mathsf{T}}(s) f(s)| ds &\geq \int_{s_i}^{s_i+T_0} |g^{\mathsf{T}}(s_i) f(s)| ds - \int_{s_i}^{s_i+T_0} |[g(s) - g(s_i)]^{\mathsf{T}} f(s)| ds \\ &\geq T_0 \delta_1 \epsilon_1 - \delta_2 T_0^2 \bar{f}. \end{aligned}$$

Since δ_2 can be arbitrarily small,

$$\int_{s_i}^{s_i+T_0} |g^{\mathsf{T}}(s)f(s)|ds \geq \epsilon_2$$

for some positive ϵ_2 independent of δ_3. Thus, there exists a time $\bar{s}_i \in [s_i, s_i + T_0]$ such that

$$\delta_3 \geq |g^{\mathsf{T}}(\bar{s}_i)f(\bar{s}_i)| \geq \epsilon_2/T_0.$$

Noting δ_3 can be arbitrarily small leads to a contradiction. The proof is thus completed. □

This lemma gives the convergence condition of the function $g(t)$ to the origin based on the asymptotic condition (i) of $\dot{g}(t)$ and the PE condition (ii) of $f(t)$. Both conditions are indispensable as illustrated in the following examples.

Example 2.9 Let $f(t) = [\cos\omega t \quad \sin\omega t]^{\mathsf{T}}$ and $g(t) = [-\sin\omega t \quad \cos\omega t]^{\mathsf{T}}$. The condition $\lim_{t\to\infty} g^{\mathsf{T}}(t)f(t) = 0$ obviously holds and $f(t)$ is PE. However, $\lim_{t\to\infty} g(t) = 0$ is not true because $\lim_{t\to\infty} \dot{g}(t) = 0$ is not.

Example 2.10 Consider a continuously differentiable signal $g(t) = ca(t)$ for a constant vector $c \in \mathbb{R}^2$ and function $a(t)$ satisfying $\lim_{t\to\infty} \dot{a}(t) = 0$. Suppose $\lim_{t\to\infty} g^{\mathsf{T}}(t)f(t) = 0$ where $f(t)$ is a bounded piecewise continuous function.

If $f(t)$ is PE, we have $\lim_{t\to\infty} g(t) = 0$ for any c by Lemma 2.4.

If $f(t)$ is not PE, e.g., $f(t) = [\cos\omega t \quad \cos\omega t]^{\mathsf{T}}$, then $\lim_{t\to\infty} g(t) = 0$ is not necessarily true. For example, when $c = [1 \quad -1]^{\mathsf{T}}$ and $a(t) = 1$, the condition $\lim_{t\to\infty} g^{\mathsf{T}}(t)f(t) = 0a(t) = 0$ still holds. But $\lim_{t\to\infty} g(t) = [1 \quad -1]^{\mathsf{T}} \neq 0$.

2.4 Input-to-State Stability

In the previous sections, we have reviewed various stability concepts of the nonlinear systems described by (1.1) and (2.6), respectively. In this section, we will further consider the stability of the control systems described by (1.4) and (2.7). Since the response of the system (1.4) or (2.7) is excited not only by the initial state $x(t_0)$ but also by the input $u(t)$, we need to generalize the stability concepts about an equilibrium point to the so-called *input-to-state stability* of the system (2.7) while keeping in mind that the system (1.4) can be viewed as a special case of (2.7) by having $d(t) = t$.

Again, we assume the function $f(x, u, d(t))$ is piecewise continuous in d and locally Lipschitz in $\text{col}(x, u)$, and the function $d(t)$ is piecewise continuous in t. And we use the notation L_∞^m to denote the set of all piecewise continuous bounded functions $u : [t_0, \infty) \mapsto \mathbb{R}^m$ with the supermum norm

$$\left\| u_{[t_0,\infty)} \right\| := \sup_{t \geq t_0} \| u(t) \|.$$

For convenience, we also denote the supermum norm of the truncation of $u(t)$ in $[t_1, t_2]$ with $t_0 \leq t_1 \leq t_2$ as follows,

$$\left\| u_{[t_1,t_2]} \right\| := \sup_{t_1 \leq t \leq t_2} \| u(t) \|.$$

Definition 2.8 The system (2.7) is said to be *input-to-state stable (ISS)* if there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ, independent of t_0, such that for any initial state $x(t_0)$ and any input function $u \in L_\infty^m$, the solution $x(t)$ exists and satisfies

$$\| x(t) \| \leq \max \left\{ \beta(\| x(t_0) \|, t - t_0), \gamma \left(\left\| u_{[t_0,t]} \right\| \right) \right\}, \ \forall t \geq t_0. \tag{2.23}$$

Since the control system (2.7) involves the uncertainty $d(t)$, the functions β and γ in Definition 2.8 may or may not depend on $d(t)$. If the functions β and γ can be made to be independent of the uncertainty $d(t)$, then we have the following robust input-to-state stability concept.

Definition 2.9 The system (2.7) is said to be *robustly input-to-state stable (RISS)* if it is ISS in the sense of Definition 2.8 with β and γ independent of $d(t) \in \mathbb{D}$.

Remark 2.5 We note that the functions β and γ are independent of t_0 in the definition of ISS or RISS. In other words, the concepts ISS and RISS implicitly include the fact that they are uniformly with respect to the initial time t_0. For an RISS system (2.7), when the input u is held at zero, the solution starting from any initial state $x(t_0)$ for any initial time t_0 satisfies

$$\| x(t) \| \leq \beta(\| x(t_0) \|, t - t_0), \ \forall t \geq t_0.$$

Thus, the equilibrium point at the origin of the unforced system $\dot{x} = f(x, 0, d(t))$ is RUGAS.

Remark 2.6 In (2.23), since, for any $x(t_0)$ and any t_0, $\beta(\| x(t_0) \|, t - t_0) \to 0$ as $t \to \infty$, one has

$$\lim_{t \to \infty} \left\| x_{[t,\infty)} \right\| \leq \gamma \left(\left\| u_{[t_0,\infty)} \right\| \right).$$

Due to this inequality, the class $\mathcal{K}$ function γ is called a *gain function* of (2.7).

Remark 2.7 There is an equivalent way to characterize the ISS property of (2.7) as follows. There exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function γ such that for any initial state $x(t_0)$ and any input function $u \in L_\infty^m$, the solution $x(t)$ exists and satisfies

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0) + \gamma\left(\left\|u_{[t_0,t]}\right\|\right), \ \forall t \ge t_0.$$

This equivalence follows from the fact that $\max\{\beta, \gamma\} \le \beta + \gamma \le \max\{2\beta, 2\gamma\}$ for any pair $\beta \ge 0$, $\gamma \ge 0$.

The Lyapunov's direct theorem can also be generalized to analyze the ISS property of a system as described below.

Definition 2.10 Let $V : \mathbb{R}^n \times [t_0, \infty) \mapsto \mathbb{R}^+$ be a continuously differentiable function. It is called an *ISS-Lyapunov function* for the system (2.7) if there exist class $\mathcal{K}_\infty$ functions $\bar{\alpha}$, $\underline{\alpha}$, α, and a class $\mathcal{K}$ function ρ, such that

$$\underline{\alpha}(\|x\|) \le V(x,t) \le \bar{\alpha}(\|x\|)$$
$$\dot{V}(x,t) \le -\alpha(\|x\|), \ \forall \|x\| \ge \rho(\|u\|)$$

for all $x \in \mathbb{R}^n$, $u \in L_\infty^m$, $t \ge t_0$, and $d \in \mathbb{D}$.

Theorem 2.6 *If the system (2.7) has an ISS-Lyapunov function, then it is RISS with a gain function $\underline{\alpha}^{-1} \circ \bar{\alpha} \circ \rho$, i.e., there exist a class $\mathcal{KL}$ function β and a class $\mathcal{K}$ function $\gamma = \underline{\alpha}^{-1} \circ \bar{\alpha} \circ \rho$ such that for any initial state $x(t_0) \in \mathbb{R}^n$ and any input function $u \in L_\infty^m$, the solution $x(t)$ of (2.7) exists and satisfies (2.23).*

The proof of Theorem 2.6 can be found in [2] (see the proofs of Theorems 4.18 and 4.19). Suppose $V : \mathbb{R}^n \times [t_0, \infty) \mapsto \mathbb{R}^+$ is a continuously differentiable function, for all $x \in \mathbb{R}^n$, $u \in L_\infty^m$, $t \ge t_0$, and $d \in \mathbb{D}$, the derivative of V along the trajectory of $\dot{x} = f(x, u, d(t))$ satisfies

$$\dot{V}(x,t) \le -\alpha(\|x\|) + \sigma(\|u\|) \tag{2.24}$$

where α is some class $\mathcal{K}_\infty$ function and σ some class $\mathcal{K}$ function. Let

$$\rho(s) = \alpha^{-1}(k\sigma(s))$$

with $k > 1$. Then

$$\|x\| \ge \rho(\|u\|) \ \ \Rightarrow \ \ \sigma(\|u\|) \le \frac{1}{k}\alpha(\|x\|).$$

So, (2.24) gives

$$\dot{V}(x,t) \le -\frac{k-1}{k}\alpha(\|x\|), \ \ \forall \|x\| \ge \rho(\|u\|)$$

for all $x \in \mathbb{R}^n$, $u \in L_\infty^m$, $t \ge t_0$, and $d \in \mathbb{D}$. Thus, $V(x,t)$ is an ISS Lyapunov function of (2.7). As a result, we obtain the following result.

Theorem 2.7 *Consider the system (2.7). If there exists a continuously differentiable function $V : \mathbb{R}^n \times [t_0, \infty) \mapsto \mathbb{R}^+$ such that, for some class $\mathcal{K}_\infty$ functions $\bar{\alpha}$, $\underline{\alpha}$, α, and some class $\mathcal{K}$ function σ,*

$$\begin{aligned} &\underline{\alpha}(\|x\|) \le V(x,t) \le \bar{\alpha}(\|x\|) \\ &\dot{V}(x,t) \le -\alpha(\|x\|) + \sigma(\|u\|) \end{aligned} \tag{2.25}$$

for all $x \in \mathbb{R}^n$, $u \in L_\infty^m$, $t \ge t_0$, and $d \in \mathbb{D}$, then the system (2.7) is RISS with a gain function $\underline{\alpha}^{-1} \circ \bar{\alpha} \circ \alpha^{-1} \circ k\sigma$ for any $k > 1$.

To simplify the presentation, we use the following notation

$$V(x,t) \sim \{\underline{\alpha}, \bar{\alpha}, \alpha, (\sigma_1, \cdots, \sigma_m) \mid \dot{x} = f(x,u,d)\} \tag{2.26}$$

to mean the following statement: there exist some class $\mathcal{K}_\infty$ functions $\bar{\alpha}$, $\underline{\alpha}$, α, and some class $\mathcal{K}$ functions σ_i, $i = 1, \cdots m$, such that,

$$\begin{aligned} \underline{\alpha}(\|x\|) &\le V(x,t) \le \bar{\alpha}(\|x\|) \\ \dot{V}(x,t) &\le -\alpha(\|x\|) + \sum_{i=1}^m \sigma_i(\|u_i\|) \end{aligned}$$

for all $x \in \mathbb{R}^n$, $u \in L_\infty^m$, $t \ge t_0$, and $d \in \mathbb{D}$. In particular, for a single input system $x = f(x,u,d)$, the notation (2.26) reduces to a simpler form

$$V(x,t) \sim \{\underline{\alpha}, \bar{\alpha}, \alpha, \sigma \mid \dot{x} = f(x,u,d)\}.$$

Example 2.11 Consider the system

$$\dot{x} = A(t)x + G(u,t), \quad t \ge t_0 \ge 0 \tag{2.27}$$

where $G(u,t)$ is a continuous function satisfying, for all $u \in \mathbb{R}$ and all $t \ge t_0$, $\|G(u,t)\| \le q(\|u\|)$ for some class $\mathcal{K}$ function q. Suppose the system $\dot{x} = A(t)x$ is UAS, i.e., there exist symmetric positive definite matrices $Q(t)$ and $P(t)$ satisfying $0 < \beta_1 I \le Q(t) \le \beta_2 I, \forall t \ge 0$ and $0 < \alpha_1 I \le P(t) \le \alpha_2 I, \forall t \ge 0$, such that

$$\dot{P}(t) + P(t)A(t) + A(t)^\top P(t) = -Q(t).$$

Let $V(x,t) = x^\top P(t)x$. Then, along the trajectory of (2.27),

$$\begin{aligned} \dot{V}(x,t) &\le -\|Q(t)\|\|x\|^2 + 2x^\top P(t)G(u,t) \\ &\le -(\|Q(t)\| - 1/\epsilon)\|x\|^2 + \epsilon\|P(t)G(u,t)\|^2 \\ &\le -(\beta_1 - 1/\epsilon)\|x\|^2 + \epsilon\alpha_2^2 q^2(\|u\|). \end{aligned}$$

Let ϵ be such that $l = \beta_1 - 1/\epsilon > 0$ and let $\sigma(\|u\|) = \epsilon\alpha_2^2 q^2(\|u\|)$. Then, we have

$$\dot{V}(x,t) < -l\|x\|^2 + \sigma(\|u\|). \tag{2.28}$$

Thus, the system (2.27) is ISS with a gain function

$$\gamma(s) = \sqrt{\frac{\alpha_2 k}{\alpha_1 l}\sigma(s)} = \sqrt{\frac{\alpha_2 k\epsilon}{\alpha_1 l}}\alpha_2 q(s)$$

for any $k > 1$.

Example 2.12 As a special case of the above example, a linear time-invariant system

$$\dot{x} = Ax + Bu, \quad t \geq 0$$

where A is a Hurwitz matrix is ISS. Also, since $\|Bu\| \leq b\|u\|$ for some $b > 0$, the gain function is $\gamma(s) = \sqrt{\alpha_2 k\epsilon/(\alpha_1 l)}\alpha_2 bs$, which is a linear function.

Example 2.13 The following scalar system $\dot{x} = -x + xu$ is not ISS. In fact, let $u(t) = 2$ for all $t \geq 0$. Then the response of the system with $x(0) = x_0$ is $x(t) = e^t x_0$, which shows that the inequality (2.23) cannot hold.

Next, we further introduce two other concepts for the system (2.7) as follows.

Definition 2.11 The system (2.7) is said to have the *robustly globally stable (RGS)* property, and the *robustly asymptotic gain (RAG)* property, respectively, if there exist class $\mathcal{K}$ functions γ_0 and γ, independent of $d(t)$, such that for any initial time t_0, any initial state $x(t_0) \in \mathbb{R}^n$, any $d(t) \in \mathbb{D}$, and any input function $u \in L_\infty^m$, the solution $x(t)$ exists and satisfies

$$\left\|x_{[t_0,\infty)}\right\| \leq \max\left\{\gamma_0(\|x(t_0)\|), \gamma\left(\left\|u_{[t_0,\infty)}\right\|\right)\right\},$$

and, respectively,

$$\lim_{t\to\infty}\left\|x_{[t,\infty)}\right\| \leq \gamma\left(\lim_{t\to\infty}\left\|u_{[t,\infty)}\right\|\right).$$

These two concepts are of particular interest to autonomous control systems, e.g., the system (2.7) with $d(t) = constant$, since it is possible to show, for autonomous control systems, the RISS property is equivalent to the RGS property plus the RAG property (see, e.g., [3]), i.e.,

$$\text{RISS} \iff \text{RGS} + \text{RAG}.$$

Example 2.14 Consider the linear time-invariant system

$$\dot{x} = Ax + Bu, \quad t \geq 0 \tag{2.29}$$

where A is Hurwitz and $u(t)$ is piecewise continuous in t and $\lim_{t\to\infty} u(t) = 0$. By Example 2.12, this system is ISS, and is thus of the asymptotic gain property. Therefore, for any initial condition $x(0)$, $\lim_{t\to\infty} x(t) = 0$.

However, for a non-autonomous control system (2.7), this equivalence does not hold any more. Specifically, the implication

$$\text{RISS} \Rightarrow \text{RGS} + \text{RAG}$$

is true, but the other direction is not, i.e.,

$$\text{RISS} \not\Leftarrow \text{RGS} + \text{RAG},$$

as shown in the following Example.

Example 2.15 Consider a non-autonomous system

$$\dot{x} = -\frac{x-u}{1+t}, \;\; x \in \mathbb{R}, \;\; u \in L^1_\infty. \tag{2.30}$$

It can be verified that, for any initial state $x(t_0)$ with any initial time $t_0 \geq 0$, the solution of (2.30) is

$$x(t) = \frac{1+t_0}{1+t} x(t_0) + \frac{1}{1+t} \int_{t_0}^{t} u(\tau) d\tau, \;\; \forall t \geq t_0.$$

On one hand,

$$|x(t)| \leq |x(t_0)| + \left\| u_{[t_0,\infty)} \right\|,$$

hence, $\left\| x_{[t_0,\infty)} \right\| \leq \max\left\{2|x(t_0)|, 2\left\| u_{[t_0,\infty)} \right\|\right\}$. That is, the system (2.30) has RGS property. On the other hand, for any $\epsilon > 0$, there exists $T_1 \geq t_0$ such that

$$\left\| u_{[T_1,\infty)} \right\| \leq \lim_{t\to\infty} \left\| u_{[t,\infty)} \right\| + \epsilon.$$

And there exists $T_2 \geq T_1$ such that

$$\frac{1+T_1}{1+T_2} x(T_1) \leq \epsilon.$$

Then, for any time $t \geq T_2$,

$$\begin{aligned} |x(t)| &\leq \frac{1+T_1}{1+t} x(T_1) + \left\| u_{[T_1,\infty)} \right\| \\ &\leq \epsilon + \lim_{t\to\infty} \left\| u_{[t,\infty)} \right\| + \epsilon \end{aligned}$$

hence, $\lim_{t\to\infty} \left\|x_{[t,\infty)}\right\| \leq \lim_{t\to\infty} \left\|u_{[t,\infty)}\right\| + 2\epsilon$. Letting $\epsilon \to 0$ yields that the system (2.30) has the RAG property.

However, the system (2.30) is not RISS. If this were not the case, then

$$|x(t)| \leq \max\left\{\beta(|x(t_0)|, t-t_0), \gamma\left(\left\|u_{[t_0,t]}\right\|\right)\right\}, \ \forall t \geq t_0.$$

for some class $\mathcal{KL}$ function β and class $\mathcal{K}$ function γ, independent of t_0. Let $u(t) = 1$ for all $t \geq t_0$, and $x(t_0) = 2\gamma(1)$. Then,

$$|x(t)| = \frac{1+t_0}{1+t}2\gamma(1) + \frac{t-t_0}{1+t} \leq \max\{\beta(2\gamma(1), t-t_0), \gamma(1)\}.$$

Choose a finite real number s^* satisfying $\beta(2\gamma(1), s^*) \leq \gamma(1)$. Then

$$|x(t_0+s^*)| = \frac{1+t_0}{1+t_0+s^*}2\gamma(1) + \frac{s^*}{1+t_0+s^*} \leq \gamma(1),$$

hence, $(1+t_0)/(1+t_0+s^*) < 1/2$, i.e., $t_0 < s^* - 1$, which contradicts that t_0 is an arbitrary nonnegative real number.

2.5 Changing Supply Function

The ISS Lyapunov function $V(x, t)$ is also called a *supply function* or a *storage function*, and the pair (α, σ) is called a *supply pair*. The ISS Lyapunov function is not unique. It is possible to use the changing supply function technique to generate an alternative ISS Lyapunov function with exploitable property. In most scenarios encountered in this book, it is assumed that the range of uncertainties is represented by a *compact set* $\mathbb{D}$. In these scenarios, we usually consider an ISS Lyapunov function $V(x)$ not explicitly depending on t.

Lemma 2.5 (Changing Supply Function) *Suppose the system $\dot{x} = f(x, u, d)$ has an ISS Lyapunov function $V(x)$, i.e.,*

$$V(x) \sim \{\underline{\alpha}, \bar{\alpha}, \alpha, \sigma \mid \dot{x} = f(x, u, d)\}. \tag{2.31}$$

Let α' be a class $\mathcal{K}_\infty$ function such that $\alpha'(s) = \mathcal{O}[\alpha(s)]$ as $s \to 0^+$.[3] *Then the system $\dot{x} = f(x, u, d)$ has another ISS Lyapunov function $V'(x)$, i.e.,*

$$V'(x) \sim \{\underline{\alpha}', \bar{\alpha}', \alpha', \sigma' \mid \dot{x} = f(x, u, d)\}. \tag{2.32}$$

---

[3] The notation $\alpha'(s) = \mathcal{O}[\alpha(s)]$ as $s \to 0^+$ means $\limsup_{s\to 0^+} \left[\alpha'(s)/\alpha(s)\right] < \infty$.

In particular, the class $\mathcal{K}_\infty$ functions $\bar{\alpha}'$ and $\underline{\alpha}'$ and the class $\mathcal{K}$ function σ' are given by Algorithm 2.1.

Proof Let $\mathcal{SN}$ be the set of smooth non-decreasing functions defined over $[0, \infty)$ that satisfy $\rho(s) > 0,\ \forall s > 0$ for $\rho \in \mathcal{SN}$. Let

$$V'(x) = \int_0^{V(x)} \rho(s)ds \tag{2.33}$$

where $\rho \in \mathcal{SN}$. The statement (2.31) implies that

$$\underline{\alpha}'(\|x\|) \leq \int_0^{\underline{\alpha}(\|x\|)} \rho(s)ds \leq V'(x) \leq \int_0^{\bar{\alpha}(\|x\|)} \rho(s)ds \leq \bar{\alpha}'(\|x\|) \tag{2.34}$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}'$ and $\bar{\alpha}'$.

We now show that, along the trajectory of $\dot{x} = f(x, u, d)$,

$$\begin{aligned}\dot{V}'(x) &\leq \rho(V(x))[-\alpha(\|x\|) + \sigma(\|u\|)] \\ &\leq -\frac{1}{2}\rho(\underline{\alpha}(\|x\|))\alpha(\|x\|) + \rho(\bar{\alpha}(\alpha^{-1}(2\sigma(\|u\|))))\sigma(\|u\|).\end{aligned}$$

In fact, we consider the following two cases for the second inequality.

(i) $\alpha(\|x\|)/2 \geq \sigma(\|u\|)$: In this case, the claim follows from the fact that $\rho(V(x))[-\alpha(\|x\|) + \sigma(\|u\|)]$ is bounded from above by $-\rho(V(x))\alpha(\|x\|)/2$, and hence bounded from above by $-\rho\left(\underline{\alpha}(\|x\|)\right)\alpha(\|x\|)/2$.

(ii) $\alpha(\|x\|)/2 < \sigma(\|u\|)$: In this case, the following inequalities hold

$$\rho(V(x)) \leq \rho(\bar{\alpha}(\|x\|)) \leq \rho(\bar{\alpha}(\alpha^{-1}(2\sigma(\|u\|)))).$$

Since $\alpha'(s) = \mathcal{O}[\alpha(s)]$ as $s \to 0^+$, by Lemma 11.2 in the Appendix, it is always possible to find a function ρ such that

$$\frac{1}{2}\rho(\underline{\alpha}(s))\alpha(s) \geq \alpha'(s). \tag{2.35}$$

Also, there exists a class $\mathcal{K}$ function σ' such that

$$\sigma'(s) \geq \rho(\bar{\alpha}(\alpha^{-1}(2\sigma(s))))\sigma(s). \tag{2.36}$$

The proof is thus completed. □

Algorithm 2.1

INPUT: $\underline{\alpha}, \bar{\alpha}, \alpha, \sigma, \alpha'$
OUTPUT: $\underline{\alpha}', \bar{\alpha}', \sigma'$
STEP 1: Pick an $\mathcal{SN}$ function ρ satisfying (2.35).
STEP 2: Find the class $\mathcal{K}_\infty$ functions $\underline{\alpha}'$ and $\bar{\alpha}'$ from (2.34).
STEP 3: Find the class $\mathcal{K}$ function σ' from (2.36).
STEP 4: END

Corollary 2.1 *Suppose the system $\dot{x} = f(x, u, d)$ has an ISS Lyapunov function $V(x)$, i.e.,*

$$V(x) \sim \{\underline{\alpha}, \bar{\alpha}, \alpha, \sigma \mid \dot{x} = f(x, u, d)\}.$$

Then, for any smooth function Δ, the system $\dot{x} = f(x, u, d)$ has another ISS Lyapunov function $V'(x)$ such that

$$\begin{aligned} \underline{\alpha}'(\|x\|) &\le V'(x) \le \bar{\alpha}'(\|x\|) \\ \dot{V}'(x) &\le -\Delta(x)\alpha(\|x\|) + \varkappa(u)\sigma(\|u\|). \end{aligned} \tag{2.37}$$

for a smooth function $\varkappa$. In particular, the class $\mathcal{K}_\infty$ functions $\bar{\alpha}'$ and $\underline{\alpha}'$ and the smooth function $\varkappa$ are given by Algorithm 2.2.

Proof Following the proof of Lemma 2.5, it suffices to choose a function $\rho \in \mathcal{SN}$ such that

$$\frac{1}{2}\rho(\underline{\alpha}(\|x\|)) \ge \Delta(x) \tag{2.38}$$

and to choose a smooth function $\varkappa$ such that

$$\varkappa(u) \ge \rho(\bar{\alpha}(\alpha^{-1}(2\sigma(\|u\|)))). \tag{2.39}$$

The proof is thus completed. □

Algorithm 2.2

INPUT: $\underline{\alpha}, \bar{\alpha}, \alpha, \sigma, \Delta$
OUTPUT: $\underline{\alpha}', \bar{\alpha}', \varkappa$
STEP 1: Pick an $\mathcal{SN}$ function ρ satisfying (2.38).
STEP 2: Find the class $\mathcal{K}_\infty$ functions $\underline{\alpha}'$ and $\bar{\alpha}'$ from (2.34).
STEP 3: Find the smooth function $\varkappa$ from (2.39).
STEP 4: END

In many applications, we would like the supply pair α and σ to have the following properties, i.e.,

$$\limsup_{s\to 0^+} \frac{s^2}{\alpha(s)} < \infty, \ \limsup_{s\to 0^+} \frac{\sigma(s)}{s^2} < \infty. \tag{2.40}$$

For convenience, we make the following explicit assumption.

Assumption 2.1 The system $\dot{x} = f(x, u, d)$ has an ISS Lyapunov function $V(x)$, i.e.,

$$V(x) \sim \{\underline{\alpha}, \bar{\alpha}, \alpha, \sigma \mid \dot{x} = f(x, u, d)\}$$

and α and σ satisfy (2.40).

Remark 2.8 Let $\alpha(s) = \sum_{i=1}^{n} a_i s^{r_i}$ be a polynomial with $a_i \neq 0$ and $r_1 < \cdots < r_n$. Then, the condition $\limsup_{s\to 0^+} s^2/\alpha(s) < \infty$ is satisfied if and only if $r_1 \leq 2$ and the condition $\limsup_{s\to 0^+} \alpha(s)/s^2 < \infty$ is satisfied if and only if $r_1 \geq 2$.

Remark 2.9 Assumption 2.1 is slightly stronger than requiring the system $\dot{x} = f(x, u, d)$ be RISS viewing x as the state and u as the input. The RISS property only implies the asymptotic stability of the equilibrium point $x = 0$ of the undriven subsystem with $u = 0$. However, Assumption 2.1 may imply the exponential stability of the equilibrium point of the undriven subsystem if all the functions $\underline{\alpha}$, $\bar{\alpha}$, and α take the quadratic form.

Remark 2.10 Under Assumption 2.1, there exist smooth functions $\alpha_0(x)$ and $\sigma_0(u)$ such that

$$\alpha_0(x)\alpha(\|x\|) \geq \|x\|^2, \ \sigma_0(u)\|u\|^2 \geq \sigma(\|u\|). \tag{2.41}$$

On one hand, since α satisfies (2.40), there exits a constant $l_1 \geq 1$ such that $\alpha(\|x\|) \geq \|x\|^2/l_1^2$ for $\|x\| \leq 1$, and since α is of class $\mathcal{K}_\infty$, there exists a constant $l_2 > 0$ such that $\alpha(\|x\|) \geq l_2$ for $\|x\| \geq 1$. As a result, the first inequality of (2.41) holds for any α_0 satisfying

$$\alpha_0(x) \geq l_1^2 + \frac{1}{l_2}\|x\|^2.$$

On the other hand, since σ satisfies (2.40), we can define a function $l : [0, \infty) \mapsto [0, \infty)$ such that $l(s) = \sigma(s)/s^2, \ \forall s > 0$ and $l(0) = \lim_{s\to 0^+} l(s)$. Let $\sigma_0(u)$ be a smooth function such that $\sigma_0(u) \geq l(\|u\|)$. Then

$$\sigma_0(u)\|u\|^2 \geq l(\|u\|)\|u\|^2 = \sigma(\|u\|).$$

Under Assumption 2.1, Corollary 2.1 can be further specialized to the following corollary.

Corollary 2.2 *Under Assumption 2.1, for any smooth function $\Delta(x)$, there exists another ISS Lyapunov function $V'(x)$ satisfying*

$$\begin{aligned} \underline{\alpha}'(\|x\|) \le V'(x) \le \bar{\alpha}'(\|x\|) \\ \dot{V}'(x) \le -\Delta(x)\|x\|^2 + \varkappa(u)\|u\|^2. \end{aligned} \tag{2.42}$$

for a smooth function $\varkappa$. In particular, the class $\mathcal{K}_\infty$ functions $\bar{\alpha}'$ and $\underline{\alpha}'$ and the smooth function $\varkappa$ are given by Algorithm 2.3.

Proof Under Assumption 2.1, we can find two smooth functions $\alpha_0(x)$ and $\sigma_0(u)$ such that (2.41) is satisfied. Letting

$$\bar{\Delta}(x) \ge \Delta(x)\alpha_0(x) \tag{2.43}$$

gives

$$\bar{\Delta}(x)\alpha(\|x\|) \ge \Delta(x)\alpha_0(x)\alpha(\|x\|) \ge \Delta(x)\|x\|^2. \tag{2.44}$$

By Corollary 2.1, for the smooth function $\bar{\Delta}(x)$, there exists some smooth function $\bar{\varkappa}$ such that the Lyapunov function $V'(x)$ defined in Corollary 2.1 satisfies

$$\dot{V}'(x) \le -\bar{\Delta}(x)\alpha(\|x\|) + \bar{\varkappa}(u)\sigma(\|u\|)$$

which yields (2.42) upon letting

$$\varkappa(u) = \bar{\varkappa}(u)\sigma_0(u) \tag{2.45}$$

and using (2.41). □

Algorithm 2.3

INPUT: $\underline{\alpha}, \bar{\alpha}, \alpha, \sigma, \Delta$
OUTPUT: $\underline{\alpha}', \bar{\alpha}', \varkappa$
STEP 1: Pick two smooth functions $\alpha_0(x)$ and $\sigma_0(u)$ satisfying (2.41).
STEP 2: Pick the smooth function $\bar{\Delta}(x)$ satisfying (2.43).
STEP 3: Call $(\underline{\alpha}', \bar{\alpha}', \bar{\varkappa})$ = ALGORITHM 2.2$(\underline{\alpha}, \bar{\alpha}, \alpha, \sigma, \bar{\Delta})$.
STEP 4: Find the smooth function $\varkappa$ from (2.45).
STEP 5: END

Remark 2.11 In Corollary 2.1, if $\alpha(\|x\|)$ is a quadratic function, say, $\alpha(\|x\|) = a\|x\|^2$ for some $a > 0$. Then, by letting Δ be a constant and $V'(x) = \Delta V(x)$, we have

$$\dot{V}'(x) \le -a\Delta\|x\|^2 + \varkappa(u)\|u\|^2$$

where $\varkappa(u)$ is a smooth function such that $\varkappa(u)\|u\|^2 \geq \Delta\sigma(\|u\|)$. In particular, if $\sigma(\|u\|)$ is also a quadratic function, say, $\sigma(\|u\|) = b\|u\|^2$ for some $b > 0$, then $\varkappa(u) = b\Delta$ is a constant.

As an application of Corollary 2.2, we consider the global robust stabilization problem of the following class of nonlinear systems

$$\begin{aligned} \dot{z} &= q(z, x, d) \\ \dot{x} &= f(z, x, d) + b(d)u \end{aligned} \tag{2.46}$$

where $z \in \mathbb{R}^n$ and $x \in \mathbb{R}$ are the state variables, $u \in \mathbb{R}$ is the input, and $d : [t_0, \infty) \mapsto \mathbb{D}$ is a piecewise continuous function with $\mathbb{D}$ a compact subset of $\mathbb{R}^l$. The functions q and f are sufficiently smooth[4] with $q(0, 0, d) = 0$ and $f(0, 0, d) = 0$ for all $d \in \mathbb{D}$.

We need the following two assumptions.

Assumption 2.2 The function $b(d)$ is away from zero, e.g., $b(d) > 0, \forall d \in \mathbb{D}$.

Assumption 2.3 The subsystem $\dot{z} = q(z, x, d)$ has an ISS Lyapunov function $V(z)$, i.e.,

$$V(z) \sim \{\underline{\alpha}, \bar{\alpha}, \alpha, \sigma \mid \dot{z} = q(z, x, d)\}$$

and

$$\limsup_{s \to 0^+} \frac{s^2}{\alpha(s)} < \infty, \quad \limsup_{s \to 0^+} \frac{\sigma(s)}{s^2} < \infty.$$

In particular, the functions $\underline{\alpha}$, $\bar{\alpha}$, α and σ are known.

Theorem 2.8 *Consider the system (2.46) with a prescribed compact set $\mathbb{D}$. Under Assumptions 2.2 and 2.3, there exist a controller*

$$u = -\rho(x)x + \bar{u} \tag{2.47}$$

and an ISS Lyapunov function $W(z, x)$ satisfying

$$\underline{\beta}(\|col(z, x)\|) \leq W(z, x) \leq \bar{\beta}(\|col(z, x)\|)$$

for some class $\mathcal{K}_\infty$ functions $\underline{\beta}$ and $\bar{\beta}$, and, along the trajectory of the closed-loop system,

$$\dot{W}(z, x) \leq -\|z\|^2 - \|x\|^2 + \|\bar{u}\|^2. \tag{2.48}$$

As a result, the controller (2.47) with $\bar{u} = 0$ globally robustly stabilizes the system (2.46). In particular, the function ρ is given in Algorithm 2.4.

---

[4] A sufficiently smooth function means a function whose k-th derivatives exist for a sufficiently large integer k.

Proof Since $f(z, x, d)$ is a sufficiently smooth function, using (11.13) of the Appendix, one has

$$|f(z, x, d)| \leq m_1(z)\|z\| + m_2(x)|x|, \ \forall d \in \mathbb{D} \tag{2.49}$$

for some smooth positive functions m_1 and m_2. Let

$$\Delta(z) \geq 1 + m_1^2(z). \tag{2.50}$$

By Corollary 2.2, there exists a continuously differentiable function $V'(z, t)$ satisfying $\underline{\alpha}'(\|x_1\|) \leq V'(z, t) \leq \bar{\alpha}'(\|z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_1'$ and $\bar{\alpha}_1'$, and, along the trajectory of $\dot{z} = q(z, x, d)$,

$$\dot{V}'(z) \leq -\Delta(z)\|z\|^2 + \varkappa(x)x^2 \tag{2.51}$$

for a smooth function $\varkappa$. Since $b(d) > 0, \ \forall d \in \mathbb{D}$, there exist two constants $\bar{b}$ and $\underline{b}$ such that $\bar{b} \geq b(d) \geq \underline{b}, \ \forall d \in \mathbb{D}$. Then, define the function ρ such that

$$\rho(x) \geq [\varkappa(x) + m_2(x) + 5/4]/\underline{b} + \bar{b}/4 \tag{2.52}$$

and an ISS Lyapunov function candidate for the closed-loop system:

$$W(z, x) = V_1'(z) + x^2/2.$$

Direct calculation shows that the derivative of $W(z, x)$ along the trajectory of the closed-loop satisfies:

$$\begin{aligned}\dot{W}(z, x) &\leq -\Delta(z)\|z\|^2 + \varkappa(x)x^2 + x(f(z, x, d) + b(-\rho(x)x + \bar{u}))\\ &\leq -\Delta(z)\|z\|^2 + m_1^2(z)\|z\|^2\\ &\quad + x^2[\varkappa(x) + 1/4 + m_2(x) - b\rho(x) + b^2/4] + \bar{u}^2\\ &\leq -\|z\|^2 - \|x\|^2 + \bar{u}^2.\end{aligned}$$

The proof is thus completed by choosing the class $\mathcal{K}_\infty$ functions $\underline{\beta}$ and $\bar{\beta}$, using Lemma 11.3 of the Appendix, such that

$$\begin{aligned}\underline{\beta}(\|\text{col}(z, x)\|) &\leq \underline{\alpha}_1'(\|z\|) + x^2/2\\ \bar{\beta}(\|\text{col}(z, x)\|) &\geq \bar{\alpha}_1'(\|z\|) + x^2/2.\end{aligned}$$

□

Algorithm 2.4

INPUT: $f, b, \underline{\alpha}, \bar{\alpha}, \alpha, \sigma, \mathbb{D}$
OUTPUT: ρ
STEP 1: Find the functions m_1 and m_2 from (2.49).

STEP 2: Pick the function Δ from (2.50) and call

$$(\underline{\alpha}', \bar{\alpha}', \varkappa) = \text{ALGORITHM 2.3 } (\underline{\alpha}, \bar{\alpha}, \alpha, \sigma, \Delta).$$

STEP 3: Calculate the function ρ from (2.52).
STEP 4: END

Remark 2.12 The function ρ selected in (2.52) is to satisfy (2.48) and thus solving the stabilization problem of the system (2.46). In fact, to solve the stabilization problem of the system (2.46), it suffices to pick

$$\rho(x) \geq [\varkappa(x) + m_2(x) + 5/4]/\underline{b} \tag{2.53}$$

so that

$$\begin{aligned}\dot{W}(z, x) &\leq -\Delta(z)\|z\|^2 + \varkappa(x)x^2 + x(f(z, x, d) + b(-\rho(x)x)) \\ &\leq -\Delta(z)\|z\|^2 + m_1^2(z)\|z\|^2 + x^2[\varkappa(x) + 1/4 + m_2(x) - b\rho(x)] \\ &\leq -\|z\|^2 - \|x\|^2.\end{aligned}$$

Remark 2.13 From (2.52), it can be seen that the validity of Theorem 2.8 requires that the functions $\varkappa$ and m_2 as well as the constants $\underline{b}$ and $\bar{b}$ be known precisely. This is possible since $\mathbb{D}$ is assumed to be a known compact set. The case where $\mathbb{D}$ is not a known compact set cannot be handled by Theorem 2.8, and will be studied in the next section.

Remark 2.14 If $b(d) < 0, \ \forall d \in \mathbb{D}$, Theorem 2.8 still works by rewriting the second equation of (2.46) as $\dot{x} = f(z, x, d) + \hat{b}(d)\hat{u}$ where $\hat{b}(d) = -b(d)$ and $\hat{u} = -u$.

Example 2.16 Consider a second order system

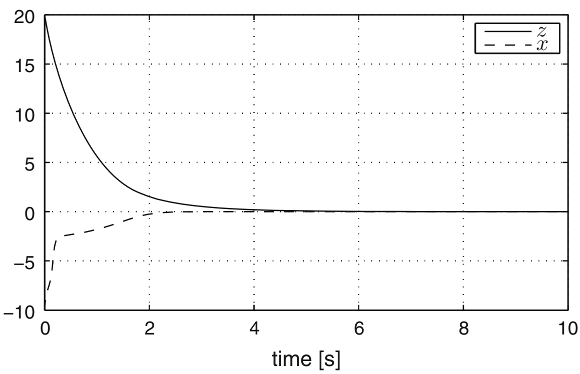

Fig. 2.2 Profile of aysmptoically stable state trajectories of the closed-loop system in Example 2.16

$$\dot{z} = -z + x$$
$$\dot{x} = w_1 z \sin x + w_2 x^3 + u$$

where w_1 and w_2 are unknown parameters with $|w_1| \leq 2$ and $|w_2| \leq 1$. We will find a controller u for the global robust stabilization problem and a corresponding Lyapunov function of the closed-loop system.

First, it can be verified that the derivative of $V(z) = z^2$, along the trajectory of $\dot{z} = -z + x$, satisfies

$$\dot{V}(z) \leq -z^2 + x^2.$$

Assumption 2.3 is satisfied with $\underline{\alpha}(s) = \bar{\alpha}(s) = \alpha(s) = \sigma(s) = s^2$.

Observe that

$$|w_1 z \sin x + w_2 x^3| \leq 2|z| + x^2|x|,$$

so (2.49) is true for $m_1(z) = 2$ and $m_2(x) = x^2$. Using the inequality (2.50) gives $\Delta(z) = 1 + m_1^2 = 5$. Since Δ is constant, and both α and σ are quadratic, by Corollary 2.1 and Remark 2.11, letting $V'(z) = \Delta z^2$ shows (2.51) is satisfied with $\varkappa(x) = 5$. Now, we are ready to pick the following function according to (2.53)

$$\rho(x) = 5 + x^2 + 5/4 = x^2 + 6.25,$$

which gives the controller

$$u = -x^3 - 6.25x. \tag{2.54}$$

Moreover, the closed-loop system has a Lyapunov function

$$W(z, x) = V'(z) + x^2/2 = 5z^2 + x^2/2$$

whose derivative, along the state trajectory of the closed-loop system, satisfies

$$\dot{W}(z, x) \leq -5z^2 + 5x^2 + x(w_1 z \sin x + w_2 x^3 - x^3 - 6.25x) \leq -z^2 - x^2.$$

The global robust stabilization problem is thus solved. The controller (2.54) is designed for $|w_1| \leq 2$ and $|w_2| \leq 1$, and the simulation is conducted with $w_1 = 1.8$ and $w_2 = 1$. The initial state values are $z(0) = 20$ and $x(0) = -10$. The state of the closed-loop system converges to the origin as shown in Fig. 2.2. If the uncertainties are out of this range, the controller may fail as illustrated in Fig. 2.3 with $w_1 = 1.8$ but $w_2 = 2$.

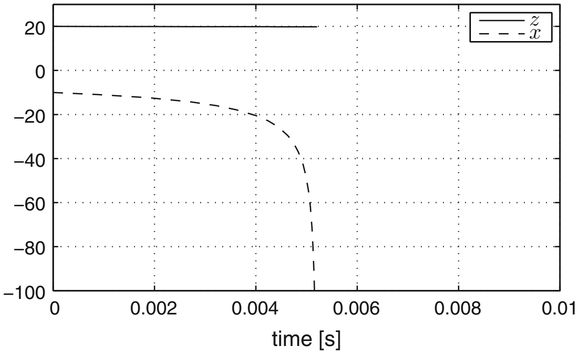

Fig. 2.3 Profile of unstable state trajectories of the closed-loop system in Example 2.16

2.6 Universal Adaptive Control

So far, we have assumed that the range of the uncertainty $d(t)$ belongs to a compact set $\mathbb{D}$ whose bound is known. In many cases, the range of $\mathbb{D}$ is unknown or $d(t)$ can be arbitrarily large, the robust control approach studied in the previous section cannot handle such uncertainty. In this section, we will further consider extending Theorem 2.8 to the case where the range of the uncertainty $d(t)$ is unknown or $d(t)$ can be arbitrarily large. In Theorem 2.8, it is known that for all $d \in \mathbb{D}$, there exist two known constants $\bar{b}$ and $\underline{b}$ such that $\bar{b} \geq b(d) \geq \underline{b}, \forall, d \in \mathbb{D}$. In this section, we assume b is an arbitrary unknown positive constant. More specifically, (2.46) is rewritten as follows

$$\begin{aligned} \dot{z} &= q(z, x, d) \\ \dot{x} &= f(z, x, d) + bu, \ \ b > 0. \end{aligned} \tag{2.55}$$

We first modify Assumption 2.1 to the following.

Assumption 2.4 The system $\dot{x} = f(x, u, d)$ has an ISS Lyapunov function $V(x)$, i.e.,

$$V(x) \sim \{\underline{\alpha}, \bar{\alpha}, \alpha, \hat{\sigma} \mid \dot{x} = f(x, u, d)\}$$

and

$$\limsup_{s \to 0^+} \frac{s^2}{\alpha(s)} < \infty, \ \ \limsup_{s \to 0^+} \frac{\hat{\sigma}(s)}{s^2} < \infty.$$

Moreover, the functions $\underline{\alpha}$, $\bar{\alpha}$, and α are known and the function $\hat{\sigma}$ is known up to a constant factor in the sense that there exist an unknown constant p and a known function σ such that $\hat{\sigma} = p\sigma$.

Remark 2.15 Assumption 2.4 is weaker than Assumption 2.1 since it allows the function $\hat{\sigma}$ to be known up to a constant factor. This assumption is more realistic when the range of $\mathbb{D}$ is unknown. For example, consider a scalar system $\dot{x} = -x + du$ where $u \in \mathbb{R}$ and d is an unknown constant. Let $V(x) = x^2$. Then, along the trajectory of the system $\dot{x} = -x + du$, we have

$$\dot{V}(x) \leq -x^2 + pu^2 \tag{2.56}$$

for $p = d^2$. Letting $\alpha(s) = s^2$ and $\hat{\sigma}(s) = ps^2$ shows that Assumption 2.4 is satisfied, but Assumption 2.1 is not satisfied since p is unknown. In general, when the compact set $\mathbb{D}$ is unknown, all the functions $\underline{\alpha}, \bar{\alpha}, \alpha, \hat{\sigma}$ may only be known up to a constant factor. This more general case will be handled in Chap. 6.

Corresponding to Assumption 2.4, we can also modify Corollary 2.2 to the following.

Corollary 2.3 *Under Assumption 2.4, for any smooth function* Δ*, there exists another ISS Lyapunov function* $V'(x)$ *satisfying*

$$\begin{aligned} \underline{\alpha}'(\|x\|) \leq V'(x) &\leq \bar{\alpha}'(\|x\|) \\ \dot{V}'(x) &\leq -\Delta(x)\|x\|^2 + p'\varkappa(u)\|u\|^2 \end{aligned} \tag{2.57}$$

for some unknown positive constant p' *and some known smooth function* $\varkappa$*. In particular, the class* $\mathcal{K}_\infty$ *functions* $\bar{\alpha}'$ *and* $\underline{\alpha}'$ *and the smooth function* $\varkappa$ *are given by Algorithm 2.5.*

Proof As shown in the proof of Lemma 2.5, for any smooth function $\Delta(x)$, there exists some ISS Lyapunov function $V'(x)$ for $\dot{x} = f(x, u, d)$ satisfying $\underline{\alpha}'(\|x\|) \leq V'(x) \leq \bar{\alpha}'(\|x\|)$ and the following inequality:

$$\dot{V}'(x) \leq -\frac{1}{2}\rho(\underline{\alpha}(\|x\|))\alpha(\|x\|) + p\rho(\bar{\alpha}(\alpha^{-1}(2p\sigma(\|u\|))))\sigma(\|u\|).$$

for any $\rho \in \mathcal{SN}$. By Remark 2.10, there exist smooth functions $\alpha_0(x)$ and $\sigma_0(u)$ such that

$$\alpha_0(x)\alpha(\|x\|) \geq \|x\|^2, \ \sigma_0(u)\|u\|^2 \geq \sigma(\|u\|). \tag{2.58}$$

From the proof of Corollary 2.2, there exists a smooth function $\bar{\Delta}(x)$ satisfying (2.43).

Pick a function $\rho \in \mathcal{SN}$ such that

$$\frac{1}{2}\rho(\underline{\alpha}(\|x\|)) \geq \bar{\Delta}(x). \tag{2.59}$$

Then we have,

$$\dot{V}'(x) \le -\bar{\Delta}(x)\alpha(\|x\|) + p\rho(\bar{\alpha}(\alpha^{-1}(2p\sigma(\|u\|))))\sigma(\|u\|). \tag{2.60}$$

By part (i) of Lemma 11.1 in the Appendix, one has

$$\rho(\bar{\alpha}(\alpha^{-1}(2\hat{\sigma}(\|u\|)))) = \rho(\bar{\alpha}(\alpha^{-1}(2p\sigma(\|u\|)))) \le c(p)\bar{\varkappa}(u) \tag{2.61}$$

for some smooth functions $c(p) \ge 0$ and $\bar{\varkappa}(u) \ge 0$.

Letting p' be any unknown positive constant satisfying $p' \ge c(p)p$, and using (2.43) and (2.61) in (2.60) gives

$$\dot{V}'(x) \le -\Delta(x)\alpha_0(x)\alpha(\|x\|) + p'\bar{\varkappa}(u)\sigma(\|u\|).$$

Letting

$$\varkappa(u) = \bar{\varkappa}(u)\sigma_0(u) \tag{2.62}$$

and using (2.58) completes the proof. □

Algorithm 2.5

INPUT: $\underline{\alpha}, \bar{\alpha}, \alpha, \sigma, \Delta$
OUTPUT: $\underline{\alpha}', \bar{\alpha}', \varkappa$
STEP 1: Pick the functions $\alpha_0(x)$ and $\sigma_0(u)$ satisfying (2.58).
STEP 2: Pick the smooth function $\bar{\Delta}(x)$ satisfying (2.43).
STEP 3: Pick an $\mathcal{SN}$ function ρ satisfying (2.59).
STEP 4: Find the class $\mathcal{K}_\infty$ functions $\underline{\alpha}'$ and $\bar{\alpha}'$ from (2.34).
STEP 5: Find the smooth function $\bar{\varkappa}$ from (2.61).
STEP 6: Find the smooth function $\varkappa$ from (2.62).
STEP 7: END

As pointed out in Remark 2.13, Theorem 2.8 cannot handle the system (2.46) when $\mathbb{D}$ is not a known compact set. We now modify Theorem 2.8 by using a so-called *universal adaptive control* technique. For this purpose, we modify Assumption 2.3 to the following.

Assumption 2.5 The subsystem $\dot{z} = q(z, x, d)$ has an ISS Lyapunov function $V(z)$, i.e.,

$$V(z) \sim \{\underline{\alpha}, \bar{\alpha}, \alpha, \hat{\sigma} \mid \dot{z} = q(z, x, d)\}$$

and

$$\limsup_{s\to 0^+} \frac{s^2}{\alpha(s)} < \infty, \quad \limsup_{s\to 0^+} \frac{\hat{\sigma}(s)}{s^2} < \infty.$$

Moreover, the functions $\underline{\alpha}$, $\bar{\alpha}$, and α are known and the function $\hat{\sigma}$ is known up to a constant factor in the sense that there exist an unknown constant p and a known function σ such that $\hat{\sigma} = p\sigma$.

Example 2.17 Consider a linear system

$$\dot{z} = Az + B(d)x \tag{2.63}$$

where $z \in \mathbb{R}^n$ and $x \in \mathbb{R}$ are the state variables, $d \in \mathbb{D}$ is the system uncertainty for an unknown compact set $\mathbb{D}$, and A is a Hurwitz matrix. Now, let P be a symmetric positive matrix such that $PA + A^\top P = -I$. Since $d \in \mathbb{D}$ for a compact set $\mathbb{D}$, we can pick a positive number $p \geq 2\|PB(d)\|^2$ which is not necessarily known because it depends on the size of $\mathbb{D}$. Let $V(z) = z^\top Pz$. Then its derivative along the system trajectory satisfies

$$\begin{aligned}\dot{V}(z) &= -\|z\|^2 + 2z^\top PB(d)x \leq -\|z\|^2/2 + 2\|PB(d)\|^2 x^2 \\ &\leq -\alpha(\|z\|) + p\sigma(|x|)\end{aligned}$$

where $\alpha(s) = s^2/2$ and $\sigma(s) = s^2$ are known functions. So, Assumption 2.5 is satisfied for the system (2.63).

Theorem 2.9 *Consider the system (2.55) with any unknown compact set* $\mathbb{D}$*. Under Assumption 2.5, there exists a controller*

$$\begin{aligned}u &= -k\rho(x)x + \bar{u} \\ \dot{k} &= \lambda\rho(x)x^2, \ \lambda > 0\end{aligned} \tag{2.64}$$

such that the closed-loop system has an ISS Lyapunov function $W(z, x, \hat{k})$ *where* $\hat{k} = k - k^*$ *for some constant* $k^* > 0$*, satisfying*

$$\underline{\beta}(\|col(z, x, \hat{k})\|) \leq W(z, x, \hat{k}) \leq \bar{\beta}(\|col(z, x, \hat{k})\|)$$

for some class $\mathcal{K}_\infty$ *functions* $\underline{\beta}$ *and* $\bar{\beta}$*, and, along the trajectory of the closed-loop system,*

$$\dot{W}(z, x, \hat{k}) \leq -\|z\|^2 - \|x\|^2 + \|\bar{u}\|^2.$$

As a result, the controller (2.64) with $\bar{u} = 0$ *globally stabilizes the system (2.55). In particular, the function* ρ *is given in Algorithm 2.6.*

Proof By Corollary 2.3, for any given smooth function $\Delta(z) \geq 0$, there exists a continuously differentiable function $V'(z)$ satisfying $\underline{\alpha}'(\|z\|) \leq V'(z) \leq \bar{\alpha}'(\|z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}'$ and $\bar{\alpha}'$, such that along the trajectory of the system $\dot{z} = q(z, x, d)$,

$$\dot{V}'(z) \leq -\Delta(z)\|z\|^2 + p'\varkappa(x)x^2 \tag{2.65}$$

for some unknown positive number p' and a known smooth function $\varkappa$. By Corollary 11.1 of the Appendix, there exist a positive number c, depending on the size of $\mathbb{D}$, and two positive and sufficiently smooth known functions m_1 and m_2, such that

$$|f(z,x,d)| \leq cm_1(z)\|z\| + cm_2(x)|x|,\ \forall d \in \mathbb{D}. \tag{2.66}$$

Let $\hat{k} = k - k^*$ with k^* a positive number to be specified later. Direct calculation shows that, along the trajectory of the closed-loop system, the derivative of

$$U(z,x) = V'(z) + x^2/2$$

satisfies,

$$\begin{aligned}\dot{U}(z,x) &\leq -\Delta(z)\|z\|^2 + p'\varkappa(x)x^2 + x[f(z,x,d) + bu]\\ &\leq -\Delta(z)\|z\|^2 + m_1^2(z)\|z\|^2 + x^2[p'\varkappa(x) + c^2/4 + cm_2(x)\\ &\quad + b^2/4 - bk\rho(x)] + \bar{u}^2\\ &= -\Delta(z)\|z\|^2 + m_1^2(z)\|z\|^2 + x^2[p'\varkappa(x) + c^2/4 + cm_2(x)\\ &\quad + b^2/4 - bk^*\rho(x)] + \bar{u}^2 - b\hat{k}\rho(x)x^2.\end{aligned} \tag{2.67}$$

In (2.67), let

$$\Delta(z) \geq 1 + m_1^2(z) \tag{2.68}$$

$$\rho(x) \geq \max\{\varkappa(x), m_2(x), 1\} \tag{2.69}$$

and

$$k^* \geq (1 + p' + c^2/4 + c)/b + b/4. \tag{2.70}$$

One has

$$\dot{U}(z,x) \leq -\|z\|^2 - \|x\|^2 + \bar{u}^2 - b\hat{k}\rho(x)x^2. \tag{2.71}$$

Define a Lyapunov function candidate as follows:

$$W(z,x,\hat{k}) = U(z,x) + b\hat{k}^2/(2\lambda).$$

Direct calculation shows that the derivative of $W(z,x,\hat{k})$ along the trajectory of the closed-loop system satisfies,

$$\begin{aligned}\dot{W}(z,x,\hat{k}) &\leq -\|z\|^2 - \|x\|^2 + \bar{u}^2 - b\hat{k}\rho(x)x^2 + b(k-k^*)\dot{k}/\lambda\\ &\leq -\|z\|^2 - \|x\|^2 + \bar{u}^2.\end{aligned}$$

The proof is thus completed by choosing the class $\mathcal{K}_\infty$ functions $\underline{\beta}$ and $\bar{\beta}$, using Lemma 11.3 of the Appendix, such that

$$\begin{aligned}\underline{\beta}(\|\text{col}(z, x, \hat{k})\|) &\leq \underline{\alpha}'(\|z\|) + x^2/2 + b\hat{k}^2/(2\lambda)\\ \bar{\beta}(\|\text{col}(z, x, \hat{k})\|) &\geq \bar{\alpha}'(\|z\|) + x^2/2 + b\hat{k}^2/(2\lambda).\end{aligned} \tag{2.72}$$

□

Algorithm 2.6

INPUT: $f, \underline{\alpha}, \bar{\alpha}, \alpha, \sigma$
OUTPUT: ρ
STEP 1: Find the functions m_1 and m_2 from (2.66).
STEP 2: Pick the function Δ from (2.68) and call

$$(\underline{\alpha}', \bar{\alpha}', \varkappa) = \text{ALGORITHM 2.5 } (\underline{\alpha}, \bar{\alpha}, \alpha, \sigma, \Delta).$$

STEP 3: Calculate the function ρ from (2.69).
STEP 4: END

Remark 2.16 If the size of the uncertainty $\mathbb{D}$ is known, then a real number k^* dominating the inequality (2.70) is known. One can pick a sufficiently large constant gain $k \geq k^*$ to be the controller gain. However, if the size of the uncertainty $\mathbb{D}$ is unknown, then k^* is unknown, either. One has to use a dynamic gain governed by (2.64). From (2.64), it can be seen that if the gain k is not large enough to achieve $\lim_{t\to\infty} x(t) = 0$, then it will increase until $\lim_{t\to\infty} x(t) = 0$. This type of adaptive approach for tuning the controller gain is called *universal adaptive control* or *self-tuning adaptive control*.

Example 2.18 Consider a second order nonlinear system

$$\begin{aligned}\dot{z} &= -z + w_3 x\\ \dot{x} &= w_1 z \sin x + w_2 x^3 + u\end{aligned}$$

which was studied in Example 2.16 assuming $w_3 = 1$ and the size of the unknown parameters w_1 and w_2 are known. Here, we consider the more general case where w_1, w_2, and w_3 can be any unknown real numbers. First, it can be verified that the derivative of $V(z) = z^2$, along the trajectory of $\dot{z} = -z + w_3 x$, satisfies

$$\dot{V}(z) \leq -z^2 + px^2$$

where $p \geq w_3^2$ is any unknown constant. Thus, Assumption 2.5 is satisfied with $\underline{\alpha}(s) = \bar{\alpha}(s) = \alpha(s) = s^2$ and $\hat{\sigma}(s) = ps^2$.

Next, we note that

$$|w_1 z \sin x + w_2 x^3| \leq c(|z| + x^2|x|)$$

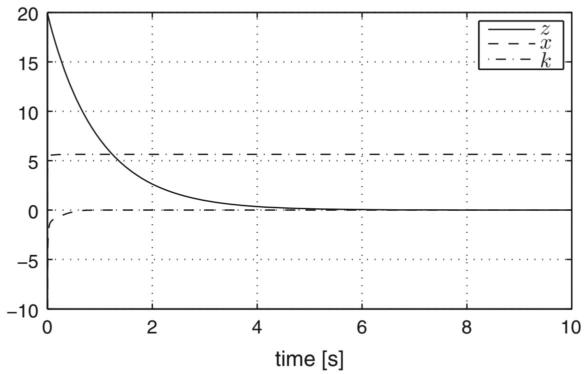

Fig. 2.4 Profile of state trajectories of the closed-loop system in Example 2.18

where $c \geq \max\{|w_1|, |w_2|\}$, that is, (2.66) is true for $m_1 = 2$ and $m_2(x) = x^2$. Let $\Delta(z) = 1 + m_1^2 = 5$. Since Δ is constant, and both α and $\hat{\sigma}$ are quadratic, by Corollary 2.3 and Remark 2.11, letting $V'(z) = \Delta z^2$ shows (2.65) is satisfied with $\varkappa(x) = 5$. Now, pick the following function according to (2.69)

$$\rho(x) = x^2 + 5 \geq \max\{5, x^2, 1\},$$

which leads to the controller (2.64). The performance of the controller is simulated and illustrated in Fig. 2.4 with $w_1 = 1.8$, $w_2 = 2$, and $w_3 = 1$. The initial state values are $z(0) = 20$, $x(0) = -10$, and $k(0) = 0$. It can be seen that both x and z approach 0 asymptotically while k approaches a finite constant asymptotically.

2.7 Small Gain Theorem

In this section, we introduce the small gain theorem to analyze the property of two inter-connected ISS systems. Let us consider the following two systems Σ_1 and Σ_2,

$$\begin{aligned} \Sigma_1 : \quad \dot{x}_1 &= f_1(x_1, u_1, u_c, d), \\ \Sigma_2 : \quad \dot{x}_2 &= f_2(x_2, u_2, u_c, d), \;\; t \geq t_0 \end{aligned} \tag{2.73}$$

where, for $i = 1, 2$, $x_i \in \mathbb{R}^{n_i}$ is the state, $u_i \in \mathbb{R}^{m_i}$, $u_c \in \mathbb{R}^{m_c}$ are the inputs of the subsystem Σ_i, and the function $f_i(x_i, u_i, u_c, d(t))$ is piecewise continuous in d and locally Lipschitz in $\mathrm{col}(x_i, u_i, u_c)$ and $d(t) : [t_0, \infty) \mapsto \mathbb{D} \subset \mathbb{R}^l$ is piecewise continuous in t for a compact set $\mathbb{D}$.

Suppose $m_1 = n_2$ and $m_2 = n_1$, and consider the following connection (see Fig. 2.5),

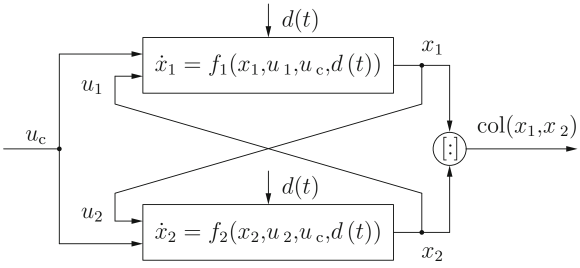

Fig. 2.5 Inter-connection of two ISS systems

$$\begin{aligned} u_1 &= x_2 \\ u_2 &= x_1. \end{aligned} \tag{2.74}$$

Under (2.74), the system (2.73) can be put in a compact form

$$\dot{x} = f(x, u_c, d)$$

with

$$x = \mathrm{col}(x_1, x_2),\ \ f(x, u_c, d) = \mathrm{col}(f_1(x_1, x_2, u_c, d),\ f_2(x_2, x_1, u_c, d)).$$

We will introduce two versions of the small gain theorem. The first version is formulated in terms of the ISS Lyapunov function of the individual subsystems, and this version will be frequently used in nonlinear controller design in the subsequent chapters.

Theorem 2.10 (Small Gain Theorem in terms of ISS Lyapunov Functions) *For $i = 1, 2$, assume the subsystem Σ_i of (2.73) is RISS with an ISS Lyapunov function $V_i(x_i)$, i.e.,*

$$V_i(x_i) \sim \{\underline{\alpha}_i, \bar{\alpha}_i, \alpha_i, (\sigma_i, \varsigma_i) \mid \dot{x}_i = f_i(x_i, u_i, u_c, d)\}.$$

Further assume

$$\alpha_1(s) - \sigma_2(s) \ge \delta_1(s),\ \ \alpha_2(s) - \sigma_1(s) \ge \delta_2(s),\ \ \forall s \ge 0 \tag{2.75}$$

for class $\mathcal{K}_\infty$ functions δ_i, $i = 1, 2$. Then the system (2.73) under the connection (2.74) is RISS with an ISS Lyapunov function $V(x)$, i.e.,

$$V(x) \sim \{\underline{\alpha}, \bar{\alpha}, \alpha, \varsigma \mid \dot{x} = f(x, u_c, d)\}.$$

Proof Let $V(x) = V_1(x_1) + V_2(x_2)$. Under (2.75), we have

$$\begin{aligned}\underline{\alpha}_1(\|x_1\|) + \underline{\alpha}_2(\|x_2\|) &\leq V(x) \leq \bar{\alpha}_1(\|x_1\|) + \bar{\alpha}_2(\|x_2\|)\\ \dot{V}(x) &\leq -\delta_1(\|x_1\|) - \delta_2(\|x_2\|) + \varsigma_1(\|u_c\|) + \varsigma_2(\|u_c\|).\end{aligned}$$

By Lemma 11.3 of the Appendix, there exist functions $\underline{\alpha}$, $\bar{\alpha}$, α, and ς such that

$$\begin{aligned}\underline{\alpha}(\|x\|) &\leq \underline{\alpha}_1(\|x_1\|) + \underline{\alpha}_2(\|x_2\|)\\ \bar{\alpha}(\|x\|) &\geq \bar{\alpha}_1(\|x_1\|) + \bar{\alpha}_2(\|x_2\|)\\ \alpha(\|x\|) &\leq \delta_1(\|x_1\|) + \delta_2(\|x_2\|)\\ \varsigma(\|u_c\|) &\geq \varsigma_1(\|u_c\|) + \varsigma_2(\|u_c\|).\end{aligned}$$

The proof is thus completed. □

Example 2.19 Consider the following system

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + x_1 x_2\\ \dot{x}_2 &= x_1^2 - a x_2 + u\end{aligned} \tag{2.76}$$

where a is a real parameter. This system results from the connection (2.74) of the following two systems

$$\dot{x}_1 = -x_1^3 + x_1 u_1 \tag{2.77}$$

$$\dot{x}_2 = u_2^2 - a x_2 + u. \tag{2.78}$$

Let $V_1(x_1) = x_1^2/2$. We first show that $V_1(x_1)$ is an ISS-Lyapunov function for the system (2.77) with state x_1 and input u_1. Indeed, the derivative of $V(x_1)$ along the trajectory of (2.77) is

$$\dot{V}_1(x_1) \leq -x_1^4 + x_1^2|u_1| \leq -x_1^4/2 + u_1^2/2.$$

In other words, one has

$$V_1(x_1) \sim \{\underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \sigma_1 \mid \dot{x}_1 = -x_1^3 + x_1 u_1\}$$

for $\alpha_1(s) = s^4/2$ and $\sigma_1(s) = s^2/2$.

Next we consider the system (2.78) with state x_2 and input (u_2, u). Let $V_2(x_2) = x_2^2/2$. Then the derivative of $V_2(x_2)$ along (2.78) is

$$\begin{aligned}\dot{V}_2(x_2) &\leq |x_2|(|u_2|^2 - a|x_2| + |u|)\\ &\leq (1/(4\epsilon_1))u_2^4 + \epsilon_1 x_2^2 - a x_2^2 + \epsilon_2 x_2^2 + (1/(4\epsilon_2))u^2\end{aligned}$$

for any $\epsilon_1, \epsilon_2 > 0$. In other words, one has

$$V_2(x_2) \sim \{\underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, (\sigma_2, \varsigma_2) \mid \dot{x}_2 = u_2^2 - ax_2 + u\}$$

for $\alpha_2(s) = (a - \epsilon_1 - \epsilon_2)s^2$, $\sigma_2(s) = (1/(4\epsilon_1))s^4$, and $\varsigma_2(s) = (1/(4\epsilon_2))s^2$.

Obviously, if $a > 1$, there exist $\epsilon_1, \epsilon_2 > 0$ to satisfy

$$1/2 > 1/(4\epsilon_1), \ \ a - \epsilon_1 - \epsilon_2 > 1/2, \tag{2.79}$$

which implies the small gain condition (2.75). By Theorem 2.10, the inter-connected system, i.e., the original system (2.76) is ISS and admits an ISS Lyapunov function $V(x_1, x_2) = V_1(x_1) + V_1(x_2)$.

In Theorem 2.10, the inequalities in (2.75) are the small gain conditions. In real applications, the functions α_i and σ_i are usually modified using the changing supply function technique to make the conditions (2.75) satisfied. The following result will be used in robust controller design (see Chap. 4 and some other chapters).

Theorem 2.11 *Consider a nonlinear system*

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u, d) \\ \dot{x}_2 &= f_2(x_2, x_1, u, d). \end{aligned} \tag{2.80}$$

Assume both subsystems are RISS with ISS Lyapunov functions

$$\begin{aligned} V_1(x_1) &\sim \{\underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \sigma_1) \mid \dot{x}_1 = f_1(x_1, u, d)\}. \\ V_2(x_2) &\sim \{\underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, (\varsigma, \sigma_2) \mid \dot{x}_2 = f_2(x_2, x_1, u, d)\}. \end{aligned}$$

Suppose the function α_i, σ_i, and ς satisfy the following properties:

$$\limsup_{s\to 0^+} \frac{s^2}{\alpha_i(s)} < \infty, \ \ \limsup_{s\to 0^+} \frac{\sigma_i(s)}{s^2} < \infty, \ \ i = 1, 2, \ \ \limsup_{s\to 0^+} \frac{\varsigma(s)}{s^2} < \infty.$$

Then there exists an ISS Lyapunov function $V(x)$, with $x = col(x_1, x_2)$, satisfying

$$\underline{\alpha}(\|x\|) \le V(x) \le \bar{\alpha}(\|x\|) \tag{2.81}$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\bar{\alpha}$, and, along the trajectory of (2.80),

$$\dot{V}(x) \le -\|x\|^2 + \varkappa(u)\|u\|^2 \tag{2.82}$$

for some smooth function $\varkappa$.

Proof We first consider the x_1-subsystem. By Corollary 2.2, for any smooth function Δ, there exists an ISS Lyapunov function $V_1'(x_1)$ satisfying

$$\underline{\alpha}_1'(\|x_1\|) \leq V_1'(x_1) \leq \bar{\alpha}_1'(\|x_1\|)$$
$$\dot{V}_1'(x_1) \leq -\Delta(x_1)\|x_1\|^2 + \varkappa_1(u)\|u\|^2$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_1'$ and $\bar{\alpha}_1'$ and some smooth function $\varkappa_1$. Then, we consider the x_2-subsystem. By Corollary 2.2 again, there exists an ISS Lyapunov function $V_2'(x_2)$ satisfying

$$\underline{\alpha}_2'(\|x_2\|) \leq V_2'(x_2) \leq \bar{\alpha}_2'(\|x_2\|)$$
$$\dot{V}_2'(x_2) \leq -\|x_2\|^2 + \varkappa(x_1, u)\|\text{col}(x_1, u)\|^2$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_2'$ and $\bar{\alpha}_2'$ and some smooth function $\varkappa$. Moreover, we have

$$\varkappa(x_1, u)\|\text{col}(x_1, u)\|^2 \leq \varkappa_2(x_1)\|x_1\|^2 + \varkappa_3(u)\|u\|^2$$

for some smooth functions $\varkappa_2$ and $\varkappa_3$. Let $\Delta(x_1) = \varkappa_2(x_1)+1$ and $V(x) = V_1'(x_1)+V_2'(x_2)$. One has (2.82) for any smooth function

$$\varkappa(u) \geq \varkappa_1(u) + \varkappa_3(u).$$

By Lemma 11.3 of the Appendix, there exist class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\bar{\alpha}$ such that

$$\underline{\alpha}(\|x\|) \leq \underline{\alpha}_1'(\|x_1\|) + \underline{\alpha}_2'(\|x_2\|)$$
$$\bar{\alpha}(\|x\|) \geq \bar{\alpha}_1'(\|x_1\|) + \bar{\alpha}_2'(\|x_2\|).$$

The inequalities in (2.81) are thus proved. □

Corollary 2.4 *Consider a nonlinear system*

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, u, d) \\ \dot{x}_2 &= Ax_2 + \phi(x_1, u, d) \end{aligned} \tag{2.83}$$

where the matrix A is Hurwitz and the function ϕ is sufficiently smooth with $\phi(0, 0, d) = 0$ for all $d \in \mathbb{D}$ with $\mathbb{D}$ a compact set. Assume the x_1-subsystem is RISS with an ISS Lyapunov function $V_1(x_1)$, i.e.,

$$V_1(x_1) \sim \{\underline{\alpha}_1, \bar{\alpha}_1, \alpha_1, \sigma_1) \mid \dot{x}_1 = f_1(x_1, u, d)\},$$

and the functions α_1 and σ_1 satisfy the following properties:

$$\limsup_{s\to 0^+} \frac{s^2}{\alpha_1(s)} < \infty, \ \limsup_{s\to 0^+} \frac{\sigma_1(s)}{s^2} < \infty.$$

Then (2.83) has an ISS Lyapunov function $V(x)$ with $x = col(x_1, x_2)$ satisfying

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \bar{\alpha}(\|x\|)$$

for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\bar{\alpha}$, and, along the trajectory of (2.83),

$$\dot{V}(x) \leq -\|x\|^2 + \varkappa(u)\|u\|^2$$

for a smooth function $\varkappa$.

Proof By Theorem 2.11, it suffices to show the x_2-subsystem is RISS with an ISS Lyapunov function $V_2(x_2)$, i.e.,

$$V_2(x_2) \sim \{\underline{\alpha}_2, \bar{\alpha}_2, \alpha_2, (\varsigma, \sigma_2) \mid \dot{x}_2 = f_2(x_2, x_1, u, d)\}$$

and the function α_2, σ_2, and ς satisfy the following properties:

$$\limsup_{s\to 0^+} \frac{s^2}{\alpha_2(s)} < \infty, \ \limsup_{s\to 0^+} \frac{\sigma_2(s)}{s^2} < \infty, \ \limsup_{s\to 0^+} \frac{\varsigma(s)}{s^2} < \infty.$$

Let $V_2(x_2) = x_2^\top P x_2$ where P is a symmetric positive definite matrix satisfying the Lyapunov equation

$$PA + A^\top P = -I.$$

It can be seen that

$$\underline{\alpha}_2(\|x_2\|) = \lambda_{\min}\|x_2\|^2/2 \leq V_2(x_2) \leq \lambda_{\max}\|x_2\|^2 \leq \bar{\alpha}_2(\|x_2\|)$$

where $\lambda_{\min}$ and $\lambda_{\max}$ are the minimal and maximal eigenvalues of P, respectively. Since ϕ is sufficiently smooth with $\phi(0, 0, d) = 0$, by Corollary 11.1 of the Appendix,

$$|\phi(x_1, u, d)| \leq m_1(\|x_1\|)\|x_1\| + m_2(|u|)|u|, \ \forall d \in \mathbb{D},$$

for some smooth functions m_1 and m_2. Then, the derivative of $V_2(x_2)$ along the trajectory of the x_2-subsystem of (2.83) satisfies

$$\begin{aligned}\dot{V}_2(x_2) &= -\|x_2\|^2 + 2x_2^\top P\phi(x_1, u, d)\\ &\leq -\|x_2\|^2/2 + 4\|P\|^2 m_1^2(\|x_1\|)\|x_1\|^2 + 4\|P\|^2 m_2^2(|u|)u^2.\end{aligned}$$

Thus, the proof is completed with

$$\alpha_2(s) = s^2/2, \ \varsigma(s) \geq 4\|P\|^2 m_1^2(s)s^2, \ \sigma_2(s) \geq 4\|P\|^2 m_2^2(s)s^2.$$

□

The small gain theorem can also be given in terms of gain functions of the individual subsystems. The proof of the following theorem is given in the Appendix.

Theorem 2.12 (Small Gain Theorem) *For $i = 1, 2$, assume the subsystem Σ_i of (2.73) is RISS viewing x_i as state, $col(u_i, u_c)$ as input, i.e., there exist class $\mathcal{KL}$ functions β_i, class $\mathcal{K}$ functions γ_i^x, γ_i^u, independent of t_0 and $d(t)$, such that, for any initial state $x_i(t_0)$, and any input function $col(u_i, u_c) \in L_\infty^{m_i+m_c}$, the solution $x_i(t)$ of Σ_i exists and satisfies*

$$\|x_i(t)\| \le \max\left\{\beta_i(\|x_i(t_0)\|, t - t_0), \gamma_i^x\left(\left\|u_{i[t_0,t]}\right\|\right), \gamma_i^u\left(\left\|u_{c[t_0,t]}\right\|\right)\right\}, \quad \forall t \ge t_0. \tag{2.84}$$

Further assume

$$\gamma_1^x \circ \gamma_2^x(s) < s, \ \forall s > 0. \tag{2.85}$$

Then the system (2.73) under the connection (2.74) is RISS viewing $x = col(x_1, x_2)$ as state and u_c as input, i.e.,

$$\|x(t)\| \le \max\left\{\beta(\|x(t_0)\|, t - t_0), \gamma\left(\left\|u_{c[t_0,t]}\right\|\right)\right\}, \ \forall t \ge t_0, \tag{2.86}$$

for some class $\mathcal{KL}$ function β, and any class $\mathcal{K}$ function γ satisfying

$$\gamma(s) \ge \max\left\{2\gamma_1^x \circ \gamma_2^u(s), 2\gamma_1^u(s), 2\gamma_2^x \circ \gamma_1^u(s), 2\gamma_2^u(s)\right\}, \quad \forall s > 0. \tag{2.87}$$

Corollary 2.5 *Consider the nonlinear system*

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, u, d)\\ \dot{x}_2 &= f_2(x_2, x_1, u, d).\end{aligned}$$

Assume both the subsystems are RISS viewing x_1 as state, u as input, and viewing x_2 as state, $col(x_1, u)$ as input, respectively, Then, the overall system is RISS viewing $x = col(x_1, x_2)$ as state and u as input with its gain function given by any class $\mathcal{K}$ function γ satisfying

$$\gamma(s) \ge \max\left\{2\gamma_1^u(s), 2\gamma_2^u(s), 2\gamma_2^x \circ \gamma_1^u(s)\right\}, \quad \forall s > 0 \tag{2.88}$$

Proof Note that the inequality (2.84) holds for $i = 1, 2$ with any class $\mathcal{K}$ function γ_1^x (noting $u_2 = x_1$ and $u_c = u$). In particular, let

$$\gamma_1^x(s) = \min\left\{(\gamma_2^x)^{-1}(s)/2, s\right\}.$$

Then, the inequality (2.84) holds with $i = 1$ and the inequality (2.85) holds with this γ_1^x. Thus, the inequality (2.87) reduces to (2.88). □

2.8 Notes and References

Most materials in this chapter are standard and can be found in many textbooks on nonlinear systems and control, e.g., [2, 4–8]. Theorems 11.1 and 11.2 (given in the Appendix) on the existence and uniqueness of the initial value solution to a nonlinear system can be found in Sect. 3.1 of [2]. A series of stability concepts are introduced in Sect. 2.1. The definitions are consistent with those commonly used in literature. When uncertainties are taken into consideration, the robust version of various stability concepts are introduced in Sect. 2.2. Some of the tools for adaptive control introduced in Sect. 2.3 can also been found in textbooks [6, 9–12]. Lemma 2.1 (Barbalat's Lemma) can be found in [2, 8]. Theorem 2.5 (LaSalle-Yoshizawa Theorem) due to LaSalle [13] and Yoshizawa [14] is from Theorem 2.1 of [6]. Lemma 2.3 is from Lemma 3.4 of [15]. Lemma 2.4 is taken from [16]. It can be viewed as an alternative form of Lemma 1 in [17] and can also be directly derived from Lemma 2 of [18]. For the linear adaptive control systems, the parameter convergence condition is established by showing, under the PE condition, the closed-loop system which is typically a linear time-varying system is uniform exponentially stable. However, for nonlinear adaptive control systems, the PE condition may not guarantee the uniform exponential stability of the closed-loop system which is typically a nonlinear time-varying system, see, e.g., [19]. Lemma 2.4 is of interest in that it only depends on the characteristics of the signals $g(t)$ and $f(t)$ without assuming that these signals are governed by some dynamical systems as in the literature of adaptive control of linear systems. Thus, it may also apply to the adaptive control of nonlinear systems. The notion of ISS discussed in Sect. 2.4 was first proposed by Sontag in [20–25], etc. It has become an effective tool in the analysis and design of nonlinear control systems. Theorems 2.6 and 2.7 are of particular interest for designing control laws for nonlinear systems. The time-invariant version of Theorems 2.6 and 2.7 can also be found in [5]. The technique of changing supply function was developed in [26]. Some variants of this technique are introduced in Sect. 2.5 and will be used in the subsequent chapters. The universal adaptive control technique in Sect. 2.6 has been used for handling static uncertainty with unknown boundary in, e.g., [9, 10, 27–29]. The small gain theorem was established in [30–32] in terms of a general inter-connection of two nonlinear subsystems. In Sect. 2.7, a more clear-cut version of the small gain theorem is introduced for a simpler inter-connection of two nonlinear subsystems. This simplified version is taken from [33].

2.9 Problems

Problem 2.1 Simulate the following systems starting from different initial conditions. Observe the stability, asymptotic stability, and globally asymptotic stability properties of the equilibrium point $x = 0$.

(a) $\dot{x} = ax + bx^2 + cx^3, a, b, c \in \mathbb{R}$;
(b) $\ddot{y} - \mu(1 - y^2)\dot{y} + y = 0, \mu \in \mathbb{R}, x = \text{col}(y, \dot{y})$;

(c) $\dot{r} = r(1-r)$, $\dot{\theta} = \sin^2(\theta/2)$, $r = \sqrt{x_1^2 + x_2^2}$, $\theta = \arctan(x_2/x_1)$, $x = \mathrm{col}(x_1, x_2)$;
(d) $\dot{x}_1 = -x_1 + 4x_2$, $\dot{x}_2 = -x_1 - x_2^3$, $x = \mathrm{col}(x_1, x_2)$;
(e) $\dot{x}_1 = x_2$, $\dot{x}_2 = x_1 - \mathrm{sat}(2x_1 + x_2)$, $x = \mathrm{col}(x_1, x_2)$, where the saturation function is defined as

$$\mathrm{sat}(s) = \begin{cases} s, |s| \le 1 \\ 1, s > 1 \\ -1, s < -1 \end{cases}.$$

Problem 2.2 Determine if the following functions belong to class $\mathcal{K}$ function, class $\mathcal{K}_\infty$ function, or class $\mathcal{KL}$ function.

(a) $\gamma(r) = 2r + r^2$, $r \in [0, \infty)$;
(b) $\gamma(r) = e^r$, $r \in [0, \infty)$;
(c) $\gamma(r) = \arctan(r)$, $r \in [0, 1)$;
(d) $\gamma(r, s) = re^{-2s}$, $r \in [0, \infty)$, $s \in [0, \infty)$;
(e) $\gamma(r, s) = (r^3 + r)/(s + 1)$, $r \in [0, \infty)$, $s \in [0, \infty)$.

Problem 2.3 Find the Jacobian matrices for the following systems at the origin $x = 0$ and determine the stability of their equilibrium points at the origin.

(a) $\dot{x} = -\sin x$;
(b) $\dot{x} = -x^3$;
(c) $\dot{x}_1 = \sin x_2$, $\dot{x}_2 = -x_1 + x_1 x_2$, $x = \mathrm{col}(x_1, x_2)$;
(d) $\dot{x} = \begin{bmatrix} 5x_2 \\ 4x_1^2 - 2\sin(x_2 x_3) \\ x_2 x_3 \end{bmatrix}$, $x = \mathrm{col}(x_1, x_2, x_3)$.

Problem 2.4 Use Lyapunov's linearization theorem to determine the stability of the equilibrium point of the mechanical system at the origin

$$m\ddot{y} + c\dot{y} + k_1 y + k_3 y^3 = 0,\ m, c > 0,\ k_1, k_2 \in \mathbb{R}.$$

Problem 2.5 Show the nonlinear system $\dot{x} = -c(x, t)$ is UGAS if

$$xc(x, t) \ge \alpha(|x|),\ \forall x \in \mathbb{R}$$

for a class $\mathcal{K}$ function α.

Problem 2.6 Consider the following system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 - x_1^3 - x_2^3. \end{aligned}$$

(a) Find the Jacobian matrix at the origin; what does it say about the stability of the equilibrium point?

(b) Use Lyapunov's direct theorem to determine the stability of the equilibrium point at the origin.
Hint:

$$V(x_1, x_2) = \frac{x_2^2}{2} + \int_0^{x_1} (s + s^3) ds$$

Problem 2.7 Use Lyapunov's direct theorem to determine the stability of the equilibrium point at the origin for the systems in Problem 2.3.

Problem 2.8 Determine if $x = 0$ is an equilibrium point for the following systems, where $d(t)$ represents external disturbance.

(a) $\dot{x} = \sin x + d(t)$;
(b) $\dot{x} = -x + d(t)x^3$;
(c) $\dot{x}_1 = x_2,\ \dot{x}_2 = d(t)x_1^2 + x_2,\ x = \text{col}(x_1, x_2)$;
(d) $\dot{x} = (x + \cos x - 1)/(1 + d(t))$.

Problem 2.9 Find the Jacobian matrices for the following systems at $(x, d) = (0, 0)$ and determine the stability of their equilibrium points at $x = 0$, where d represents an unknown parameter.

(a) $\dot{x} = -(1 + d) \sin x$;
(b) $\dot{x} = -x^3/(2 + \sin(d))$;
(c) $\dot{x}_1 = \sin x_2,\ \dot{x}_2 = -x_1 + dx_1x_2, x = \text{col}(x_1, x_2)$;
(d) $\dot{x} = \begin{bmatrix} dx_2 \\ 4x_1^2 - 2\sin(x_2x_3) \\ x_2x_3 \end{bmatrix}, x = \text{col}(x_1, x_2, x_3)$.

Problem 2.10 Use Lyapunov's direct theorem to determine the stability of the equilibrium point at the origin for the systems in Problem 2.9.

Problem 2.11 Use LaSalle-Yoshizawa Theorem to show that the equilibrium point of the system in Problem 2.6 is GAS.

Problem 2.12 Consider the nonlinear system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 + (1 + d(t))x_2 \end{aligned}$$

where $d(t)$ represents external disturbance satisfying $|d(t)| < 0.1$. Use LaSalle-Yoshizawa Theorem (hint: $V(x_1, x_2) = 1 - \cos x_1 + x_2^2/2$) to show

$$\lim_{t \to \infty} x_2(t) = 0$$

for all initial state values $x_1(0), x_2(0) \in \mathbb{R}$.

Problem 2.13 Find ISS-Lyapunov functions for the following systems where $d(t)$ represents external disturbance.

(a) $\dot{x} = -|d(t)|x - x^3 + u$;
(b) $\dot{x} = -x^3 - x^2 u$;
(c) $\dot{x} = \begin{bmatrix} -x_1 - x_2 \\ x_1 - x_2^3 + d(t)u \end{bmatrix}, \ x = \operatorname{col}(x_1, x_2), \ 1 < d(t) < 2.$

Problem 2.14 Find the gain functions for the RISS systems in Problem 2.13.

Problem 2.15 The function V defined in Theorem 2.7 is an ISS Lyapunov function for the system (2.7). If the class $\mathcal{K}_\infty$ function α is relaxed by a continuous positive definite function α (not required to be unbounded), the function V is called an integral input-to-state stable (iISS) Lyapunov function for the system (2.7). The system (2.7) is said to be iISS if there exists an iISS Lyapunov function [34]. Show that the following systems are not ISS but iISS using the suggested iISS Lyapunov functions.

(a) $\dot{x} = -x + xu^2$, $V(x) = \ln(1 + x^2)$;
(b) $\dot{x} = -\arctan x + u$, $V(x) = x \arctan x$.

Problem 2.16 Write the ISS-Lyapunov functions in Problem 2.13 in the form

$$V(x) \sim \{\underline{\alpha}, \bar{\alpha}, \alpha, \sigma \mid \dot{x} = f(x, u, d)\}.$$

For each function Δ given below, find another ISS Lyapunov function $V'(x)$ satisfying (2.37) and the corresponding functions $\bar{\alpha}'$, $\underline{\alpha}'$ and $\varkappa$.

(a) $\Delta(x) = 2$;
(b) $\Delta(x) = \|x\|^2$;
(c) $\Delta(x) = 3\|x\|^2 + 2\|x\|^4$.

Problem 2.17 Determine if the functions $V'(x)$ given in Problem 2.16 satisfies (2.42). If yes, find the corresponding functions $\bar{\alpha}'$, $\underline{\alpha}'$ and $\varkappa$.

Problem 2.18 Suppose the system (2.7) is iISS with an iISS Lyapunov function $V(x)$ satisying (2.25) where α is a continuous positive definite function. Let α' be a continuous positive definite function such that $\alpha'(s) = \mathcal{O}[\alpha(s)]$ as $s \to 0^+$ and $\limsup_{s\to\infty} \left[\alpha'(s)/\alpha(s)\right] < \infty$ if α is not of class $\mathcal{K}_\infty$. Show that there exists another iISS Lyapunov function $V'(x)$ satisfying $\underline{\alpha}'(\|x\|) \leq V'(x) \leq \bar{\alpha}'(\|x\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}'$ and $\bar{\alpha}'$ such that, along the trajectory of (2.7),

$$\dot{V}'(x) \leq -\alpha'(\|x\|) + \varkappa(u)$$

for some positive definite function $\varkappa$.

Problem 2.19 Consider the system (2.83) where the matrix A is Hurwitz and the function ϕ is sufficiently smooth with $\phi(0,0,d) = 0$ for all $d \in \mathbb{D}$ with $\mathbb{D}$ a compact set. Assume the x_1-subsystem is iISS with an iISS Lyapunov function $V_1(x_1)$ satisfying $\underline{\alpha}_1(\|x_1\|) \leq V_1(x_1) \leq \bar{\alpha}_1(\|x_1\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}_1$ and $\bar{\alpha}_1$ such that, along the trajectory of the x_1-subsystem,

$$\dot{V}(x_1) \leq -\alpha_1(\|x_1\|) + \sigma_1(\|u\|)$$

for some positive definite function α_1 and some class $\mathcal{K}$ function σ_1 satisfying $\limsup_{s\to 0^+}\left[\sigma_1(s)/s^2\right] < \infty$.

Let ϕ be a continuous positive definite function such that $\phi(s) = \mathcal{O}[\alpha_1(s)]$ as $s \to 0^+$ and $\limsup_{s\to\infty}[\phi(s)/\alpha_1(s)] < \infty$ if α_1 is not of class $\mathcal{K}_\infty$. Show that there exists an iISS Lyapunov function $V(x)$ with $x = \mathrm{col}(x_1, x_2)$, satisfying $\underline{\alpha}(\|x\|) \leq V(x) \leq \bar{\alpha}(\|x\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\bar{\alpha}$ such that, along the trajectory of (2.83),

$$\dot{V}(x) \leq -\phi^2(\|x\|) + \varkappa(u)\|u\|^2$$

for some smooth function $\varkappa$.

Problem 2.20 Consider the system (2.46) with $\mathbb{D}$ a known compact set and $b(d)>0$. Assume the z-subsystem is iISS with an iISS Lyapunov function $V(z)$ satisfying $\underline{\alpha}(\|z\|) \leq V(z) \leq \bar{\alpha}(\|z\|)$ for some class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\bar{\alpha}$ such that, along the trajectory of the z-subsystem,

$$\dot{V}(z) \leq -\alpha(\|z\|) + \sigma(|x|)$$

for some positive definite function α and some class $\mathcal{K}$ function σ satisfying $\limsup_{s\to 0^+}\left[\sigma(s)/s^2\right] < \infty$.

Let m_1 and m_2 be some smooth positive functions such that

$$|f(z,x,d)| \leq m_1(\|z\|)\|z\| + m_2(|x|)|x|, \ \forall d \in \mathbb{D}.$$

Moreover, $m_1(s)s = \mathcal{O}[\alpha(s)]$ and $\limsup_{s\to\infty}[m_1(s)s/\alpha(s)] < \infty$ if α is not of class $\mathcal{K}_\infty$. Show that there exists a controller of the form $u = -\rho(x)x$ that globally robustly stabilizes the system (2.46).

Problem 2.21 For the inter-connected system

$$\dot{x}_1 = f_1(x_1, x_2, d(t))$$
$$\dot{x}_2 = f_2(x_1, x_2, d(t))$$

where $d(t)$ represents external disturbance. Assume the system satisfies

$$\|x_1(t)\| \leq \max\left\{\beta_1(\|x_1(t_0)\|, t - t_0), \gamma_1\left(\left\|x_{2[t_0,t]}\right\|\right)\right\}$$
$$\|x_2(t)\| \leq \max\left\{\beta_2(\|x_2(t_0)\|, t - t_0), \gamma_2\left(\left\|x_{1[t_0,t]}\right\|\right)\right\}, \ \forall t \geq t_0$$

for some class $\mathcal{KL}$ functions β_1 and β_2 and some class $\mathcal{K}$ functions γ_1 and γ_2 given below. Use the small gain theorem to determine the asymptotic stability of the system for the following cases.

(a) $\gamma_1(s) = 2$, $\gamma_2(s) = 0.4$;
(b) $\gamma_1(s) = 0.8s^2$, $\gamma_2(s) = \sqrt{s}$;
(c) $\gamma_1(s) = \alpha^{-1}(s)$, $\gamma_2(s) = s^2$ for $\alpha(s) = 2s^2 + s^4$.

Problem 2.22 For each of the systems given below with state $x = \text{col}(x_1, x_2)$, input u, and disturbance $1 < d(t) < 2$, find an ISS Lyapunov function $V(x)$ satisfying

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \bar{\alpha}(\|x\|)$$
$$\dot{V}(x) \leq -\|x\|^2 + \varkappa(u)\|u\|^2.$$

Calculate the class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\bar{\alpha}$ and the smooth function $\varkappa$.

(a) $\dot{x}_1 = -x_1 + u, \ \dot{x}_2 = -x_2 + d(t)x_1^2 + u$;
(b) $\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} u, \ \dot{x}_2 = -x_2 + d(t)\|x_1\|^2 + u$;
(c) $\dot{x}_1 = -x_1 + u, \ \dot{x}_2 = -\begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} (d(t)x_1^2 + u)$.

Problem 2.23 For each of the systems given below with state $x = \text{col}(x_1, x_2)$, input u, and disturbance $1 < d(t) < 2$, design a controller $u = -\rho(x_2)x_2 + \bar{u}$ and find an ISS Lyapunov function $V(x)$ satisfying

$$\underline{\alpha}(\|x\|) \leq V(x) \leq \bar{\alpha}(\|x\|)$$
$$\dot{V}(x) \leq -\|x\|^2 + \|\bar{u}\|^2.$$

Calculate the class $\mathcal{K}_\infty$ functions $\underline{\alpha}$ and $\bar{\alpha}$.

(a) $\dot{x}_1 = -x_1 + x_2, \ \dot{x}_2 = d(t)x_1x_2 + u$;
(b) $\dot{x}_1 = -x_1 + d(t)x_2, \ \dot{x}_2 = -x_2^3 + d(t)u$;
(c) $\dot{x}_1 = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ 2 \end{bmatrix} x_2^2, \ \dot{x}_2 = d(t)\sin x_2 + u$.

Problem 2.24 For each of the systems in Problem 2.23, design a controller u to solve the global stabilization problem for $d(t) \in \mathbb{D}$ when $\mathbb{D}$ is an arbitrarily large compact set.

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