Chapter 2
Preliminaries on Transition Functions and Their Invariant Probabilities

Our goal in this chapter is to introduce the transition functions and to discuss various related notions and known basic results that will be used throughout the book. Also in this chapter, we discuss several examples of transition functions which have a didactic purpose in the sense that we present these examples only to use them to illustrate the results of the book.

In Sect. 2.1, after going over the definition of a transition function, we define the family of Markov pairs generated by a transition function and discuss several basic properties of this family. We also present various types of transition functions. In a similar manner as in the case of transition probabilities (see Sect. 1.1) we define the orbits and the orbit-closures under the action of a transition function and study basic properties of these orbits and orbit-closures.

In Sect. 2.2, we discuss and list various examples of transition functions. The transition functions that we will consider here are of two kinds: transition functions defined by one-parameter semigroups or one-parameter groups of measurable functions, and transition functions defined by one-parameter convolution semigroups of probability measures. We should keep in mind that the transition functions discussed in this section are by no means the only important ones; the reason for our choice of examples is that they are better suited to illustrate the results of the book.

In the last section (Sect. 2.3), we discuss basic facts about invariant probabilities of transition functions.

2.1 Transition Functions

As pointed out in the abstract of this chapter, in this section we introduce and review basic properties of the transition functions and of certain concepts closely related to transition functions.

The transition functions under consideration in this book are often called homogeneous transition functions in probability theory and have their origin in the

R. Zaharopol, *Invariant Probabilities of Transition Functions*, Probability and Its Applications 44, DOI 10.1007/978-3-319-05723-1_2,

theory of continuous-time time-homogeneous Markov processes. These transition functions are discussed in virtually every book that deals with continuous-time Markov processes (see, for instance, Sections 36 and 42 of Bauer [8], Section A2 of Appendix in Beznea and Boboc [11], Chapter 1 of Blumenthal and Getoor [14], Section 18 of Borovkov [15], Section 1.2 of Chung and Walsh [18], Section 4.2 of Deuschel and Stroock [29], Chapters 2 and 3 of Vol. 1 of Dynkin [31], Chapter 4 of Ethier and Kurtz [35], the last three sections of Chapter 1 of Fukushima, Oshima and Takeda [36], Chapters 1 and 2 of Gihman and Skorohod [39], Section 2.4 of Hida [47], Section 1.4 of Mandl [69], Chapter 2 of Marcus and Rosen [71], Sections 9.3 and 10.3 of Meyer [76], Section 6.4 of Rao [88], Chapter 3 of Revuz and Yor [98], Section 3.1 of Rogers and Williams [100], Exercise 4.3.55 and Section 7.4 of Stroock [119], and Section 3.2 of Taira [124]). Later, it was noticed that, under very general conditions, one can associate a (homogeneous) transition function to any flow or semiflow. Thus, most results about transition functions are relevant to both the theory of continuous-time time-homogeneous Markov processes on one hand, and to ergodic theory and dynamical systems on the other (this fact is discussed in Section 8.4 of Dunford and Schwartz [30] and in Chapter 13 of Yosida [138]).

Throughout this section and the entire chapter we will use the notations established in Chap. 1.

Let (X, d) be a locally compact separable metric space, and let $\mathbb{T}$ stand for either the additive metric group $\mathbb{R}$ of all real numbers, where the metric $d_{\mathbb{R}}$ on $\mathbb{R}$ is the usual one defined in terms of the absolute value ($d_{\mathbb{R}}(s,t) = |s-t|$ for every $s \in \mathbb{R}$ and $t \in \mathbb{R}$), or else the additive metric semigroup $[0, +\infty)$, where the distance on $[0, +\infty)$ is the restriction of $d_{\mathbb{R}}$ to $[0, +\infty) \times [0, +\infty)$.

A family $(P_t)_{t\in\mathbb{T}}$ of transition probabilities defined on (X, d) is called a *transition function on* (X, d) if it has the property that

$$P_{s+t}(x, A) = \int_X P_s(y, A) P_t(x, \mathrm{d}y) \tag{2.1.1}$$

for every $s \in \mathbb{T}$, $t \in \mathbb{T}$, $x \in X$, and $A \in \mathcal{B}(X)$.

The reader has no doubt recognized the similarity between the above equality (2.1.1) and the equality that appears in (b) in Proposition 1.1.2. The equality (2.1.1) above is called the *Chapman-Kolmogorov equation for transition functions*, or the *continuous-time Chapman-Kolmogorov equation*, or, if there is no danger of confusion, simply, the *Chapman-Kolmogorov equation.*

The transition functions are also known as *Markov transition families in continuous time.*

Let $(P_t)_{t\in\mathbb{T}}$ be a transition function.

If $\mathbb{T} = \mathbb{R}$, and we want to emphasize that $\mathbb{T} = \mathbb{R}$, we will call $(P_t)_{t\in\mathbb{R}}$ an $\mathbb{R}$-transition function. Similarly, if $\mathbb{T} = [0, +\infty)$, we will sometimes call $(P_t)_{t\in[0,+\infty)}$ a $[0, +\infty)$-transition function.

Given an $\mathbb{R}$-transition function $(P_t)_{t\in\mathbb{R}}$, it is obvious that $(P_t)_{t\in[0,+\infty)}$ is also a transition function. We call $(P_t)_{t\in[0,+\infty)}$ the *restriction of* $(P_t)_{t\in\mathbb{R}}$ *to* $[0, +\infty)$.

The reader familiar only with the theory of continuous-time Markov processes may wonder why we define $\mathbb{R}$-transition functions when continuous-time Markov processes generate only $[0, +\infty)$-transition functions. The reason for allowing the time $\mathbb{T}$ to be the entire real line is that, as we will see in the next section, where we discuss examples of transition functions, the flows that appear in the study of dynamical systems generate in a very natural manner $\mathbb{R}$-transition functions, and we want the results of this book to be of use in both the theory of continuous time Markov processes and the theory of dynamical systems.

As before, let $(P_t)_{t\in\mathbb{T}}$ be a transition function.

As discussed in Sect. 1.1, for every $t \in \mathbb{T}$, the transition probability P_t generates a Markov pair (S_t, T_t) as follows: $S_t : B_b(X) \to B_b(X)$ is defined by

$$S_t f(x) = \int_X f(y) P_t(x, \mathrm{d}y)$$

for every $f \in B_b(X)$ and $x \in X$, and $T_t : \mathcal{M}(X) \to \mathcal{M}(X)$ is defined by

$$T_t \mu(A) = \int_X P_t(x, A) \mathrm{d}\mu(x)$$

for every $\mu \in \mathcal{M}(X)$ and $A \in \mathcal{B}(X)$. We say that $((S_t, T_t))_{t\in\mathbb{T}}$ is the *family of Markov pairs defined* (or *generated*) *by* $(P_t)_{t\in\mathbb{T}}$.

As in the case of transition functions, given a family $((S_t, T_t))_{t\in\mathbb{R}}$ of Markov pairs defined by a transition function $(P_t)_{t\in\mathbb{R}}$, we call $((S_t, T_t))_{t\in[0,+\infty)}$ the *restriction of* $((S_t, T_t))_{t\in\mathbb{R}}$ *to* $[0, +\infty)$. Clearly, the restriction $((S_t, T_t))_{t\in[0,+\infty)}$ of $((S_t, T_t))_{t\in\mathbb{R}}$ to $[0, +\infty)$ is the family of Markov pairs defined by the transition function $(P_t)_{t\in[0,+\infty)}$, which is the restriction of $(P_t)_{t\in\mathbb{R}}$ to $[0, +\infty)$.

Since the transition probability that defines a Markov pair is unique (see the comment made after Lemma 1.1.1), it follows that if the transition function $(P_t)_{t\in\mathbb{T}}$ defines a family $((S_t, T_t))_{t\in\mathbb{T}}$ of Markov pairs, then $(P_t)_{t\in\mathbb{T}}$ is the unique transition function with this property; that is, if $(P'_t)_{t\in\mathbb{T}}$ is another transition function that defines $((S_t, T_t))_{t\in\mathbb{T}}$, then $P_t = P'_t$ for every $t \in \mathbb{T}$.

In the next proposition we discuss an important property of the operators S_t, $t \in \mathbb{T}$, and T_t, $t \in \mathbb{T}$, that appear in the family of Markov pairs $((S_t, T_t))_{t\in\mathbb{T}}$ defined by a transition function.

Proposition 2.1.1. *Let* $(P_t)_{t\in\mathbb{T}}$ *be a transition function and let* $((S_t, T_t))_{t\in\mathbb{T}}$ *be the family of Markov pairs defined by* $(P_t)_{t\in\mathbb{T}}$. *Then* $S_{r+t} = S_r S_t$ *and* $T_{r+t} = T_r T_t$ *for every* $r \in \mathbb{T}$ *and* $t \in \mathbb{T}$; *that is, the families* $(S_t)_{t\in\mathbb{T}}$ *and* $(T_t)_{t\in\mathbb{T}}$ *are one-parameter semigroups of operators if* $\mathbb{T} = [0, +\infty)$, *and one-parameter groups of operators if* $\mathbb{T} = \mathbb{R}$.

Proof. We first prove that $S_{r+t} = S_r S_t$ for every $r \in \mathbb{T}$ and $t \in \mathbb{T}$.

To this end, let $r \in \mathbb{T}$ and $t \in \mathbb{T}$.

Using (i) of Lemma 1.1.1 and the Chapman-Kolmogorov equation we obtain that

$$\begin{aligned} S_{r+t}\mathbf{1}_A(x) &= P_{r+t}(x, A) = \int_X P_t(y, A)P_r(x, \mathrm{d}y) \\ &= \int_X S_t\mathbf{1}_A(y)P_r(x, \mathrm{d}y) = S_r S_t \mathbf{1}_A(x) \end{aligned}$$

for every $A \in \mathcal{B}(X)$ and $x \in X$. Accordingly, it follows that $S_{r+t} f = S_r S_t f$ for every real-valued simple measurable function f on X.

Now, if $f \in B_b(X)$ is not necessarily a simple function, then for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists a simple function $g \in B_b(X)$ such that $\|f - g\| < \frac{\varepsilon}{2}$. Taking into consideration that $S_{r+t} g = S_r S_t g$ and that S_u, $u \in \mathbb{T}$ are positive contractions of $B_b(X)$, we obtain that

$$\begin{aligned} \|S_{r+t} f - S_r S_t f\| &\le \|S_{r+t} f - S_{r+t} g\| + \|S_r S_t g - S_r S_t f\| \\ &= \|S_{r+t}(f - g)\| + \|S_r S_t(f - g)\| \\ &\le \|f - g\| + \|f - g\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

We have therefore proved that $\|S_{r+t} f - S_r S_t f\| < \varepsilon$ for every $f \in B_b(X)$ and $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Thus, $S_{r+t} = S_r S_t$.

We now prove that $T_{r+t} = T_r T_t$ for every $r \in \mathbb{T}$ and $t \in \mathbb{T}$.

Thus, let $r \in \mathbb{T}$ and $t \in \mathbb{T}$.

Using the equality (1.1.3) and the fact that $S_{r+t} = S_r S_t$, that we have just proved, we obtain that

$$\begin{aligned} T_{r+t}\mu(A) &= \langle \mathbf{1}_A, T_{r+t}\mu \rangle = \langle S_{r+t}\mathbf{1}_A, \mu \rangle = \langle S_r S_t \mathbf{1}_A, \mu \rangle \\ &= \langle \mathbf{1}_A, T_r T_t \mu \rangle = T_r T_t \mu(A) \end{aligned}$$

for every $\mu \in \mathcal{M}(X)$ and $A \in \mathcal{B}(X)$. □

Note that according to the definitions of one-parameter semigroups and one-parameter groups (see Sect. A.1) these one-parameter semigroups and one-parameter groups are semigroup and group homomorphisms from $[0, +\infty)$ and $\mathbb{R}$ to a semigroup H, respectively. However, in Proposition 2.1.1 we did not specify the codomain H for $(S_t)_{t \in \mathbb{T}}$ and $(T_t)_{t \in \mathbb{T}}$. We did so because any semigroup of linear operators from $B_b(X)$ to $B_b(X)$, where the algebraic operation that defines the semigroup structure is the composition of operators, can play the role of H for $(S_t)_{t \in \mathbb{T}}$ provided that each S_t belongs to the semigroup, $t \in \mathbb{T}$, and, similarly, any semigroup of linear operators from $\mathcal{M}(X)$ to $\mathcal{M}(X)$ which is a semigroup with respect to the composition of operators can be used as H if T_t belongs to the semigroup for every $t \in \mathbb{T}$.

So far, after defining the transition functions, we have considered the families of Markov pairs generated by these transition functions and we have discussed some of their properties. A natural question at this point is: given two families $(S_t)_{t\in\mathbb{T}}$ and $(T_t)_{t\in\mathbb{T}}$ of positive linear contractions of $B_b(X)$ and $\mathcal{M}(X)$, respectively, under what conditions does there exist a transition function $(P_t)_{t\in\mathbb{T}}$ such that (S_t, T_t) is the Markov pair defined by P_t for every $t \in \mathbb{T}$? In the next proposition we discuss such conditions. In the proposition and, unless stated explicitly otherwise, throughout the book, the one-parameter semigroups or groups of operators are semigroup or group homomorphisms from $[0, +\infty)$ or $\mathbb{R}$, respectively, to a semigroup H of operators, where the semigroup structure of H is defined by the composition of operators.

Proposition 2.1.2. *Assume that $(S_t)_{t\in\mathbb{T}}$, $S_t : B_b(X) \to B_b(X)$ for every $t \in \mathbb{T}$, is a one-parameter semigroup or group of positive contractions of $B_b(X)$ if $\mathbb{T} = [0, +\infty)$ or $\mathbb{T} = \mathbb{R}$, respectively, and assume that $S_t\mathbf{1}_X = \mathbf{1}_X$ for every $t \in \mathbb{T}$. Also, let $(T_t)_{t\in\mathbb{T}}$, $T_t : \mathcal{M}(X) \to \mathcal{M}(X)$ for every $t \in \mathbb{T}$, be a one-parameter semigroup or group of positive contractions if $\mathbb{T} = [0, +\infty)$ or $\mathbb{T} = \mathbb{R}$, respectively, and assume that $\langle S_t f, \mu\rangle = \langle f, T_t\mu\rangle$ for every $f \in B_b(X)$, $\mu \in \mathcal{M}(X)$, and $t \in \mathbb{T}$. Then there exists a unique transition function $(P_t)_{t\in\mathbb{T}}$ such that $((S_t, T_t))_{t\in\mathbb{T}}$ is the family of Markov pairs defined by $(P_t)_{t\in\mathbb{T}}$.*

Proof. First note that, under the conditions of the proposition, for every $t \in \mathbb{T}$, the positive contraction T_t is a Markov operator because $T_t\mu(X) = \langle \mathbf{1}_X, T_t\mu\rangle = \langle S_t\mathbf{1}_X, \mu\rangle = \mu(X)$ for every $\mu \in \mathcal{M}(X)$, so $\|T_t\mu\| = \|\mu\|$ for every $\mu \in \mathcal{M}(X)$, $\mu \geq 0$.

For every $t \in \mathbb{T}$, let $P_t : X \times \mathcal{B}(X) \to \mathbb{R}$ be defined by $P_t(x, A) = S_t\mathbf{1}_A(x)$ for every $x \in X$ and $A \in \mathcal{B}(X)$.

Taking into consideration that the unicity of a transition function which has the property that $((S_t, T_t))_{t\in\mathbb{T}}$ is the family of Markov pairs defined by the transition function follows from the discussion that precedes Proposition 2.1.1, we obtain that in order to prove the proposition, we have to prove that the following three assertions are true:

(a) P_t is a transition probability for every $t \in \mathbb{T}$.
(b) (S_t, T_t) is the Markov pair defined by P_t for every $t \in \mathbb{T}$.
(c) The Chapman-Kolmogorov equation for transition functions holds true for $(P_t)_{t\in\mathbb{T}}$.

Proof of (a). Let $t \in \mathbb{T}$.

For every $A \in \mathcal{B}(X)$, it follows that $P_t(x, A) = S_t\mathbf{1}_A(x)$ for every $x \in X$; since $S_t\mathbf{1}_A$ belongs to $B_b(X)$, we obtain that the map $x \mapsto P_t(x, A)$ from X to $\mathbb{R}$ is measurable.

Let $x \in X$. Since T_t is a Markov operator, it follows that $T_t\delta_x$ is a probability measure. Since $P_t(x, A) = \langle S_t\mathbf{1}_A, \delta_x\rangle = \langle \mathbf{1}_A, T_t\delta_x\rangle = T_t\delta_x(A)$ for every $A \in \mathcal{B}(X)$, we obtain that the map $A \mapsto P_t(x, A)$ from $\mathcal{B}(X)$ to $\mathbb{R}$ is a probability measure.

Thus, P_t is a transition probability.

Proof of (b). Let $t \in \mathbb{T}$.

Since $S_t \mathbf{1}_A(x) = \int_X \mathbf{1}_A(y) P_t(x, \mathrm{d}y)$ for every $A \in \mathcal{B}(X)$ and $x \in X$, it follows that $S_t f(x) = \int_X f(y) P_t(x, \mathrm{d}y)$ for every simple measurable real-valued function f and $x \in X$. Using the fact that S_t is a linear contraction of $B_b(X)$, taking into consideration that the set of all measurable real-valued simple functions defined on X is dense in $B_b(X)$, and using the Lebesgue dominated convergence theorem, we obtain that $S_t f(x) = \int_X f(y) P_t(x, \mathrm{d}y)$ for every $f \in B_b(X)$ and $x \in X$. Finally, since

$$T_t \mu(A) = \langle S_t \mathbf{1}_A, \mu \rangle = \int_X S_t \mathbf{1}_A(x) \mathrm{d}\mu(x) = \int_X P_t(x, A) \mathrm{d}\mu(x)$$

for every $\mu \in \mathcal{M}(X)$ and $A \in \mathcal{B}(X)$, it follows that (S_t, T_t) is the Markov pair defined by P_t.

Proof of (c). Taking into consideration that for every $x \in X$ and $t \in \mathbb{T}$ the probability measures $T_t \delta_x$ and $\mu_x^{(t)}$ are equal, where $\mu_x^{(t)}$ is defined by $\mu_x^{(t)}(A) = P_t(x, A)$ for every $A \in \mathcal{B}(X)$, we obtain that

$$\begin{aligned} P_{r+t}\delta_x(A) &= T_{r+t}\delta_x(A) = \langle \mathbf{1}_A, T_r T_t \delta_x \rangle \\ &= \langle S_r \mathbf{1}_A, T_t \delta_x \rangle = \int_X S_r \mathbf{1}_A(y) \mathrm{d}T_t \delta_x(y) \\ &= \int_X P_r(y, A) P_t(x, \mathrm{d}y) \end{aligned}$$

for every $r \in \mathbb{T}$, $t \in \mathbb{T}$, $x \in X$, and $A \in \mathcal{B}(X)$.

Thus, $(P_t)_{t \in \mathbb{T}}$ is a transition function. □

Let $(P_t)_{t \in \mathbb{T}}$ be a transition function defined on (X, d).

We say that $(P_t)_{t \in \mathbb{T}}$ satisfies the *standard measurability assumption* (*s.m.a.*) if, for every $A \in \mathcal{B}(X)$, the map $(t, x) \mapsto P_t(x, A)$, $(t, x) \in \mathbb{T} \times X$, is jointly measurable with respect to t and x; that is, the map is measurable with respect to the Borel σ-algebra on $\mathbb{R}$ and the product σ-algebra $\mathcal{L}(\mathbb{T}) \otimes \mathcal{B}(X)$, where $\mathcal{L}(\mathbb{T})$ is the σ-algebra of all Lebesgue measurable subsets of $\mathbb{T}$.

The s.m.a. is a rather common assumption, so common that it is sometimes incorporated in the definition of a transition function (see, for instance, p. 156 of Ethier and Kurtz [35]). We will use it frequently.

Our goal now is to discuss several useful reformulations of the s.m.a. To this end, we need the following two lemmas:

Lemma 2.1.3. *Let $\mathcal{A}$ be a collection of subsets of X, and assume that $\mathcal{A}$ satisfies the following three conditions:*

(a) $\mathcal{A}$ is closed under the formation of finite disjoint unions; that is, for every $n \in \mathbb{N}$ and n disjoint subsets $A_1, A_2, \cdots, A_n$ of X such that $A_i \in \mathcal{A}$ for every $i = 1, 2, \cdots, n$, it follows that $\cup_{j=1}^n A_j \in \mathcal{A}$.

(b) $\mathcal{A}$ *is closed under the formation of proper differences; that is, if* $A \in \mathcal{A}$ *and* $B \in \mathcal{A}$*, and* $A \subseteq B$*, then* $B \setminus A \in \mathcal{A}$*.*
(c) $\mathcal{A}$ *is a monotone class; that is, if* $(A_n)_{n\in\mathbb{N}}$ *is a sequence of elements of* $\mathcal{A}$ *and if* $(A_n)_{n\in\mathbb{N}}$ *is monotone (the fact that* $(A_n)_{n\in\mathbb{N}}$ *is monotone means that* $(A_n)_{n\in\mathbb{N}}$ *is either increasing (*$A_n \subseteq A_{n+1}$ *for every* $n \in \mathbb{N}$*), or else decreasing (*$A_n \supseteq A_{n+1}$ *for every* $n \in \mathbb{N}$*)), then* $\lim_{n\to+\infty} A_n \in \mathcal{A}$*, where* $\lim_{n\to+\infty} A_n = \cup_{n=1}^{\infty} A_n$ *if* $(A_n)_{n\in\mathbb{N}}$ *is increasing and* $\lim_{n\to+\infty} A_n = \cap_{n=1}^{\infty} A_n$ *if* $(A_n)_{n\in\mathbb{N}}$ *is decreasing.*

If the compact subsets of X *belong to* $\mathcal{A}$*, then* $\mathcal{B}(X) \subseteq \mathcal{A}$*.*

For a proof of the lemma, see Lemma 2.1 of [146].

Lemma 2.1.4. *Let* G *be an open subset of* X*. Then there exists an increasing sequence* $(f_n)_{n\in\mathbb{N}}$ *of elements of* $C_0(X)$ *such that* $f_n \geq 0$ *for every* $n \in \mathbb{N}$ *and such that the sequence of functions* $(f_n)_{n\in\mathbb{N}}$ *converges pointwise to* $\mathbf{1}_G$ *(that is, for every* $x \in X$*, the sequence of real numbers* $(f_n(x))_{n\in\mathbb{N}}$ *converges to 1 if* $x \in G$*, and to 0 if* $x \notin G$*).*

Proof. Let G be an open subset of X. Since the assertion of the lemma is obviously true if G is the empty set, we may and do assume that G is nonempty. The restriction d_G of the metric d to $G \times G$ is a metric on G, and (G, d_G) is a locally compact separable metric space in its own right. Thus, using Proposition 1.1.3 of [143] we obtain that (G, d_G) is σ-compact, so there exists an increasing sequence $(K_n)_{n\in\mathbb{N}}$ of compact subsets of G (the sets K_n, $n \in \mathbb{N}$, are compact in both the topology defined by the metric d_G on G and the topology defined by d on X) such that $G = \cup_{n=1}^{\infty} K_n$.

Since K_n, $n \in \mathbb{N}$, are compact subsets of X in the topology defined by d on X, we can use Proposition 7.1.8, p. 199, of Cohn's book [20] in order to obtain that there exists a $g_n \in C_0(X)$ (actually, the function g_n can be chosen with compact support) such that $\mathbf{1}_{K_n} \leq g_n \leq \mathbf{1}_G$ for every $n \in \mathbb{N}$.

Set $f_1 = g_1$, and $f_n = \sup_{1\leq i\leq n} g_i$ for every $n \in \mathbb{N}$, $n \geq 2$. Then $f_n \in C_0(X)$ for every $n \in \mathbb{N}$, and the sequence $(f_n)_{n\in\mathbb{N}}$ is increasing. Moreover, $(f_n)_{n\in\mathbb{N}}$ converges pointwise to $\mathbf{1}_G$ because if $x \in X \setminus G$, then $f_n(x) = 0$ for every $n \in \mathbb{N}$, and if $x \in G$, then there exists an $n_x \in \mathbb{N}$ such that $x \in K_n$ for every $n \in \mathbb{N}$, $n \geq n_x$, so $f_n(x) = 1$ for every $n \in \mathbb{N}$, $n \geq n_x$. □

In the next proposition we discuss the reformulations of the s.m.a. that we mentioned before Lemma 2.1.3.

Proposition 2.1.5. *Let* $(P_t)_{t\in\mathbb{T}}$ *be a transition function defined on* (X, d) *and let* $((S_t, T_t))_{t\in\mathbb{T}}$ *be the family of Markov pairs defined by* $(P_t)_{t\in\mathbb{T}}$*. The following assertions are equivalent:*

(a) *For every* $f \in C_0(X)$*, the real-valued map* $(t, x) \mapsto S_t f(x)$ *for every* $(t, x) \in \mathbb{T} \times X$ *is jointly measurable with respect to* t *and* x*.*
(b) $(P_t)_{t\in\mathbb{T}}$ *satisfies the s.m.a.*
(c) *For every* $f \in B_b(X)$*, the real-valued map* $(t, x) \mapsto S_t f(x)$ *for every* $(t, x) \in \mathbb{T} \times X$ *is jointly measurable with respect to* t *and* x*.*

(*d*) *For every* $f \in C_b(X)$, *the real-valued map* $(t,x) \mapsto S_t f(x)$ *for every* $(t,x) \in \mathbb{T} \times X$ *is jointly measurable with respect to* t *and* x.

Proof. $(a) \Rightarrow (b)$. Set

$$\mathcal{A} = \left\{ A \in \mathcal{B}(X) \;\middle|\; \begin{array}{l} \text{the real-valued map } (t,x) \mapsto P_t(x,A) \text{ for} \\ \text{every } (t,x) \in \mathbb{T} \times X \text{ is measurable with respect} \\ \text{to the Borel } \sigma\text{-algebra on } \mathbb{R} \text{ and } \mathcal{L}(\mathbb{T}) \otimes \mathcal{B}(X) \end{array} \right\}.$$

The proof of the implication will be completed if we show that $\mathcal{A} = \mathcal{B}(X)$. To this end, we will use Lemma 2.1.3.

Note that $\mathcal{A}$ satisfies conditions (a), (b) and (c) of Lemma 2.1.3.

We now show that the compact subsets of X belong to $\mathcal{A}$.

For every $f \in B_b(X)$, let $\psi_f : \mathbb{T} \times X \to \mathbb{R}$ be defined by $\psi_f(t,x) = S_t f(x)$ for every $(t,x) \in \mathbb{T} \times X$.

Since $\psi_{\mathbf{1}_A}(t,x) = S_t \mathbf{1}_A(x) = P_t(x,A)$ and $\psi_{\mathbf{1}_{X \setminus A}}(t,x) = 1 - P_t(x,A)$ for every $A \in \mathcal{B}(X)$ and every $(t,x) \in \mathbb{T} \times X$, it follows that if $A \in \mathcal{A}$, then $X \setminus A$ belongs to $\mathcal{A}$, as well. Therefore, taking into consideration that every compact subset of X is closed, we infer that in order to prove that the compact subsets of X belong to $\mathcal{A}$, it is enough to show that the open subsets of X belong to $\mathcal{A}$.

Thus, let G be an open subset of X. Since the empty set $\emptyset$ belongs to $\mathcal{A}$ (because $P_t(x,\emptyset) = 0$ for every $(t,x) \in \mathbb{T} \times X$), we may and do assume that $G \neq \emptyset$.

By Lemma 2.1.4 there exists an increasing sequence $(f_n)_{n\in\mathbb{N}}$ of elements of $C_0(X)$ that converges pointwise to $\mathbf{1}_G$ and such that $f_n \geq 0$ for every $n \in \mathbb{N}$.

Since we assume that (a) is true, it follows that ψ_{f_n} is jointly measurable with respect to t and x for every $n \in \mathbb{N}$.

Since $\psi_{f_n}(t,x) = S_t f_n(x) = \langle S_t f_n, \delta_x \rangle = \langle f_n, T_t \delta_x \rangle = \int_X f_n(y)\, \mathrm{d}T_t\delta_x(y)$ for every $n \in \mathbb{N}$, and since, by the monotone convergence theorem, the sequence $(\int_X f_n(y)\, \mathrm{d}T_t\delta_x(y))_{n\in\mathbb{N}}$ converges to $\int_X \mathbf{1}_G(y)\, \mathrm{d}T_t\delta_x(y)$, it follows that $(\psi_{f_n}(t,x))_{n\in\mathbb{N}}$ converges to $\psi_{\mathbf{1}_G}(t,x)$ for every $(t,x) \in \mathbb{T} \times X$.

Thus, $\psi_{\mathbf{1}_G}$ is jointly measurable with respect to t and x. Since $\psi_{\mathbf{1}_G}(t,x) = S_t \mathbf{1}_G(x) = P_t(x,G)$ for every $(t,x) \in \mathbb{T} \times X$, it follows that $G \in \mathcal{A}$.

Since $\mathcal{A}$ satisfies all the conditions of Lemma 2.1.3, using the lemma we obtain that (b) holds true.

$(b) \Rightarrow (c)$. Assume that (b) is true.

In terms of the functions ψ_f, $f \in B_b(X)$, the fact that $(P_t)_{t\in\mathbb{T}}$ satisfies the s.m.a. means that the function $\psi_{\mathbf{1}_A}$ is jointly measurable with respect to t and x for every $A \in \mathcal{B}(X)$. It is easy to see that, in this case ψ_f is jointly measurable with respect to t and x whenever f is a simple measurable real-valued function on X.

Now, if $f \in B_b(X)$ is not necessarily simple, then there exists a sequence $(f_n)_{n\in\mathbb{N}}$ of simple functions, $f_n \in B_b(X)$ for every $n \in \mathbb{N}$, such that $(f_n)_{n\in\mathbb{N}}$ converges uniformly to f.

Since S_t, $t \in \mathbb{T}$, are linear contractions of $B_b(X)$, and since $\psi_g(t,x) = S_t g(x)$ for every $g \in B_b(X)$ and every $(t,x) \in \mathbb{T} \times X$, it follows that $(\psi_{f_n})_{n\in\mathbb{N}}$ converges pointwise (on $\mathbb{T} \times X$) to ψ_f. Since, for every $n \in \mathbb{N}$, ψ_{f_n} is jointly measurable with respect to t and x, we obtain that ψ_f is jointly measurable with respect to t and x, as well.

$(c) \Rightarrow (a)$ is obvious.

$(d) \Leftrightarrow (a)$. The equivalence of (a) and (d) is obtained using the fact that (a) and (c) are equivalent, which we just proved, and the inclusions $C_0(X) \subseteq C_b(X) \subseteq B_b(X)$, which are obvious. □

Proposition 2.1.5 has the following consequence:

Corollary 2.1.6. *Let $(P_t)_{t\in\mathbb{T}}$ be a transition function defined on (X, d), assume that $(P_t)_{t\in\mathbb{T}}$ satisfies the s.m.a., and let $((S_t, T_t))_{t\in\mathbb{T}}$ be the family of Markov pairs defined by $(P_t)_{t\in\mathbb{T}}$. Then, for every $x \in X$, and $f \in B_b(X)$, the function $\psi_f^{(x)} : \mathbb{T} \to \mathbb{R}$ defined by $\psi_f^{(x)}(t) = S_t f(x)$ for every $t \in \mathbb{T}$ is measurable with respect to the Borel σ-algebra on $\mathbb{R}$ and the σ-algebra $\mathcal{L}(\mathbb{T})$ of all Lebesgue measurable subsets of $\mathbb{T}$.*

Proof. Since $(P_t)_{t\in\mathbb{T}}$ satisfies the s.m.a. we can use $(b) \Rightarrow (c)$ of Proposition 2.1.5 in order to conclude that the functions ψ_f, $f \in B_b(X)$, are jointly measurable with respect to t and x. Taking into consideration that $\psi_f^{(x)}(t) = \psi_f(t, x)$ for every $(t, x) \in \mathbb{T} \times X$ and $f \in B_b(X)$ and using a well-known fact from measure theory (see, for instance, Lemma 5.1.1, p. 155, of Cohn [20]) we obtain that the assertion of the corollary is true. □

Let $(P_t)_{t\in\mathbb{T}}$ be a transition function defined on (X, d), and let $((S_t, T_t))_{t\in\mathbb{T}}$ be the family of Markov pairs defined by $(P_t)_{t\in\mathbb{T}}$.

We say that $(P_t)_{t\in\mathbb{T}}$ (or $((S_t, T_t))_{t\in\mathbb{T}}$, or $(S_t)_{t\in\mathbb{T}}$) is $C_0(X)$*-pointwise continuous* (or, simply, *pointwise continuous*) if, for every $f \in C_0(X)$ and $x \in X$, the real-valued function $t \mapsto S_t f(x)$, $t \in \mathbb{T}$, is continuous. Naturally, if X is compact, we might use also the term $C(X)$*-pointwise continuous.*

Although somewhat unusual, pointwise continuity is a surprisingly weak condition. It is satisfied by most transition functions associated to continuous-time Markov processes (for instance, all the transition functions associated with the Markov processes defined by the interacting particle systems discussed in Liggett's monographs [65] and [66] are pointwise continuous), by most transition functions defined by flows and semiflows, and by the transition functions defined by the one-parameter convolution semigroups of probability measures discussed in this book. We will have to impose this condition in order to obtain various results in the last three chapters in this book.

We say that $(P_t)_{t\in\mathbb{T}}$ is a *Feller transition function* if P_t is a Feller transition probability for every $t \in \mathbb{T}$. Thus, $(P_t)_{t\in\mathbb{T}}$ is a Feller transition function if $S_t f \in C_b(X)$ for every $f \in C_b(X)$ and $t \in \mathbb{T}$. Naturally, if $(P_t)_{t\in\mathbb{T}}$ is a Feller transition function, then $((S_t, T_t))_{t\in\mathbb{T}}$ is said to be the *family of Markov-Feller pairs defined by* $(P_t)_{t\in\mathbb{T}}$. The Feller transition functions have nice properties and we will discuss these transition functions in-depth in Chap. 6.

An interesting family of Feller transition functions are the $C_0(X)$-equicontinuous (or, simply, equicontinuous) transition functions. The transition function $(P_t)_{t\in\mathbb{T}}$ (or $((S_t, T_t))_{t\in\mathbb{T}}$ or $(S_t)_{t\in\mathbb{T}}$) is said to be $C_0(X)$-*equicontinuous* (or *equicontinuous*) if the set of functions $\{S_t f | t \in \mathbb{T}\}$ is equicontinuous for every $f \in C_0(X)$; thus, $(P_t)_{t\in\mathbb{T}}$ is equicontinuous if and only if for every $f \in C_0(X)$, for every convergent sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X, and for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists an $n_\varepsilon \in \mathbb{N}$ such that $|S_t f(x_n) - S_t f(x)| < \varepsilon$ for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$, and every $t \in \mathbb{T}$, where $x = \lim_{n\to\infty} x_n$. If the transition probability $(P_t)_{t\in\mathbb{T}}$ is equicontinuous, then, for every $f \in C_0(X)$ and $t \in \mathbb{T}$, the function $S_t f$ is continuous and bounded; therefore, using Proposition 1.1.5, we obtain that $(P_t)_{t\in\mathbb{T}}$ is a Feller transition function. We will discuss various properties of equicontinuous transition functions in the last two sections of Chap. 6.

Let $x \in X$. Set $\mathcal{O}(x) = \bigcup_{t\in\mathbb{T}} \operatorname{supp}(T_t\delta_x)$ and $\mathcal{O}^{(F)}(x) = \bigcup_{\substack{t\in\mathbb{T}\\ t\geq 0}} \operatorname{supp}(T_t\delta_x)$. The sets $\mathcal{O}(x)$ and $\mathcal{O}^{(F)}(x)$ are called the *orbit* and the *forward orbit of* x *under the action of* $(T_t)_{t\in\mathbb{T}}$ (or $(P_t)_{t\in\mathbb{T}}$ or $((S_t, T_t))_{t\in\mathbb{T}}$), respectively. The closures $\overline{\mathcal{O}(x)}$ and $\overline{\mathcal{O}^{(F)}(x)}$ of $\mathcal{O}(x)$ and $\mathcal{O}^{(F)}(x)$ in the topology defined by the metric d on X are called the *orbit-closure* and the *forward orbit-closure of* x *under the action of* $(T_t)_{t\in\mathbb{T}}$ (or $(P_t)_{t\in\mathbb{T}}$ or $((S_t, T_t))_{t\in\mathbb{T}}$), respectively. For a subset A of X, set $\mathcal{O}(A) = \bigcup_{x\in A} \mathcal{O}(x)$ and $\mathcal{O}^{(F)}(A) = \bigcup_{x\in A} \mathcal{O}^{(F)}(x)$. The sets $\mathcal{O}(A)$, $\overline{\mathcal{O}(A)}$, $\mathcal{O}^{(F)}(A)$, and $\overline{\mathcal{O}^{(F)}(A)}$ are called the *orbit*, the *orbit-closure*, the *forward orbit*, and the *forward orbit-closure of* A *under the action of* $(T_t)_{t\in\mathbb{T}}$ (or $(P_t)_{t\in\mathbb{T}}$, or $((S_t, T_t))_{t\in\mathbb{T}}$), respectively. Note that, if $\mathbb{T} = [0, +\infty)$, then the orbits and orbit-closures are forward orbits and forward orbit-closures, respectively, and vice versa. Thus, the study of forward orbits and forward orbit-closures is of interest only when $\mathbb{T} = \mathbb{R}$.

The transition function $(P_t)_{t\in\mathbb{T}}$ (or $((S_t, T_t))_{t\in\mathbb{T}}$ or $(T_t)_{t\in\mathbb{T}}$) is said to be *minimal* or *forward minimal* if, for every $x \in X$, the orbit or the forward orbit of x under the action of $(P_t)_{t\in\mathbb{T}}$ is dense in X, respectively. Clearly, the study of the forward minimality of $(P_t)_{t\in\mathbb{T}}$ is of interest only when $\mathbb{T} = \mathbb{R}$. Also obvious is the fact that the forward minimality of $(P_t)_{t\in\mathbb{T}}$ always implies the minimality of $(P_t)_{t\in\mathbb{T}}$; however, (if $\mathbb{T} = \mathbb{R}$) the minimality of a transition function does not imply its forward minimality in general as we will show using a very simple example in the next section.

It is often of interest to know if the orbit of an element x of X under the action of the transition function $(P_t)_{t\in\mathbb{T}}$ is dense in X. For instance, if we know that the orbit of every $x \in X$ is dense in X, then we know that $(P_t)_{t\in\mathbb{T}}$ is a minimal transition function. In the next proposition we discuss a necessary and sufficient condition for an orbit to be dense in X.

In order to state the proposition, recall that if $(P_t)_{t\in\mathbb{T}}$ satisfies the s.m.a., then using Corollary 2.1.6, we obtain that, given $f \in B_b(X)$ and $x \in X$, the map $t \mapsto S_t f(x)$, $t \in \mathbb{T}$, is measurable (with respect to $\mathcal{L}(\mathbb{T})$ and the Borel σ-algebra on $\mathbb{R}$); if, in addition, $f \geq 0$, then the map is also positive, so the Lebesgue integral $\int_{\mathbb{T}} S_t f(x)\mathrm{d}t$ exists (note that the integral could be equal to $+\infty$); in particular, if $f = \mathbf{1}_A$ for some $A \in \mathcal{B}(X)$, the integral $\int_{\mathbb{T}} P_t(x, A)\mathrm{d}t$ exists.

Proposition 2.1.7. *Assume that the transition function $(P_t)_{t\in\mathbb{T}}$ satisfies the s.m.a. and is pointwise continuous, and let $x \in X$. The following assertions are equivalent:*

(a) *The orbit $\mathcal{O}(x)$ of x under the action of $(P_t)_{t\in\mathbb{T}}$ is dense in X.*
(b) *$\int_{\mathbb{T}} P_t(x, U)\,\mathrm{d}t > 0$ for every nonempty subset U of X.*

Proof. $(a) \Rightarrow (b)$ Assume that $\overline{\mathcal{O}(x)} = X$, and let U be a nonempty open subset of X. Then $U \cap \mathcal{O}(x) \neq \emptyset$, so there exists a $t_0 \in \mathbb{T}$ such that $U \cap (\operatorname{supp}(T_{t_0}\delta_x)) \neq \emptyset$. Let $y \in U \cap (\operatorname{supp}(T_{t_0}\delta_x))$.

Using Proposition 7.1.8 of Cohn's book [20], applied to the compact set $\{y\}$ and to U, we obtain that there exists an $f \in C_0(X)$ such that $f(y) = 1$, $0 \leq f \leq \mathbf{1}_U$, and such that the support $\operatorname{supp} f$ of f is included in U.

Set $U_{\frac{1}{2}} = \left\{ z \in X \mid f(z) > \frac{1}{2} \right\}$. Then $U_{\frac{1}{2}}$ is an open subset of X. Also, $U_{\frac{1}{2}} \cap (\operatorname{supp}(T_{t_0}\delta_x)) \neq \emptyset$ because $y \in U_{\frac{1}{2}} \cap (\operatorname{supp}(T_{t_0}\delta_x))$. Thus, $T_{t_0}\delta_x(U_{\frac{1}{2}}) > 0$.

Taking into consideration that $\frac{1}{2}\mathbf{1}_{U_{\frac{1}{2}}} \leq f$, we obtain that

$$0 < \frac{1}{2}(T_{t_0}\delta_x)(U_{\frac{1}{2}}) = \left\langle \frac{1}{2}\mathbf{1}_{U_{\frac{1}{2}}}, T_{t_0}\delta_x \right\rangle \leq \langle f, T_{t_0}\delta_x \rangle$$
$$= \langle S_{t_0} f, \delta_x \rangle = S_{t_0} f(x).$$

Since $(P_t)_{t\in\mathbb{T}}$ is pointwise continuous and $S_{t_0} f(x) > 0$, it follows that there exists an $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, such that $[t_0, t_0 + \varepsilon) \subseteq \mathbb{T}$ and $S_t f(x) > 0$ for every $t \in [t_0, t_0 + \varepsilon)$.

Taking into consideration that $0 \leq f \leq \mathbf{1}_U$ and that S_t, $t \in \mathbb{T}$, are positive operators, we obtain that

$$0 < \int_{t_0}^{t_0+\varepsilon} S_t f(x)\,\mathrm{d}t \leq \int_{t_0}^{t_0+\varepsilon} S_t \mathbf{1}_U(x)\,\mathrm{d}t \leq \int_{\mathbb{T}} S_t \mathbf{1}_U(x)\,\mathrm{d}t \leq \int_{\mathbb{T}} P_t(x, U)\,\mathrm{d}t.$$

$(b) \Rightarrow (a)$ Assume that (b) is true, but $\mathcal{O}(x)$ is not dense in X.

Set $U = X \setminus \overline{\mathcal{O}(x)}$. Clearly, U is a nonempty open subset of X. Since $U \cap \mathcal{O}(x) = \emptyset$, it follows that $U \cap (\operatorname{supp} T_t\delta_x) = \emptyset$ for every $t \in \mathbb{T}$. Therefore, $P_t(x, U) = T_t\delta_x(U) = 0$ for every $t \in \mathbb{T}$, so $\int_{\mathbb{T}} P_t(x, U)\,\mathrm{d}t = 0$.

We have obtained a contradiction which stems from our assumption that $\overline{\mathcal{O}(x)} \neq X$. □

The above proposition has the following obvious consequence:

Corollary 2.1.8. *Assume that the transition function $(P_t)_{t\in\mathbb{T}}$ satisfies the s.m.a. and is pointwise continuous. The following assertions are equivalent:*

(a) *$(P_t)_{t\in\mathbb{T}}$ is a minimal transition function.*
(b) *$\int_{\mathbb{T}} P_t(x,U)\,\mathrm{d}t > 0$ for every $x \in X$ and every nonempty open subset U of X.*

It is of interest to point out that a proposition similar to Proposition 2.1.7 can be stated for transition probabilities. We state it next.

Proposition 2.1.9. *Let P be a transition probability on (X,d), let (S,T) be the Markov pair defined by P, and let $x \in X$. The following assertions are equivalent:*

(a) *The orbit $\mathcal{O}^{(\mathrm{TP})}(x)$ of x under the action of P is dense in X.*
(b) *$\sum_{n=0}^{\infty} P_n(x,U) > 0$ for every nonempty open subset U of X, where P_n, $n \in \mathbb{N}$, are the (not necessarily Feller) transition probabilities defined after Lemma 1.1.1 starting with $P_1 = P$, and P_0 is the transition probability defined by $P_0(y,A) = \mathbf{1}_A(y)$ for every $y \in X$ and $A \in \mathcal{B}(X)$ (note that the Markov pair defined by P_0 consists of the identity operators on $B_b(X)$ and $\mathcal{M}(X)$).*

The proof of Proposition 2.1.9 follows along the lines of the proof of Proposition 2.1.7, but is significantly simpler.

Like Proposition 2.1.7, Proposition 2.1.9 has a consequence similar to Corollary 2.1.8 and can be combined with Proposition 1.4.6 as follows:

Corollary 2.1.10. *Let P be a transition probability on (X,d), and let (S,T) be the Markov pair defined by P. The following two assertions are equivalent:*

(a) *P is a minimal transition probability.*
(b) *$\sum_{n=0}^{\infty} P_n(x,U) > 0$ for every $x \in X$ and every nonempty open subset U of X, where P_n, $n \in \mathbb{N}\cup\{0\}$, are the transition probabilities that appear in Proposition 2.1.9.*

If P is a Feller transition probability, then each of the assertions (a) and (b) is also equivalent to:

(c) *$\sum_{n=1}^{\infty} P_n(x,U) > 0$ for every $x \in X$ and every nonempty open subset U of X, where P_n, $n \in \mathbb{N}$, are the transition probabilities that appear in assertion (b) above.*

The proof of the corollary is obtained easily using Propositions 2.1.9 and 1.4.6.

2.2 Examples

As mentioned in the abstract of this chapter, in this section we are going to present examples of transition functions. The transition functions under consideration are of two types: transition functions defined by one-parameter semigroups and one-parameter groups of measurable functions (actually, except for one example, all the transition functions of this type will be transition functions defined by semiflows and flows), and transition functions defined by one-parameter convolution semigroups of probability measures. The reader might wonder at this point why we don't also discuss examples of transition functions associated with continuous-time time-homogeneous Markov processes. The reason is that we want to use the examples in order to illustrate various aspects of the theory developed in the work, and, at the time of writing, we are able to state and prove our results only by using transition functions and their associated families of Markov pairs; even though it can be shown that many results in this volume are valid for various transition functions associated to Markov processes, the problem is that for most Markov processes of interest, their transition functions cannot be given explicitly.

The section is organized into three subsections. In the first subsection we show how to associate a transition function and a family of Markov pairs to a one-parameter semigroup or a one-parameter group of Borel measurable functions defined on a locally compact separable metric space, and we study various properties of these transition functions and families of Markov pairs. In the second subsection we list the transition functions and the corresponding families of Markov pairs of various one-parameter semigroups and one-parameter groups of Borel measurable functions that we will use to illustrate various concepts and results throughout the book. Finally, in the last subsection we discuss the transition functions and the corresponding families of Markov pairs defined by one-parameter convolution semigroups of probability measures.

2.2.1 Transition Functions Defined by One-Parameter Semigroups or Groups of Measurable Functions: General Considerations

As usual, in this book, let (X, d) be a locally compact separable metric space and let $\mathbb{T}$ stand for the additive metric group $\mathbb{R}$ or the additive metric semigroup $[0, \infty)$.

Let $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ be a one-parameter semigroup or a one-parameter group of elements of $\mathbf{B}(X)$, where $\mathbf{B}(X)$ is the semigroup of all measurable functions $f : X \to X$, where the algebraic operation that defines the semigroup structure on $\mathbf{B}(X)$ is the operation of composition of functions.

The range $\mathbf{w}(\mathbb{T}) = \{w_t | t \in \mathbb{T}\}$ of $\mathbf{w}$ is a subsemigroup or a subgroup of $\mathbf{B}(X)$ if $\mathbb{T} = [0, \infty)$ or $\mathbb{T} = \mathbb{R}$, respectively, and in both cases w_0 is a neutral element for $\mathbf{w}(\mathbb{T})$, thought of as a semigroup (if $\mathbb{T} = [0, \infty)$) or as a group (if $\mathbb{T} = \mathbb{R}$) in its own right. However, since $\mathbf{B}(X)$ is only a semigroup rather than a group, w_0 may or may not be the neutral element of $\mathbf{B}(X)$ (that is, the identity map Id_X of X). The neutral element w_0 is equal to Id_X whenever $\mathbf{w}$ is a semiflow (in the case $\mathbb{T} = [0, \infty)$) or a flow (when $\mathbb{T} = \mathbb{R}$), where we think of semiflows and flows as families of maps rather than functions defined on $\mathbb{T}\times X$. Although, most of the time, we will deal with semiflows and flows rather than the more general one-parameter semigroups and one-parameter groups of elements of $\mathbf{B}(X)$, we will however encounter an example of a one-parameter semigroup of elements of $\mathbf{B}(X)$ which is not a semiflow.

Since w_t, $t \in \mathbb{T}$, are measurable functions, we can use them to define transition probabilities as we did in Sect. 1.1 before Example 1.1.6; that is, for every $t \in \mathbb{T}$, let $P_t^{(\mathbf{w})} : X \times \mathcal{B}(X) \to \mathbb{R}$ be defined by

$$P_t^{(\mathbf{w})}(x, A) = \delta_{w_t(x)}(A) = \mathbf{1}_A(w_t(x)) \tag{2.2.1}$$

for every $x \in X$ and $A \in \mathcal{B}(X)$. Then, as pointed out in Sect. 1.1, $P_t^{(\mathbf{w})}$ is a transition probability called the transition probability defined by w_t, $t \in \mathbb{T}$.

Proposition 2.2.1. *The family $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ of transition probabilities is a transition function.*

Proof. We have to prove that $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ satisfies the Chapman-Kolmogorov equation (the equality (2.1.1)).

To this end, let $s \in \mathbb{T}$, $t \in \mathbb{T}$, $x \in X$, and $A \in \mathcal{B}(X)$. Then, using the above equality (2.2.1), we obtain that

$$\begin{aligned} P_{s+t}^{(\mathbf{w})}(x, A) &= \mathbf{1}_A(w_{s+t}(x)) = \mathbf{1}_A(w_s(w_t(x))) \\ &= P_s^{(\mathbf{w})}(w_t(x), A) = \int_X P_s^{(\mathbf{w})}(y, A)\, \mathrm{d}\delta_{w_t(x)}(y) \\ &= \int_X P_s^{(\mathbf{w})}(y, A) P_t^{(\mathbf{w})}(x, \mathrm{d}y), \end{aligned}$$

where the last equality holds true because $\delta_{w_t(x)}$ is the probability measure that corresponds to $P_t^{(\mathbf{w})}$ and to $x \in X$ that appears at (i) in the definition of a transition probability (see Sect. 1.1). □

We call $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ the *transition function defined* (or *generated*) *by* $\mathbf{w}$.

For every $t \in \mathbb{T}$, let $(S_t^{(\mathbf{w})}, T_t^{(\mathbf{w})})$ be the Markov pair defined by $P_t^{(\mathbf{w})}$. Thus, using the equalities (1.1.4) and (1.1.5) discussed in Sect. 1.1, we obtain that $S_t^{(\mathbf{w})} : B_b(X) \to B_b(X)$ is defined by

$$S_t^{(\mathbf{w})} f(x) = f(w_t(x)) \tag{2.2.2}$$

for every $f \in B_b(X)$ and $x \in X$, and $T_t^{(\mathbf{w})} : \mathcal{M}(X) \to \mathcal{M}(X)$ is defined by

$$T_t^{(\mathbf{w})} \mu(A) = \mu(w_t^{-1}(A)) \tag{2.2.3}$$

for every $\mu \in \mathcal{M}(X)$ and $A \in \mathcal{B}(X)$, whenever $t \in \mathbb{T}$.

We will also refer to the family $((S_t^{(\mathbf{w})}, T_t^{(\mathbf{w})}))_{t\in\mathbb{T}}$ of Markov pairs defined by the transition function $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ as the *family of Markov pairs defined by* $\mathbf{w}$.

Note that, by Proposition 2.1.1, the families $(S_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ and $(T_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ are one-parameter semigroups of operators if $\mathbb{T} = [0, \infty)$, and one-parameter groups of operators if $\mathbb{T} = \mathbb{R}$.

Also note that, when endowed with the operation of composition of operators, the sets $\{S_t^{(\mathbf{w})} | t \in \mathbb{T}\}$ and $\{T_t^{(\mathbf{w})} | t \in \mathbb{T}\}$ are semigroups with neutral elements $S_0^{(\mathbf{w})}$ and $T_0^{(\mathbf{w})}$ whenever $\mathbb{T} = [0, \infty)$, and are groups with neutral elements $S_0^{(\mathbf{w})}$ and $T_0^{(\mathbf{w})}$ whenever $\mathbb{T} = \mathbb{R}$, respectively. The operators $S_0^{(\mathbf{w})}$ and $T_0^{(\mathbf{w})}$ are the identity operators on $B_b(X)$ and $\mathcal{M}(X)$, respectively, if and only if w_0 is the identity map Id_X of X.

We say that $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ is a *measurable one-parameter semigroup* (if $\mathbb{T} = [0, \infty)$) or a *measurable one-parameter group* (if $\mathbb{T} = \mathbb{R}$) *of elements of* $\mathbf{B}(X)$ if the mapping $(t, x) \mapsto w_t(x)$ for every $(t, x) \in \mathbb{T} \times X$, is jointly measurable with respect to t and x in the sense that the mapping is measurable with respect to the σ-algebra $\mathcal{B}(X)$ on X and the product σ-algebra $\mathcal{L}(\mathbb{T}) \otimes \mathcal{B}(X)$ on $\mathbb{T} \times X$, where $\mathcal{L}(\mathbb{T})$ is the σ-algebra of all Lebesgue measurable subsets of $\mathbb{T}$. Note that if $\mathbf{w}$ is a semiflow or a flow, then the measurability of $\mathbf{w}$ as defined here is the same as the measurability of $\mathbf{w}$ as a semiflow or a flow as defined in Sect. A.3, before Example A.3.4.

Using formula (2.2.1) in this section, we obtain that if $\mathbf{w}$ is a measurable one-parameter semigroup or a measurable one-parameter group of elements of $\mathbf{B}(X)$, then the transition function $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ satisfies the s.m.a.

As expected, we say that $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ is *continuous in* t (or *continuous with respect to* t) if, for every $x \in X$, the mapping $t \mapsto w_t(x)$, $t \in \mathbb{T}$, is continuous with respect to the standard topology on $\mathbb{T}$ and the topology defined by the metric d on X.

In view of the equality (2.2.2) in this section, we obtain that if $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ is continuous in t, then $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ is $C_0(X)$-pointwise continuous.

We say that $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ is a *continuous one-parameter semigroup* (if $\mathbb{T} = [0, \infty)$) or a *continuous one-parameter group* (if $\mathbb{T} = \mathbb{R}$) *of elements of* $\mathbf{B}(X)$ if the mapping $(t, x) \mapsto w_t(x)$, $(t, x) \in \mathbb{T} \times X$, is jointly continuous with respect to t and x (that is, if the mapping is continuous with respect to the topology defined by the metric d on X, and the product topology on $\mathbb{T} \times X$ defined by the standard topology on $\mathbb{T}$ and the metric topology on X). Note that the notions of continuous one-parameter semigroup and continuous one-parameter group of elements of $\mathbf{B}(X)$ are natural extensions of the notions of continuous semiflow and continuous flow defined before Example A.3.4, in the sense that if $\mathbf{w}$ is a semiflow or a flow, then the continuity of $\mathbf{w}$ as defined here is the same as the continuity of $\mathbf{w}$ as a semiflow or a flow as defined in Sect. A.3.

Taking into consideration formula (2.2.2) in this section, we obtain that if $\mathbf{w}$ is a continuous one-parameter semigroup or group of elements of $\mathbf{B(X)}$, then the transition function $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ defined by $\mathbf{w}$ is a Feller transition function. Note that, since the continuity of $\mathbf{w}$ implies the measurability of $\mathbf{w}$, if $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$

is continuous, then $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ satisfies the s.m.a. Also, since the continuity of $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ implies that $\mathbf{w}$ is continuous in t, we obtain that if $\mathbf{w}$ is continuous, then $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ is $C_0(X)$-pointwise continuous.

We say that the one-parameter semigroup or group $(w_t)_{t\in\mathbb{T}}$ of elements of $\mathbf{B}(X)$ is *equicontinuous with respect to* $t \in \mathbb{T}$ if for every convergent sequence $(x_n)_{n\in\mathbb{N}}$ of elements of X and for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists an $n_\varepsilon \in \mathbb{N}$ such that $d(w_t(x_n), w_t(x)) < \varepsilon$ for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$, and every $t \in \mathbb{T}$, where $x = \lim_{n\to+\infty} x_n$. Note that the definition of the equicontinuity with respect to $t \in \mathbb{T}$ of the one-parameter semigroup or group $(w_t)_{t\in\mathbb{T}}$ of elements of $\mathbf{B}(X)$ is a natural extension of the definition of the equicontinuity with respect to $t \in \mathbb{T}$ for semiflows and flows discussed before Proposition B.4.6.

Using the fact that every $f \in C_0(X)$ is uniformly continuous and the equality (2.2.2) of this section, we obtain that if $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ is a one-parameter semigroup or group of elements of $\mathbf{B}(X)$ that is equicontinuous with respect to $t \in \mathbb{T}$, then the transition function $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ defined by $\mathbf{w}$ is $C_0(X)$-equicontinuous.

As in the case of semiflows and flows (see the discussion following Proposition A.3.3, in the slightly more general case of one-parameter semigroups and one-parameter groups of elements of $\mathbf{B}(X)$, given such a one-parameter semigroup or group $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ and $x \in X$ we call the sets $\mathcal{O}(x) = \{w_t(x) \mid t \in \mathbb{T}\}$, $\overline{\mathcal{O}(x)}$, $\mathcal{O}^{(\mathrm{F})}(x) = \{w_t(x) \mid t \in \mathbb{T}, t \geq 0\}$ and $\overline{\mathcal{O}^{(\mathrm{F})}(x)}$ the *orbit*, *orbit-closure*, *forward orbit* and *forward orbit-closure* of x under the action of $\mathbf{w}$, respectively. If the action of $\mathbf{w}$ has to be emphasized we use the notations $\mathcal{O}_{\mathbf{w}}(x)$, $\overline{\mathcal{O}_{\mathbf{w}}(x)}$, $\mathcal{O}_{\mathbf{w}}^{(\mathrm{F})}(x)$ and $\overline{\mathcal{O}_{\mathbf{w}}^{(\mathrm{F})}(x)}$ instead of $\mathcal{O}(x)$, $\overline{\mathcal{O}(x)}$, $\mathcal{O}^{(\mathrm{F})}(x)$ and $\overline{\mathcal{O}^{(\mathrm{F})}(x)}$, respectively. If we consider the transition function $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ defined by $\mathbf{w}$, then the orbit, orbit-closure, forward orbit and forward orbit-closure of x under the action of $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ as defined in the previous section (Sect. 2.1) are precisely the orbit, orbit-closure, forward orbit and forward orbit-closure of x under the action of $\mathbf{w}$, respectively, as defined in this section. Accordingly, the notions of orbit, orbit-closure, forward orbit and forward orbit-closure under the action of a transition function as defined in Sect. 2.1 are natural and fairly significant extensions of the corresponding notions for semiflows and flows that have appeared in ergodic theory and dynamical systems and that we discussed after Proposition A.3.3.

2.2.2 Transition Functions Defined by Specific One-Parameter Semigroups or Groups of Measurable Functions

As pointed out at the beginning of the section, in this subsection we discuss in detail several transition functions and families of Markov pairs defined by one-parameter semigroups or groups of elements of $\mathbf{B}(X)$ for various locally compact separable metric spaces (X, d). In order to warm up, we start with a flow which, in all likelihood, is the simplest example of a flow.

Example 2.2.2. Let $X = \mathbb{R}$, where $\mathbb{R}$ is endowed with its usual metric d defined by $d(x, y) = |x - y|$ for every $x \in \mathbb{R}$ and $y \in \mathbb{R}$.

For every $t \in \mathbb{R}$, let $w_t : \mathbb{R} \to \mathbb{R}$ be defined by $w_t(x) = t + x$ for every $x \in \mathbb{R}$. Clearly, $\mathbf{w} = (w_t)_{t\in\mathbb{R}}$ is a flow defined on $(\mathbb{R}, d)$.

Let $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{w}$. Then using the equality (2.2.1) of this section, we obtain that

$$P_t^{(\mathbf{w})}(x, A) = \delta_{t+x}(A) = \mathbf{1}_A(t + x) = \mathbf{1}_{A-t}(x)$$

for every $t \in \mathbb{R}$, $x \in \mathbb{R}$, and $A \in \mathcal{B}(\mathbb{R})$.

Let $((S_t^{(\mathbf{w})}, T_t^{(\mathbf{w})}))_{t\in\mathbb{R}}$ be the family of Markov pairs defined by $\mathbf{w}$. Then, for every $t \in \mathbb{R}$, the operator $S_t^{(\mathbf{w})} : B_b(\mathbb{R}) \to B_b(\mathbb{R})$ is defined by $S_t^{(\mathbf{w})} f(x) = f(t + x)$ for every $f \in B_b(\mathbb{R})$ and $x \in \mathbb{R}$, and the operator $T_t^{(\mathbf{w})} : \mathcal{M}(\mathbb{R}) \to \mathcal{M}(\mathbb{R})$ is defined by $T_t^{(\mathbf{w})}\mu(A) = \mu(A - t)$ for every $\mu \in \mathcal{M}(\mathbb{R})$ and every Borel subset A of $\mathbb{R}$.

Note that since the flow $\mathbf{w}$ is continuous, it follows that $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is a Feller transition probability, and $(S_t^{(\mathbf{w})}, T_t^{(\mathbf{w})})$ is a Markov-Feller pair for every $t \in \mathbb{R}$.

Since $\mathbf{w}$ is continuous, it follows that $\mathbf{w}$ is also measurable, so $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ satisfies the s.m.a. Again using the continuity of $\mathbf{w}$, we obtain that $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is $C_0(X)$-pointwise continuous.

Note that $\mathcal{O}(x) = \overline{\mathcal{O}(x)} = \mathbb{R}$ and $\mathcal{O}^{(F)}(x) = \overline{\mathcal{O}^{(F)}(x)} = [x, +\infty)$ for every $x \in \mathbb{R}$, so we see that the forward orbit of $x \in \mathbb{R}$ is a proper subset of the orbit of x. Note also that $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is an example of a minimal transition function that fails to be forward minimal. ∎

We now discuss an example of a one-parameter semigroup of elements of $\mathbf{B}(X)$ which is not a semiflow and which has the property that the transition function defined by the one-parameter semigroup fails to be Feller.

Example 2.2.3. Let $X = [0, 1]$ and consider on X the usual metric d defined by the absolute value as follows: $d(x, y) = |x - y|$ for every $x \in [0, 1]$ and $y \in [0, 1]$. Clearly, (X, d) is a compact metric space.

Now let $\mathbf{w} = (w_t)_{t\in[0,+\infty)}$ be the family of mappings defined as follows: $w_t : X \to X$,

$$w_t(x) = \begin{cases} \frac{x}{2^t} \text{ if } 0 < x < 1 \\ 1 \text{ if } x = 0 \text{ or } x = 1 \end{cases}$$

for every $t \in [0, +\infty)$.

Using the definition of $\mathbf{w}$ and by studying the cases $x \in (0, 1)$ and $x \in \{0, 1\}$ separately we obtain that $w_{s+t}(x) = w_s(w_t(x))$ for every $s \in [0, +\infty)$, $t \in [0, +\infty)$, and $x \in [0, 1]$. Thus, $\mathbf{w} = (w_t)_{t\in[0,+\infty)}$ is a one-parameter semigroup of elements of $\mathbf{B}([0, 1])$. However, note that $\mathbf{w}$ is not a semiflow because w_0 is not the identity map $Id_{[0,1]}$ of $[0, 1]$.

The reader has probably already noticed that the example discussed here is a "continuous-time" version of Example 1.1.6.

Let $(P_t^{(\mathbf{w})})_{t\in[0,+\infty)}$ be the transition function defined by $\mathbf{w}$.

We will now prove that $(P_t^{(\mathbf{w})})_{t\in[0,+\infty)}$ satisfies the s.m.a. As pointed out earlier in this section, in order to prove that $(P_t^{(\mathbf{w})})_{t\in[0,+\infty)}$ satisfies the s.m.a., it is enough to prove that the following assertion holds true:

Assertion. The one-parameter semigroup $\mathbf{w}$ is measurable.

Proof of Assertion. We have to prove that the mapping $\varphi : [0, +\infty)\times[0, 1] \to [0, 1]$ defined by $\varphi(t, x) = w_t(x)$ for every $(t, x) \in [0, +\infty) \times [0, 1]$ is measurable with respect to the Borel σ-algebra $\mathcal{B}([0, 1])$ and the product σ-algebra $\mathcal{L}([0, +\infty)) \otimes \mathcal{B}([0, 1])$, where $\mathcal{L}([0, +\infty))$ is the σ-algebra of all Lebesgue measurable subsets of $[0, +\infty)$.

To this end, set $E = [0, +\infty) \times (0, 1)$ and $F = [0, +\infty) \times \{0, 1\}$. Note that E and F are measurable subsets of $[0, +\infty) \times [0, 1]$ (in the sense that both E and F belong to $\mathcal{L}([0 + \infty)) \otimes \mathcal{B}([0, 1])$). Note also that the restriction $\varphi_{|E}$ of φ to E is a continuous function (with respect to the standard topology on $[0, 1]$ (the topology defined by the metric d on $[0, 1]$) and the topology induced on E by the standard topology on $\mathbb{R}^2$), so $\varphi_{|E}^{-1}(B)$ belongs to $\mathcal{L}([0, +\infty)) \otimes \mathcal{B}([0, 1])$ whenever B is a Borel subset of $[0, 1]$.

Now let B be a Borel subset of $[0, 1]$. If $1 \in B$, then $\varphi^{-1}(B) = \varphi_{|E}^{-1}(B) \cup F$, so $\varphi^{-1}(B)$ belongs to $\mathcal{L}([0, +\infty)) \otimes \mathcal{B}([0, 1])$; if $1 \notin B$, then $\varphi^{-1}(B)$ belongs to $\mathcal{L}([0, +\infty)) \otimes \mathcal{B}([0, 1])$, as well, because $\varphi^{-1}(B) = \varphi_{|E}^{-1}(B)$.

Since $\varphi^{-1}(B) \in \mathcal{L}([0, +\infty)) \otimes \mathcal{B}([0, 1])$ for every $B \in \mathcal{B}([0, 1])$, it follows that $\mathbf{w}$ is measurable. □

Since the above assertion holds true, we obtain that $(P_t^{(\mathbf{w})})_{t\in[0,+\infty)}$ satisfies the s.m.a.

Now, let $((S_t^{(\mathbf{w})}, T_t^{(\mathbf{w})}))_{t\in[0,+\infty)}$ be the family of Markov pairs defined by $\mathbf{w}$.

Using the equality (2.2.2) of this section and the fact that the map $t \mapsto w_t(x)$, $t \in [0, +\infty)$, is continuous for every $x \in [0, 1]$, we obtain that the map $t \mapsto S_t f(x)$, $t \in [0, +\infty)$, is continuous for every continuous function f and every $x \in [0, 1]$. Thus, $(P_t^{(\mathbf{w})})_{t\in[0,+\infty)}$ is pointwise continuous.

Again using the equality (2.2.2) of this section and the fact that w_t fails to be continuous at $x = 0$ for every $t \in [0, +\infty)$, we obtain that $(P_t^{(\mathbf{w})})_{t\in[0,+\infty)}$ is not a Feller transition function. ■

Example 2.2.4 (The Transition Function of the Flow of the Rotations of the Unit Circle). Let $X = \mathbb{R}/\mathbb{Z}$, where $\mathbb{R}/\mathbb{Z}$ is the commutative compact metric group known as the unit circle discussed in Example A.2.8. Also, let $\mathbf{w} = (w_t)_{t\in\mathbb{R}}$ be the flow of the rotations of the unit circle defined in Example A.3.4 and let $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{w}$.

Using the equality (2.2.1) of this section, we obtain that

$$P_t^{(\mathbf{w})}(\hat{x}, A) = \delta_{\widehat{t+x}}(A) = \mathbf{1}_A(\widehat{t+x}) = \mathbf{1}_{A-\hat{t}}(\hat{x})$$

for every $t \in \mathbb{R}$, $\hat{x} \in \mathbb{R}/\mathbb{Z}$, and every Borel subset A of $\mathbb{R}/\mathbb{Z}$. Note that we use the "hat" notation here because it is significantly more convenient.

Let $((S_t^{(\mathbf{w})}, T_t^{(\mathbf{w})}))_{t\in\mathbb{R}}$ be the family of Markov pairs defined by $\mathbf{w}$. Using the equalities (2.2.2) and (2.2.3) of this section, we obtain that $S_t^{(\mathbf{w})} f(\hat{x}) = f(\widehat{t+x})$ for every $t \in \mathbb{R}$, $f \in B_b(\mathbb{R}/\mathbb{Z})$, and $\hat{x} \in \mathbb{R}/\mathbb{Z}$, and that $T_t^{(\mathbf{w})}\mu(A) = \mu(A - \hat{t})$ for every $t \in \mathbb{R}$, $\mu \in \mathcal{M}(\mathbb{R}/\mathbb{Z})$, and $A \in \mathcal{B}(\mathbb{R}/\mathbb{Z})$.

Note that for every $a \in \mathbb{R}$, $P_a^{(\mathbf{w})} = P_{\hat{a}}$, $S_a^{(\mathbf{w})} = S_{\hat{a}}$, and $T_a^{(\mathbf{w})} = T_{\hat{a}}$, where $P_{\hat{a}}$ and $(S_{\hat{a}}, T_{\hat{a}})$ are the transition probability and the Markov pair, respectively, defined and discussed in Example 1.1.7.

Since $\mathbf{w}$ is a continuous flow, it follows that $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ satisfies the s.m.a. (because the continuity of $\mathbf{w}$ implies that $\mathbf{w}$ is measurable), and that $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is a pointwise continuous Feller transition function.

It is easy to see that the flow $\mathbf{w}$ is equicontinuous with respect to $t \in \mathbb{R}$; therefore, the transition function $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is equicontinuous.

Finally, note that since $\mathbf{w}$ is both minimal and forward minimal, it follows that $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is minimal and forward minimal, as well. ■

Example 2.2.5 (The Rectilinear Flow on the Torus and its Transition Function). Let $n \in \mathbb{N}, n \geq 2$, let $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} = (v_1, v_2, \cdots, v_n)$, let $\mathbb{R}^n/\mathbb{Z}^n$ be the n-dimensional torus defined in Example A.2.9, let $\mathbf{w} = (w_t)_{t\in\mathbb{R}}$ be the rectilinear flow on $\mathbb{R}^n/\mathbb{Z}^n$ with velocity $\mathbf{v}$ defined in Example A.3.5, and let $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{w}$.

In view of the equality (2.2.1) of this section and using the "hat" notation for the elements of $\mathbb{R}^n/\mathbb{Z}^n$, we obtain that

$$P_t^{(\mathbf{w})}(\hat{\mathbf{x}}, A) = \delta_{\widehat{t\mathbf{v}+\mathbf{x}}}(A) = \mathbf{1}_A(\widehat{t\mathbf{v}+\mathbf{x}}) = \mathbf{1}_{A-\widehat{t\mathbf{v}}}(\hat{\mathbf{x}})$$

for every $t \in \mathbb{R}$, $\hat{\mathbf{x}} \in \mathbb{R}^n/\mathbb{Z}^n$, and every Borel subset A of $\mathbb{R}^n/\mathbb{Z}^n$.

If $((S_t^{(\mathbf{w})}, T_t^{(\mathbf{w})}))_{t\in\mathbb{R}}$ is the family of Markov pairs defined by $\mathbf{w}$, then using the equalities (2.2.2) and (2.2.3) of this section, we obtain that $S_t^{(\mathbf{w})} f(\hat{\mathbf{x}}) = f(t\mathbf{v} \hat{+} \mathbf{x})$ for every $t \in \mathbb{R}$, $f \in B_b(\mathbb{R}^n/\mathbb{Z}^n)$ and $\hat{\mathbf{x}} \in \mathbb{R}^n/\mathbb{Z}^n$, and that $T_t^{(\mathbf{w})}\mu(A) = \mu(A - t\hat{\mathbf{v}})$ for every $t \in \mathbb{R}$, $\mu \in \mathcal{M}(\mathbb{R}^n/\mathbb{Z}^n)$ and $A \in \mathcal{B}(\mathbb{R}^n/\mathbb{Z}^n)$.

Note that (for $t = 1$) $P_1^{(\mathbf{w})} = P_{\mathbf{v}}$, $S_1^{(\mathbf{w})} = S_{\mathbf{v}}$, and $T_1^{(\mathbf{w})} = T_{\mathbf{v}}$, where $P_{\mathbf{v}}$ and $(S_{\mathbf{v}}, T_{\mathbf{v}})$ are the transition probability and the Markov pair discussed in Example 1.1.8, respectively.

Since the rectilinear flow on the torus is continuous and equicontinuous with respect to $t \in \mathbb{R}$, it follows that $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ satisfies the s.m.a. and is a pointwise continuous equicontinuous Feller transition function.

Note that in general $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is neither forward minimal nor minimal. However, if the entries $v_1, v_2, \cdots, v_n$ are rationally independent, then $\mathbf{w}$ is minimal (see Example A.3.5 for details), so $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is minimal, as well, in this case. ■

Example 2.2.6 (Transition Functions of Geodesic Flows). Our goal here is to discuss the transition functions of the geodesic flow on $\mathrm{PSL}(2,\mathbb{R})$, which is defined in Example A.3.6, and the geodesic flows on certain spaces of cosets of $\mathrm{PSL}(2,\mathbb{R})$, which are defined in Example B.1.8.

(a) (The Geodesic Flow on $\mathrm{PSL}(2,\mathbb{R})$ and its Transition Function). Let $\mathrm{PSL}(2,\mathbb{R})$ be the locally compact separable metrizable group defined in Example A.2.10. Since the elements of $\mathrm{PSL}(2,\mathbb{R})$ are cosets (of $L = \{\mathbf{I}_2, -\mathbf{I}_2\}$ in $\mathrm{SL}(2,\mathbb{R})$), we will use the "hat" notation when dealing with these elements.

Let $\mathbf{w} = (w_t)_{t\in\mathbb{R}}$ be the geodesic flow on $\mathrm{PSL}(2,\mathbb{R})$. Thus, for every $t \in \mathbb{R}$, $w_t : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ is the function defined by $w_t(\hat{g}) = \hat{g}\hat{g}_t$ for every $\hat{g} \in \mathrm{PSL}(2,\mathbb{R})$, where $g_t = \begin{pmatrix} e^{\frac{t}{2}} & 0 \\ 0 & e^{-\frac{t}{2}} \end{pmatrix}$.

Let $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{w}$. Then, using the equality (2.2.1) of this section, we obtain that

$$P_t^{(\mathbf{w})}(\hat{g}, A) = \delta_{\hat{g}\hat{g}_t}(A) = \mathbf{1}_A(\hat{g}\hat{g}_t) = \mathbf{1}_{A(\hat{g}_t)^{-1}}(\hat{g})$$

for every $t \in \mathbb{R}$, $\hat{g} \in \mathrm{PSL}(2,\mathbb{R})$ and every Borel subset A of $\mathrm{PSL}(2,\mathbb{R})$.

Now, let $((S_t^{(\mathbf{w})}, T_t^{(\mathbf{w})}))_{t\in\mathbb{R}}$ be the family of Markov pairs defined by $\mathbf{w}$. Then, using formula (2.2.2) of this section, we obtain that $S_t^{(\mathbf{w})} f(\hat{g}) = f(\hat{g}\hat{g}_t)$ for every $t \in \mathbb{R}$, $f \in B_b(\mathrm{PSL}(2,\mathbb{R}))$, and $\hat{g} \in \mathrm{PSL}(2,\mathbb{R})$; also, in view of the equality (2.2.3) of this section, we obtain that $T_t^{(\mathbf{w})}\mu(A) = \mu(A(\hat{g}_t)^{-1})$ for every $t \in \mathbb{R}$, $\mu \in \mathcal{M}(\mathrm{PSL}(2,\mathbb{R}))$, and $A \in \mathcal{B}(\mathrm{PSL}(2,\mathbb{R}))$.

Note that if we set $h = g_s$ for some $s \in \mathbb{R}$, then $P_s^{(\mathbf{w})} = P_h$, $S_s^{(\mathbf{w})} = S_h$, and $T_s^{(\mathbf{w})} = T_h$, where P_h and (S_h, T_h) are the transition probability and the Markov pair discussed in Example 1.1.9.

Since, as shown in Example A.3.6, $\mathbf{w}$ is a continuous flow, it follows that $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ satisfies the s.m.a. and is a $C_0(\mathrm{PSL}(2,\mathbb{R}))$-pointwise continuous Feller transition function.

(b) The Transition Functions of the Geodesic Flows on Certain Spaces of Cosets of $\mathrm{PSL}(2,\mathbb{R})$. Let Γ be a lattice in $\mathrm{PSL}(2,\mathbb{R})$, and let $\mathbf{w}^{(\Gamma)} = (w_t^{(\Gamma)})_{t\in\mathbb{R}}$ be the geodesic flow on $(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$ defined in Example B.1.8. Thus, for every $t \in \mathbb{R}$, $w_t^{(\Gamma)} : (\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}} \to (\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$ is defined by $w_t^{(\Gamma)}(\Gamma\hat{x}) = \Gamma\hat{x}\hat{g}_t$ for every $\Gamma\hat{x} \in (\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$, where $\hat{g}_t$ is the element of $\mathrm{PSL}(2,\mathbb{R})$ defined in (a) of this example and we continue to use the "hat" notation for the elements of $\mathrm{PSL}(2,\mathbb{R})$.

Let $(P_t^{(\mathbf{w}^{(\Gamma)})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{w}^{(\Gamma)}$. Using formula (2.2.1) of this section, we obtain that

$$P_t^{(\mathbf{w}^{(\Gamma)})}(\Gamma\hat{x}, A) = \delta_{\Gamma\hat{x}\hat{g}_t}(A) = \mathbf{1}_A(\Gamma\hat{x}\hat{g}_t) = \mathbf{1}_{A(\hat{g}_t)^{-1}}(\Gamma\hat{x})$$

for every $t \in \mathbb{R}$, $\Gamma\hat{x} \in (\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$, and every Borel subset A of $(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$.

If $(S_t^{(\mathbf{w}^{(\Gamma)})}, T_t^{(\mathbf{w}^{(\Gamma)})})_{t\in\mathbb{R}}$ is the family of Markov pairs defined by $\mathbf{w}^{(\Gamma)}$, then using formula (2.2.2) of this section, we obtain that $S_t^{(\mathbf{w}^{(\Gamma)})} f(\Gamma\hat{x}) = f(\Gamma\hat{x}\hat{g}_t)$ for every $t \in \mathbb{R}$, $f \in B_b((\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}})$, and $\Gamma\hat{x} \in (\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$, and using formula (2.2.3) also of this section, we obtain that $T_t^{(\mathbf{w}^{(\Gamma)})}\mu(A) = \mu(A(\hat{g}_t)^{-1})$ for every $t \in \mathbb{R}$, $\mu \in \mathcal{M}((\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}})$, and $A \in \mathcal{B}((\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}})$.

Note that if we set $h = g_s$ for some $s \in \mathbb{R}$, then $P_s^{(\mathbf{w}^{(\Gamma)})} = P_h^{(\mathrm{R})}$, $S_s^{(\mathbf{w}^{(\Gamma)})} = S_h^{(\mathrm{R})}$, and $T_s^{(\mathbf{w}^{(\Gamma)})} = T_h^{(\mathrm{R})}$ where $P_h^{(\mathrm{R})}$ and $(S_h^{(\mathrm{R})}, T_h^{(\mathrm{R})})$ are the transition probability and the Markov pair discussed in Example 1.1.10.

Since, as pointed out in Example B.1.8, $\mathbf{w}^{(\Gamma)}$ is a continuous flow, it follows that $(P_t^{(\mathbf{w}^{(\Gamma)})})_{t\in\mathbb{R}}$ satisfies the s.m.a. and is a $C_0((\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}})$-pointwise continuous Feller transition function. ■

Example 2.2.7 (Transition Functions of Horocycle Flows). We will now discuss the transition functions of the horocycle flows defined in Appendices A and B. Thus, we will consider the horocycle flows on $\mathrm{PSL}(2,\mathbb{R})$ discussed in Example A.3.7 and the horocycle flows on spaces of cosets of $\mathrm{PSL}(2,\mathbb{R})$ and $\mathrm{SL}(2,\mathbb{R})$ that are described in Example B.1.9.

(a) The Horocycle Flows on $\mathrm{PSL}(2,\mathbb{R})$. As usual in this book, we will use the "hat" notation when dealing with elements of $\mathrm{PSL}(2,\mathbb{R})$.

Let $\mathbf{v}^{(1)} = (v_t^{(1)})_{t\in\mathbb{R}}$ and $\mathbf{v}^{(2)} = (v_t^{(2)})_{t\in\mathbb{R}}$ be the two horocycle flows on $\mathrm{PSL}(2,\mathbb{R})$ defined in Example A.3.7. Thus, for every $i = 1, 2$ and $t \in \mathbb{R}$, the function $v_t^{(i)}$: $\mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ is defined by $v_t^{(i)}(\hat{h}) = \hat{h}\hat{h}_t^{(i)}$ for every $\hat{h} \in \mathrm{PSL}(2,\mathbb{R})$, where $h_t^{(1)} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ and $h_t^{(2)} = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}$.

Let $(P_t^{(\mathbf{v}^{(i)})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{v}^{(i)}$, $i = 1, 2$. Using formula (2.2.1) of this section, we obtain that

$$P_t^{(\mathbf{v}^{(i)})}(\hat{h}, A) = \delta_{\hat{h}\hat{h}_t^{(i)}}(A) = \mathbf{1}_A(\hat{h}\hat{h}_t^{(i)}) = \mathbf{1}_{A(\hat{h}_t^{(i)})^{-1}}(\hat{h})$$

for every $i = 1, 2$, $t \in \mathbb{R}$, $\hat{h} \in \mathrm{PSL}(2,\mathbb{R})$, and every Borel subset A of $\mathrm{PSL}(2,\mathbb{R})$.

If $i \in \{1, 2\}$ and $((S_t^{(\mathbf{v}^{(i)})}, T_t^{(\mathbf{v}^{(i)})}))_{t\in\mathbb{R}}$ is the family of Markov pairs defined by $\mathbf{v}^{(i)}$, then using formula (2.2.2) of this section, we obtain that $S_t^{(\mathbf{v}^{(i)})} f(\hat{h}) = f(\hat{h}\hat{h}_t^{(i)})$ for every $t \in \mathbb{R}$, $f \in B_b(\mathrm{PSL}(2,\mathbb{R}))$, and $\hat{h} \in \mathrm{PSL}(2,\mathbb{R})$, and using formula (2.2.3) also of this section, we obtain that $T_t^{(\mathbf{v}^{(i)})}\mu(A) = \mu(A(\hat{h}_t^{(i)})^{-1})$ for every $t \in \mathbb{R}$, $\mu \in \mathcal{M}(\mathrm{PSL}(2,\mathbb{R}))$, and $A \in \mathcal{B}(\mathrm{PSL}(2,\mathbb{R}))$.

Note that $P_s^{(\mathbf{v}^{(i)})} = P_{h_s^{(i)}}$, $S_s^{(\mathbf{v}^{(i)})} = S_{h_s^{(i)}}$, and $T_s^{(\mathbf{v}^{(i)})} = T_{h_s^{(i)}}$, where $P_{h_s^{(i)}}$ and $(S_{h_s^{(i)}}, T_{h_s^{(i)}})$ are the transition probability and the Markov pair defined by $u_{h_s^{(i)}}$ in Example 1.1.9, respectively, whenever $s \in \mathbb{R}$ and $i = 1$ or 2.

Since, as mentioned in Example A.3.7, the flows $\mathbf{v}^{(1)}$ and $\mathbf{v}^{(2)}$ are continuous, if follows that $(P_t^{(\mathbf{v}^{(1)})})_{t\in\mathbb{R}}$ and $(P_t^{(\mathbf{v}^{(2)})})_{t\in\mathbb{R}}$ satisfy the s.m.a. and are $C_0(\mathrm{PSL}(2,\mathbb{R}))$-pointwise continuous Feller transition functions.

(*b*) Horocycle Flows on Spaces of Cosets of PSL$(2,\mathbb{R})$. Let Γ be a lattice in PSL$(2,\mathbb{R})$, and let $\bar{\mathbf{v}}^{(1\Gamma)} = (\bar{v}_t^{(1\Gamma)})_{t\in\mathbb{R}}$ and $\bar{\mathbf{v}}^{(2\Gamma)} = (\bar{v}_t^{(2\Gamma)})_{t\in\mathbb{R}}$ be the two horocycle flows defined in (*a*) of Example B.1.9. Thus, for every $j \in \{1,2\}$ and $t \in \mathbb{R}$, the mapping $\bar{v}_t^{(j\Gamma)} : (\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}} \to (\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$ is defined by $\bar{v}_t^{(j\Gamma)}(\Gamma\hat{x}) = \Gamma\hat{x}\hat{h}_t^{(j)}$ for every element $\Gamma\hat{x}$ of $(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$, where $\hat{h}_t^{(j)}$ is the element of PSL$(2,\mathbb{R})$ that appears in the definition of the horocycle flows on PSL$(2,\mathbb{R})$ in (*a*) of this example.

Let $(P_t^{(\bar{\mathbf{v}}^{(j\Gamma)})})_{t\in\mathbb{R}}$ be the transition function defined by the flow $\bar{\mathbf{v}}^{(j\Gamma)}$, $j = 1$ or 2. In view of the formula (2.2.1) of this section, we obtain that

$$P_t^{(\bar{\mathbf{v}}^{(j\Gamma)})}(\Gamma\hat{x}, A) = \delta_{\Gamma\hat{x}\hat{h}_t^{(j)}}(A) = \mathbf{1}_A(\Gamma\hat{x}\hat{h}_t^{(j)}) = \mathbf{1}_{A(\hat{h}_t^{(j)})^{-1}}(\Gamma\hat{x})$$

for every $j \in \{1,2\}$, $t \in \mathbb{R}$, $\Gamma\hat{x} \in (\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$ and every Borel subset A of $(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$.

If $((S_t^{(\bar{\mathbf{v}}^{(j\Gamma)})}, T_t^{(\bar{\mathbf{v}}^{(j\Gamma)})}))_{t\in\mathbb{R}}$ is the family of Markov pairs defined by $\bar{\mathbf{v}}^{(j\Gamma)}$, $j = 1, 2$, then using formula (2.2.2) of this section, we obtain that $S_t^{(\bar{\mathbf{v}}^{(j\Gamma)})}f(\Gamma\hat{x}) = f(\Gamma\hat{x}\hat{h}_t^{(j)})$ for every $j = 1, 2$, $t \in \mathbb{R}$, $f \in B_b((\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}})$ and $\Gamma\hat{x} \in (\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$, and by formula (2.2.3) also of this section, we obtain that $T_t^{(\bar{\mathbf{v}}^{(j\Gamma)})}\mu(A) = \mu(A(\hat{h}_t^{(j)})^{-1})$ for every $j \in \{1,2\}$, $t \in \mathbb{R}$, $\mu \in \mathcal{M}((\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}})$, and $A \in \mathcal{B}((\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}})$.

Observe that $P_t^{(\bar{\mathbf{v}}^{(j\Gamma)})} = P_{h_t^{(j)}}^{(\mathrm{R})}$, $S_t^{(\bar{\mathbf{v}}^{(j\Gamma)})} = S_{h_t^{(j)}}^{(\mathrm{R})}$, and $T_t^{(\bar{\mathbf{v}}^{(j\Gamma)})} = T_{h_t^{(j)}}^{(\mathrm{R})}$, where $P_{h_t^{(j)}}^{(\mathrm{R})}$ and $(S_{h_t^{(j)}}^{(\mathrm{R})}, T_{h_t^{(j)}}^{(\mathrm{R})})$ are the transition probability and the Markov pair defined by $u_{h_t^{(j)}}^{(\mathrm{R})}$ in Example 1.1.10, respectively, whenever $t \in \mathbb{R}$ and $j = 1$ or 2.

Since, as pointed out in (*a*) of Example B.1.9, the two flows $\bar{\mathbf{v}}^{(j\Gamma)}$, $j = 1$ or 2, are continuous, it follows that both $(P_t^{(\bar{\mathbf{v}}^{(1\Gamma)})})_{t\in\mathbb{R}}$ and $(P_t^{(\bar{\mathbf{v}}^{(2\Gamma)})})_{t\in\mathbb{R}}$ satisfy the s.m.a. and are $C_0((\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}})$-pointwise continuous Feller transition functions.

Using the result of Hedlund [41] that we mentioned in (*a*) of Example B.1.9 (see also Theorem 1.9 in Chapter 4 of Bachir Bekka and Mayer [10]), we obtain that for every $\Gamma\hat{x} \in (\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$ and $j \in \{1,2\}$, either $\Gamma\hat{x}$ is a periodic point for the flow $\bar{\mathbf{v}}^{(j\Gamma)}$, or else the orbit of $\Gamma\hat{x}$ under the action of the transition function

$(P_t^{(\bar{\mathbf{v}}^{(j\Gamma)})})_{t\in\mathbb{R}}$ is dense in $(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_\mathrm{R}$. If Γ is cocompact, the two transition functions defined by the horocycle flows $\bar{\mathbf{v}}^{(j\Gamma)}$, $j = 1, 2$, are minimal.

(c) Horocycle Flows on Spaces of Cosets of $\mathrm{SL}(2,\mathbb{R})$. Let Γ be a lattice in $\mathrm{SL}(2,\mathbb{R})$, and let $\mathbf{v}^{(j\Gamma\mathrm{L})} = (v_t^{(j\Gamma\mathrm{L})})_{t\in\mathbb{R}}$ and $\mathbf{v}^{(j\Gamma\mathrm{R})} = (v_t^{(j\Gamma\mathrm{R})})_{t\in\mathbb{R}}$, $j = 1$ or 2, be the four horocycle flows defined in (b) of Example B.1.9.

Thus, for every $j = 1$ or 2, and every $t \in \mathbb{R}$, the function

$$v_t^{(j\Gamma\mathrm{L})} : (\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{L} \to (\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{L}$$

is defined by $v_t^{(j\Gamma\mathrm{L})}(x\Gamma) = h_t^{(j)}x\Gamma$ for every $x\Gamma \in (\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{L}$, and the function

$$v_t^{(j\Gamma\mathrm{R})} : (\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{R} \to (\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{R}$$

is defined by $v_t^{(j\Gamma\mathrm{R})}(\Gamma x) = \Gamma x h_t^{(j)}$ for every $\Gamma x \in (\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{R}$, where $h_t^{(j)}$ is the element of $\mathrm{SL}(2,\mathbb{R})$ defined in (a) of this example.

Let $(P_t^{(\mathbf{v}^{(j\Gamma\mathrm{L})})})_{t\in\mathbb{R}}$ and $((S_t^{(\mathbf{v}^{(j\Gamma\mathrm{L})})}, T_t^{(\mathbf{v}^{(j\Gamma\mathrm{L})})}))_{t\in\mathbb{R}}$ be the transition function and the family of Markov pairs defined by the flow $\mathbf{v}^{(j\Gamma\mathrm{L})}$, respectively, for every $j \in \{1, 2\}$. Then using the formulas (2.2.1)–(2.2.3) of this section, we obtain that

$$P_t^{(\mathbf{v}^{(j\Gamma\mathrm{L})})}(x\Gamma, A) = \delta_{h_t^{(j)}x\Gamma}(A) = \mathbf{1}_A(h_t^{(j)}x\Gamma) = \mathbf{1}_{(h_t^{(j)})^{-1}A}(x\Gamma)$$

for every $j \in \{1, 2\}$, $t \in \mathbb{R}$, $x\Gamma \in (\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{L}$, and every Borel subset A of $(\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{L}$; $S_t^{(\mathbf{v}^{(j\Gamma\mathrm{L})})}f(x\Gamma) = f(h_t^{(j)}x\Gamma)$ for every $j \in \{1, 2\}$, $t \in \mathbb{R}$, $f \in B_b((\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{L})$, and $x\Gamma \in (\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{L}$; $T_t^{(\mathbf{v}^{(j\Gamma\mathrm{L})})}\mu(A) = \mu((h_t^{(j)})^{-1}A)$ for every $j \in \{1, 2\}$, $t \in \mathbb{R}$, $\mu \in \mathcal{M}((\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{L})$, and $A \in \mathcal{B}((\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{L})$.

Similarly, let $(P_t^{(\mathbf{v}^{(j\Gamma\mathrm{R})})})_{t\in\mathbb{R}}$ and $((S_t^{(\mathbf{v}^{(j\Gamma\mathrm{R})})}, T_t^{(\mathbf{v}^{(j\Gamma\mathrm{R})})}))_{t\in\mathbb{R}}$ be the transition function and the family of Markov pairs defined by $\mathbf{v}^{(j\Gamma\mathrm{R})}$, respectively, for every $j \in \{1, 2\}$. Then

$$P_t^{(\mathbf{v}^{(j\Gamma\mathrm{R})})}(\Gamma x, A) = \delta_{\Gamma x h_t^{(j)}}(A) = \mathbf{1}_A(\Gamma x h_t^{(j)}) = \mathbf{1}_{A(h_t^{(j)})^{-1}}(\Gamma x)$$

for every $j \in \{1, 2\}$, $t \in \mathbb{R}$, $\Gamma x \in (\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{R}$ and $A \in \mathcal{B}((\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{R})$; $S_t^{(\mathbf{v}^{(j\Gamma\mathrm{R})})}f(\Gamma x) = f(\Gamma x h_t^{(j)})$ for every $j \in \{1, 2\}$, $t \in \mathbb{R}$, $f \in B_b((\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{R})$, and $\Gamma x \in (\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{R}$; $T_t^{(\mathbf{v}^{(j\Gamma\mathrm{R})})}\mu(A) = \mu(A(h_t^{(j)})^{-1})$ for every $j \in \{1, 2\}$, $t \in \mathbb{R}$, $\mu \in \mathcal{M}((\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{R})$, and $A \in \mathcal{B}((\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{R})$.

Note that $P_t^{(\mathbf{v}^{(j\Gamma\mathrm{L})})} = P_{(h_t^{(j)})}^{(\mathrm{L})}$, $S_t^{(\mathbf{v}^{(j\Gamma\mathrm{L})})} = S_{(h_t^{(j)})}^{(\mathrm{L})}$, $T_t^{(\mathbf{v}^{(j\Gamma\mathrm{L})})} = T_{(h_t^{(j)})}^{(\mathrm{L})}$, $P_t^{(\mathbf{v}^{(j\Gamma\mathrm{R})})} = P_{(h_t^{(j)})}^{(\mathrm{R})}$, $S_t^{(\mathbf{v}^{(j\Gamma\mathrm{R})})} = S_{(h_t^{(j)})}^{(\mathrm{R})}$ and $T_t^{(\mathbf{v}^{(j\Gamma\mathrm{R})})} = T_{(h_t^{(j)})}^{(\mathrm{R})}$, where $P_{(h_t^{(j)})}^{(\mathrm{L})}$ and $(S_{(h_t^{(j)})}^{(\mathrm{L})}, T_{(h_t^{(j)})}^{(\mathrm{L})})$ are the transition probability and the Markov pair defined by the mapping $u_{h_t^{(j)}}^{(\mathrm{L})}$: $(\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{L} \to (\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{L}$ discussed in Example 1.1.11, and, similarly,

$P^{(\mathrm{R})}_{(h_t^{(j)})}$ and $(S^{(\mathrm{R})}_{(h_t^{(j)})}, T^{(\mathrm{R})}_{(h_t^{(j)})})$ are the transition probability and the Markov pair defined by the function $u^{(\mathrm{R})}_{h_t^{(j)}} : (\mathrm{SL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}} \to (\mathrm{SL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$ also defined in Example 1.1.11, respectively, whenever $t \in \mathbb{R}$ and $j = 1$ or 2.

Since, as mentioned in (*b*) of Example B.1.9, the four horocycle flows under consideration here are continuous, it follows that the transition functions defined by these flows satisfy the s.m.a. and are pointwise continuous Feller transition functions. ■

Example 2.2.8 (Transition Functions of Exponential Flows on Spaces of Cosets of SL$(n,\mathbb{R})$*).* Let $n \in \mathbb{N}$, $n \geq 2$, let M be a closed subgroup of $\mathrm{SL}(n,\mathbb{R})$, let $\mathbf{A}$ be a trace zero $n \times n$ matrix, and let $\mathbf{u} = (u_t)_{t\in\mathbb{R}}$ and $\mathbf{v} = (v_t)_{t\in\mathbb{R}}$ be the exponential flows on $(\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{L}}$ and $(\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{R}}$, respectively, defined by $\mathbf{A}$. (See Sect. B.4.2 for details on these flows and some of the terminology used in this example.)

Thus, for every $t \in \mathbb{R}$,

$$u_t : (\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{L}} \to (\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{L}}$$

is defined by $u_t(\mathbf{X}M) = \exp(t\mathbf{A})\mathbf{X}M$ for every $\mathbf{X}M \in (\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{L}}$ and

$$v_t : (\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{R}} \to (\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{R}}$$

is defined by $v_t(M\mathbf{X}) = M\mathbf{X}\exp(t\mathbf{A})$ for every $M\mathbf{X} \in (\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{R}}$.

Let $(P_t^{(\mathbf{u})})_{t\in\mathbb{R}}$ and $((S_t^{(\mathbf{u})}, T_t^{(\mathbf{u})}))_{t\in\mathbb{R}}$ be the transition function and the family of Markov pairs defined by $\mathbf{u}$, respectively.

Then,

$$P_t^{(\mathbf{u})}(\mathbf{X}M, E) = \delta_{\exp(t\mathbf{A})\mathbf{X}M}(E) = \mathbf{1}_E(\exp(t\mathbf{A})\mathbf{X}M) = \mathbf{1}_{\exp(-t\mathbf{A})E}(\mathbf{X}M)$$

for every $t \in \mathbb{R}$, $\mathbf{X}M \in (\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{L}}$, and every Borel subset E of $(\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{L}}$; $S_t^{(\mathbf{u})} f(\mathbf{X}M) = f(\exp(t\mathbf{A})\mathbf{X}M)$ for every $t \in \mathbb{R}$, $f \in B_b((\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{L}})$, and $\mathbf{X}M \in (\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{L}}$; $T_t^{(\mathbf{u})}\mu(E) = \mu(\exp(-t\mathbf{A})E)$ for every $t \in \mathbb{R}$, $\mu \in \mathcal{M}((\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{L}})$, and $E \in \mathcal{B}((\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{L}})$.

Similarly, let $(P_t^{(\mathbf{v})})_{t\in\mathbb{R}}$ and $(S_t^{(\mathbf{v})}, T_t^{(\mathbf{v})})_{t\in\mathbb{R}}$ be the transition function and the family of Markov pairs defined by $\mathbf{v}$, respectively.

Then,

$$P_t^{(\mathbf{v})}(M\mathbf{X}, E) = \delta_{M\mathbf{X}\exp(t\mathbf{A})}(E) = \mathbf{1}_E(M\mathbf{X}\exp(t\mathbf{A})) = \mathbf{1}_{E\exp(-t\mathbf{A})}(M\mathbf{X})$$

for every $t \in \mathbb{R}$, $M\mathbf{X} \in (\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{R}}$, and $E \in \mathcal{B}((\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{R}})$; $S_t^{(\mathbf{v})} f(M\mathbf{X}) = f(M\mathbf{X}\exp(t\mathbf{A}))$ for every $t \in \mathbb{R}$, $f \in B_b((\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{R}})$, and $M\mathbf{X} \in (\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{R}}$; $T_t^{(\mathbf{v})}\mu(E) = \mu(E\exp(-t\mathbf{A}))$ for every $t \in \mathbb{R}$, $\mu \in \mathcal{M}((\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{R}})$, and $E \in \mathcal{B}((\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{R}})$.

Observe that if M is a lattice in $\mathrm{SL}(n,\mathbb{R})$, and if we set $\Gamma = M$ and $h = \exp(A)$, then $P_1^{(\mathbf{u})} = P_{(h)}^{(\mathrm{L})}$, $(S_1^{(\mathbf{u})}, T_1^{(\mathbf{u})}) = (S_{(h)}^{(\mathrm{L})}, T_{(h)}^{(\mathrm{L})})$, $P_1^{(\mathbf{v})} = P_{(h)}^{(\mathrm{R})}$, and $(S_1^{(\mathbf{v})}, T_1^{(\mathbf{v})}) = (S_{(h)}^{(\mathrm{R})}, T_{(h)}^{(\mathrm{R})})$, where the transition probabilities $P_{(h)}^{(\mathrm{L})}$ and $P_{(h)}^{(\mathrm{R})}$, and the Markov pairs $(S_{(h)}^{(\mathrm{L})}, T_{(h)}^{(\mathrm{L})})$ and $(S_{(h)}^{(\mathrm{R})}, T_{(h)}^{(\mathrm{R})})$ are defined by the maps $u_h^{(\mathrm{L})}$ and $u_h^{(\mathrm{R})}$, respectively, and are discussed in Example 1.1.11.

Since, as pointed out in Sect. B.4.2, the flows $\mathbf{u}$ and $\mathbf{v}$ are continuous, it follows that $(P_t^{(\mathbf{u})})_{t\in\mathbb{R}}$ and $(P_t^{(\mathbf{v})})_{t\in\mathbb{R}}$ satisfy the s.m.a. and are pointwise continuous Feller transition functions.

Note that if $\mathbf{v}$ is a unipotent flow, we obtain from one of Ratner's theorems stated in Sect. B.4.2 (Theorem B.4.10) that if M is a lattice in $\mathrm{SL}(n,\mathbb{R})$, then, for every $M\mathbf{X} \in (\mathrm{SL}(n,\mathbb{R})/M)_{\mathrm{R}}$, the orbit-closure of $M\mathbf{X}$ under the action of $(P_t^{(\mathbf{v})})_{t\in\mathbb{R}}$ (which is the same as the orbit-closure of $M\mathbf{X}$ under the action of $\mathbf{v}$) is homogeneous (see Sect. B.4.2 for the definition of a homogeneous orbit-closure under the action of a flow). ■

Example 2.2.9 (Semiflows on $\mathbb{S}_n$ and $\mathbb{P}_n$ and their Transition Functions). Our goal here is to discuss the transition functions and the families of Markov pairs generated by the two kinds of semiflows defined in Sect. B.4.1.

(a) Transition Functions of Semiflows on $\mathbb{S}_n$. Let $n \in \mathbb{N}$, $n \geq 2$, let $\mathbf{A} \in \mathbb{S}_n$, and let $\pi^{(\mathbf{A})} = (\pi_t^{(\mathbf{A})})_{t\in[0,+\infty)}$ be the semiflow defined in Sect. B.4.1. Thus, $\pi_t^{(\mathbf{A})}(\mathbf{X}) = (\exp_{\mathrm{s}}(t\mathbf{A}))\mathbf{X} = e^{-t}(\exp(t\mathbf{A}))\mathbf{X}$ for every $t \in [0,+\infty)$ and $\mathbf{X} \in \mathbb{S}_n$.

Let $(P_t^{(\pi^{(\mathbf{A})})})_{t\in[0,+\infty)}$ and $((S_t^{(\pi^{(\mathbf{A})})}, T_t^{(\pi^{(\mathbf{A})})}))_{t\in[0,+\infty)}$ be the transition function and the family of Markov pairs defined by $\pi^{(\mathbf{A})}$, respectively.

Then

$$P_t^{(\pi^{(\mathbf{A})})}(\mathbf{X}, E) = \delta_{e^{-t}(\exp(t\mathbf{A}))\mathbf{X}}(E) = \mathbf{1}_E(e^{-t}(\exp(t\mathbf{A}))\mathbf{X}) \\ = \mathbf{1}_{(e^{-t}(\exp(t\mathbf{A})))^{-1}E}(\mathbf{X})$$

for every $t \in [0,+\infty)$, $\mathbf{X} \in \mathbb{S}_n$, and every Borel subset E of $\mathbb{S}_n$, where, of course, the subset $(e^{-t}(\exp(t\mathbf{A})))^{-1}E$ of $\mathbb{S}_n$ is defined using formula (A.1.4); that is,

$$(e^{-t}\exp(t\mathbf{A}))^{-1}E = \{\mathbf{Y} \in \mathbb{S}_n \mid e^{-t}(\exp(t\mathbf{A}))\mathbf{Y} \in E\}$$

(note that $(e^{-t}\exp(t\mathbf{A}))^{-1}E$ is a Borel subset of $\mathbb{S}_n$ because $\mathbb{S}_n$ is a topological semigroup). We also obtain that $S_t^{(\pi^{(\mathbf{A})})}f(\mathbf{X}) = f(e^{-t}(\exp(t\mathbf{A}))\mathbf{X})$ for every $t \in [0,+\infty)$, $f \in B_b(\mathbb{S}_n)$, and $\mathbf{X} \in \mathbb{S}_n$, and that $T_t^{(\pi^{(\mathbf{A})})}\mu(E) = \mu((e^{-t}(\exp(t\mathbf{A})))^{-1}E)$ for every $t \in [0,+\infty)$, $\mu \in \mathcal{M}(\mathbb{S}_n)$, and $E \in \mathcal{B}(\mathbb{S}_n)$, where, as above, the meaning of the set $(e^{-t}(\exp(t\mathbf{A})))^{-1}E$ is given by the equality (A.1.4).

Note that, given $s \in [0,+\infty)$, if we set $\mathbf{B} = e^{-s}\exp(s\mathbf{A})$, then $P_s^{(\pi^{(\mathbf{A})})} = P_{\mathbf{B}}$, $S_s^{(\pi^{(\mathbf{A})})} = S_{\mathbf{B}}$, and $T_s^{(\pi^{(\mathbf{A})})} = T_{\mathbf{B}}$, where $P_{\mathbf{B}}$ and $(S_{\mathbf{B}}, T_{\mathbf{B}})$ are the transition probability and the Markov pair, respectively, defined by the function $u_{\mathbf{B}}$ discussed in Example 1.1.12.

Since, as pointed out when we defined $\pi^{(\mathbf{A})}$, after Proposition B.4.5, the semiflow $\pi^{(\mathbf{A})}$ is continuous, it follows that $(P_t^{(\pi^{(\mathbf{A})})})_{t\in[0,+\infty)}$ satisfies the s.m.a. and is a $C(\mathbb{S}_n)$-pointwise continuous Feller transition function.

Since, by Proposition B.4.6, the semiflow $\pi^{(\mathbf{A})}$ is equicontinuous with respect to $t \in [0, +\infty)$, using the general discussion about transition functions defined by one-parameter semigroups or groups of elements of $\mathbf{B}(X)$ in this section before Example 2.2.2, we obtain that the transition function $(P_t^{(\pi^{(\mathbf{A})})})_{t\in[0,+\infty)}$ is equicontinuous.

(*b*) Transition Functions of Semiflows on $\mathbb{P}_n$. Let $n \in \mathbb{N}$, $n \geq 2$, and let $\mathbf{A} \in \mathbb{S}_n$. Recall (see the discussion preceding Proposition B.4.7) that $\mathbb{P}_n$ is the compact metric space of all column stochastic vectors in $\mathbb{R}^n$ endowed with the metric d defined by $d(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} |x_i - y_i|$ for every column stochastic vectors $\mathbf{x} = (x_1, x_2, \cdots, x_n)^{\mathrm{T}}$ and $\mathbf{y} = (y_1, y_2, \cdots, y_n)^{\mathrm{T}}$. Let $\varphi^{(\mathbf{A})} = (\varphi_t^{(\mathbf{A})})_{t\in[0,+\infty)}$ be the semiflow discussed in Proposition B.4.7. Thus, $\varphi_t^{(\mathbf{A})}(\mathbf{x}) = e^{-t}\exp(t\mathbf{A})\mathbf{x}$ for every $t \in [0, +\infty)$, and $\mathbf{x} \in \mathbb{P}_n$.

Now, let $(P_t^{(\varphi^{(\mathbf{A})})})_{t\in[0,+\infty)}$ and $((S_t^{(\varphi^{(\mathbf{A})})}, T_t^{(\varphi^{(\mathbf{A})})}))_{t\in[0,+\infty)}$ be the transition function and the family of Markov pairs defined by $\varphi^{(\mathbf{A})}$, respectively.

Then,

$$P_t^{(\varphi^{(\mathbf{A})})}(\mathbf{x}, E) = \delta_{e^{-t}\exp(t\mathbf{A})\mathbf{x}}(E) = \mathbf{1}_E(e^{-t}\exp(t\mathbf{A})\mathbf{x}) = \mathbf{1}_{(e^{-t}\exp(t\mathbf{A}))^{-1}E}(\mathbf{x})$$

for every $t \in [0, +\infty)$, $\mathbf{x} \in \mathbb{P}_n$, and every Borel subset E of $\mathbb{P}_n$, where

$$(e^{-t}\exp(t\mathbf{A}))^{-1}E = \{\mathbf{y} \in \mathbb{P}_n \mid e^{-t}\exp(t\mathbf{A})\mathbf{y} \in E\};$$

$$S_t^{(\varphi^{(\mathbf{A})})} f(\mathbf{x}) = f(e^{-t}\exp(t\mathbf{A})\mathbf{x})$$

for every $t \in [0, +\infty)$, $f \in C(\mathbb{P}_n)$ and $\mathbf{x} \in \mathbb{P}_n$; $T_t^{(\varphi^{(\mathbf{A})})}\mu(E) = \mu((e^{-t}\exp(t\mathbf{A}))^{-1}E)$ for every $t \in [0, +\infty)$; $\mu \in \mathcal{M}(\mathbb{P}_n)$, and $E \in \mathcal{B}(\mathbb{P}_n)$, where $(e^{-t}\exp(t\mathbf{A}))^{-1}E$ has the same meaning as in the above definition of $P_t^{(\varphi^{(\mathbf{A})})}(\mathbf{x}, E)$.

Observe that if we set $\mathbf{B} = e^{-s}\exp(s\mathbf{A})$ for some $s \in [0, +\infty)$, then $P_s^{(\varphi^{(\mathbf{A})})} = P_{(\mathbf{B})}$, $S_s^{(\varphi^{(\mathbf{A})})} = S_{(\mathbf{B})}$, and $T_s^{(\varphi^{(\mathbf{A})})} = T_{(\mathbf{B})}$, where $P_{(\mathbf{B})}$ and $(S_{(\mathbf{B})}, T_{(\mathbf{B})})$ are the transition probability and the Markov pair, respectively, defined by the function $v_{\mathbf{B}}$ discussed in Example 1.1.13.

Taking into consideration that, by Proposition B.4.7, the semiflow $\varphi^{(\mathbf{A})}$ is continuous, we obtain that $(P_t^{(\varphi^{(\mathbf{A})})})_{t\in[0,+\infty)}$ satisfies the s.m.a. and is a $C(\mathbb{P}_n)$-pointwise continuous Feller transition function. Using Proposition B.4.7 again, we obtain that $\varphi^{(\mathbf{A})}$ is equicontinuous with respect to $t \in [0, +\infty)$, so the transition function $(P_t^{(\varphi^{(\mathbf{A})})})_{t\in[0,+\infty)}$ is equicontinuous. ∎

2.2.3 Transition Functions Defined by One-Parameter Convolution Semigroups of Probability Measures

As mentioned at the beginning of the section, we will now discuss transition functions defined by one-parameter convolution semigroups of probability measures.

To this end, let $(H, \cdot, d)$ be a locally compact separable metric semigroup and assume that H has a neutral element. We will denote by e the neutral element of H.

Let $(\mu_t)_{t\in[0,+\infty)}$ be a one-parameter convolution semigroup of probability measures defined on $(H, \mathcal{B}(H))$.

For every $t \in [0, +\infty)$ let P_t and (S_t, T_t) be the transition probability and the Markov pair defined by μ_t (see Example 1.1.16). Thus, for every $t \in [0, +\infty)$, P_t, S_t and T_t are defined as follows: $P_t(x, A) = (\mu_t * \delta_x)(A)$ for every $x \in H$ and $A \in \mathcal{B}(H)$, $S_t f(x) = \int_H f(zx)\, \mathrm{d}\mu_t(z)$ for every $f \in B_b(H)$ and $x \in H$, and $T_t \nu = \mu_t * \nu$ for every $\nu \in \mathcal{M}(H)$.

As expected, it turns out that the family $(P_t)_{t\in[0,+\infty)}$ is a transition function and, therefore, $((S_t, T_t))_{t\in[0,+\infty)}$ is the family of Markov pairs defined by $(P_t)_{t\in[0,+\infty)}$. We discuss the details in the next proposition.

Proposition 2.2.10. *The families* $(T_t)_{t\in[0,+\infty)}$ *and* $(S_t)_{t\in[0,+\infty)}$ *are one-parameter semigroups of operators, and* $(P_t)_{t\in[0,+\infty)}$ *is a transition function.*

Proof. The fact that $(T_t)_{t\in[0,+\infty)}$ is a one-parameter semigroup of operators was discussed at the end of Example 1.1.16 and is easy to see since

$$\begin{aligned} T_{u+t}\nu &= \mu_{u+t} * \nu = \mu_u * \mu_t * \nu \\ &= T_u(\mu_t * \nu) = T_u T_t \nu \end{aligned}$$

for every $u \in [0, +\infty)$, $t \in [0, +\infty)$, and $\nu \in \mathcal{M}(H)$.

Since

$$\begin{aligned} S_{u+t} f(x) &= \langle S_{u+t} f, \delta_x \rangle = \langle f, T_{u+t}\delta_x \rangle \\ &= \langle f, T_u T_t \delta_x \rangle = \langle f, T_t T_u \delta_x \rangle \\ &= \langle S_u S_t f, \delta_x \rangle \\ &= S_u S_t f(x) \end{aligned}$$

for every $u \in [0, +\infty)$, $t \in [0, +\infty)$, $f \in B_b(H)$, and $x \in H$, it follows that $(S_t)_{t\in[0,+\infty)}$ is also a one-parameter semigroups of operators.

Since $S_t \mathbf{1}_X = \mathbf{1}_X$ for every $t \in [0, +\infty)$ and since $\langle S_t f, \mu \rangle = \langle f, T_t \mu \rangle$ for every $t \in [0, +\infty)$, $f \in B_b(H)$, and $\mu \in \mathcal{M}(H)$, it follows that we can use Proposition 2.1.2 in order to infer that there exists a unique transition function that defines the family of Markov pairs $((S_t, T_t))_{t\in[0,+\infty)}$.

Taking into consideration that, as pointed out after Lemma 1.1.1, a Markov pair cannot be defined by two distinct transition probabilities, we obtain that $(P_t)_{t\in[0,+\infty)}$ is a transition function and defines the family of Markov pairs $((S_t, T_t))_{t\in[0,+\infty)}$. □

We will often refer to the transition function $(P_t)_{t\in[0,+\infty)}$ and the family $((S_t, T_t))_{t\in[0,+\infty)}$ discussed in Proposition 2.2.10 as the *transition function* and the *family of Markov pairs defined by* $(\mu_t)_{t\in[0,+\infty)}$.

As pointed out in Example 1.1.16, the transition probabilities constructed there define Markov-Feller pairs; consequently, if $(P_t)_{t\in[0,+\infty)}$ is the transition function defined by $(\mu_t)_{t\in[0,+\infty)}$, then $(P_t)_{t\in[0,+\infty)}$ is a Feller transition function.

Our goal now is to discuss a fairly general condition for the transition function $(P_t)_{t\in[0,+\infty)}$ defined by $(\mu_t)_{t\in[0,+\infty)}$ to satisfy the s.m.a. and to be $C_0(H)$-pointwise continuous. To this end, we need the following two lemmas.

Lemma 2.2.11. *Let K be a compact subset of H, let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence of elements of H, and set $x = \lim_{n\to\infty} x_n$. Then, for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists an $n_\varepsilon \in \mathbb{N}$ such that $d(yx_n, yx) < \varepsilon$ for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$, and every $y \in K$.*

For a proof of the lemma, see Lemma 4.1 of [144].

Lemma 2.2.12. *Let (X, d) be a locally compact separable metric space, let $(\mu_n)_{n\in\mathbb{N}}$ be a sequence of probability measures, $\mu_n \in \mathcal{M}(X)$ for every $n \in \mathbb{N}$, and assume that $(\mu_n)_{n\in\mathbb{N}}$ converges in the weak* topology of $\mathcal{M}(X)$ to a probability measure μ', $\mu' \in \mathcal{M}(X)$. Then the sequence $(\mu_n)_{n\in\mathbb{N}}$ is tight and $\{\mu'\}\cup\{\mu_n | n \in \mathbb{N}\}$ is a tight set of probability measures.*

For a proof of the above lemma, see, for instance, Observation on p. 96 of [143].

Note that, in the lemma, (X, d) is not necessarily a topological semigroup.

Proposition 2.2.13. *Let $(\mu_t)_{t\in[0,+\infty)}$ be a weak* continuous one-parameter convolution semigroup of probability measures defined on $(H, \mathcal{B}(H))$ (for the definition of the weak* continuity of such one-parameter convolution semigroups, see the beginning of Sect. B.3), and let $(P_t)_{t\in[0,+\infty)}$ and $((S_t, T_t))_{t\in[0,+\infty)}$ be the transition function and the family of Markov pairs defined by $(\mu_t)_{t\in[0,+\infty)}$, respectively. Then $(P_t)_{t\in[0,+\infty)}$ satisfies the s.m.a. and is $C_0(H)$-pointwise continuous.*

Proof. We will first prove that, for every $f \in C_0(H)$, the real valued mapping $(t, x) \mapsto S_t f(x)$, $(t, x) \in [0, +\infty) \times H$, is continuous with respect to the standard topology on $\mathbb{R}$ and the product topology $\mathcal{T}([0, +\infty)) \otimes \mathcal{T}(H)$ on $[0, +\infty) \times H$, where $\mathcal{T}([0, +\infty))$ and $\mathcal{T}(H)$ are the standard topology on $[0, +\infty)$ (the topology induced on $[0, +\infty)$ by the standard topology on $\mathbb{R}$) and the metric topology on H, respectively.

To this end, let $f \in C_0(H)$. Since the assertion is obviously true if $f = 0$, we may and do assume that $f \neq 0$.

Since, by Lemma B.2.5, the topology $\mathcal{T}([0,+\infty)) \otimes \mathcal{T}(H)$ is metrizable, it follows that in order to prove the continuity of the map $(t,x) \mapsto S_t f(x)$, $(t,x) \in [0,+\infty) \times H$, it is enough to prove that if $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ are convergent sequences of elements of $[0,+\infty)$ and H, respectively, then the sequence $(S_{t_n} f(x_n))_{n\in\mathbb{N}}$ converges to $S_{t'} f(x')$, where $t' = \lim_{n\to\infty} t_n$ and $x' = \lim_{n\to\infty} x_n$.

Thus, let $(t_n)_{n\in\mathbb{N}}$ and $(x_n)_{n\in\mathbb{N}}$ be two convergent sequences of elements of $[0,+\infty)$ and H, respectively, and set $t' = \lim_{n\to\infty} t_n$ and $x' = \lim_{n\to\infty} x_n$.

We have to prove that for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists an $n_\varepsilon \in \mathbb{N}$ such that $|S_{t_n} f(x_n) - S_{t'} f(x')| < \varepsilon$ for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$.

Accordingly, let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$.

Since $(\mu_t)_{t\in[0,+\infty)}$ is a weak* continuous one-parameter convolution semigroup, it follows that the sequence $(\mu_{t_n})_{n\in\mathbb{N}}$ converges in the weak* topology of $\mathcal{M}(H)$ to $\mu_{t'}$. Thus, using Lemma 2.2.12, we obtain that the set $\{\mu_{t'}\} \cup \{\mu_{t_n} \mid n \in \mathbb{N}\}$ is tight. Accordingly, there exists a compact subset K_1 of H such that $\mu_{t'}(H \setminus K_1) < \frac{\varepsilon}{4\|f\|}$ and $\mu_{t_n}(H \setminus K_1) < \frac{\varepsilon}{4\|f\|}$ for every $n \in \mathbb{N}$.

Let $K_2 = \{x'\} \cup \{x_n \mid n \in \mathbb{N}\}$.

Since K_2 is a compact subset of H, it follows that $K_1K_2 = \{yx \mid y \in K_1, x \in K_2\}$ is also a compact subset of H because the algebraic operation that defines the metric semigroup structure on H is continuous.

Since f is a real-valued continuous function on H, it follows that the restriction of f to K_1K_2 is uniformly continuous, so there exists a $\delta \in \mathbb{R}$, $\delta > 0$, such that $|f(z_1) - f(z_2)| < \frac{\varepsilon}{4}$ whenever $z_1 \in K_1K_2$, $z_2 \in K_1K_2$, and $d(z_1,z_2) < \delta$.

By Lemma 2.2.11, there exists an $n'_\varepsilon \in \mathbb{N}$ such that $d(yx_n, yx') < \delta$ for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$ and every $y \in K_1$.

Let $f_{x'} : H \to \mathbb{R}$ be defined by $f_{x'}(y) = f(yx')$ for every $y \in H$. Clearly, $f_{x'}$ is a continuous bounded function; that is, $f_{x'} \in C_b(H)$. Since the sequence $(\mu_{t_n})_{n\in\mathbb{N}}$ weak* converges to $\mu_{t'}$, and since $\mu_{t'}$ is a probability measure, it follows that $(\mu_{t_n})_{n\in\mathbb{N}}$ is also $C_b(H)$-weak* convergent to $\mu_{t'}$ (see the discussion preceding Proposition 1.1.5, or p. 71 of Högnäs and Mukherjea [48]). Accordingly, the sequence $(\langle f_{x'}, \mu_{t_n}\rangle)_{n\in\mathbb{N}}$ converges to $\langle f_{x'}, \mu_{t'}\rangle$, so there exists an $n_\varepsilon \in \mathbb{N}$, $n_\varepsilon \geq n'_\varepsilon$, such that $|\langle f_{x'}, \mu_{t_n}\rangle - \langle f_{x'}, \mu_{t'}\rangle| < \frac{\varepsilon}{4}$ for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$.

Using the definitions of $f_{x'}$, n_ε, K_1, δ and n'_ε, we obtain that

$$\left|S_{t_n} f(x_n) - S_{t'} f(x')\right| \leq \left|S_{t_n} f(x_n) - S_{t_n} f(x')\right| + \left|S_{t_n} f(x') - S_{t'} f(x')\right|$$

$$= \left|\int_H f(yx_n)\,\mathrm{d}\mu_{t_n}(y) - \int_H f(yx')\,\mathrm{d}\mu_{t_n}(y)\right|$$

$$+ \left|\int_H f(yx')\,\mathrm{d}\mu_{t_n}(y) - \int_H f(yx')\,\mathrm{d}\mu_{t'}(y)\right|$$

$$\leq \left|\int_{K_1} f(yx_n)\,\mathrm{d}\mu_{t_n}(y) - \int_{K_1} f(yx')\,\mathrm{d}\mu_{t_n}(y)\right|$$

$$
\begin{aligned}
&+ \left| \int_{H\setminus K_1} f(yx_n)\,\mathrm{d}\mu_{t_n}(y) - \int_{H\setminus K_1} f(yx')\,\mathrm{d}\mu_{t_n}(y) \right| + |\langle f_{x'}, \mu_{t_n}\rangle - \langle f_{x'}, \mu_{t'}\rangle| \\
&< \int_{K_1} \left| f(yx_n) - f(yx') \right| \mathrm{d}\mu_{t_n}(y) + \int_{H\setminus K_1} \left| f(yx_n) - f(yx') \right| \mathrm{d}\mu_{t_n}(y) + \frac{\varepsilon}{4} \\
&\le \frac{\varepsilon}{4} + 2\,\|f\|\,\mu_{t_n}(H \setminus K_1) + \frac{\varepsilon}{4} \\
&< \frac{\varepsilon}{4} + 2\,\|f\|\,\frac{\varepsilon}{4\,\|f\|} + \frac{\varepsilon}{4} = \frac{\varepsilon}{4} + \frac{\varepsilon}{2} + \frac{\varepsilon}{4} = \varepsilon
\end{aligned}
$$

for every $n \in \mathbb{N}$, $n \ge n_\varepsilon$.

We have therefore proved that, for every $f \in C_0(H)$, the map $(t,x) \mapsto S_t f(x)$, $(t,x) \in [0,+\infty) \times H$ is continuous with respect to the standard topology on $\mathbb{R}$ and the topology $\mathcal{T}([0,+\infty)) \otimes \mathcal{T}(H)$ on $[0,+\infty) \times H$. Thus, it is easy to see now that $(P_t)_{t\in[0,+\infty)}$ is $C_0(H)$-pointwise continuous.

In order to prove that $(P_t)_{t\in[0,+\infty)}$ satisfies the s.m.a., we will use a type of argument similar to that used in the proof of Proposition B.2.1. That is, using D32, p. 348 of Cohn [20] as in Proposition B.2.1, we obtain that H has a countable basis for its topology; since $[0,+\infty)$ has a countable basis for its topology, as well, it follows that we can use Proposition 7.6.2, p. 242 of Cohn [20] in order to infer that the σ-algebra, say $\mathcal{A}$, generated by $\mathcal{B}([0,+\infty)) \times \mathcal{B}(H)$ is equal to the Borel σ-algebra $\mathcal{B}([0,+\infty) \times H)$ generated by the open subsets of $[0,+\infty) \times H$ in the product topology of $[0,+\infty) \times H$. Since we have just proved that, for every $f \in C_0(H)$, the map $(t,x) \mapsto S_t f(x)$, $(t,x) \in [0,+\infty) \times H$, is continuous with respect to the product topology of $[0,+\infty) \times H$ (and the standard topology on $\mathbb{R}$), we obtain (using Proposition 2.1.1, p. 31 of Neveu [81]) that the map is measurable with respect to $\mathcal{B}([0,+\infty) \times H)$ (and $\mathcal{B}(\mathbb{R})$), so it is measurable with respect to $\mathcal{A}$ and $\mathcal{B}(\mathbb{R})$. Since $\mathcal{A} \subseteq \mathcal{L}([0,+\infty)) \otimes \mathcal{B}(H)$ (because $\mathcal{B}([0,+\infty)) \subseteq \mathcal{L}([0,+\infty))$), and using Proposition 2.1.5, we obtain that $(P_t)_{t\in[0,+\infty)}$ satisfies the s.m.a. □

We will now discuss a family of one-parameter convolution semigroups of probability measures and the transition functions that these one-parameter convolution semigroups define. We will often use these transition functions to illustrate various results in the book, especially in Chap. 7.

Example 2.2.14. Let $\mu \in \mathcal{M}(H)$ be a probability measure.

For every $t \in [0,+\infty)$, set $\mu_t = \exp_{\mathrm{s}}(t\mu)$; that is, $\mu_t = e^{-t}\exp(t\mu) = e^{-t}\sum_{k=0}^{\infty} \frac{t^k}{k!}\mu^k$.

By Propositions B.3.2 and B.3.3, $(\mu_t)_{t\in[0,+\infty)}$ is a weak* continuous one-parameter convolution semigroup of probability measures. Therefore, for every $t \in [0,+\infty)$, we can consider the transition probability P_t and the Markov pair (S_t, T_t) defined by μ_t as we did after Example 2.2.9 (see also Example 1.1.16), and the resulting families $(P_t)_{t\in[0,+\infty)}$ and $((S_t, T_t))_{t\in[0,+\infty)}$ are the transition function and the family of Markov pairs defined by $(\mu_t)_{t\in[0,+\infty)}$.

As pointed out before Lemma 2.2.11, $(P_t)_{t\in[0,+\infty)}$ is a Feller transition function and, by Proposition 2.2.13, $(P_t)_{t\in[0,+\infty)}$ satisfies the s.m.a. and is $C_0(H)$-pointwise continuous. ■

In view of the notions defined in Sect. 2.1 and of the manner in which we analyzed the examples discussed so far in this section, a natural question comes to mind: can one find a "nice enough" condition on a probability measure $\mu \in \mathcal{M}(H)$ that will guarantee that the transition function defined by $(\mu_t)_{t\in[0,+\infty)}$ is $C_0(H)$-equicontinuous, where $(\mu_t)_{t\in[0,+\infty)}$ is the exponential one-parameter convolution semigroup of probability measures defined by μ and discussed in Example 2.2.14? In turns out that if μ is an equicontinuous probability measure as defined before Example 1.4.29, then $(P_t)_{t\in[0,+\infty)}$ is a $C_0(H)$-equicontinuous transition function. We discuss the details in the next proposition.

Proposition 2.2.15. *Let $\mu \in \mathcal{M}(H)$ be an equicontinuous probability measure, let $(\mu_t)_{t\in[0,+\infty)}$ be the exponential one-parameter convolution semigroup of probability measures defined by μ, and let $(P_t)_{t\in[0,+\infty)}$ be the transition function defined by $(\mu_t)_{t\in[0,+\infty)}$. Then $(P_t)_{t\in[0,+\infty)}$ is a $C_0(H)$-equicontinuous transition function.*

Proof. Let $\mu \in \mathcal{M}(H)$, $(\mu_t)_{t\in[0,+\infty)}$, and $(P_t)_{t\in[0,+\infty)}$ be as in the proposition. Also let $((S_t, T_t))_{t\in[0,+\infty)}$ be the family of Markov pairs defined by $(P_t)_{t\in[0,+\infty)}$.

We have to prove that for every $f \in C_0(H)$, for every convergent sequence $(x_n)_{n\in\mathbb{N}}$ of elements of H, and for every $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$, there exists an $n_\varepsilon \in \mathbb{N}$ such that $|S_t f(x_n) - S_t f(x)| < \varepsilon$ for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$, and every $t \in [0, +\infty)$, where $x = \lim_{n\to\infty} x_n$.

To this end, let $f \in C_0(H)$, let $(x_n)_{n\in\mathbb{N}}$ be a convergent sequence of elements of H, set $x = \lim_{n\to\infty} x_n$, and let $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$. Also, let P_μ and (S_μ, T_μ) be the transition probability and the Markov-Feller pair defined by μ (see Example 1.1.16).

Since we assume that μ is an equicontinuous probability measure, we obtain that there exists an $n_\varepsilon \in \mathbb{N}$ such that

$$\left|S_\mu^k f(x_n) - S_\mu^k f(x)\right| < \frac{\varepsilon}{2} \tag{2.2.4}$$

for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$, and every $k \in \mathbb{N} \cup \{0\}$.

Using the definition of T_μ we obtain that

$$S_\mu^k f(y) = \left\langle S_\mu^k f, \delta_y \right\rangle = \left\langle f, T_\mu^k \delta_y \right\rangle = \langle f, \mu^k * \delta_y \rangle \tag{2.2.5}$$

for every $k \in \mathbb{N} \cup \{0\}$ and $y \in H$.

Since the sequence $(e^{-t} \sum_{k=0}^{l} \frac{t^k}{k!} \mu^k)_{l\in\mathbb{N}\cup\{0\}}$ converges in the norm topology of $\mathcal{M}(H)$ to μ_t, since the convolution operation defines a structure of topological semigroup on $\mathcal{M}(H)$ with respect to the norm topology of $\mathcal{M}(H)$ (so the operation of convolution is continuous with respect to the norm topology of $\mathcal{M}(H)$), and since the norm convergence implies the weak* convergence of a sequence of elements of $\mathcal{M}(H)$, we obtain that

$$\begin{aligned} S_t f(y) &= \langle S_t f, \delta_y \rangle = \langle f, T_t \delta_y \rangle = \langle f, \mu_t * \delta_y \rangle \\ &= \left\langle f, (e^{-t} \sum_{k=0}^{\infty} \frac{t^k}{k!} \mu^k) * \delta_y \right\rangle = \lim_{l \to \infty} \left\langle f, (e^{-t} \sum_{k=0}^{l} \frac{t^k}{k!} \mu^k) * \delta_y \right\rangle \\ &= \lim_{l \to \infty} e^{-t} \sum_{k=0}^{l} \frac{t^k}{k!} \langle f, \mu^k * \delta_y \rangle = \lim_{l \to \infty} e^{-t} \sum_{k=0}^{l} \frac{t^k}{k!} S_\mu^k f(y) \end{aligned} \tag{2.2.6}$$

for every $y \in H$ and $t \in [0, +\infty)$.

Using the above expressions (2.2.6), (2.2.5) and (2.2.3) (in this order), we obtain that:

$$\begin{aligned} |S_t f(x_n) - S_t f(x)| &= \left| \lim_{l \to \infty} e^{-t} \sum_{k=0}^{l} \frac{t^k}{k!} S_\mu^k f(x_n) - \lim_{l \to \infty} e^{-t} \sum_{k=0}^{l} \frac{t^k}{k!} S_\mu^k f(x) \right| \\ &= \lim_{l \to \infty} e^{-t} \left| \sum_{k=0}^{l} \frac{t^k}{k!} (S_\mu^k f(x_n) - S_\mu^k f(x)) \right| \\ &\leq \limsup_{l \to \infty} e^{-t} \sum_{k=0}^{l} \frac{t^k}{k!} \left| S_\mu^k f(x_n) - S_\mu^k f(x) \right| \\ &\leq \frac{\varepsilon}{2} e^{-t} \limsup_{l \to \infty} \sum_{k=0}^{l} \frac{t^k}{k!} = \frac{\varepsilon}{2} e^{-t} \lim_{l \to \infty} \sum_{k=0}^{l} \frac{t^k}{k!} \\ &= \frac{\varepsilon}{2} e^{-t} e^t = \frac{\varepsilon}{2} < \varepsilon \end{aligned}$$

for every $n \in \mathbb{N}$, $n \geq n_\varepsilon$, and every $t \in [0, +\infty)$.

Accordingly, $(P_t)_{t \in [0,+\infty)}$ is an equicontinuous transition function. □

It is important to realize that the transition functions discussed in this subsection are significantly different from the transition functions defined by one-parameter semigroups or groups of measurable functions.

For instance, if $(P_t)_{t \in \mathbb{T}}$ and $((S_t, T_t))_{t \in \mathbb{T}}$ are the transition function and the family of Markov pairs defined by a one-parameter semigroup or group $(w_t)_{t \in \mathbb{T}}$ of elements of $\mathbf{B}(X)$ for some locally compact separable metric space (X, d), respectively, then the probability measures $T_t \delta_x$, $t \in \mathbb{T}$, $x \in X$, are all Dirac measures and their supports are singletons (sets that contain exactly one element each). By contrast, if $(P_t)_{t \in [0,+\infty)}$ and $((S_t, T_t))_{t \in [0,+\infty)}$ are the transition function and the family of Markov pairs defined by an exponential one-parameter convolution semigroup $(\mu_t)_{t \in [0,+\infty)}$ of probability measures defined by a probability measure

$\mu \in \mathcal{M}(H)$ where H is a locally compact separable metric semigroup with unit, and if $A = \bigcup_{n=0}^{\infty} \text{supp}(\mu^n)$, then, for every $t \in [0, +\infty)$ and $x \in H$, the support of $T_t\delta_x$ is $\overline{Ax}$.

2.3 Invariant Probability Measures

After having introduced the transition functions in Sect. 2.1 and after having discussed several examples in Sect. 2.2, our goal in this section is to introduce the second main object studied in this book, namely the invariant probability measures of transition functions.

Let (X, d) be a locally compact separable metric space, and, as always in this book, let $\mathbb{T}$ stand for $\mathbb{R}$ or the interval $[0, +\infty)$.

Let $(P_t)_{t\in\mathbb{T}}$ be a transition function on (X, d), and let $((S_t, T_t))_{t\in\mathbb{T}}$ be the family of Markov pairs defined by $(P_t)_{t\in\mathbb{T}}$.

Given $\mu \in \mathcal{M}(X)$, we say that μ is an *invariant element for* $(P_t)_{t\in\mathbb{T}}$ (or *for* $((S_t, T_t))_{t\in\mathbb{T}}$, or *for* $(T_t)_{t\in\mathbb{T}}$) if $T_t\mu = \mu$ for every $t \in \mathbb{T}$.

Since the zero measure in $\mathcal{M}(X)$ is always an invariant element for $(P_t)_{t\in\mathbb{T}}$, the interesting situation is when $(P_t)_{t\in\mathbb{T}}$ also has nonzero invariant elements. Using arguments perfectly similar to those used for a Markov-Feller pair (and the associated Feller transition probability) at the beginning of the subsection *Invariant Probabilities of Markov-Feller Operators* of Section 1.2 on p. 17 of [143], we obtain that $(P_t)_{t\in\mathbb{T}}$ has nonzero invariant elements if and only if $(P_t)_{t\in\mathbb{T}}$ has invariant probability measures; that is, if and only if there exists a $\mu \in \mathcal{M}(X)$, $\mu \geq 0$, $\|\mu\| = 1$, such that $T_t\mu = \mu$ for every $t \in \mathbb{T}$.

As in the case of Markov-Feller pairs, when dealing with a transition function $(P_t)_{t\in\mathbb{T}}$, in order to understand the structure of the set of all invariant elements of $(P_t)_{t\in\mathbb{T}}$, in most cases it is enough to understand the structure of the set of invariant probabilities for $(P_t)_{t\in\mathbb{T}}$. Thus, in this book, we will mostly be interested in studying invariant probability measures for transition functions.

Note that if $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ is a semiflow or a flow on (X, d), and if $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ is the transition function defined by $\mathbf{w}$ (see the beginning of Sect. 2.2.1 for details), then $\mu \in \mathcal{M}(X)$ is an invariant element for $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ if and only if μ is an invariant element for $\mathbf{w}$ as defined in Appendix B before Example B.1.2.

Let us use the above observation to list a few examples of invariant probabilities.

Example 2.3.1. Let $\mathbf{w} = (w_t)_{t\in\mathbb{R}}$ be the flow of the rotations of the unit circle $\mathbb{R}/\mathbb{Z}$, and let $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ be the transition function of $\mathbf{w}$. Then the Haar-Lebesgue probability measure on $\mathbb{R}/\mathbb{Z}$ is an invariant probability for $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ because, as pointed out in Example B.1.6, the Haar-Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ is an invariant probability measure for $\mathbf{w}$. ■

Example 2.3.2. Let $n \in \mathbb{N}$, $n \geq 2$, let $\mathbf{v} \in \mathbb{R}^n$, let $\mathbf{w} = (w_t)_{t\in\mathbb{R}}$ be the rectilinear flow on the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$ with velocity $\mathbf{v}$, and let $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{w}$ (see Example 2.2.5). Then the Haar-Lebesgue probability measure on $\nu_{\mathbb{R}^n/\mathbb{Z}^n}$ on $\mathbb{R}^n/\mathbb{Z}^n$ is an invariant probability measure for $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ because, as discussed in Example B.1.7, $\nu_{\mathbb{R}^n/\mathbb{Z}^n}$ is an invariant probability for $\mathbf{w}$. ■

Example 2.3.3. Let Γ be a lattice in $\mathrm{PSL}(2,\mathbb{R})$, let $\mathbf{w}^{(\Gamma)} = (w_t^{(\Gamma)})_{t\in\mathbb{R}}$ be the geodesic flow on $(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_\mathrm{R}$ defined in Example B.1.8, and let $(P_t^{(\mathbf{w}^{(\Gamma)})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{w}^{(\Gamma)}$ (see (b) of Example 2.2.6). Then the standard $\mathrm{PSL}(2,\mathbb{R})$-invariant probability measure $\nu_{(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_\mathrm{R}}$ is an invariant measure for $\mathbf{w}^{(\Gamma)}$, so $\nu_{(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_\mathrm{R}}$ is also an invariant probability for $(P_t^{(\mathbf{w}^{(\Gamma)})})_{t\in\mathbb{R}}$. If $\overline{\mathbf{v}}^{(j\Gamma)} = (v_t^{(j\Gamma)})_{t\in\mathbb{R}}$, $j = 1,2$, are the horocycle flows defined in (a) of Example B.1.9 and $(P_t^{(\overline{\mathbf{v}}^{(j\Gamma)})})_{t\in\mathbb{R}}$, $j = 1,2$, are the two corresponding transition functions (see (b) of Example 2.2.7), then $\nu_{(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_\mathrm{R}}$ is an invariant probability for both transition functions $(P_t^{(\overline{\mathbf{v}}^{(j\Gamma)})})_{t\in\mathbb{R}}$, $j = 1,2$, because $\nu_{(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_\mathrm{R}}$ is an invariant measure for $\mathbf{v}^{(j\Gamma)}$, $j = 1,2$. ■

Example 2.3.4. Let $n \in \mathbb{N}$, $n \geq 2$, and let Γ be a lattice in $\mathrm{SL}(n,\mathbb{R})$. Let $\mathbf{w} = (w_t)_{t\in\mathbb{R}}$ be one of the four horocycle flows defined on $(\mathrm{SL}(n,\mathbb{R})/\Gamma)_\mathrm{S}$ where S stands for L or R, and $n = 2$ (see (b) of Example B.1.9 and (c) of Example 2.2.7), or assume that $\mathbf{w} = (w_t)_{t\in\mathbb{R}}$ is one of the exponential flows defined on $(\mathrm{SL}(n,\mathbb{R})/\Gamma)_\mathrm{S}$, where, again, S stands for L or R (see Sect. B.4.2 and Example 2.2.8). Now, let $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{w}$ on the corresponding space $(\mathrm{SL}(n,\mathbb{R})/\Gamma)_\mathrm{S}$; thus, $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is defined as in (c) of Example 2.2.7 if $\mathbf{w}$ is a horocycle flow, or $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is defined as in Example 2.2.8 if $\mathbf{w}$ is one of the exponential flows discussed in Sect. B.4.2. In all these cases the corresponding $\mathrm{SL}(n,\mathbb{R})$-invariant probability measure $\nu_{(\mathrm{SL}(n,\mathbb{R})/\Gamma)_\mathrm{S}}$ is an invariant probability for $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ because $\nu_{(\mathrm{SL}(n,\mathbb{R})/\Gamma)_\mathrm{S}}$ is an invariant measure for $\mathbf{w}$, where, of course, $(\mathrm{SL}(n,\mathbb{R})/\Gamma)_\mathrm{S}$ is the space on which $\mathbf{w}$ and $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ are defined. ■

Let $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ be a one-parameter semigroup or a one-parameter group of elements of $\mathbf{B}(X)$. As expected, an element μ of $\mathcal{M}(X)$ is said to be an *invariant element for* (or *of*) $\mathbf{w}$ if $\mu(w_t^{-1}(A)) = \mu(A)$ for every $t \in \mathbb{T}$ and $A \in \mathcal{B}(X)$. Clearly, the notion of an invariant element of a one-parameter semigroup or one-parameter group of elements of $\mathbf{B}(X)$ is a natural extension of the corresponding notion for a semiflow or a flow in the sense that if $\mathbf{w}$ is a semiflow or a flow and $\mu \in \mathcal{M}(X)$, then μ is an invariant element of $\mathbf{w}$ in the sense of the definition given in Appendix B before Example B.1.2, where we think of $\mathbf{w}$ as a semiflow or a flow, if and only if μ is an invariant element of $\mathbf{w}$, thought of as a one-parameter semigroup or a one-parameter group of elements of $\mathbf{B}(X)$, respectively.

Note that if $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ is a one-parameter semigroup or a one-parameter group of elements of $\mathbf{B}(X)$, if $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ is the transition function defined by $\mathbf{w}$ and if $\mu \in \mathcal{M}(X)$, then μ is an invariant element of $\mathbf{w}$ if and only if μ is an invariant element of $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$. The next example deals with such a situation.

Example 2.3.5. Let $\mathbf{w} = (w_t)_{t\in[0,+\infty)}$ be the one-parameter semigroup of elements of $\mathbf{B}([0,1])$ defined in Example 2.2.3 (note that, as pointed out there, $\mathbf{w}$ is not a semiflow), and let $(P_t^{(\mathbf{w})})_{t\in[0,+\infty)}$ be the transition function defined by $\mathbf{w}$. Then the Dirac measure δ_1 concentrated at 1 is an invariant probability measure for $(P_t^{(\mathbf{w})})_{t\in[0,+\infty)}$ because δ_1 is an invariant probability for $\mathbf{w}$. ■

We conclude this discussion of examples of invariant probabilities for transition functions by discussing certain invariant probabilities of transition functions defined by exponential one-parameter convolution semigroups of probability measures.

Example 2.3.6. Let $(H, \cdot, d)$ be a locally compact separable metric semigroup, and assume that H has a neutral element. Also, let $\mu \in \mathcal{M}(H)$ be a probability measure, let $(\mu_t)_{t\in[0,+\infty)}$ be the exponential one-parameter convolution semigroup of probability measures defined by μ (see the first paragraph after Proposition B.3.2), and let $(P_t)_{t\in[0,+\infty)}$ and $((S_t, T_t))_{t\in[0,+\infty)}$ be the transition function and the family of Markov pairs defined by $(\mu_t)_{t\in[0,+\infty)}$ (see Sect. 2.2.3).

Now let $\nu \in \mathcal{M}(H)$ be a probability measure which satisfies the Choquet-Deny equation $\mu * \nu = \nu$ (see the equality (1.4.1) and Section 1 of [144]). Then using the fact that $\mathcal{M}(H)$ is a Banach algebra when endowed with the operation of convolution (see Proposition B.2.4), we obtain that $e^{-t}\exp(t\mu) * \nu = \nu$ for every $t \in [0,+\infty)$. Accordingly, $T_t\nu = \nu$ for every $t \in [0,+\infty)$; that is, ν is an invariant probability measure for $(P_t)_{t\in[0,+\infty)}$. ■

If $\mu \in \mathcal{M}(X)$ is an invariant element for a transition function $(P_t)_{t\in\mathbb{T}}$, then, for every $t \in \mathbb{T}$, μ is an invariant element for the transition probability P_t, but if μ is an invariant element of one of the transition probabilities, say P_{t_0}, $t_0 \in \mathbb{T}$, that make up a transition function $(P_t)_{t\in\mathbb{T}}$, then, in general, μ is not an invariant element for the transition function (for instance, if $\mathbf{w} = (w_t)_{t\in\mathbb{R}}$ is the flow of the rotations of the unit circle $\mathbb{R}/\mathbb{Z}$ defined in Example A.3.4, and if $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ and $((S_t, T_t))_{t\in\mathbb{R}}$ are the transition function and the family of Markov pairs defined by $\mathbf{w}$, respectively (see Example 2.2.4), then, for every $s \in \mathbb{Z}$, $T_s^{(\mathbf{w})}$ is the identity operator on $\mathcal{M}(\mathbb{R}/\mathbb{Z})$, so the Dirac measure $\delta_{\hat{0}}$ concentrated at $\hat{0}$ is an invariant probability measure for the transition probability $P_s^{(\mathbf{w})}$ even though $\delta_{\hat{0}}$ is not an invariant element for the transition function $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ because if $s \in \mathbb{R}\setminus\mathbb{Z}$, then $\delta_{\hat{0}}$ is not an invariant measure for $P_s^{(\mathbf{w})}$). However, if $\mathbb{T} = \mathbb{R}$ and μ is an invariant element of the restriction $(P_t)_{t\in[0,+\infty)}$ of a transition function $(P_t)_{t\in\mathbb{R}}$ to $[0,+\infty)$, then μ is an invariant element for $(P_t)_{t\in\mathbb{R}}$. For future reference, we discuss this fact in detail in the next proposition.

Proposition 2.3.7. *Let $(P_t)_{t\in\mathbb{R}}$ be a transition function defined on (X, d) and let $\mu \in \mathcal{M}(X)$. The following assertions are equivalent:*

(a) *μ is an invariant element for $(P_t)_{t\in\mathbb{R}}$.*
(b) *μ is an invariant element of the restriction $(P_t)_{t\in[0,+\infty)}$ of $(P_t)_{t\in\mathbb{R}}$ to $[0,+\infty)$.*

Proof.

$(a) \Rightarrow (b)$ is obvious.

$(b) \Rightarrow (a)$. Let $((S_t, T_t))_{t\in\mathbb{R}}$ be the family of Markov pairs defined by $(P_t)_{t\in\mathbb{R}}$ and let $((S_t, T_t))_{t\in[0,+\infty)}$ be the restriction of $((S_t, T_t))_{t\in\mathbb{R}}$ to $[0, +\infty)$. Taking into consideration that we assume that $T_t\mu = \mu$ for every $t \geq 0$ and using the fact that $(T_t)_{t\in\mathbb{R}}$ is a one-parameter group of operators (see Proposition 2.1.1), we obtain that $T_t\mu = T_t(T_{-t}\mu) = T_0\mu = \mu$ for every $t \in \mathbb{R}$, $t < 0$.

Thus, μ is an invariant element for $(P_t)_{t\in\mathbb{R}}$. □

Note that a calculation similar to that used in the proof of $(b) \Rightarrow (a)$ in the above proposition can be used to show that if $(P_t)_{t\in\mathbb{R}}$ is a transition function which has the property that T_0 is the identity operator on $\mathcal{M}(X)$, where (S_0, T_0) is the Markov pair defined by the transition probability P_0, and if μ is an element of $\mathcal{M}(X)$, then, μ is an invariant element of the transition probability P_t if and only if μ is an invariant element of P_{-t} for every $t \in \mathbb{R}$. Therefore, for such a transition function $(P_t)_{t\in\mathbb{R}}$ and for $\mu \in \mathcal{M}(X)$, the following assertions are equivalent:

(a) μ is an invariant element for $(P_t)_{t\in\mathbb{R}}$.
(b) μ is an invariant element for $(P_t)_{t\in[0,+\infty)}$.
(c) For every $t \in \mathbb{R}$, $t > 0$, μ is an invariant element for the transition probability P_t.
(d) For every $t \in \mathbb{R}$, $t \leq 0$, μ is an invariant element for the transition probability P_t.
(e) For every $t \in \mathbb{R}$, $t < 0$, μ is an invariant element for the transition probability P_t.

Note that among the transition functions for which the above discussion makes sense are the transition functions defined by flows.

As in the case of transition probabilities, given a transition function $(P_t)_{t\in\mathbb{T}}$ defined on (X, d) and the family of Markov pairs $((S_t, T_t))_{t\in\mathbb{T}}$ generated by $(P_t)_{t\in\mathbb{T}}$, we say that $(P_t)_{t\in\mathbb{T}}$ (or $((S_t, T_t))_{t\in\mathbb{T}}$, or $(T_t)_{t\in\mathbb{T}}$) is *uniquely ergodic* if $(P_t)_{t\in\mathbb{T}}$ has nonzero invariant elements and has only one invariant probability measure. We say that $(P_t)_{t\in\mathbb{T}}$ (or $((S_t, T_t))_{t\in\mathbb{T}}$, or $(T_t)_{t\in\mathbb{T}}$) is *strictly ergodic* if $(P_t)_{t\in\mathbb{T}}$ is uniquely ergodic and the support of the unique invariant probability measure is the entire space X. Note that if $(P_t)_{t\in\mathbb{T}}$ is uniquely ergodic and if $\mu \in \mathcal{M}(X)$ is the unique invariant probability of $(P_t)_{t\in\mathbb{T}}$, then any invariant element $\nu \in \mathcal{M}(X)$ of $(P_t)_{t\in\mathbb{T}}$ is of the form $\nu = a\mu$ for some $a \in \mathbb{R}$.

As in the case of semiflows and flows (see the discussion preceding Example B.1.6), given a one-parameter semigroup or a one-parameter group $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ of elements of $\mathbf{B}(X)$, we say that $\mathbf{w}$ is *uniquely ergodic* if $\mathbf{w}$ has exactly one invariant probability measure. Clearly, the notion of unique ergodicity of one-parameter semigroups or groups of elements of $\mathbf{B}(X)$ is a natural extension of the corresponding notion for semiflows and flows, respectively. Also obvious is the fact that $\mathbf{w}$ is uniquely ergodic if and only if the transition function $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ defined by $\mathbf{w}$ is uniquely ergodic. The one-parameter semigroup or group $\mathbf{w}$ of elements of $\mathbf{B}(X)$ is said to be *strictly ergodic* if $\mathbf{w}$ is uniquely ergodic and the support of the unique invariant probability of $\mathbf{w}$ is the entire space X. Plainly, $\mathbf{w}$ is strictly ergodic if and only if the transition function $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ defined by $\mathbf{w}$ is strictly ergodic.

Example 2.3.8. Let $\mathbf{w} = (w_t)_{t\in\mathbb{R}}$ be the flow of the rotations of the unit circle $\mathbb{R}/\mathbb{Z}$, and let $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{w}$. Then, as pointed out in Example B.1.6, $\mathbf{w}$ is a uniquely ergodic flow and the Haar-Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ is the unique invariant probability of $\mathbf{w}$. Using the observations made before this example, we obtain that $\mathbf{w}$ and $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ are strictly ergodic and their unique invariant probability is the Haar-Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ (note that the fact that the Haar-Lebesgue measure on $\mathbb{R}/\mathbb{Z}$ is invariant for $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ was already pointed out in Example 2.3.1). ■

Example 2.3.9. Let $n \in \mathbb{N}$, $n \geq 2$, let $\mathbf{v} \in \mathbb{R}^n$, $\mathbf{v} = (v_1, v_2, \cdots, v_n)$, let $\mathbf{w} = (w_t)_{t\in\mathbb{R}}$ be the rectilinear flow on the n-dimensional torus $\mathbb{R}^n/\mathbb{Z}^n$ with velocity $\mathbf{v}$ defined in Example A.3.5, and let $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{w}$ (see Example 2.2.5).

As pointed out in Example B.1.7, the Haar-Lebesgue measure $\nu_{\mathbb{R}^n/\mathbb{Z}^n}$ on $\mathbb{R}^n/\mathbb{Z}^n$ is an invariant probability measure for $\mathbf{w}$, and $\mathbf{w}$ is uniquely ergodic if and only if the numbers $v_1, v_2, \cdots, v_n$ are rationally independent (see also Section 3.1 of Cornfeld, Fomin and Sinai's monograph [22]). Consequently, $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is uniquely ergodic if and only if the numbers $v_1, v_2, \cdots, v_n$ are rationally independent, and in this case $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is strictly ergodic because the unique invariant probability measure of $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is $\nu_{\mathbb{R}^n/\mathbb{Z}^n}$. ■

Example 2.3.10. Let Γ be a cocompact lattice in $\mathrm{SL}(2,\mathbb{R})$, let $\mathbf{v}^{(j\Gamma\mathrm{S})} = (v_t^{(j\Gamma\mathrm{S})})_{t\in\mathbb{R}}$ be one of the four horocycle flows considered at (c) of Example 2.2.7, where $j = 1$ or 2, and $\mathrm{S} = \mathrm{L}$ or R, and let $(P_t^{(\mathbf{v}^{(j\Gamma\mathrm{S})})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{v}^{(j\Gamma\mathrm{S})}$ (see (c) of Example 2.2.7). Then using the result of Furstenberg [37] discussed in (b) of Example B.1.9, we obtain that $(P_t^{(\mathbf{v}^{(j\Gamma\mathrm{S})})})_{t\in\mathbb{R}}$ is uniquely ergodic because, by the above-mentioned result of Furstenberg, the flow $\mathbf{v}^{(j\Gamma\mathrm{S})}$ is uniquely ergodic. Since, as pointed out in (b) of Example B.1.9, the standard $\mathrm{SL}(2,\mathbb{R})$-invariant probability measure $\nu_{(\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{S}}$ is an invariant measure for $\mathbf{v}^{(j\Gamma\mathrm{S})}$, it follows that $\nu_{(\mathrm{SL}(2,\mathbb{R})/\Gamma)_\mathrm{S}}$ is an invariant probability for $(P_t^{(\mathbf{v}^{(j\Gamma\mathrm{S})})})_{t\in\mathbb{R}}$, as well, so $(P_t^{(\mathbf{v}^{(j\Gamma\mathrm{S})})})_{t\in\mathbb{R}}$ is strictly ergodic. ■

Among the nonzero invariant measures of transition functions one can single out various kinds of such measures. For our purposes in this book, the most important type of nonzero invariant measures are the ergodic invariant probability measures.

In a similar manner as in the case of transition probabilities, given a transition function $(P_t)_{t\in\mathbb{T}}$ and a finite nonzero Borel measure μ on (X, d) such that μ is invariant for $(P_t)_{t\in\mathbb{T}}$, we say that μ is an *ergodic measure* if there is *no* Borel measurable subset A of X such that $\mu(A) > 0$ and $\mu(X \setminus A) > 0$, and such that the measures $\mu_1 : \mathcal{B}(X) \to \mathbb{R}$ and $\mu_2 : \mathcal{B}(X) \to \mathbb{R}$ defined by $\mu_1(B) = \mu(A \cap B)$ and $\mu_2(B) = \mu((X \setminus A) \cap B)$ for every $B \in \mathcal{B}(X)$ are both invariant for $(P_t)_{t\in\mathbb{T}}$.

If $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ is a one-parameter semigroup or a one-parameter group of elements of $\mathbf{B}(X)$, then a nonzero finite measure μ on $(X, \mathcal{B}(X))$ which is invariant for $\mathbf{w}$ is said to be *ergodic* if μ is ergodic for $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$, where, as usual, $(P_t^{(\mathbf{w})})_{t\in\mathbb{R}}$ is the transition function defined by $\mathbf{w}$. Note that the notion of ergodic measure defined here for one-parameter semigroups and one-parameter groups of elements of $\mathbf{B}(X)$ is a natural extension of the corresponding notion for semiflows and flows, respectively, defined in Appendix B before Example B.1.6.

It is possible to define a notion of ergodicity that is valid for measures that are not necessarily finite or invariant (see, for instance, p. 2 of Bachir Bekka and Mayer's monograph [10]). The more general notion is a natural extension of the notion of ergodic measure defined above. However, in this book we deal only with nonzero finite invariant ergodic measures, which will be probability measures most of the time.

There are many ways to define the ergodicity of a nonzero finite invariant measure and, generally, these definitions are equivalent if the settings are the same. For instance, we will see in Sect. 6.1 that if $\mathbf{w} = (w_t)_{t\in\mathbb{T}}$ is either a measurable flow, or a measurable semiflow which has the property that w_t is a surjective function for every $t \in [0, +\infty)$, and if $\mu \in \mathcal{M}(X)$ is an invariant probability measure for the transition function $(P_t^{(\mathbf{w})})_{t\in\mathbb{T}}$ defined by $\mathbf{w}$, then μ is an ergodic measure in the sense of the definition given in this section if and only if, in the terminology of Cornfeld, Fomin and Sinai's monograph [22], the dynamical system defined by $\mathbf{w}$ and μ is ergodic (see p. 14 of [22]); also in Sect. 6.1 it will be shown that if $\mathbf{w} = (w_t)_{t\in[0,+\infty)}$ is a measurable semiflow and $\mu \in \mathcal{M}(X)$ is a nonzero invariant measure, then μ is an ergodic measure as defined in this section if and only if, in the words of Stroock's book [119], p. 315, $\mathbf{w}$ is ergodic (with respect to μ).

Note that if $(P_t)_{t\in\mathbb{T}}$ is a transition function defined on (X, d), and if, for some $x \in X$, the Dirac measure δ_x is an invariant measure for $(P_t)_{t\in\mathbb{T}}$, then δ_x is ergodic. For instance, if $\mathbf{w}$ is the one-parameter semigroup defined in Example 2.2.3, and if $(P_t^{(\mathbf{w})})_{t\in[0,+\infty)}$ is the transition function generated by $\mathbf{w}$, then, as pointed out in Example 2.3.5, δ_1 is an invariant probability for $(P_t^{(\mathbf{w})})_{t\in[0,+\infty)}$, so δ_1 is an invariant ergodic measure for $(P_t^{(\mathbf{w})})_{t\in[0,+\infty)}$.

Observe that using the notion of ergodicity of a measure as defined in this section, we obtain that the unique invariant probability measure of a uniquely ergodic transition function is an ergodic measure. Actually, in Chap. 6 we will see that the following converse of the above observation holds true, as well: if a transition function $(P_t)_{t\in\mathbb{T}}$ has exactly one invariant ergodic probability measure, then $(P_t)_{t\in\mathbb{T}}$ is uniquely ergodic. Thus, since the invariant probability measure of a uniquely ergodic transition function is an ergodic measure, it follows that all the invariant probabilities of the uniquely ergodic transition functions discussed in Examples 2.3.8–2.3.10 are ergodic.

Of course, transition functions that are not uniquely ergodic can have invariant ergodic probability measures, as well (actually, as a straightforward consequence of the results obtained in Chaps. 5 and 6, we will see that every transition function that has nonzero invariant elements, also has invariant ergodic probability measures). Below are a few notable examples of invariant ergodic probabilities of transition functions that are not uniquely ergodic.

Example 2.3.11. Let Γ be a lattice in $\mathrm{PSL}(2,\mathbb{R})$, let $\mathbf{w}^{(\Gamma)} = (w_t^{(\Gamma)})_{t\in\mathbb{R}}$ be the geodesic flow on $(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}$ (see Example B.1.8), and let $(P_t^{(\mathbf{w}^{(\Gamma)})})_{t\in\mathbb{R}}$ be the transition function defined by $\mathbf{w}^{(\Gamma)}$. Then, as pointed out in Example B.1.8, the standard $\mathrm{PSL}(2,\mathbb{R})$-invariant probability measure $\nu_{(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}}$ is an invariant ergodic measure for $\mathbf{w}^{(\Gamma)}$ by a result of Hedlund [41]; therefore, $\nu_{(\mathrm{PSL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}}$ is an invariant ergodic probability measure for $(P_t^{(\mathbf{w}^{(\Gamma)})})_{t\in\mathbb{R}}$, as well. Note that, as mentioned in Example B.1.8, the geodesic flow is not uniquely ergodic, so $(P_t^{(\mathbf{w}^{(\Gamma)})})_{t\in\mathbb{R}}$ is not uniquely ergodic either. ■

Example 2.3.12. Let Γ be a lattice in $\mathrm{SL}(2,\mathbb{R})$, let $\mathbf{v}^{(j\Gamma\mathrm{L})}$ and $\mathbf{v}^{(j\Gamma\mathrm{R})}$, $j = 1, 2$, be the four horocycle flows defined in (b) of Example B.1.9, and let $(P_t^{(\mathbf{v}^{(j\Gamma\mathrm{L})})})_{t\in\mathbb{R}}$ and $(P_t^{(\mathbf{v}^{(j\Gamma\mathrm{R})})})_{t\in\mathbb{R}}$, $j = 1, 2$, be the corresponding four transition functions defined by these flows. Assume also that the four flows have periodic points (this happens if $\Gamma = \mathrm{SL}(2,\mathbb{Z})$, for instance; for details, see (b) of Example B.1.9). It follows that the invariant ergodic probability measures of the four transition functions are the invariant ergodic probabilities of the corresponding flows; that is, using a result of Dani [23] discussed in (b) of Example B.1.9, we obtain that these invariant ergodic probability measures are precisely the standard $\mathrm{SL}(2,\mathbb{R})$-invariant probabilities $\nu_{(\mathrm{SL}(2,\mathbb{R})/\Gamma)_{\mathrm{L}}}$ (for $(P_t^{(\mathbf{v}^{(j\Gamma\mathrm{L})})})_{t\in\mathbb{R}}$, $j = 1, 2$), and $\nu_{(\mathrm{SL}(2,\mathbb{R})/\Gamma)_{\mathrm{R}}}$ (for $(P_t^{(\mathbf{v}^{(j\Gamma\mathrm{R})})})_{t\in\mathbb{R}}$, $j = 1, 2$), and the invariant ergodic probability measures whose supports are the orbits of the periodic points of the corresponding flows. ■

In the next proposition, we discuss a property of (nonzero) ergodic measures that is similar to the property of the invariant elements of an $\mathbb{R}$-transition function discussed in Proposition 2.3.7.

Proposition 2.3.13. *Let $(P_t)_{t\in\mathbb{R}}$ be a transition function defined on (X, d), and let $\mu \in \mathcal{M}(X)$, $\mu \geq 0$, $\mu \neq 0$. The following assertions are equivalent:*

(a) *μ is an invariant ergodic measure for $(P_t)_{t\in\mathbb{R}}$.*
(b) *μ is an invariant ergodic measure for the restriction $(P_t)_{t\in[0,+\infty)}$ of $(P_t)_{t\in\mathbb{R}}$ to $[0, +\infty)$.*

Proof.

$(a) \Rightarrow (b)$ Since we assume that μ is an invariant measure for $(P_t)_{t\in\mathbb{R}}$, we obtain that μ is an invariant measure for $(P_t)_{t\in[0,+\infty)}$, as well.

Now, if we assume that μ is not an ergodic measure for $(P_t)_{t\in[0,+\infty)}$, then there exist two mutually singular nonzero measures ν_1 and ν_2 such that $\mu = \nu_1 + \nu_2$ and such that ν_1 and ν_2 are invariant measures for $(P_t)_{t\in[0,+\infty)}$. By Proposition 2.3.7,

ν_1 and ν_2 are also invariant measures for $(P_t)_{t\in\mathbb{R}}$. We have obtained a contradiction because we assume that (a) holds true.

$(b) \Rightarrow (a)$ Since we assume that μ is an invariant measure for $(P_t)_{t\in[0,+\infty)}$, using Proposition 2.3.7 again, we obtain that μ is also invariant for $(P_t)_{t\in\mathbb{R}}$. The proof of the implication is completed by noting that if we assume that μ fails to be an ergodic measure for $(P_t)_{t\in\mathbb{R}}$, then, obviously, μ fails to be ergodic for $(P_t)_{t\in[0,+\infty)}$, as well. □

We conclude the section (and the chapter) with a brief discussion of a notion that will appear frequently in Chaps. 5 and 6.

Let $(P_t)_{t\in\mathbb{T}}$ be a transition function defined on (X, d), and let $((S_t, T_t))_{t\in\mathbb{T}}$ be the family of Markov pairs defined by $(P_t)_{t\in\mathbb{T}}$.

A Borel measurable subset A of X is said to be a *set of maximal probability for* $(P_t)_{t\in\mathbb{T}}$ (or *for* $((S_t, T_t))_{t\in\mathbb{T}}$, or *for* $(T_t)_{t\in\mathbb{T}}$) if either $(P_t)_{t\in\mathbb{T}}$ does not have invariant probability measures, or else every invariant probability measure of $(P_t)_{t\in\mathbb{T}}$ is concentrated on A.

Thus, when studying the set of all invariant probabilities of a transition function $(P_t)_{t\in\mathbb{T}}$ in connection with the structure of the space (X, d) on which $(P_t)_{t\in\mathbb{T}}$ is defined, the sets of maximal probability are the sets "where the action is" and, naturally, we would like to find sets of maximal probability as "small" as possible. The relation of inclusion $\subseteq$ defined on $\mathcal{B}(X)$ is an order relation and $(\mathcal{B}(X), \subseteq)$ is a lattice; therefore, it makes sense to ask if the collection

$$\mathcal{A} = \{A \in \mathcal{B}(X) \mid A \text{ is a set of maximal probability for } (P_t)_{t\in\mathbb{T}}\}$$

has a minimum or at least an infimum in $\mathcal{B}(X)$. However, after a moment of reflection we realize that usually $\mathcal{A}$ has neither a minimum nor an infimum. In Chaps. 5 and 6 we will obtain various sets of maximal probability for transition functions that have invariant probabilities, sets which, for most purposes, are small enough.

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http://www.springer.com/978-3-319-05722-4

Invariant Probabilities of Transition Functions
Zaharopol, R.
2014, XVIII, 389 p., Hardcover
ISBN: 978-3-319-05722-4