Models for Quantifying Mantle Melting Processes

R. N. Singh and A. Manglik

> *Claude Allegre stubbornly passed on to his students the habit of turning his perception of any geological process into equations that could eventually be tested against measurements….*
>
> Albarede (1995)

Abstract Partial melting of mantle and crustal rocks is an important process for the genesis of a suite of igneous rocks seen at the surface of the earth. These rocks preserve the imprints of the complex physico-chemical processes in the earth's interior in the form of their distinct end-member geochemical and isotopic compositions. Spatial and temporal variations in temperature, pressure, fluid mass and concentration of chemical species basically control the petrological property of rocks. This chapter describes basic framework of petrological modelling approach to quantify the deeper processes. Most frequently used equations for geotherm construction for continental and oceanic lithosphere, degree of partial melting and its distribution with depth due to perturbation in geotherms, partition of trace elements and radioactive elements in various partial melting models, crustal evolution and chemical geodynamics models are presented with their derivations.

1 Introduction

The present structure and composition of the earth has been arrived at by physico-chemical processes which are involved in the cooling of the earth. Within the earth, thermal conduction is not an efficient mechanism for bringing heat to the surface from the deep interior. Heat convection, supported by convective instability arguments, is a more favored mechanism. Convection brings materials from the deep interior to the base of the lithosphere leading to their partial melting as a result of decompression. Melts having lesser density rise towards the surface and bring deeper heat to the surface and near-surface rocks, from where the heat is

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transported by conduction process. Thus, partial melting is a vital process for earth's evolution. Why and how such a process takes place needs quantitative answers. It should be appreciated that reliable knowledge is gained by confronting data with models. Enough knowledge exists to use principles and their mathematical forms to understand what has undergone with the earth and also fathom what lies ahead. As geology is a historical science, reductionism, philosophically speaking, will not be able to trace computationally all geological trajectories, connecting all space-time events, holistically. What can then be done reasonably? We should reduce the whole problem into subsets of characteristic processes and model each one separately to arrive at understanding of these underlying processes. Based on such knowledge, one can then construct geological history of an event or a location or the whole earth. In an interesting book titled "*Melting the Earth: The history of ideas on volcanic eruptions*" (Sigurdsson 1999), historical evolution of the idea of the melting in the earth has been discussed. Starting from Greeks to current understanding of the pressure release melting has been narrated in this book. Ideally, melting can occur in the earth due to the following processes:

(a) Temperature changes.
(b) Pressure changes.
(c) Composition changes.
(d) Melting temperature depression by the presence of volatiles, especially water.

All the above are possible, but pressure release melting is the most dominant process.

Melting can provide good estimate of the thermal condition at its source regions. We thus need to know the distribution of the temperature, pressure and composition with depth, and the relationship of melt generation with changes in these distributions. To further constrain the physico-chemical processes, we need to model how chemical species partition amongst competing phases and components. Albarede (1995) and Shaw (2006) have given an excellent theoretical treatment of the subject.

2 Geotherm

Calculation of partial melting requires the knowledge of the geotherm in the lithosphere. In the lithosphere heat transport takes place via heat conduction and the sources of the heat are radiogenic heat sources and heat flow from the mantle. Here the relevant balance law is energy conservation and constitutive law is the Fourier law. We shall derive thermal structure for both continental and oceanic regions. McKenzie et al. (2005) have formulated comprehensive thermal model of both continental and oceanic lithospheres including the effects of temperature dependence of physical properties.

2.1 Continental Lithosphere

Thermal structure is determined using heat flow and heat generation data. Worldwide analysis of these data sets has yielded following general inferences:

(1) Cratons show low surface heat flow values whereas orogens show high values. There has been a decrease in the heat flow values over the geological history. Radiogenic heat too decreases with time.
(2) There is scatter in the data over mean values. This scatter can be due to a variety of causes such as hydrothermal circulation.
(3) Radiogenic heat sources in general contribute about 40 % to the surface heat flow and the remaining is attributed to the deeper sources.
(4) Surface heat flow and heat generation are linearly related. The slope of such a curve gives the depth scale of the radiogenic heat and intercept with heat flow axis gives heat flow from the interior.

The last mentioned relationship is used to derive depth dependence of the radiogenic heat in the crust. The steady state temperature is given by the solution of the following heat conduction equation:

$$\frac{d}{dz}\left(K(T)\frac{dT}{dz}\right) + A(z) = 0, \tag{1}$$

with the following boundary conditions:

$$T = T_S, \quad \text{at} \quad z = 0; \qquad K(T)\frac{dT}{dz} = Q_S, \quad \text{at} \quad z = 0. \tag{2}$$

Here T denotes temperature, K the thermal conductivity, $A(z)$ the radiogenic heat distribution, and z the depth. The variation of the radiogenic heat with depth is given by:

$$A(z) = A_0\exp(-z/d). \tag{3}$$

For a homogeneous layer with constant K and A, the solution of Eq. (1) yields the following expression for temperature distribution:

$$T(z) = T_S + Q_S\frac{z}{K} - A\frac{z^2}{2K}. \tag{4}$$

For a stratified crustal model, thermal structure can be obtained by using the following relationship between temperature T_n and heat flux Q_n at the base of the nth layer having uniform properties (K_n and A_n) and thickness $h_n(z_n - z_{n-1})$ with temperature T_{n-1} and heat flux and Q_{n-1} at the surface of the layer:

$$T_n = T_{n-1} + Q_{n-1} \frac{z_n - z_{n-1}}{K_n} - A_n \frac{(z_n - z_{n-1})^2}{2K_n}, \tag{5}$$

$$Q_n = Q_{n-1} + A_n(z_n - z_{n-1}). \tag{6}$$

A_n can be chosen to represent any general form of radiogenic heat in the crust. Given values of the concentrations (c_i in ppm) of isotopes of U, Th and K and their decay energies (e_i in J/kg s), the radiogenic heat is given by:

$$A = \rho \sum_i e_i c_i \tag{7}$$

The decay energies of U, Th and K are respectively 9.66×10^{-2}, 2.65×10^{-2} and 3.58×10^{-6} mW/kg. For most rocks, A varies within the range of 0.008–8.0 μW/m^3. The base of the lithosphere is assumed to have temperature of about 1,330 °C. The above formalism can be used to obtain the steady state thermal regime of the lithosphere.

Sometimes the normal heat production in the crust cannot yield partial melt such as in granite formation which is a major constituent of the continental crust. In this case we need to resort to generating transient thermal regimes in the lithosphere. Tectonic deformation and erosion can lead to redistribution of heat sources in the crust which can produce higher temperature transiently. One can also assume horizontal flows of material which can transport heat to longer distances and lead to partial melting. The mathematical model to construct such regimes will require solutions of the following advection-diffusion equation:

$$\rho C_p' \left(\frac{\partial T}{\partial t} + V.\nabla T \right) = \nabla.(K.\nabla T) + A, \tag{8}$$

with suitable initial and boundary conditions. Here, V is advection velocity in the medium, ρ is the density and C_p' is the isobaric specific heat capacity.

2.2 Oceanic Lithosphere

Thermal structure of the oceanic lithosphere in its simplest form is constructed by using a cooling half space model. Mathematically, we need to solve the following problem:

$$\rho C_p' \frac{\partial T}{\partial t} = K \frac{\partial^2 T}{\partial z^2}; \qquad T(0,t) = T_S, \qquad T(z \to \infty, t) = T_m, \qquad T(z,0) = T_m. \tag{9}$$

Here K, T_m and T_S represent thermal conductivity, mantle temperature, and surface temperature (other variables are already defined). Solution of this equation with the above initial and boundary conditions is expressed in terms of error function as (Turcotte and Schubert 2002):

$$\frac{T(z,t) - T_S}{T_m - T_S} = \text{erf}\left(\frac{z}{\sqrt{4Kt/\rho C_p'}}\right). \tag{10}$$

If the oceanic lithosphere is spreading at the rate v, then time can be related to horizontal distance from the ridge as $t = x/v$. Thus, the horizontal and vertical variation in the temperature is given by:

$$\frac{T(z,x) - T_S}{T_m - T_S} = \text{erf}\left(\frac{z}{\sqrt{4Kx/\rho C_p' v}}\right). \tag{11}$$

For the error function the following approximate formula can be used (Abramowitz and Stegun 1964):

$$\text{erf}(x) \approx 1 - \left(a_1 t + a_2 t^2 + a_3 t^3\right)\exp\left(-x^2\right), \tag{12}$$

where

$$t = \frac{1}{(1 + 0.47047x)}, \quad a_1 = 0.34802, \quad a_2 = -0.09587, \quad a_3 = 0.74785.$$

Given the values of parameters appearing in the above equation, thermal structure can be easily determined. Variation of heat flow and subsidence with age of the oceanic lithosphere is given by (Turcotte and Schubert 2002):

$$Q = K\frac{T_m - T_S}{\sqrt{\pi Kt/\rho C_p'}}, \qquad w = \frac{2\rho_m\alpha(T_m - T_S)}{\rho_m - \rho_S}\sqrt{Kt/\pi\rho C_p'} \tag{13}$$

3 Pressure Distribution Within Lithosphere

At depths, stresses can be decomposed into isotropic pressure and deviatoric stresses. Except near the surface region, stresses can be taken largely as pressure P which is governed by the following equation:

$$\frac{dP}{dz} = \rho g. \tag{14}$$

Here P, ρ and g are pressure, density of rocks and acceleration due to gravity. Integrating this equation gives $P(z)$ as:

$$P(z) = \int\limits_0^z \rho g dz. \tag{15}$$

If the material properties of a given crustal column are constant, then we have:

$$P(z) = \rho g z. \tag{16}$$

Taking value of crustal thickness as 40 km, density as 2,800 kg m^{-3} and acceleration due to gravity as 9.80 ms^{-2}, the pressure at the base of the crust is $P \approx 1.1$ GPa ≈ 11 kbar. For some geological problems it would be desirable to know both hydrostatic (given above) and deviatoric stresses, required for both compressive and extensional regimes. We would then need to know all components of stress regime which requires much more involved process of calculations.

4 Degree of Partial Melting

As a parcel of rock rises upward, it experiences a decrease in pressure but its temperature does not get equilibrated with the surrounding material. This adiabatic decompression leads to partial melting as the rock parcel crosses the solidus curve. The slope of the solidus curve or of any equilibrium boundary for a reaction is defined by the Claussius-Clapeyron relation (Ganguly and Saxena 1987). This relationship is derived for one component system in two phases by equating Gibbs free energy of both phases, G_1 and G_2 at pressures and temperatures (P, T) and $(P + dP, T + dT)$. We, thus, have along the phase separation boundary:

$$G_1 = G_2 \quad \text{and} \quad G_1 + dG_1 = G_2 + dG_2, \tag{17}$$

which gives:

$$dG_1 = dG_2. \tag{18}$$

Change in the Gibbs free energy (defined as $G = E + PV - TS$) is obtained as:

$$dG = dE + PdV + VdP - TdS - SdT. \tag{19}$$

Here, E is internal energy, V is volume and S is entropy. Using first and second laws of thermodynamics ($dE = TdS - PdV$), we get

$$dG = VdP - SdT. \tag{20}$$

From Eqs. (18) and (20), we get:

$$V_1 dP - S_1 dT = V_2 dP - S_2 dT. \tag{21}$$

This gives the Claussius-Clayperon relation as:

$$\left(\frac{dP}{dT}\right) = \frac{S_2 - S_1}{V_2 - V_1} \equiv \frac{\Delta S}{\Delta V}. \tag{22}$$

where ΔS and ΔV represent change in the entropy and volume, respectively, due to reaction which in this case is the transformation of a solid assemblage to melt (Ganguly and Saxena 1987). For olivine taking the values of ΔV and ΔS as 0.434 J/MPa/g and 0.362 J/K/g, respectively, we get dT/dP as ~ 120 K/GPa. This needs to be compared with the adiabatic gradient followed by a parcel of mantle material ascending to the surface. Under equilibrium condition in which the entropy of a system is conserved:

$$dS = \left(\frac{\partial S}{\partial T}\right)_P dT + \left(\frac{\partial S}{\partial P}\right)_T dP. \tag{23}$$

Partial derivatives in the above equation can be written in terms of heat capacity C_P and coefficient of thermal expansion α as:

$$dS = \left(\frac{C_P}{T}\right) dT - \alpha V dP = \left(\frac{C_P}{T}\right) dT - \frac{\alpha}{\rho} dP. \tag{24}$$

Equating dS as zero, the adiabatic gradient can be obtained as:

$$\frac{dT}{dP} = \frac{\alpha T}{\rho C_P}, \tag{25}$$

where α *is* the coefficient of thermal expansion. This is sometimes referred as isentropic gradient to distinguish it from adiabatic gradient of system that is subjected to internal entropy production (Ganguly 2008). Using Eq. (16), this equation can be reduced to:

$$\frac{dT}{dz} = \frac{g \alpha T}{C_P}. \tag{26}$$

The above equation is used to find the adiabatic gradient for a given rock type and, thus, the depth at which partial melting starts once the solidus curve is intersected. We can estimate the adiabatic gradient for olivine using the values of α, C_p, V and T as 2.7×10^{-5} K^{-1}, 193J/(K-mol), 43.8 J/(MPa-mol), and 1650 K, respectively. This gives the adiabatic gradient as ~ 10 K GPa^{-1}, which is much smaller than inverse of the Claussius-Clayperon slope obtained earlier. Thus, a parcel of the mantle material rising along the adiabatic gradient will experience partial melting.

Fig. 1 A sketch of *solidus* and *adiabatic curves*
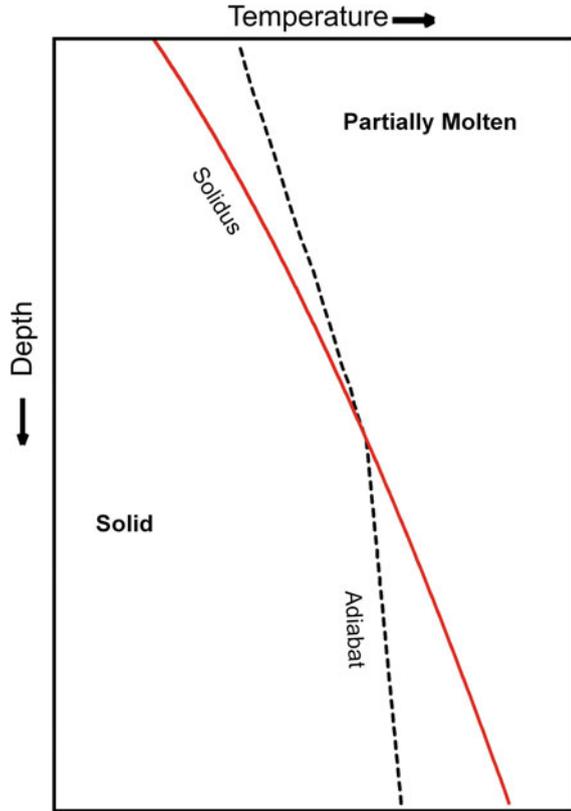
Partial melting of the rock adds another effect, the release of latent heat due to which further rise of the rock parcel deviates from the adiabatic path given by Eq. (26). Some parameterized models for partial melting taking into consideration these effects have been proposed (Fig. 1).

Recently, Ganguly (2005) has derived the following expression for temperature gradient for the irreversible adiabatic decompression of isoviscous materials rising from great depths in the mantle:

$$\left(\frac{dT}{dz}\right)_{Q(ir)} = \frac{\rho_r}{\rho}\left(\frac{dT}{dz}\right)_S + \frac{g}{C_p}\left(1 - \frac{\rho_r}{\rho}\right) - \frac{2}{C_p}\left(\frac{du}{dz}\right). \qquad (27)$$

Here, ρ_r is density of rising mantle rock and u is vertical velocity (positive downward). Ganguly (2005) showed that the temperature of rising mantle rocks will be higher than given by adiabatic gradient depending on the value of the ratio ρ_r/ρ. When $\rho_r/\rho < 0.94$ the ascending mantle rocks would be heated up instead of cooling under adiabatic condition leading to increased melt productivity.

Equations (22) and (25) set the conditions for the onset of partial melting. A quantity of interest is the amount of melt generated when the mantle material

rises after it intersects the solidus. For this, we follow the changes in the entropy as the material moves along the solidus. It is given by:

$$\frac{dS^s}{dP} = \frac{C_p^s}{T}\left(\frac{dT}{dP}\right)_{2\varnothing} - \alpha^s V^s. \tag{28}$$

Here, subscript $2\varnothing$ indicates that the gradient is along the solidus and superscripts s and l refer to solid and liquid phase, respectively. If solid and melt together have entropy S_0, then we have:

$$S_0 = fS^l + (1-f)S^s, \tag{29}$$

where f is melt fraction. From the above equation, we get:

$$f = \frac{S_0 - S^s}{S^l - S^s} = \frac{S_0 - S^s}{\Delta S}. \tag{30}$$

Differentiating Eq. (30) with respect to P and substituting Eq. (28) we get:

$$\left(\frac{\partial f}{\partial P}\right)_S = -\frac{1}{\Delta S}\left[\frac{C_p^s}{T}\left(\frac{\partial T}{\partial P}\right)_{2\varnothing} - \alpha^s V^s\right]. \tag{31}$$

This equation can further be evaluated by using solidus gradient equation as:

$$\left(\frac{\partial f}{\partial P}\right)_S = -\frac{1}{\Delta S}\left[\frac{C_p^s}{T}\frac{\Delta V}{\Delta S} - \alpha^s V^s\right]. \tag{32}$$

We can evaluate the changes in melt fraction with pressure using data for heat capacity, change in volume and entropy at fusion, coefficient of thermal expansion and temperature. Calculations show that melt fraction increases with decrease in pressure.

Besides above modelling, following relationship has been used in the literature (Ahern and Turcotte 1979) to calculate the degree of partial melting, f:

$$f = a\{\exp(b(T - cz - d)) - 1\}, \tag{33}$$

where $a = 0.4$, $b = 3.65 \times 10^{-3} K^{-1}$, $c = 3.0 \times 10^{-3} K\,m^{-1}$, and $d = 1{,}100\,°C$. This relationship does not hold for very low degree ($<5\,\%$) of partial melting. McKenzie and Bickle (1988) and Ellam (1992) have given the following expression for calculating f:

$$f - 0.5 = T' + \left(T'^2 - 0.25\right)\left(0.4256 + 2.988T'\right), \tag{34a}$$

$$f = 2.7157 \times 10^{-7}z^3 - 5.0715 \times 10^{-5}z^2 - 3.7816 \times 10^{-4}z + 0.30929, \tag{34b}$$

where $T' = (T - (T_s + T_l)/2)/(T_l - T_S) \cdot T_S$ and T_l denote solidus and liquidus temperatures. The second relationship has been used to quantify partial melting due to rise of hot mantle plume (Ellam 1992).

There has been discussion in the literature on the variation of melting with depth. McKenzie (1984) and McKenzie and Onions (1991) have argued for decrease in degree of melting with decrease in pressure whereas Ahern and Turcotte (1979) argue for increase in melting as pressure decreases. Asimov et al. (1997) have argued for increase in melt productivity with decrease in pressure both for batch and fractional melting under isentropic condition.

5 Changes in the Temperature After Melt Extraction

We shall take a simple two-layered model of lithosphere with no radiogenic heat and thermal structure as discussed in Foucher et al. (1982):

$$T(z) = \begin{cases} \frac{T_m z}{H}, & 0 \leq z \leq H \\ T_m + G(z - H), & z \geq H \end{cases}. \tag{35}$$

In this thermal model, the lower region of the lithosphere has a constant thermal gradient, denoted by G, connecting to adiabatic gradient of mantle instead of having a constant temperature.

After stretching of the lithosphere by a factor β the thermal structure changes to:

$$T = \begin{cases} T'_m(\beta/H)z, & 0 \leq z \leq H/\beta \\ T'_m + G(z - H/\beta), & z \geq H/\beta \end{cases}, \tag{36}$$

where $T'_m = T_m - G(H - H/\beta)$. With partial melting, the above expression modifies to:

$$T = \begin{cases} T'_m(\beta/H)z - fL/C, & 0 \leq z \leq H/\beta \\ T'_m + G(z - H/\beta) - fL/C, & z \geq H/\beta \end{cases}. \tag{37}$$

Here, L and C are latent heat of fusion and specific heat, respectively. f is function of temperature. After its substitution, we can obtain the thermal structure of the stretched lithosphere.

6 Depth Distribution of Partial Melt

The distribution of degree of partial melt with depth can be calculated by substituting the expression for temperature T given by Eq. (36) into equation for degree of partial melting given by Eqs. (33) and (34a) (Foucher et al. 1982). This yields an

equation for f as the function of depth. This calculation can be done numerically. It is seen that the top of the melt zone is at the depth H/β and bottom is at the depth where degree of partial melting is zero ($z = H_f$). This is given by:

$$H_f = \frac{T_m - GH - d}{c - G} \quad \text{for} \quad \beta > H/H_f. \tag{38}$$

The degree of partial melting increases from zero at H_f to maximum at the depth of H/β as:

$$f = f_0 \left(1 - \frac{z}{H_f}\right). \tag{39}$$

We can also calculate the thickness of the layer (H_b) formed by emplacement of melts as liquid igneous body (density ρ_{al}) as:

$$H_b = \frac{\rho_a}{\rho_{al}} \int_{H/\beta}^{H_f} f_0 \left(1 - \frac{z}{H_f}\right) dz. \tag{40}$$

Here, ρ_a is the density of asthenosphere. This expression is evaluated as:

$$H_b = \frac{\rho_a}{2\rho_{al}} \left(1 - \frac{\beta_c}{\beta}\right)^2 H_f f_0 z, \tag{41}$$

where $\beta_c = H/H_f$.

7 Partition Coefficients for Trace Elements

Trace elements have been found to trace the melting processes. These elements follow principles of dilute solutions. Their distributions amongst phases are easily characterized as these elements do not interact with each other and follow Henry's law. The distribution in any two phases is described in terms of partition function. The total mass of elements is conserved within all phases. Elements remaining in solid or released in melt are characterized by their partition coefficient values. Shaw (2006), Albarede (1995), and Ganguly (2008) can be referred for original references.

If mass of component i in a phase α is m_i, with phase mass being M_i, the concentration of component i in this phase is defined as $c_i^\alpha = m_i/M_i$. For total number of components being p, we have:

$$\sum_{i=1}^{p} c_i^\alpha = 1. \tag{42}$$

Let concentration of a component i in phases α and β be denoted by c_i^{α} and c_i^{β}, respectively. The partition coefficient is defined as:

$$D_i^{\beta-\alpha} = c_i^{\beta}/c_i^{\alpha}. \tag{43}$$

A bulk partition coefficient can be defined for use in petrology as rocks contain a large number of minerals which are involved in partial melting.

Let mineral proportions of a rock before partial melting be denoted by X_i^0 and after melting by X_i. The concentration of element in the rock is defined as:

$$c_S = \sum_i c_i X_i. \tag{44}$$

The concentration in the melt can be defined using partition coefficient (D^{i-m}) when mineral and melt are in equilibrium as:

$$c_l = c_i/D^{i-m}. \tag{45}$$

Combining the above two equations, we get:

$$c_S = c_l \sum_i D^{i-m} X_i. \tag{46}$$

Considering total mass of rock before melting as M_0, after melting the rock has mass as M and melt as L. If f is the melt fraction, we have:

$$L = f M_0 \quad \text{and} \quad M = (1-f)M_0. \tag{47}$$

We now write for each mineral:

$$M X_i = M_0 X_i^0 - L p_i, \tag{48}$$

where p_i is the proportion of mineral i in the melt. Thus, total concentration of an element in the solid is given by:

$$c_S = c_l \sum_i \frac{\left(D^{i-m} X_i^0 - p_i D^{i-m} f\right)}{(1-f)}. \tag{49}$$

Thus, the bulk partition coefficient D is given by:

$$D = (D_0 - Pf)/(1-f), \tag{50}$$

where $D_0 = \sum_i D^{i-m} X_i^0, P = \sum_i p_i D^{i-m}$.

The above formalism is for non-modal melting. For modal melting, we have $X_i^0 = p_i$. In this case, we have $D_0 = P = D$. Elements having small values of $D(<1)$, are called incompatible and others compatible elements. Incompatible and compatible elements partition preferentially into melt and solids, respectively.

8 Batch Melting

Consider mass of solid M_0 with concentration of some element as c_S^0. After partial melting, let the volume of the melt be L with concentration of the element as c_l and in the residual solid c_S with mass as M. Total melt fraction attains equilibrium with the solid and only then it is extracted fully. Thus, we have the following relationship (Shaw 2006; Ganguly 2008):

$$M_0 = L + M, \quad c_S^0 M_0 = c_l L + c_S M. \tag{51}$$

Let us now define melt fraction and partition coefficient as:

$$f = \frac{L}{M_0}, \qquad D = \frac{c_S}{c_l}. \tag{52}$$

Using these relationships, we find the concentrations in liquid and residual solid as (Fig. 2):

$$c_l = \frac{c_s^0}{f + D(1-f)}, \qquad c_S = \frac{D c_S^0}{f + D(1-f)}. \tag{53}$$

This is also called equilibrium melting. Figure 2 shows graphically the relationship (Eq.53(i)) for various values of D. Here total melt remains in contact with matrix. For non-modal melting, above equations are changed as:

$$c_l = \frac{c_S^0}{D_0 + f(1-P)}, \qquad c_S = \frac{c_S^0(D_0 - Pf)}{(1-f)(D_0 + f(1-P))}. \tag{54}$$

9 Fractional Melting

In this case, melt fraction is extracted from the solid as soon as it is formed or exceeds a critical limit. Here we treat the first case. Let the mass of solid converted to melt be given by dM with concentration c_l. We can then write (Shaw 2006; Ganguly 2008):

$$c_l dM = d(c_S M), \quad \frac{c_S dM}{D} = c_S dM + M dc_S, \quad \frac{dc_S}{c_S} = \left(\frac{1}{D} - 1\right) dM/M. \tag{55}$$
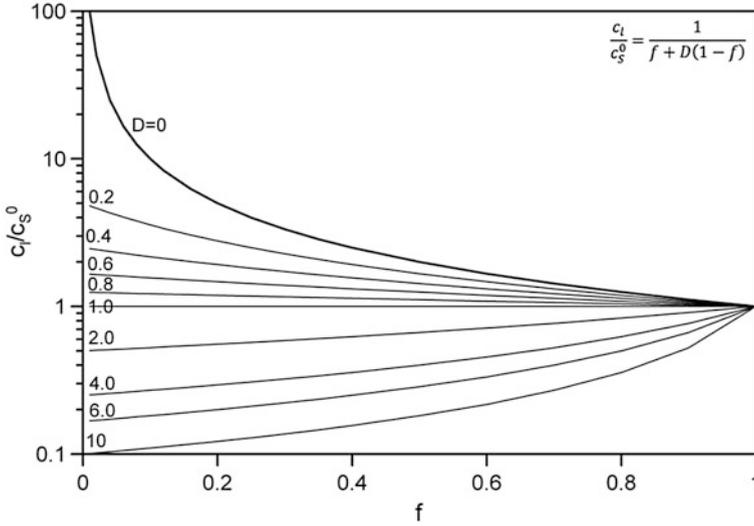
Fig. 2 Relative concentration of trace elements in liquid for partial melting by batch melting model described by Eq. (53)

Integrating the above equation, we get:

$$c_l = \left(\frac{c_S^0}{D}\right)(1-X)^{\frac{(1-D)}{D}}, \quad c_S = \left(c_S^0\right)(1-X)^{(1-D)/D}. \tag{56}$$

Here $X = L/M_0$. This is the amount of melt fraction extracted from the sources. Average concentration in melt is obtained by averaging over 0–X (Fig. 3):

$$\bar{c}_l = (1/X) \int_0^X c_l dX = c_S^0 \left(1 - (1-X)^{1/D}\right)/X. \tag{57}$$

These two melting models, batch and fractional, are the end member models. For non-modal melting, we get:

$$c_l = \left(c_S^0/D_0\right)(1 - PX/D_0)^{(1-P)/P}, \quad c_S = c_S^0(1 - PX/D_0)^{1/P}/(1-X). \tag{58}$$

The average concentration over all X is given by:

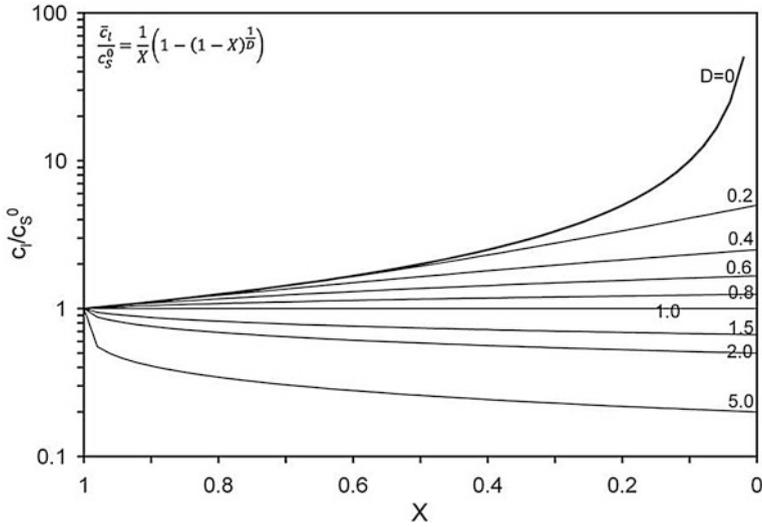$$\bar{c}_l = c_S^0 \left(1 - (1 - PX/D_0)^{1/P}\right)/X. \tag{59}$$

Fig. 3 Relative average concentration of trace elements in liquid for partial melting by fractional melting model described by Eq. (57)

10 Continuous Melting Model

McKenzie (1985) and Williams and Gill (1989) discussed a model wherein a fraction of partial melt remains always within the matrix. Let m be the melting rate, i.e., the amount of the matrix transformed into melt per unit time. We then have;

$$\frac{d}{dt}(\rho_l \phi c_l) = c_s m, \qquad \frac{d}{dt}(\rho_S(1 - \phi)c_S) = -c_l m. \qquad (60)$$

Combining these two equations, we get:

$$\frac{dc_l}{dt} = -\alpha c_l. \qquad (61)$$

where

$$\alpha = \frac{(1 - D)mF}{(\phi \rho_l)}, \qquad F = \left[1 + \frac{D\rho_S(1 - \phi)}{(\rho_l \phi)}\right]^{-1}.$$

We then have:

$$c_l = c_l^0 e^{-\alpha t}. \qquad (62)$$

Here, c_l^0 is the initial composition of the melt. McKenzie (1985) has shown that the extracted melt, denoted by X, is related to the melt production rate as:

$$X = 1 - \exp(-rt), \qquad r = m/(\rho_l \phi + \rho_S(1 - \phi)). \tag{63}$$

From here, one can get expression for t as:

$$t = -\log(1 - X)/r. \tag{64}$$

Thus, the concentration c_l is written as:

$$c_l = c_l^0(1 - X)^{G(1-D)}, \qquad G = \frac{\rho_l \phi + \rho_S(1 - \phi)}{\rho_l \phi + \rho_S(1 - \phi)D}. \tag{65}$$

It can also be shown that:

$$c_l^0 = Gc_0, \tag{66}$$

where c_0 is the initial concentration in the source. We then get:

$$c_l = c_0 D(1 - X)^{G(1-D)}. \tag{67}$$

This is the expression for the continuous melt. This can be integrated over $[0, X]$ to get an expression for aggregated continuous melt as:

$$\bar{c}_l = c_0 G \frac{1 - (1 - X)^{G(1-D)+1}}{X(1 + G(1 - D))}. \tag{68}$$

From above two equations expressions for batch melting ($X = 0$ and $\phi = f$) and fractional melting ($\phi = 0$) can be derived.

11 Fractionation of Radioactive Elements

We shall now present expressions for the radioactive element distributions in melts and matrix following McKenzie (1985) and Williams and Gill (1989). Let the concentration and half-life of elements be denoted by c and λ respectively. Let $c_S^P(c_l^P)$ represent concentration of parent element in solid (melt). We have the following equations for distribution of parent element in solid and melt as:

$$\frac{d}{dt}\left((1 - \phi)\rho_S c_S^P\right) = -c_l^P m, \qquad \frac{d}{dt}\left(\phi \rho_l c_l^P\right) = -c_S^P m. \tag{69}$$

Combining these two equations, we get:

$$((1 - \phi)\rho_S D^P + \phi\rho_l)\frac{dc_l^P}{dt} = (D^P - 1)mc_l^P, \tag{70}$$

where $D^P = c_S^P/c_l^P$. This equation can be further simplified to:

$$\frac{dc_l^P}{dt} = -\alpha_P c_l^P, \tag{71}$$

where

$$\alpha_P = \frac{(1 - D^P)mF_P}{(\rho_l\phi)}, \qquad F_P = \left[1 + \frac{(1 - \phi)\rho_S}{\phi\rho_l}D^P\right]^{-1}. \tag{72}$$

For daughter element, $c_S^D(c_l^D)$, it is necessary to include the decay as its life is small compared to the half-life of the parents. We have:

$$\frac{d}{dt}\{(1 - \phi)\rho_S c_S^D + \phi\rho_l c_l^D\} = c_S^D m - c_l^D m + \lambda_P\{(1 - \phi)\rho_S c_S^P + \phi\rho_l c_l^P\} \\ - \lambda_D\{(1 - \phi)\rho_S c_S^D + \phi\rho_l c_l^D\}. \tag{73}$$

This equation can further be reduced to:

$$\frac{d}{dt}c_l^D = -\lambda_D c_l^D + \lambda_P a_P - a_D, \tag{74}$$

where

$$\lambda_D = \frac{(1 - D^P)mF_D}{\phi\rho_l}, \qquad a_P = \frac{\lambda_P c_l^D F_P}{F_D}, \qquad a_D = \lambda_D c_l^D. \tag{75}$$

We have now two equations for both parent and daughter elements distribution in the melt. We can derive from them equations for batch and continuous melting.

11.1 Batch Melting

Here, $m = 0$. Thus, we have in terms of activity $a_p\ (= \lambda_p c^p)$:

$$\frac{da_P}{dt} = 0, \qquad \frac{da_D}{dt} = \lambda_D\left(\frac{F_D}{F_P}a_P - a_D\right). \tag{76}$$

The solutions of the above equations are:

$$a_P = A, \quad a_D = Be^{-\lambda_D t} + \frac{F_D}{F_P} a_P. \tag{77}$$

For $r = a_D/a_P$, we have the following expression from above equations:

$$r = Be^{-\lambda_D t} + F_D/F_P. \tag{78}$$

At $t = 0$ we have:

$$r(t = 0) = r_0 = \left(\frac{A}{a_P} + r_e\right), \qquad r_e = F_D/F_P \tag{79}$$

We thus get:

$$r = r_0 e^{-\lambda_D t} + r_e\left(1 - e^{-\lambda_D t}\right). \tag{80}$$

Thus, constant of decay of r depends upon λ_D. When t tends to infinity, $r = r_e$. This value will be 1 only if $D^D = D^P$.

11.2 Continuous Melting

Here m and ϕ are constant. We, therefore, have:

$$\frac{da_P}{dt} = -\alpha_P a_P, \qquad \frac{da_D}{dt} = -\alpha_D a_D + \lambda_D\left(\frac{F_D}{F_P} a_P - a_D\right). \tag{81}$$

The solutions of these equations are:

$$\begin{aligned} a_P &= a_P(0) + e^{-\alpha_P t} \\ a_D &= a_D(0)e^{-(\lambda_D + \alpha_D)t} + \frac{\lambda_D F_D}{(\lambda_D + \alpha_D - \alpha_P)F_P} a_P(0)\left(e^{-\alpha_P t} - e^{-(\lambda_D + \alpha_D)t}\right). \end{aligned} \tag{82}$$

Numerical applications of the above formalism have been given in Williams and Gill (1989).

12 Chemical Geodynamics

Chemical fractionations in the earth have involved major geodynamical phenomena. Thus, trace element evolutions, which can characterize fractionation phenomena, can also yield the nature of the geodynamical processes. Issues such as mass, duration, and time of the crust extracted from the mantle can be estimated using the evolution of trace elements in these reservoirs. Rb-Sr and Sm-Nd data in the crust, MORB and OIBs can be used for this purpose. MORB and OIBs can give estimates of depleted and undepleted reservoirs. Observations of isotopic systems define normalized isotope ratio (ε) and fractionation factor (f) as (Turcotte and Schubert 2002):

$$\varepsilon = \left[\left(\frac{c^{D*}/c^D}{c_S^{D*}/c_S^D} \right) - 1 \right] \times 10^4, \quad f = \left(\frac{c^P/c^D}{c_S^P/c_S^D} \right) - 1. \tag{83}$$

Here, c^P and C^{D*} refer to mole densities of parent and daughter isotope and c^D refers to reference isotope of daughter element. Normalization is done with respect to bulk silicate earth (denoted by subscript S). The expressions for spontaneous separation of crustal reservoir (τ_c) and ratio of mass of crust and mantle reservoirs (M_c/M_m) are given (Turcotte and Schubert 2002) as:

$$\tau_c = \frac{\varepsilon_{mp}}{Q f_{mp}}, \frac{M_c}{M_m} = \left[D_{si} \left(1 - \frac{f_{cp}}{f_{mp}} \right) - 1 \right]^{-1},$$
$$Q = 10^4 \frac{c_{s0}^P}{c_{s0}^{D*}} \lambda, D_{si} = \frac{<c_c^{D*}>}{<c_s^{D*}>} = \frac{<c_c^D>}{<c_{s0}^D>}. \tag{84}$$

Here, p refers to present values and $\langle \ldots \rangle$ refers to enrichment factor at the time of crust separation from mantle. Subscripts c and m refer to crust and mantle, respectively. λ is the decay constant. This formalism has been applied using Sm-Nd and Rb-Sr isotope systems.

13 Crustal Evolution

Crust has been generated by mantle melting. Further layered structure of the crust has been formed due to intra-crustal melting. Attempts have been made to carry out population dynamics of crustal elements which have been generated over the earth's history and survived in spite of ongoing erosion and subduction. We shall give below a highly simplified treatment of the evolution of crust. Present volume of the crust is 7×10^9 km^3, having grown over time period of 3.8 Ga. Denoting volume of crust as, $V(t)$ at time t, the conservation equation for V can be written as (Gurnis and Davis 1985):

$$\frac{dV}{dt} = \dot{V}_g - \dot{V}_r, \tag{85}$$

where $\dot{V}_g$ and $\dot{V}_r$ denote rates of growth and removal. We need to relate rates of growth and removal with the existing volume of continental crust and other crust modifying processes such as subduction and erosion. Following forms for these two rates have been considered:

$$\dot{V}_g = C_g \dot{A}^\alpha, \qquad \dot{V}_r = C_r \dot{A}^\beta V^\nu. \tag{86}$$

Here C_g, C_r, α, β, and ν are constants and $\dot{A}$ equals the average plate velocity times ridge length. At present $\dot{A} = 3$ km^3/yr. Heat flow (q) is proportional to square root of plate velocity. So we can take $q \propto \sqrt{\dot{A}}$. Thus, the equation for V is:

$$\frac{dV}{dt} = C_g q^{2\alpha} - C_r q^{2\beta} V^\nu. \tag{87}$$

Given the time dependence of q as:

$$q(t) = \exp(-\lambda t), \quad \lambda = \ln 2 (\tau_{1/2}), \quad \tau_{1/2} = 2 - 4\text{Ga}, \tag{88}$$

the form of $V(t)$, the volume of crust, can be determined. Following special cases have been solved in Gurnis and Davis (1985):

(i) No recycling $(C_r = 0, \ \alpha = 1)$:

$$V(t) = (C_g/2\lambda)(1 - e^{-2\lambda t}). \tag{89}$$

(ii) Constant recycling rate $(\alpha = 1, \ \beta = 0 = \nu)$:

$$V(t) = (C_g/2\lambda)(1 - e^{-2\lambda t}) - C_r t. \tag{90}$$

(iii) Rate dependent recycling $(\nu = 0, \ \beta \neq 0)$:

$$V(t) = \frac{C_g - C_r}{2\lambda}(1 - e^{-2\lambda t}), \quad \text{for } \beta = 1. \tag{91}$$

$$V(t) = (C_g/2\lambda)(1 - e^{-2\lambda t}) - (C_r/4\lambda)(1 - e^{-4\lambda t}), \quad \text{for } \beta = 2 \tag{92}$$

(iv) Volume dependent removal rate $(\alpha = 1, \nu = 1)$:

$$V(t) = \frac{C_g}{2\lambda - C_r} \left(e^{-C_r t} - e^{-2\lambda t} \right), \qquad \text{for } \beta = 0. \tag{93}$$

$$V(t) = \frac{C_g}{C_r} \left(1 - e^{-C_r t} \exp\left(\frac{C_r}{2\lambda} e^{-2\lambda t} \right) \right), \quad \text{for } \beta = 1. \tag{94}$$

These equations can be used to find evolution of the volume of the crust.

14 Melt Extraction Velocity

As pressure release melting is the main process for partial melt generation, it is necessary to know how mantle upwells (passive, active or both). Melt distribution and flow can be envisioned as porous flow or channel flow. In case of porous flow models, the velocities are found to be of the order of few meters per year whereas faster velocities are possible in the case of channel flows. Which one of these is more reasonable for MORB or other types of melt sources can be determined by looking into signatures of short time processes. Stracke et al. (2006) have analyzed U-series decay isotopes in the Icelandic lavas and found evidences of short times for melt transport to the surface. Thus, channel flow model fits better with available data.

15 Mantle Heterogeneities

When we characterize mantle and crust as reservoirs, we take them to be homogeneous, ignoring small-scale heterogeneities. It has been shown that mantle convection assimilates the subducted materials with chaotic mixing. Evidences of such stirring have also been seen in the rocks brought at the surface. Quantification of such processes will require constructing suitable mathematical models and confronting with data. There are two ways to make progress. One can make fully physics based models. But problem here is that there are too many physical and chemical parameters to be known for making progress, the uncertainty of result being decided by least known parameter. With current level of development this is not profitable approach. Another approach is to use statistical methods. One can here theoretically obtain the probability distribution of isotopic ratios and confront them with observed data to know the nature of controlling parameters, such as fraction of melt, average duration between melting and nature of mantle stirring. Significant progress has been made in this direction. Rudge et al. (2005) have derived a comprehensive statistical framework, extending formulations and ideas of several previous workers (Allegre and Lewin 1995a, b; Slater et al. 2001; Kelllog et al. 2002; Meinborn and Anderson 2003), and applied it to model the

isotopic variability of MORB. Melt fraction is found to be 0.5 % and average time since parcel last melted as 1.4–2.4 Ga.

16 Concluding Remarks

Modelling of the partial melting processes in general requires the study of multiphase media including matrix deformation. The detailed petrological modelling will need to consider space- and time- variations of temperature, stress, fluid mass and concentration of chemical species which are coupled (Sleep 1974; McKenzie 1984; Ribe 1987; Dobran 2001; Steefel et al. 2005). The physico-chemical processes will be described by a coupled set of nonlinear partial differential equations. It is seen that the nonlinear systems show extremely complex behavior such as bifurcation, deterministic chaos, self-organization and spatio-temporal chaos. Petrological complexity as observed in the field and also required to interpret the geochemical and geophysical data needs an approach as mentioned above. Several critical problems have been solved in the literature.

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