Preface

Since its original development in the mid-nineties by Terry Lyons, culminating in the landmark paper [Lyo98], the theory of rough paths has grown into a mature and widely applicable mathematical theory, and there are by now several monographs dedicated to the subject, notably Lyons–Qian [LQ02], Lyons et al [LCL07] and Friz–Victoir [FV10b]. So why do we believe that there is room for yet another book on this matter? Our reasons for writing this book are twofold.

First, the theory of rough paths has gathered the reputation of being difficult to access for “mainstream” probabilists because it relies on some non-trivial algebraic and/or geometric machinery. It is true that if one wishes to apply it to signals of arbitrary roughness, the general theory relies on several objects (in particular on the Hopf-algebraic properties of the free tensor algebra and the free nilpotent group embedded in it) that are unfamiliar to most probabilists. However, in our opinion, some of the most interesting applications of the theory arise in the context of stochastic differential equations, where the driving signal is Brownian motion. In this case, the theory simplifies dramatically and essentially no non-trivial algebraic or geometric objects are required at all. This simplification is certainly not novel. Indeed, early notes by Lyons, and then of Davie and Gubinelli, all took place in this simpler setting (which allows to incorporate Brownian motion and Lévy’s area). However, it does appear to us that all these ideas can nowadays be put together in unprecedented simplicity, and we made a conscious choice to restrict ourselves to this simpler case throughout most of this book.

The second and main raison d’être of this book is that the scope of the theory has expanded dramatically over the past few years and that, in this process, the point of view has slightly shifted from the one exposed in the aforementioned monographs. While Lyons’ theory was built on the integration of 1-forms, Gubinelli gave a natural extension to the integration of so-called “controlled rough paths”. As a benefit, differential equations driven by rough paths can now be solved by fixed point arguments in linear Banach spaces which contain a sufficiently accurate (second order) local description of the solution.

This shift in perspective has first enabled the use of rough paths to provide solution theories for a number of classically ill-posed stochastic partial differential equations
with one-dimensional spatial variables, including equations of Burgers type and the KPZ equation. More recently, the perspective which emphasises linear spaces containing sufficiently accurate local descriptions modelled on some (rough) input, spurred the development of the theory of “regularity structures” which allows to give consistent interpretations for a number of ill-posed equations, also in higher dimensions. It can be viewed as an extension of the theory of controlled rough paths, although its formulation is somewhat different. In the last chapters of this book, we give a short and rather informal (i.e. very few proofs) introduction to that theory, which in particular also sheds new light on some of the definitions of the theory of rough paths.

This book does not have the ambition to provide an exhaustive description of the theory of rough paths, but rather to complement the existing literature on the subject. As a consequence, there are a number of aspects that we chose not to touch, or to do so only barely. One omission is the study of rough paths of arbitrarily low regularity: we do provide hints at the general theory at the end of several chapters, but these are self-contained and can be skipped without impacting the understanding of the rest of the book. Another serious omission concerns the systematic study of signatures, that is the collection of all iterated integrals over a fixed interval associated to a sufficiently regular path, providing an intriguing nonlinear characterisation.

We have used several parts of this book for lectures and mini-courses. In particular, over the last years, the material on rough paths was given repeatedly by the first author at TU Berlin (Chapters 1-12, in the form of a 4h/week, full semester lecture for an audience of beginning graduate students in stochastics) and in some mini-courses (Vienna, Columbia, Rennes, Toulouse; e.g. Chapters 1-5 with a selection of further topics). The material of Chapters 13-15 originates in a number of minicourses by the second author (Bonn, ETHZ, Toulouse, Columbia, XVII Brazilian School of Probability, 44th St. Flour School of Probability, etc). The “KPZ and rough paths” summer school in Rennes (2013) was a particularly good opportunity to try out much of the material here in joint mini-course form – we are very grateful to the organisers for their efforts. Chapters 13-15 are, arguably, a little harder to present in a classroom. Jointly with Paul Gassiat, the first author gave this material as full lecture at TU Berlin (with examples classes run by Joscha Diehl, and more background material on Schwartz distributions, Hölder spaces and wavelet theory than what is found in this book); we also started to use consistently colours on our handouts. We felt the resulting improvement in readability was significant enough to try it out also in the present book and take the opportunity to thank Jörg Sixt from Springer for making this possible, aside from his professional assistance concerning all other aspects of this book project. We are very grateful for all the feedback we received from participants at all theses courses. Furthermore, we would like to thank Bruce Driver, Paul Gassiat, Massimilliano Gubinelli, Terry Lyons, Etienne Pardoux, Jeremy Quastel and Hendrik Weber for many interesting discussions on how to present this material. In addition, Khalil Chouk, Joscha Diehl and Sebastian Riedel kindly offered to partially proofread the final manuscript.

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