Chapter 2
The space of rough paths

Abstract We define the space of (Hölder continuous) rough paths, as well as the subspace of “geometric” rough paths which preserve the usual rules of calculus. The latter can be interpreted in a natural way as paths with values in a certain nilpotent Lie group. At the end of the chapter, we give a short discussion showing how these definitions should be generalized to treat paths of arbitrarily low regularity.

2.1 Basic definitions

In this section, we give a practical definition of the space of Hölder continuous rough paths. Our choice of Hölder spaces is chiefly motivated by our hope that most readers will already be familiar with the classical Hölder spaces from real analysis. We could in the sequel have replaced “$\alpha$-Hölder continuous” by “finite $p$-variation” for $p = 1/\alpha$ in many statements. This choice would also have been quite natural, due to the fact that one of our primary goals will be to give meaning to integrals of the form $\int f(X) \, dX$ or solutions to controlled differential equations of the form $dY = f(Y) \, dX$ for rough paths $X$. The value of such an integral/solution does not depend on the parametrisation of $X$, which dovetails nicely with the fact that the $p$-variation of a function is also independent of its parametrisation. This motivated its choice in the original development of the theory. In some other applications however (like the solution theory to rough stochastic partial differential equations developed in [Hai11b, HW13, Hai13] and more generally the theory of regularity structures [Hai14c]), parametrisation-independence is lost and the choice of Hölder norms is more natural.

A rough path on an interval $[0, T]$ with values in a Banach space $V$ then consists of a continuous function $X : [0, T] \to V$, as well as a continuous “second order process” $\mathcal{X} : [0, T]^2 \to V \otimes V$, subject to certain algebraic and analytical conditions. Regarding the former, the behaviour of iterated integrals, such as (2.2) below, suggests to impose the algebraic relation (“Chen’s relation”).

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\[ X_{s,t} - X_{s,u} - X_{u,t} = X_{s,u} \otimes X_{u,t} , \] (2.1)

which we assume to hold for every triple of times \((s, u, t)\). Since \(X_{t,t} = 0\), it immediately follows (take \(s = u = t\)) that we also have \(X_{t,t} = 0\) for every \(t\). As already mentioned in the introduction, one should think of \(X\) as postulating the value of the quantity

\[ \int_s^t X_{s,r} \otimes dX_r \overset{\text{def}}{=} X_{s,t} , \] (2.2)

where we take the right hand side as a definition for the left hand side. (And not the other way around!) We insist (cf. Exercise 2.7 below) that as a consequence of (2.1), knowledge of the path \(t \mapsto (X_{0,t}, X_{0,t})\) already determines the entire second order process \(X\). In this sense, the pair \((X, X)\) is indeed a path, and not some two-parameter object, although it is often more convenient to consider it as one. If \(X\) is a smooth function and we read (2.2) from right to left, then it is straightforward to verify (see Exercise 2.6 below) that the relation (2.1) does indeed hold. Furthermore, one can convince oneself that if \(f \mapsto \int f \, dX\) denotes any form of “integration” which is linear in \(f\), has the property that \(\int_s^t dX_r = X_{s,t}\), and is such that \(\int_s^t f(r) \, dX_r + \int_t^u f(r) \, dX_r = \int_s^u f(r) \, dX_r\) for any admissible integrand \(f\), and if we use such a notion of “integral” to define \(X\) via (2.2), then (2.1) does automatically hold. This makes it a very natural postulate in our setting.

Note that the algebraic relations (2.1) are by themselves not sufficient to determine \(X\) as a function of \(X\). Indeed, for any \(V \otimes V\)-valued function \(F\), the substitution \(X_{s,t} \mapsto X_{s,t} + F_t - F_s\) leaves the left hand side of (2.1) invariant. We will see later on how one should interpret such a substitution. It remains to discuss what are the natural analytical conditions one should impose for \(X\). We are going to assume that the path \(X\) itself is \(\alpha\)-Hölder continuous, so that \(|X_{s,t}| \lesssim |t - s|^\alpha\). The archetype of an \(\alpha\)-Hölder continuous function is one which is self-similar with index \(\alpha\), so that \(X_{\lambda s, \lambda t} \sim \lambda^\alpha X_{s,t}\).

(We intentionally do not give any mathematical definition of self-similarity here, just think of \(\sim\) as having the vague meaning of “looks like”.) Given (2.2), it is then very natural to expect \(X\) to also be self-similar, but with \(X_{\lambda s, \lambda t} \sim \lambda^{2\alpha} X_{s,t}\). This discussion motivates the following definition of our basic spaces of rough paths.

**Definition 2.1.** For \(\alpha \in (\frac{1}{2}, \frac{1}{2}],\) define the space of \(\alpha\)-Hölder rough paths (over \(V\)), in symbols \(C^\alpha([0, T], V)\), as those pairs \((X, X) =: X\) such that

\[ \|X\|_\alpha \overset{\text{def}}{=} \sup_{s \neq t \in [0, T]} \frac{|X_{s,t}|}{|t - s|^\alpha} < \infty , \quad \|X\|_{2\alpha} \overset{\text{def}}{=} \sup_{s \neq t \in [0, T]} \frac{|X_{s,t}|}{|t - s|^{2\alpha}} < \infty , \] (2.3)

and such that the algebraic constraint (2.1) is satisfied.

**Remark 2.2.** Given an arbitrary path \(X \in C^\alpha\) with values in some Banach space \(V\) it is far from obvious that this path can indeed be lifted to a rough path \((X, X) \in C^\alpha\). The Lyons–Victoir extension theorem [LV07] asserts that this can always be done provided \(\alpha \in (\frac{1}{2}, \frac{1}{2}]\), with an infinite dimensional counter example given in the case.
\( \alpha = 1/2 \). When \( \dim V < \infty \), there is no such restriction, see Proposition 13.23 below. In typical applications to stochastic processes, a “canonical” lift is constructed via probability and one does not rely on the extension theorem.

If one ignores the nonlinear constraint (2.1), there is a natural way to think of \((X, X)\) as an element in the Banach space \( C^\alpha \oplus C^2 \) of such maps with (semi-)norm \( \|X\|_\alpha + \|X\|_2 \). However, taking into account (2.1) we see that \( C^\alpha \) is not a linear space, although it is a closed subset of the aforementioned Banach space. We will need (some sort of) a norm and metric on \( C^\alpha \). The induced “natural” norm on \( C^\alpha \) given by \( \|X\|_\alpha + \|X\|_2 \) fails to respect the structure of (2.1) which is homogeneous with respect to a natural dilatation on \( C^\alpha \), given by \((X, X) \mapsto (\lambda X, \lambda^2 X)\). This suggests to introduce the \( \alpha \)-Hölder (homogeneous) rough path norm

\[
\|X\|_\alpha := \|X\|_\alpha + \sqrt{\|X\|_2},
\]

which, although not a norm in the usual sense of normed linear spaces, is the adequate concept for the rough path \( X = (X, X) \).

Note also that the quantities defined in (2.3) are merely seminorms since they vanish for constants. Most importantly, (2.3) leads to a notion of rough path metric (and then rough path topology).

**Definition 2.3.** Given rough paths \( X, Y \in C^\alpha([0, T], V) \), we define the (inhomogeneous) \( \alpha \)-Hölder rough path metric 1

\[
\varrho_\alpha(X, Y) := \sup_{s \neq t \in [0, T]} \frac{|X_{s,t} - Y_{s,t}|}{|t - s|^{\alpha}} + \sup_{s \neq t \in [0, T]} \frac{|X_{s,t} - Y_{s,t}|}{|t - s|^{2\alpha}}.
\]

The perhaps cheapest way to show convergence with respect to this rough path metric is based on interpolation: in essence, it is enough to establish pointwise convergence in conjunction with uniform “rough path” bounds of the form (2.3); see Exercise 2.9. Let us also note that \( C^\alpha([0, T], V) \) so becomes a complete, metric space; the reader is asked to work out the details in Exercise 2.11.

We conclude this part with two important remarks. First, we can ask ourselves up to which point the relations (2.1) are already sufficient to determine \( X \). Assume that we can associate to a given function \( X \) two different second order processes \( \Xi \) and \( \bar{X} \), and set \( G_{s,t} = \Xi_{s,t} - \bar{X}_{s,t} \). It then follows immediately from (2.1) that

\[
G_{s,t} = G_{u,t} + G_{s,u},
\]

so that in particular \( G_{s,t} = G_{0,t} - G_{0,s} \). Since, conversely, we already noted that setting \( \bar{X}_{s,t} = \Xi_{s,t} + F_t - F_s \) for an arbitrary continuous function \( F \) does not change the left hand side of (2.1), we conclude that \( \bar{X} \) is in general determined

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1 As was already emphasised, \( C^\alpha \) is not a linear space but is naturally embedded in the normed space of maps \( X, \bar{X} \); the definition of \( \varrho_\alpha \) makes use of this. While this may not appear intrinsic (the situation is somewhat similar to using the (restricted) Euclidean metric on \( \mathbb{R}^3 \) on the 2-sphere), the ultimate justification is that the Itô map will turn out to be locally Lipschitz continuous in \( \varrho_\alpha \).
only up to the increments of some function \( F \in C^{2\alpha}(V \otimes V) \). The choice of \( F \) does usually matter and there is in general no obvious canonical choice. However, there are important examples where such a canonical choice exists and we will see in Section 10 below that such examples are provided by a large class of Gaussian processes that in particular include Brownian motion, and more generally fractional Brownian motion for every Hurst parameter \( H > \frac{1}{4} \).

The second remark is that this construction can possibly be useful only if \( \alpha \leq \frac{1}{2} \). Indeed, if \( \alpha > \frac{1}{2} \), then a canonical choice of \( X \) is given by reading (2.2) from right to left and interpreting the left hand side by a simple Young integral [You36]. Furthermore, it is clear in this case that \( X \) must be unique, since any additional increment should be \( 2\alpha \)-Hölder continuous by (2.3), which is of course only possible if \( \alpha \leq \frac{1}{2} \). Let us stress once more however that this is not to say that \( X \) is uniquely determined by \( X \) if the latter is smooth, when it is interpreted as an element of \( \mathcal{C}^\alpha \) for some \( \alpha \leq \frac{1}{2} \). In other words, for all times \( s, t \) we have the “first order calculus” condition

\[
\text{Sym}(\mathbb{X}_{s,t}) = \frac{1}{2} \mathbb{X}_{s,t} \otimes \mathbb{X}_{s,t} .
\]

However, if we take \( X \) to be an \( n \)-dimensional Brownian path and define \( \mathbb{X} \) by Itô integration, then (2.1) still holds, but (2.5) certainly does not.
There are two natural ways to define a set of “geometric” rough paths for which (2.5) holds. On the one hand, we can define a subspace $\mathcal{C}^\alpha_g \subset \mathcal{C}^\alpha$ by stipulating that $(X, X) \in \mathcal{C}^\alpha_g$ if and only if $(X, X) \in \mathcal{C}^\alpha$ and (2.5) holds for every $s, t$. Note that $\mathcal{C}^\alpha_g$ is a closed subset of $\mathcal{C}^\alpha$. On the other hand, we have already seen that every smooth path can be lifted canonically to an element of $\mathcal{C}^\alpha$ by reading the definition (2.2) from right to left. This choice of $X$ then obviously satisfies (2.5) and we can define $\mathcal{C}^0,\alpha_g$ as the closure of lifts of smooth paths in $\mathcal{C}^\alpha$. We leave it as exercise to the reader to see that smooth paths in the definition of $\mathcal{C}^0,\alpha_g$ may be replaced by piecewise smooth paths or (piecewise) $C^1$ paths without changing the resulting space of geometric rough paths; see also Exercise 2.12.

One has the obvious inclusion $\mathcal{C}^0,\alpha_g \subset \mathcal{C}^\alpha_g$, which turns out to be strict [FV06a]. The situation is similar to the classical situation of the set of $\alpha$-Hölder continuous functions being strictly larger than the closure of smooth functions under the $\alpha$-Hölder norm. (Or the set of bounded measurable functions being strictly larger than $\mathcal{C}$, the closure of smooth functions under the supremum norm.) Also similar to the case of classical Hölder spaces, one has the converse inclusion $\mathcal{C}^\beta_g \subset \mathcal{C}^\alpha_{g,0}$ whenever $\beta > \alpha$, see Exercise 2.14. Let us finally mention that non-geometric rough paths can always be embedded in a space of geometric rough paths at the expense of adding new components; this is made precise in Exercise 2.14 and was systematically explored in [HK12].

### 2.3 Rough paths as Lie-group valued paths

We now present a very fruitful interpretation of rough paths, at least in finite dimensions, say $V = \mathbb{R}^d$. To this end, consider $X : [0, T] \to \mathbb{R}^d$, $\dot{X} : [0, T]^2 \to \mathbb{R}^d \otimes \mathbb{R}^d$ subject to (2.1) and define (with $X_{s,t} = X_t - X_s$ as usual)

$$X_{s,t} := (1, X_{s,t}, \dot{X}_{s,t}) \in \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d) \cong T^{(2)}(\mathbb{R}^d).$$ (2.6)

The space $T^{(2)}(\mathbb{R}^d)$ has an obvious (“component-wise”) vector space structure. More interestingly, for our purposes, it is a non-commutative algebra with unit element $(1, 0, 0)$ under

$$(a, b, c) \otimes (a', b', c') \overset{\text{def}}{=} (aa', ab' + a'b, ac' + a'c + b \otimes b'),$$

also known as truncated tensor algebra. This multiplicative structure is very well adapted to our needs since (2.1), combined with the obvious identity $X_{s,t} = X_{s,u} + X_{u,t}$, means precisely that (again, called “Chen’s relation”)

$$X_{s,t} = X_{s,u} \otimes X_{u,t}.$$  

Set $T^{(2)}_a(\mathbb{R}^d) = \{(a, b, c) : b \in \mathbb{R}^d, c \in \mathbb{R}^d \otimes \mathbb{R}^d\}$. As suggested in (2.6), the (affine) subspace $T^{(2)}_1(\mathbb{R}^d)$ will play a special role for us. We remark that each of its
elements has an explicit inverse given by
\[(1, b, c) \otimes (1, -b, -c + b \otimes b) = (1, -b, -c + b \otimes b) \otimes (1, b, c) = (1, 0, 0), \quad (2.7)\]
so that \( T^{(2)}_1(\mathbb{R}^d) \) is a Lie group. It follows that \( X_{s,t} = X_{0,s}^{-1} \otimes X_{0,t} \) are the natural increments of the group-valued path \( t \mapsto X_{0,t} \).

Identifying \( 1, b, c \) with elements \((1, 0, 0), (0, b, 0), (0, 0, c) \in T^{(2)}(\mathbb{R}^d)\), we may write \((1, b, c) = 1 + b + c\). Computations using “formal power series” are then possible by considering the standard basis \( \{e_i : 1 \leq i \leq d\} \subset \mathbb{R}^d \) as non-commutative variables. The usual power series \((1 + x)^{-1} = 1 - x + x^2 - \ldots\) then leads to
\[(1 + b + c)^{-1} = 1 - (b + c) + (b + c) \otimes (b + c) = 1 - b - c + b \otimes b, \]
and confirms the inverse of \( 1 + b + c \) given in (2.7). The usual power-series also suggest
\[
\log (1 + b + c) \overset{\text{def}}{=} b + c - \frac{1}{2} b \otimes b
\]
\[
\exp (b + c) \overset{\text{def}}{=} 1 + b + c + \frac{1}{2} b \otimes b \quad (2.8)
\]
and effectively allow to identify \( T^{(2)}_0(\mathbb{R}^d) \cong \mathbb{R}^d \oplus \mathbb{R}^{d \times d} \), with \( T^{(2)}_1(\mathbb{R}^d) = \exp (\mathbb{R}^d \oplus \mathbb{R}^{d \times d}) \). A Lie algebra structure is defined on \( T^{(2)}_0(\mathbb{R}^d) \) by
\[
[b + c, b' + c'] = b \otimes b' - b' \otimes b,
\]
which is nothing but the commutator associated to the non-commutative product \( \otimes \). Denote by \( \mathfrak{g}^{(2)} \subset T^{(2)}_0(\mathbb{R}^d) \) the sub-algebra generated by elements of the form \((0, b, 0)\). One can check that, as a Lie algebra, \( \mathfrak{g}^{(2)} = \mathbb{R}^d \oplus \mathfrak{so}(d) \), i.e. the linear span of \( \{e_i : 1 \leq i \leq d\} \) and \( \{e_{ij} : 1 \leq i < j \leq d\} \), where \( e_{ij} = [e_i, e_j] \). The Lie bracket of \( e_{ij} \) with any other element in \( \mathfrak{g}^{(2)} \) vanishes. Since \( \mathfrak{g}^{(2)} \) is closed under the operation \( [\cdot, \cdot] \), its image under the exponential map, \( G^{(2)}(\mathbb{R}^d) := \exp(\mathfrak{g}^{(2)}) \), is a Lie subgroup of \( T^{(2)}_1(\mathbb{R}^d) \).

We call \( G^{(2)}(\mathbb{R}^d) \) the step-2 nilpotent Lie group (with \( d \) generators). The algebraic constraint (2.5) then translates precisely to the statement that the path \( t \mapsto X_{0,t} \) (and then the increments \( X_{s,t} \)) takes values in \( G^{(2)}(\mathbb{R}^d) \).

Without going into too much details here, \( G^{(2)}(\mathbb{R}^d) \) admits a natural homogeneous “Carnot-Carathéodory norm” \( \| \cdot \|_c \) with the property, for \( x = \exp (b + c) \),
\[
\|x\|_c \simeq |b| + |c|^{1/2}, \quad (2.9)
\]
where \( \simeq \) indicates Lipschitz equivalence (with constants that may depend on the dimension \( d \)). A left-invariant metric \( d_C \), known as the Carnot-Carathéodory metric, is induced by \( \| \cdot \|_c \) so that
\[ d_C(X_s, X_t) = \|X_{s,t}\|_C \approx |X_{s,t}| + |X_{s,t}|^{1/2}. \] (2.10)

As a matter of fact, defining the “truncated signature” of a smooth path \( \gamma : [0, 1] \to \mathbb{R}^d \) by
\[
G^{(2)}(\mathbb{R}^d) \ni S^{(2)}(\gamma) = \left( 1, \int_0^1 d\gamma(t), \int_0^1 \int_0^t d\gamma(s) \otimes d\gamma(t) \right),
\]
we have the identity
\[
\|x\|_C \overset{\text{def}}{=} \inf \left\{ \int_0^1 |\dot{\gamma}(t)| \, dt : \gamma \in C^1([0, 1], \mathbb{R}^d), \right. \left. S^{(2)}(\gamma) = x \right\}.
\]

Using the homogeneous rough path norm introduced in (2.4), taking into account (2.3), we thus have
\[
\|X\|_{\alpha;[0,T]} \approx \sup_{s,t \in [0,T]} d_C(X_s, X_t) |t - s|^\alpha.
\]

and in particular the following appealing characterisation of geometric rough paths.

**Proposition 2.4.** Let \( \alpha \in \left( \frac{1}{3}, \frac{1}{2} \right) \). The following two statements are equivalent:

1. One has \( (X, X) \in \mathcal{C}_g^\alpha \), i.e. it satisfies (2.1), (2.3) and (2.5).
2. The path \( t \mapsto X_t = 1 + X_{0,t} + \mathbb{X}_{0,t} \) takes values in \( G^{(2)}(\mathbb{R}^d) \) and is \( \alpha \)-Hölder continuous with respect to the distance \( d_C \).

Without going into full detail, the above proposition, combined with the geodesic nature of the space \( G^{(2)}(\mathbb{R}^d) \), shows that geometric rough paths are essentially limits of smooth paths (“geodesic approximations” in the terminology of [FV10b]) in the rough path metric.

**Proposition 2.5.** Let \( \beta \in \left( \frac{1}{3}, \frac{1}{2} \right) \). For every \( (X, \mathbb{X}) \in \mathcal{C}_g^\beta ([0,T], \mathbb{R}^d) \), there exists a sequence of smooth paths \( X^n : [0,T] \to \mathbb{R}^d \) such that
\[
(X^n, \mathbb{X}^n) \overset{\text{def}}{=} \left( X^n, \int_0^1 X^n_{0,t} \otimes dX^n_t \right) \to (X, \mathbb{X}) \text{ uniformly on } [0,T]
\]
with uniform rough path bounds \( \sup_n \|X^n\|_\beta + \|\mathbb{X}^n\|_{2\beta} < \infty \). By interpolation, convergence holds in \( \alpha \)-Hölder rough path metric for any \( \alpha \in \left( \frac{1}{3}, \beta \right) \), namely
\[
\lim_{n \to \infty} \mathcal{L}_\alpha((X^n, \mathbb{X}^n), (X, \mathbb{X})) = 0.
\]

**2.4 Geometric rough paths of low regularity**

The interpretation given above gives a strong hint on how to construct geometric rough paths with \( \alpha \)-Hölder regularity for \( \alpha \leq \frac{1}{2} \): setting \( p = \lfloor 1/\alpha \rfloor \), one defines the \( p \)-step truncated tensor algebra \( T^{(p)}(\mathbb{R}^d) \) by
We can construct a Lie group $G(p)(\mathbb{R}^d) \subset T(p)(\mathbb{R}^d)$ as before, by setting $G(p) = \exp(g(p))$, where $g(p) \subset T_0(p)(\mathbb{R}^d)$ is the Lie algebra spanned by elements of the form $(1, b, 0, \ldots, 0)$. Again, one can construct a “homogeneous Carnot-Carathéodory metric” on $G(p)$, with a property similar to (2.9), but with the contribution coming from the $k$th level scaling like $|\cdot|^{1/k}$.

A geometric $\alpha$-Hölder rough path for arbitrary $\alpha \in (0, \frac{1}{2}]$ is then given by a function $t \mapsto X_t \in G(p)(\mathbb{R}^d)$ with $p = \lfloor 1/\alpha \rfloor$, which is $\alpha$-Hölder continuous with respect to the corresponding distance $d_C$. It is actually also possible to extend this construction to the non-geometric setting. This is algebraically somewhat more involved and requires to keep track of more than just the “iterated integrals” of the rough path $X$, see [Gub10]. Again, as in Exercise 2.14, it is possible to embed spaces of non-geometric rough paths of low regularity into a suitable space of geometric rough paths. This construction however is also much more involved in the case of very low regularities and can be found in [HK12].

2.5 Exercises

**Exercise 2.6.** Let $X$ be a smooth $V$-valued path and let $\mathbb{X}$ be given by the left hand side of (2.2), namely

$$\mathbb{X}_{s,t} = \int_s^t X_{s,r} \otimes X_r \, dr.$$ 

a) Show that $\mathbb{X}$ does indeed satisfy Chen’s relation (2.1).

b) Consider the collection of all iterated integrals over $[s, t]$, viewed as element in the tensor algebra over $V$, say

$$X_{s,t} := \left(1, X_{s,t}, X_{s,t}, \int_{s<u_1<u_2<u_3<t} dX_{u_1} \otimes dX_{u_2} \otimes dX_{u_3}, \ldots \right) \in T((V)).$$

and show that the following general form of Chen’s relation holds,

$$X_{s,t} = X_{s,u} \otimes X_{u,t}. \quad (2.11)$$

**Hint:** It suffices to consider the projection of $X_{s,t}$ to $V^\otimes n$, for an arbitrary integer $n$, given by the $n$-fold integral of $dX_{u_1} \otimes \cdots \otimes dX_{u_n}$ over the simplex $\{s < u_1 < \cdots < u_n < t\}$.

**Exercise 2.7.** It is common to define $\mathbb{X}$ on $\Delta_{0,T} := \{(s,t) : 0 \leq s \leq t \leq T\}$ rather than $[0,T]^2$. There is no difference however: if $\mathbb{X}_{s,t}$ is only defined for $s \leq t$, show that the relation (2.1) already determines the values of $\mathbb{X}_{s,t}$ for $s > t$ and give an explicit formula. In fact, show that knowledge of the path $t \mapsto (X_{0,t}, \mathbb{X}_{0,t})$ already
determines the entire second order process $X$. In this sense $(X, X)$ is indeed a path, and not some two-parameter object.

**Exercise 2.8.** Consider $s \equiv \tau_0 < \tau_1 < \cdots < \tau_N \equiv t$. Show that (2.1) implies

$$X_{s,t} = \sum_{0 \leq i < N} X_{\tau_i, \tau_{i+1}} + \sum_{0 \leq j < i < N} X_{\tau_j, \tau_{j+1}} \otimes X_{\tau_i, \tau_{i+1}} = \sum_{i=0}^{N-1} (X_{\tau_i, \tau_{i+1}} + X_{s, \tau_i} \otimes X_{\tau_i, \tau_{i+1}}).$$

(2.12)

**Exercise 2.9 (Interpolation).** Assume that $X^n \in C^{\beta}$, for $1/3 < \alpha < \beta$, with uniform bounds

$$\sup_n \|X^n\|_\beta < \infty \quad \text{and} \quad \sup_n \|X^n\|_{2\beta} < \infty$$

and uniform convergence $X^n_{s,t} \to X_{s,t}$ and $X^n_{s,t} \to X_{s,t}$, i.e. uniformly over $s, t \in [0, T]$. Show that this implies $X \in C^{\beta}$ and

$$\varrho_\alpha(X^n, X) \to 0.$$ 

Show furthermore that the assumption of uniform convergence can be weakened to pointwise convergence:

$$\forall t \in [0, T] : \ X^n_{0,t} \to X_{0,t} \quad \text{and} \quad X^n_{0,t} \to X_{0,t}.$$ 

**Solution 2.10.** Using the uniform bounds and pointwise convergence, there exists $C$ such that uniformly in $s, t$

$$|X_{s,t}| = \lim_n |X^n_{s,t}| \leq C|t-s|^\beta, \quad |X^n_{s,t}| = \lim_n |X^n_{s,t}| \leq C|t-s|^{2\beta}.$$

It readily follows that $X = (X, X) \in C^{\beta}$. In combination with the assumed uniform convergence, there exists $\varepsilon_n \to 0$, such that, uniformly in $s, t$,

$$|X_{s,t} - X^n_{s,t}| \leq \varepsilon_n, \quad |X_{s,t} - X^n_{s,t}| \leq 2C|t-s|^\beta,$$

$$|X^n_{s,t} - X_{s,t}| \leq \varepsilon_n, \quad |X^n_{s,t} - X_{s,t}| \leq 2C|t-s|^{2\beta}.$$ 

By geometric interpolation ($a \wedge b \leq a^{1-\theta} b^\theta$ when $a, b > 0$ and $0 < \theta < 1$) with $\theta = \alpha/\beta$ we have

$$|X_{s,t} - X^n_{s,t}| \leq \varepsilon_n^{1-\alpha/\beta} |t-s|^\alpha, \quad |X^n_{s,t} - X_{s,t}| \leq \varepsilon_n^{1-\alpha/\beta} |t-s|^{2\alpha},$$

and the desired $\varrho_\alpha$-convergence follows.

It remains to weaken the assumption to pointwise convergence. By Chen’s relation, pointwise convergence of $X^n_{0,t}$ for all $t$ actually implies pointwise convergence of $X^n_{s,t}$ for all $s, t$. We claim that, thanks to the uniform Hölder bounds, this implies
uniform convergence. Indeed, given $\varepsilon > 0$, pick a (finite) dissection $D$ of $[0, T]$ with small enough mesh so that $C|D|^\beta < \varepsilon/8$. Given $s, t \in [0, T]$ write $\hat{s}, \hat{t}$ for the nearest points in $D$ and note that

$$|X_{s,t} - X_{\hat{s},\hat{t}}^n| \leq |X_{\hat{s},\hat{t}} - X_{\hat{s},\hat{t}}^n| + |X_{s,\hat{s}}| + |X_{\hat{s},\hat{t}}| + |X_{t,\hat{t}}| + |X_{\hat{s},\hat{t}}^n|$$

$$\leq |X_{\hat{s},\hat{t}} - X_{\hat{s},\hat{t}}^n| + \varepsilon/2. $$

By picking $n$ large enough, $|X_{\hat{s},\hat{t}} - X_{\hat{s},\hat{t}}^n|$ can also be bounded by $\varepsilon/2$, uniformly over the (finitely many!) points in $D$, so that $X^n \to X$ uniformly. Although the second level is handled similarly, the non-additivity of $(s, t) \mapsto X_{s,t}$ requires some extra care, (2.1). For simplicity of notation only, we assume $s < \hat{s} < t = \hat{t}$ so that

$$|X_{s,t} - X_{\hat{s},\hat{t}}^n| \leq |X_{s,\hat{s}} - X_{\hat{s},\hat{t}}^n| + |X_{\hat{s},\hat{t}}| + |X_{s,\hat{s}} \otimes X_{\hat{s},\hat{t}} - X_{\hat{s},\hat{t}}^n \otimes X_{\hat{s},\hat{t}}^n|.$$ 

It remains to write the last summand as $|X_{s,\hat{s}} \otimes (X_{\hat{s},\hat{t}} - X_{\hat{s},\hat{t}}^n) - (X_{s,\hat{s}} - X_{s,\hat{s}}) \otimes X_{\hat{s},\hat{t}}^n|$ and to repeat the same reasoning as in the first level.

**Exercise 2.11.** Check that $C^\alpha([0, T], V)$ is a complete metric space under the metric $|X_0 - Y_0| + g_\alpha(X, Y)$.

Assuming that $\dim V \geq 1$ to avoid trivialities, show that $C^\alpha([0, T], V)$ is not separable. **Hint:** Reduce to the case of scalar Hölder paths on $[0, 1]$; non-separability of such spaces is well known.

**Exercise 2.12.** a) Define the space of geometric ($\alpha$-Hölder) rough paths

$$C^{0,\alpha}_g([0, T], V) \subset C^\alpha([0, T], V)$$

as the $g_\alpha$-closure of smooth paths (enhanced with their iterated Riemann integrals) in $C^\alpha([0, T], V)$. Assuming that $V$ is separable, show that $C^{0,\alpha}_g([0, T], V)$ is also separable.

b) Show that for every geometric $1/2$-Hölder rough path, $X \in C^{0,1/2}_g$, $X$ is necessarily the iterated Riemann-Stieltjes integral of the underlying path $X \in C^{0,1/2}$. (Attention, this does not mean that for every $X \in C^{0,1/2}$ the iterated Riemann-Stieltjes integral exist! A counterexample is found in [FV10b, Ex.9.14 (iii)].)

**Solution 2.13.** Let $Q$ be a countable, dense subset of $V$ and consider the space $\Lambda_n$ of paths which are piecewise linear between level-$n$ dyadic rationals $\mathbb{D}^n := \{kT/2^n : 0 \leq k \leq 2^n\}$, and, at level-$n$ dyadic points, take values in $Q$. Clearly $\Lambda = \cup \Lambda_n$ is countable for each $\Lambda_n$ is in one-to-one correspondence with the $(2^n + 1)$-fold Cartesian product of $Q$. It is easy to see that each smooth $X$ is the limit in $C^1$ of some sequence $(X^n) \subset \Lambda$. Indeed, one can take $X^n$ to be the piecewise linear dyadic approximation, modified such that $X^n|_{\mathbb{D}^n}$ takes values in $Q$ and such that $|X^n - X||_{\mathbb{D}^n} < 1/n$. By continuity of the map $X \in C^1 \mapsto (X, \int X \otimes dX) \in C^\alpha$ in the respective topologies (we could even take $\alpha = 1$), we have more than enough to assert that every lifted smooth path, $(X, \int X \otimes dX)$, is the $g_\alpha$-limit of lifted paths in $\Lambda$. It is then easy to see that every $g_\alpha$ limit point of lifted smooth path is also the $g_\alpha$-limit of lifted paths in $\Lambda$. 
Turning to the second part of the question, it is not hard to see that
\[ \mathcal{E}_{g}^{0,\alpha} \subset \left\{ X \in \mathcal{C}^{\alpha} : \sup_{s,t:|t-s|<\varepsilon} \frac{|X_{s,t}|}{|t-s|^\alpha} \to 0, \sup_{s,t:|t-s|<\varepsilon} \frac{|X_{s,t}|}{|t-s|^{2\alpha}} \to 0 \text{ as } \varepsilon \to 0 \right\}. \]

Consider now the case \( \alpha = \frac{1}{2} \) and a dissection \( \{ s = \tau_0 < \tau_1 < \cdots < \tau_N = t \} \) with mesh \( \leq \varepsilon \). It follows from Chen’s relation (2.1) that
\[
\left| X_{s,t} - \sum_{0 \leq i < n} X_{\tau_i} \otimes X_{\tau_i,\tau_{i+1}} \right| = \left| \sum_{0 \leq i < n} X_{\tau_i,\tau_{i+1}} \right| \leq C(\varepsilon) \sum_{0 \leq i < n} |\tau_{i+1} - \tau_i|^{2\alpha} = TC(\varepsilon).
\]
It follows that \( X_{s,t} \) is the limit of the above Riemann-Stieltjes sum.

**Exercise 2.14.** One can also consider “non-geometric” separable subspaces of \( \mathcal{C}^{\alpha} \). Consider \( \frac{1}{3} < \alpha < \frac{1}{2} \) (in view of the previous exercise there is no point in taking \( \alpha = \frac{1}{2} \) here) and define
\[ \mathcal{E}_{g}^{0,\alpha}([0,T],V) \subset \mathcal{C}^{\alpha}([0,T],V) \]
as the \( \varrho_{\alpha} \)-closure of smooth paths and their iterated integrals plus smooth \( V \otimes V \)-valued path increments. Show that
\[ \mathcal{E}_{g}^{0,\alpha}([0,T],V) \cong \mathcal{E}_{g}^{0,\alpha}([0,T],V) \oplus C^{0,2\alpha}([0,T],V \otimes V). \]
Define the (non-separable) space of weak geometric \( \alpha \)-Hölder rough paths, \( \mathcal{C}^{\alpha} \) as those elements \( X \in \mathcal{C}^{\alpha} \) for which \( 2 \text{ Sym } (X) = X \otimes X \). Show that \( \mathcal{E}_{g}^{0,\alpha} \) is a closed subspace of \( \mathcal{C}^{\alpha} \) and that
\[ \mathcal{C}^{\alpha}([0,T],V) \cong \mathcal{E}_{g}^{\alpha}([0,T],V) \oplus C^{2\alpha}([0,T],V \otimes V). \]
The point of this exercise is that non-geometric rough path spaces can effectively be embedded in geometric rough path spaces.

**Exercise 2.15.** At least when \( \dim V < \infty \), there is not much difference between \( \mathcal{E}_{g}^{0,\alpha} \subset \mathcal{E}_{g}^{\alpha} \) in the following sense. Let \( \frac{1}{3} < \alpha < \beta \leq \frac{1}{2} \). By using the (non-trivial!) fact that every \( X \in \mathcal{C}^{\beta} \) can be approximated uniformly by smooth paths, with uniform \( \beta \)-Hölder rough path bounds, use interpolation to see that \( X \in \mathcal{C}^{0,\alpha} \), in fact show that one has the compact embedding
\[ \mathcal{C}^{\beta} \to \mathcal{C}_{g}^{0,\alpha}. \]
Show a similar statement for non-geometric rough path spaces.

**Solution 2.16.** \( \mathcal{C}^{\beta} \subset \mathcal{C}^{0,\alpha} \) (and in fact a continuous embedding) is obvious from the interpolation exercise above. The compactness of the embedding is a consequence
of Arzela-Ascoli (use \( \dim V < \infty \)). At last the extension to non-geometric rough path spaces, is fairly straightforward using the embedding into geometric rough path spaces.

**Exercise 2.17 (Pure area rough path).** Identify \( \mathbb{R}^2 \) with the complex numbers and consider 

\[ [0, 1] \ni t \mapsto n^{-1} \exp \left( 2\pi i n^2 t \right) = X^n. \]

a) Set \( \mathbb{X}^{n}_{s,t} := \int_{s}^{t} X^n_{s,r} \otimes dX^n_{r} \). Show that, for fixed \( s < t \),

\[ X^n_{s,t} \to 0, \quad \mathbb{X}^{n}_{s,t} \to \pi(t - s) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \] (2.13)

b) Establish the uniform bounds \( \sup_n \| X^n \|_{1/2} < \infty \) and \( \sup_n \| X^n \|_1 < \infty \).

c) Conclude by interpolation that (2.13) takes place in \( \alpha \)-Hölder rough path metric \( \varrho_\alpha \) for any \( 1/3 < \alpha < 1/2 \).

**Solution 2.18.** a) Obviously, \( X^n_{s,t} = O(1/n) \to 0 \) uniformly in \( s, t \). Then

\[ \mathbb{X}^{n}_{s,t} = \frac{1}{2} X^n_{s,t} \otimes X^n_{s,t} + A^n_{s,t} = O(1/n^2) + A^n_{s,t} \]

where \( A^n_{s,t} \in \mathfrak{so}(2) \) is the antisymmetric part of \( \mathbb{X}^{n}_{s,t} \). To avoid cumbersome notation, we identify

\[ \begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix} \in \mathfrak{so}(2) \leftrightarrow a \in \mathbb{R}. \]

\( A^n_{s,t} \) then represents the signed area between the curve \( (X^n_r : s \leq r \leq t) \) and the straight chord from \( X^n_s \) to \( X^n_t \). (This is a simple consequence of Stokes theorem: the exterior derivative of the 1-form \( \frac{1}{2}(x \ dy - y \ dx) \) which vanishes along straight chords, is the volume form \( dx \wedge dy \).) With \( s < t \), \( (X^n_r : s \leq r \leq t) \) makes \( \lfloor n^2(t - s) \rfloor \) full spins around the origin, at radius \( 1/n \). Each full spin contributes area \( \pi(1/n)^2 \), while the final incomplete spin contributes some area less than \( \pi(1/n)^2 \). The total signed area, with multiplicity, is thus

\[ A^n_{s,t} = \left( n^2(t - s) + O(1) \right) \frac{\pi}{n^2} = \pi(t - s) + \frac{C_{s,t}}{n^2}, \]

where \( |C_{s,t}| \leq \pi \) uniformly in \( s, t \). It follows that

\[ \mathbb{X}^{n}_{s,t} = \pi(t - s) \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + O(1/n^2) \] (2.14)

and the claimed uniform convergence follows.

b) The following two estimates for path increments of \( n^{-1} \exp \left( 2\pi i n^2 t \right) \equiv X^n_t \) hold true:
Since $a \wedge b \leq \sqrt{ab}$, it immediately follows that
\[ |X^n_{s,t}| \leq 2|t-s|, \quad \sup_n \|X^n\|_{1/2} < \infty. \]
uniformly in $n, s, t$. In other words, the argument for the uniform bounds on $X_{s,t}$ is similar. On the one hand, we have the bound (2.14). On the other hand, we also have
\[ |X^n_{s,t}| = \int \int_{s \leq u \leq v \leq t} X^n_u \otimes X^n_v \, du \, dv \leq |\dot{X}^n|_\infty \frac{|t-s|^2}{2} \leq \frac{n^2}{2} |t-s|^2. \]
The required uniform bound on $\|X\|_1$ follows by using (2.14) for $n^2|t-s| > 1$ and the above bound for $n^2|t-s| \leq 1$.

c) The interpolation argument is left to the reader.

**Exercise 2.19 (Translation of rough paths).** Fix $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$ and $X = (X, X) \in \mathcal{C}^\alpha([0, T], \mathbb{R}^d)$. For sufficiently smooth $h : [0, T] \to \mathbb{R}^d$, the translation of $X$ in direction $h$ is given by
\[ T_h(X) \overset{\text{def}}{=} \left( X^h, X^h \right), \]
where $X^h := X + h$ and
\[ X^h_{s,t} := X_{s,t} + \int_s^t h_{s,r} \otimes dX_r + \int_s^t X_{s,r} \otimes dh_r + \int_s^t h_{s,r} \otimes dh_r. \] (2.15)
a) Assume $h$ is Lipschitz. (In particular, the last three integrals above are well-defined Riemann–Stieltjes integrals.) Show that for fixed $h$, the translation operator $T_h : X \mapsto T_h(X)$ is a continuous map from $\mathcal{C}^\alpha$ into itself.
b) The above (Lipschitz) assumption on $h$ is equivalently expressed by saying that $h \in W^{1,\infty}$, where $W^{1,q}$ denotes the space of absolutely continuous paths $h$ with derivative $h \in L^q$. Weaken the assumption on $h$ by only requiring $h \in L^q$, for suitable $q = q(\alpha)$. Show that $q = 2$ (“Cameron–Martin paths of Brownian motion”) works for all $\alpha \leq 1/2$. As a matter of fact, the integrals appearing in (2.15) make sense for every $q \geq 1$, but the resulting translated “rough path” would not necessarily lie in $\mathcal{C}^\alpha$.

**2.6 Comments**

The notion of rough path is due to Lyons and was introduced in [Lyo98]. Rather than using Hölder-type norms, the original article introduced rough paths in the $p$-variation sense for any $p \in [1, \infty)$. For $p \geq 3$ (corresponding to $\alpha < \frac{1}{3}$), this requires
additional $[p]^{th}$ order information. Various notes by Lyons preceding [Lyo98] already dealt with $\alpha$-Hölder rough paths for $\alpha \in \left(\frac{1}{3}, \frac{1}{2}\right]$.

In the recent literature, elements in $C^\alpha_g$ are actually called weakly geometric (\(\alpha\)-Hölder) rough paths. In contrast, the space of geometric rough paths $C^{0,\alpha}_g$ is, by definition, obtained via completion of smooth paths in $g_\alpha$. We do not insist on this terminology here and indeed, by Proposition 2.5 there is not much difference. In the early literature the two concepts were somewhat blurred, matters were clarified in [FV06a].
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