Euclidis ab omni naevo vindicati Liber Primus

In quo demonstratur:

duas quaslibet in eodem plano existentes rectas lineas, in quas recta quaepiam incidens duos ad easdem partes internos angulos efficiat duobus rectis minores, ad eas partes aliquando invicem coituras, si in infinitum producantur.

Pars Prima

[1] **Propositio I.**

Si duae aequales rectae (Fig. 1) *AC, BD, aequales ad easdem partes efficiant angulos cum recta AB: Dico angulos ad junctam CD aequales invicem fore.*

Demonstratur. Jungantur AD, CB. Tum considerentur triangula CAB, DBA. Constat (ex quarta primi) aequales fore bases CB, AD. Deinde considerentur triangula ACD, BDC. Constat (ex octava primi) aequales fore angulos ACD, BDC. Quod erat demonstrandum.

Propositio II.

Manente uniformi quadrilatero ABDC, latera AB, CD, bifariam dividantur (Fig. 2) *in punctis*
[2] *M, & H. Dico angulos ad junctam MH fore hinc inde rectos.*

Demonstratur. Jungantur AH, BH, atque item CM, DM. Quoniam in eo quadrilatero anguli A, & B positi sunt aequales, atque item (ex praecedente) aequales sunt anguli C, & D; constat ex quarta primi (cum alias nota sit aequalitas laterum) aequales fore in triangulis CAM, DBM, bases CM, DM; atque item, in triangulis ACH, BDH, bases AH, BH. Quare; collatis inter se triangulis CHM, DHM, ac rursum inter se triangulis AMH, BMH; constabit (ex octava primi) aequales invicem fore, atque ideo rectos angulos hinc inde ad puncta M, & H. Quod erat demonstrandum.

G. Saccheri, *Euclid Vindicated from Every Blemish,* Classic Texts in the Sciences 1,
DOI 10.1007/978-3-319-05966-2_2, © Springer International Publishing Switzerland 2014

Euclid Vindicated from every Blemish
Book One

In which is proved:

any two coplanar straight lines, falling upon which any straight makes toward the same parts two internal angles less than two right angles, at length meet each other toward those parts, if infinitely produced.

First Part

Proposition 1.

If two equal straights (Fig. 1) *AC, BD, make with the straight AB angles equal toward the same parts: I say that the angles at the join* CD *will be mutually equal.*

Proof. Join AD, CB. Then consider the triangles CAB, DBA. It follows (*Elements* I, 4) that the bases CB, AD will be equal. Then consider the triangles ACD, BDC. It follows (*Elements* I, 8) that the angles ACD, BDC will be equal. This is what was to be demonstrated.

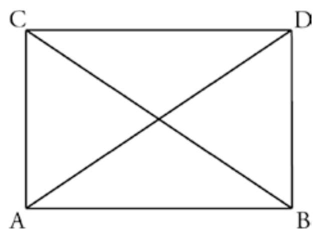

Fig. 1

Proposition 2.

Retaining the uniform quadrilateral ABCD, bisect the sides AB, CD (Fig. 2) *in the points M and H. I say the angles at the join MH will then be right.*

Proof. Join AH, BH, and likewise CM, DM. Because in this quadrilateral the angles A and B are taken equal and likewise (Proposition 1) the angles C, and D are equal; it follows (*Elements* I, 4) (noting the equality of the sides) that in the triangles CAM, DBM, the bases CM, DM will be equal; and likewise, in the triangles ACH, BDH, the bases AH, BH. Therefore; comparing the triangles CHM, DHM, and in turn the triangles AMH, BMH; it follows (*Elements* I, 8) that we have mutually equal, and therefore right, the angles at the points M, and H. This is what was to be demonstrated.

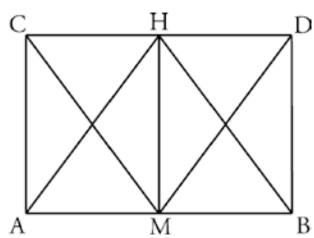

Fig. 2

Propositio III.

Si duae aequales rectae (Fig. 3) *AC, BD, perpendiculariter insistant cuivis rectae AB: Dico junctam CD aequalem fore, aut minorem, aut majorem ipsa AB, prout anguli ad eandem CD fuerint aut recti, aut obtusi, aut acuti.*

Demonstratur prima pars. Existente recto utroque angulo C, & D; sit, si fieri potest, alterutra ipsarum, ut DC, major altera BA. Sumatur in DC portio DK aequalis ipsi BA, jungaturque AK. Quoniam igitur super BD perpendiculariter insistunt aequales rectae BA, DK, aequales erunt (ex prima hujus) anguli BAK, DKA. Hoc autem absurdum est; cum angulus BAK sit ex constructione minor supposito recto BAC; & angulus DKA sit ex constructione externus, atque ideo (ex decimasexta primi) major interno, & opposito DCA, qui supponitur rectus. Non ergo alterutra praedictarum rectarum, DC, BA, est altera major, dum anguli ad junctam CD sint recti; ac propterea aequales invicem sunt. Quod erat primo
[3] loco demonstrandum.

Demonstratur secunda pars. Si autem obtusi fuerint anguli ad junctam CD, dividantur bifariam AB, & CD, in punctis M, & H, jungaturque MH. Quoniam ergo super recta MH perpendiculariter insistunt (ex praecedente) duae rectae AM, CH, poniturque ad junctam AC angulus rectus in A, non erit (ex prima hujus) recta CH aequalis ipsi AM, cum desit angulus rectus in C. Sed neque erit major: caeterum sumpta in HC portione KH aequali ipsi AM, aequales forent (ex prima hujus) anguli ad junctam AK. Hoc autem absurdum est, ut supra. Nam angulus MAK est minor recto; & angulus HKA est (ex decimasexta primi) major obtuso, qualis supponitur internus, & oppositus HCA. Restat igitur, ut CH, dum anguli ad junctam CD ponantur obtusi, minor sit ipsa AM; ac propterea prioris dupla CD minor sit posterioris dupla AB. Quod erat secundo loco demonstrandum.

Demonstratur tertia pars. Tandem vero, si acuti fuerint anguli ad junctam CD, ducta pariformiter (ex praecedente) perpendiculari MH, sic proceditur. Quoniam super recta MH perpendiculariter insistunt duae rectae AM, CH, poniturque ad junctam AC angulus rectus in A, non erit (ut supra) recta CH aequalis ipsi AM, cum desit angulus rectus in C. Sed neque erit minor: caeterum; si in HC protracta sumatur HL aequalis ipsi AM; aequales forent (ut supra) anguli ad junctam AL. Hoc autem absurdum est. Nam angulus MAL est ex constructione major supposito recto MAC; & angulus HLA est ex constructione internus, & oppositus, atque ideo minor (ex decimasexta primi) externo HCA, qui supponitur acutus. Restat igitur, ut CH, dum anguli ad junctam CD sint acuti, major sit ipsa AM, atque ideo prioris dupla CD major sit posterioris dupla AB. Quod erat tertio loco demonstrandum.

[4] Itaque constat junctam CD aequalem fore, aut minorem, aut majorem ipsa AB, prout anguli ad eandem CD fuerint aut recti, aut obtusi, aut acuti. Quae erant demonstranda.

Proposition 3.

If two equal straights (Fig. 3) *AC, BD, stand perpendicular to any straight AB: I say the join CD will be equal to, or less, or greater than, AB, according as the angles at CD are right, or obtuse, or acute.*

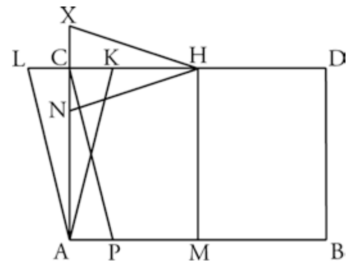

Fig. 3

Proof of the first part. Each angle C, and D, being right; suppose, if it were possible, either one of those, as DC, greater than the other BA. Take in DC the piece DK equal to BA, and join AK. Since therefore on BD stand perpendicular the equal straights BA, DK, the angles BAK, DKA will be equal (Proposition 1). But this is absurd; since the angle BAK is by construction less than the assumed right angle BAC; and the angle DKA is by construction external, and therefore (*Elements* I, 16) greater than the internal and opposite DCA, which is supposed right. Therefore neither of the aforesaid straights, DC, BA, is greater than the other, whilst the angles at the join CD are right; and therefore they are mutually equal. This is what was to be demonstrated in the first part. Proof of the second part. But if the angles at the join CD are obtuse, bisect AB, and CD, in the points M, and H, and join MH. Since therefore on the straight MH stand perpendicular (Proposition 2) the two straights AM, CH, and at the join AC is a right angle at A, the straight CH will not be (Proposition 1) equal to this AM, since a right angle is lacking at C. But neither will it be greater: otherwise in HC the piece KH being assumed equal to this AM, the angles at the join AK will be (Proposition 1) equal. But this is absurd, as above. For the angle MAK is less than a right; and the angle HKA is (*Elements* I, 16) greater than an obtuse, such as the internal and opposite HCA is supposed.[1] It remains therefore, that CH, whilst the angles at the join CD are taken obtuse, is less than this AM; and therefore CD double the former is less than AB double the latter. This is what was to be demonstrated in the second part.

Proof of the third part. Finally, however, if the angles at the join CD are acute, MH being constructed as before perpendicular (Proposition 2), we proceed thus. Since on the straight MH stand perpendicular two straights AM, CH, and at the join AC is a right angle at A, the straight CH will not be equal to this AM (as above), since the angle at C is not right. But neither will it be less: otherwise, if in HC produced HL is taken equal to this AM, the angles at the join AL will be (as above) equal. But this is absurd. For the angle MAL is by construction greater than the assumed right MAC; and the angle HLA is by construction internal, and opposite, and therefore less than (*Elements* I, 16) the external HCA, which is assumed acute. It remains therefore, that CH, whilst the angles at the join CD are acute, is greater than this AM, and therefore CD the double of the former is greater than AB the double of the latter. This is what was to be demonstrated in the third part.

Therefore it is established that the join CD will be equal to, or less, or greater than this AB, according as the angles at the same CD are right, or obtuse, or acute. This is what was to be demonstrated.

Corollarium I.

Hinc in omni quadrilatero continente tres quidem angulos rectos, & unum obtusum, aut acutum, latera adjacentia illi angulo non recto minora sunt, alterum altero, lateribus contrapositis, si ille angulus sit obtusus, majora autem, si sit acutus. Id enim demonstratum jam est de latere CH relate ad contrapositum latus AM; similique modo ostenditur de latere AC relate ad contrapositum latus MH. Cum enim rectae AC, MH, perpendiculares sint ipsi AM, nequeunt (ex prima hujus) esse invicem aequales, propter inaequales angulos ad junctam CH. Sed neque (in hypothesi anguli obtusi in C) potest quaedam AN, portio ipsius AC, aequalis esse ipsi MH, qua nimirum major sit praedicta AC: caeterum (ex eadem prima) aequales forent anguli ad junctam HN; quod est absurdum, ut supra. Rursum vero (in hypothesi anguli acuti in eo puncto C) si velis quandam AX, sumptam in AC protracta, aequalem ipsi MH, qua nimirum minor sit modo dicta AC; jam eodem titulo aequales erunt anguli ad HX; quod utique absurdum itidem est, ut supra. Restat igitur, ut in hypothesi quidem anguli obtusi in eo puncto C, latus AC minus sit contraposito latere MH; in hypothesi autem anguli acuti sit eodem majus. Quod erat intentum.

Corollarium II.

Multo autem magis erit CH major portione qualibet ipsius AM, ut puta PM, ad quam
[5] nempe juncta CP acutiorem adhuc angulum efficiat cum ipsa CH versus partes puncti H,
& obtusum (ex decimasexta primi) cum ea PM versus partes puncti M.

Corollarium III.

Rursum constat praedicta omnia aeque procedere, sive assumpta perpendicula AC, & BD, fuerint certae cujusdam apud nos longitudinis, sive sint, aut supponantur infinite parva. Quod quidem notari opportune debet in reliquis sequentibus Propositionibus.

Propositio IV.

Vicissim autem (manente figura praecedentis Propositionis) anguli ad junctam CD erunt aut recti, aut obtusi, aut acuti, prout recta CD aequalis fuerit, aut minor, aut major contraposita AB.

Demonstratur. Si enim recta CD aequalis sit contrapositae AB, & nihilominus anguli ad eandem sint aut obtusi, aut acuti; jam ipsi tales anguli eam probabunt (ex praecedente) non aequalem, sed minorem, aut majorem contraposita AB; quod est absurdum contra hypothesim. Idem uniformiter valet circa reliquos casus. Stat igitur angulos ad junctam CD esse aut rectos, aut obtusos, aut acutos, prout recta CD aequalis fuerit, aut minor, aut major contraposita AB. Quod erat demonstrandum.

Corollary 1.

Hence in every quadrilateral containing three right angles, and one obtuse, or acute,[2] the sides adjacent to this oblique angle are less respectively than the opposite sides if this angle is obtuse, but greater if it is acute. For this has just now been demonstrated of the side CH relatively to the opposite side AM; in the same way it is demonstrated of the side AC relatively to the opposite side MH. For since the straights AC, MH, are perpendicular to this AM, they cannot (Proposition 1) be mutually equal, on account of the unequal angles at the join CH. But neither (in the hypothesis of an obtuse angle at C) can a certain AN, a piece of this AC, be equal to this MH (of which certainly the aforesaid AC would then be greater): otherwise (Proposition 1) the angles at the join HN would be equal; which is absurd, as above. Again however (in the hypothesis of an acute angle at this point C), if you take a certain AX, assumed on AC produced, equal to this MH (of which certainly the just mentioned AC would be less,); now by this same title the angles at HX will be equal; which assuredly is absurd in the same way, as above. It remains therefore, that indeed in the hypothesis of an obtuse angle at this point C, the side AC is less than the opposite side MH; but in the hypothesis of an acute angle is greater than it. This is what we wanted.

Corollary 2.

But by much more will CH be greater than any piece of this AM, as for instance PM, since of course the join CP makes an angle still more acute with this CH toward the parts of the point H, and obtuse (*Elements* I, 16) with this PM toward the parts of the point M.

Corollary 3.

Again it abides that all things aforesaid equally result, whether the assumed perpendiculars AC, and BD are of some length fixed by us, are, or are supposed infinitesimal. This indeed ought opportunely to be noted in remaining subsequent Propositions.[3]

Proposition 4.

But inversely (the figure of the preceding Proposition remaining) the angles at the join CD will be right, or obtuse, or acute, according as the straight CD is equal, or less, or greater than the opposite AB.

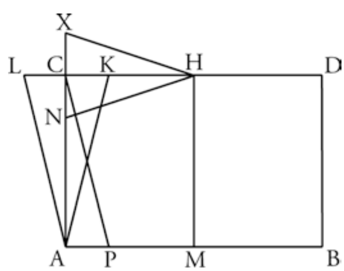

Fig. 3

Proof. For if the straight CD is equal to the opposite AB, and nevertheless the angles at it are either obtuse, or acute; now these such angles prove it (Proposition 3) not equal, but less, or greater than the opposite AB; which is absurd against the hypothesis. The same uniformly avails in regard to the remaining cases. It holds therefore that the angles at the join CD are either right, or obtuse, or acute, according as the straight CD is equal to, or less, or greater than the opposite AB. This is what was to be demonstrated.

Definitiones.

Quandoquidem (ex prima hujus) recta jungens extremitates aequalium perpendiculorum
[6] eidem rectae (quam vocabimus basim) insistentium, aequales efficit angulos cum ipsis perpendiculis; tres idcirco distinguendae sunt hypotheses circa speciem horum angulorum. Et primam quidem appellabo hypothesim anguli recti; secundam vero, & tertiam appellabo hypothesim anguli obtusi, & hypothesim anguli acuti.

Propositio V.

Hypothesis anguli recti, si vel in uno casu est vera, semper in omni casu illa sola est vera.

Demonstratur. Efficiat juncta CD (Fig. 4) angulos rectos cum duobus quibusvis aequalibus perpendiculis AC, BD, uni cuivis AB insistentibus. Erit CD (ex tertia hujus) aequalis ipsi AB. Sumantur in AC, & BD protractis duae CR, DX, aequales ipsis AC, BD; jungaturque RX. Facile ostendemus junctam RX aequalem fore ipsi AB, & angulos ad eandem rectos. Et primo quidem per superpositionem quadrilateri ABDC super quadrilaterum CDXR, adhibita communi basi CD. Deinde elegantius sic proceditur. Jungantur AD, RD. Constat (ex quarta primi) aequales fore in triangulis ACD, RCD, bases AD, RD, atque item angulos CDA, CDR, ac propterea aequales reliquos ad unum rectum, nimirum ADB, RDX. Quare rursum (ex eadem quarta primi) aequalis erit, in triangulis ADB, RDX, basis AB, basi RX. Igitur (ex praecedente) anguli ad junctam RX erunt recti, ac propterea persistemus in eadem hypothesi anguli recti.

Quoniam vero augeri similiter potest longitudo perpendiculorum in infinitum, sub eadem basi AB, consistente semper hypothesi anguli recti, demonstrandum est eandem hypothesim semper mansuram in casu cujusvis imminutionis eorundem perpendiculorum; quod quidem ita evincitur.

[7] Sumantur in AR, & BX duo quaelibet aequalia perpendicula AL, BK, jungaturque LK. Si anguli ad junctam LK recti non sint, erunt tamen (ex prima hujus) invicem aequales. Erunt igitur ex una parte, ut puta versus AB obtusi, & versus RX acuti, ut nimirum anguli hinc inde ad utrunque illorum punctorum aequales sint (ex decimatertia primi) duobus rectis. Constat autem aequalia etiam invicem esse perpendicula LR, KX, ipsi RX insistentia. Igitur (ex tertia hujus) erit LK major quidem contraposita RX, & minor contraposita AB.

Hoc autem absurdum est; cum AB, & RX ostensae sint aequales. Non ergo mutabitur hypothesis anguli recti sub quacunque imminutione perpendiculorum, dum consistat semel posita basis AB.

Sed neque immutabitur hypothesis anguli recti, sub quacunque imminutione, aut majori amplitudine basis; cum manifestum sit considerari posse ut basim quodvis perpendiculum BK, aut BX, atque ideo considerari vicissim ut perpendicula ipsam AB, & rectam aequalem contrapositam KL, aut XR.

Constat igitur hypothesim anguli recti, si vel in uno casu sit vera, semper in omni casu illam solam esse veram. Quod erat demonstrandum.

Definitions.

Since (Proposition 1) the straight joining the extremities of equal perpendiculars standing upon the same straight (which we call base), makes equal angles with these perpendiculars; therefore there are three hypotheses to be distinguished according to the species of these angles. And the first indeed I will call hypothesis of right angle: the second however, and the third I will call hypothesis of obtuse angle, and hypothesis of acute angle.

Proposition 5.

If even in a single case the hypothesis of right angle is true, always in every case it alone is true.

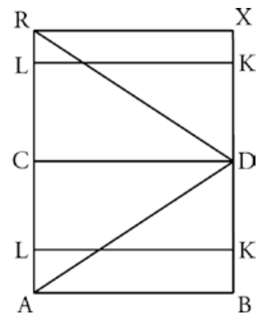

Fig. 4

Proof. Let the join CD (Fig. 4) make right angles with any two equal perpendiculars AC, BD, standing upon any straight AB. Then CD will be (Proposition 3) equal to this AB. Assume in AC, and BD produced two straights CR, DX, equal to these AC, BD; and join RX. We may easily show that the join RX will be equal to this AB, and the angles at it right. And first indeed by superposition[1] of the quadrilateral ABDC upon the quadrilateral CDXR, applied to the common base CD. Also we may proceed more elegantly thus. Join AD, RD. It follows (*Elements* I, 4) in the triangles ACD, RCD, the bases AD, RD will be equal and likewise the angles CDA, CDR, and certainly ADB, RDX because equal remainders from a right angle. Whereby in turn (*Elements* I, 4) in the triangles ADB, RDX, the base AB will be equal to the base RX. Therefore (Proposition 4) the angles at the join RX will be right, and so we abide in the same hypothesis of right angle.

Since now the length of the perpendiculars can be similarly increased infinitely, under the same base AB, the hypothesis of right angle always subsisting, it only remains to be proved that the same hypothesis will always abide in any case of diminution of those perpendiculars; which indeed is thus evinced.[2]

Assume in AR, and BX any two equal perpendiculars AL, BK, and join LK. If the angles at the join LK are not right, nevertheless (Proposition 1) they will be equal to each other. Therefore they will be toward one part, as suppose toward AB obtuse, and toward RX acute, since certainly the angles here at each of those points are (*Elements* I, 13) equal to two rights. But it also holds that the perpendiculars LR, KX, those standing upon RX, will be mutually equal. Therefore (Proposition 3) LK will be greater indeed than the opposite RX, and less than the opposite AB.

But this is absurd; because AB, and RX have been shown equal. Therefore the hypothesis of right angle is not changed by any diminution of the perpendiculars, whilst abides the once posited base AB.

But neither is the hypothesis of right angle changed for any diminution, or greater amplitude of the base; since manifestly may be considered as base any perpendicular BK, or BX, and therefore may be considered in turn as perpendiculars that AB, and the equal opposite sect KL, or XR.

Therefore is established that if even in a single case the hypothesis of right angle be true, always in every case it alone is true. This is what was to be demonstrated.

Propositio VI.

Hypothesis anguli obtusi, si vel in uno casu est vera, semper in omni casu illa sola est vera.

Demonstratur. Efficiat juncta CD (Fig. 5) angulos obtusos cum duobus quibusvis aequalibus perpendiculis AC, BD, uni cuivis rectae AB insistentibus. Erit CD (ex tertia hujus) minor ipsa AB. Sumantur in AC, BD protractis duae quaelibet invicem aequales portiones
[8] CR, DX; jungaturque RX. Jam quaero de angulis ad junctam RX, qui utique (ex prima hujus) aequales invicem erunt. Si obtusi sunt, habemus intentum. At recti non sunt; quia sic unum haberemus casum pro hypothesi anguli recti, qui nullum (ex praecedente) relinqueret locum pro hypothesi anguli obtusi. Sed neque acuti sunt. Nam sic esset RX (ex tertia hujus) major ipsa AB; ac propterea multo major ipsa CD. Hoc autem subsistere non posse sic ostenditur. Si quadrilaterum CDXR intelligatur impleri rectis abscindentibus ab ipsis CR, DX, portiones invicem aequales, implicat transiri a recta CD, quae minor est ipsa AB, ad RX eadem majorem, quin transeatur per quandam ST ipsi AB aequalem. Hoc autem absurdum esse in hac hypothesi ex eo constat; quia sic (ex quarta hujus) unus haberetur casus pro hypothesi anguli recti, qui nullum (ex praecedente) relinqueret locum hypothesi anguli obtusi. Igitur anguli ad junctam RX debent esse obtusi.

Deinde, sumptis in AC, BD, aequalibus portionibus AL, BK; simili modo ostendemus angulos ad junctam LK nequire esse acutos versus ipsam AB; quia sic illa foret major, quam AB, ac propterea multo major recta CD. Hinc autem reperiri deberet, ut supra, quaedam intermedia inter CD minorem, & LK majorem ipsa AB; intermedia, inquam, aequalis ipsi AB, quae utique, ex jam notis, omnem locum auferret hypothesi anguli obtusi. Tandem propter hanc ipsam causam recti esse nequeunt anguli ad junctam LK; ergo erunt obtusi. Igitur sub eadem basi A B, auctis, aut imminutis ad libitum perpendiculis, manebit semper hypothesis anguli obtusi.

Proposition 6.

If in even a single case the hypothesis of obtuse angle is true, always in every case it alone is true.

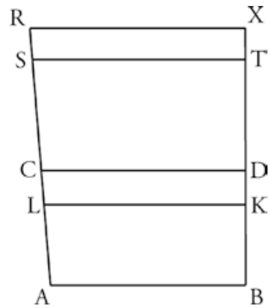

Fig. 5

Proof. Let the join CD (Fig. 5) make obtuse angles with any two equal perpendiculars AC, BD, standing upon any straight AB. Then CD will be (Proposition 3) less than this AB. Assume in AC and BD produced any two mutually equal portions CR and DX; and join RX. Now I investigate the angles at the join RX, which certainly (Proposition 1) will be mutually equal. If they are obtuse we have our assertion. But they are not right; because thus we would have a case for the hypothesis of right angle, which (Proposition 5) would leave no place for the hypothesis of obtuse angle. But neither are they acute. For thus RX would be (Proposition 3) greater than this AB; and still more therefore greater than CD itself. But that this cannot be is thus shown. If the quadrilateral CDXR is taken to be filled up by straights cutting off from these CR, DX, portions mutually equal, this implies transition from the straight CD, which is less than AB itself, to RX greater than it, verily transition[3] through a certain ST equal to this AB. But that this is absurd in the present hypothesis follows so; because thus (Proposition 4) we have a case for the hypothesis of right angle, which (Proposition 5) would leave no place for the hypothesis of obtuse angle. Therefore the angles at the join RX must be obtuse.

Then, equal portions AL, BK being assumed in AC, BD; in a similar manner we show the angles at the join LK cannot be acute toward this AB; because thus LK would be greater than AB, and still more therefore greater than the straight CD. But here would be found, as above, a certain intermediate between CD less, and LK greater than this AB; an intermediate, I say, equal to AB itself, which certainly, from what was just now observed, would take away every place for the hypothesis of obtuse angle. Finally from this very cause the angles at the join LK cannot be right; therefore they will be obtuse. Therefore with the same base AB, the perpendiculars being increased or diminished at will, the hypothesis of obtuse angle will always persist.

Springer

http://www.springer.com/978-3-319-05965-5

Euclid Vindicated from Every Blemish
Edited and Annotated by Vincenzo De Risi. Translated by G.B. Halsted and L. Allegri
Saccheri, G. - De Risi, V. (Ed.)
2014, VII, 381 p. 89 illus., 6 illus. in color., Hardcover
ISBN: 978-3-319-05965-5
A product of Birkhäuser Basel