Chapter 2
Set Theoretic Methods in Control

2.1 Set Terminology

For completeness, some standard definitions of set terminology will be introduced. For a detailed reference, the reader is referred to the book [72].

Definition 2.1 (Closed set) A set $S \subseteq \mathbb{R}^n$ is *closed* if it contains its own boundary. In other words, any point outside S has a neighborhood disjoint from S.

Definition 2.2 (Closure of a set) The *closure* of a set $S \subseteq \mathbb{R}^n$ is the intersection of all *closed sets* containing S.

Definition 2.3 (Bounded set) A set $S \subset \mathbb{R}^n$ is *bounded* if it is contained in some ball $B_R = \{x \in \mathbb{R}^n : \|x\|_2 \leq \varepsilon\}$ of finite radius $\varepsilon > 0$.

Definition 2.4 (Compact set) A set $S \subset \mathbb{R}^n$ is *compact* if it is closed and bounded.

Definition 2.5 (Support function) The support function of a set $S \subset \mathbb{R}^n$, evaluated at $z \in \mathbb{R}^n$ is defined as

$$\phi_S(z) = \sup_{x \in S} \{z^T x\}$$

2.2 Convex Sets

2.2.1 Basic Definitions

The fact that convexity is a more important property than linearity has been recognized in several domains, the optimization theory being maybe the best example [31, 106]. We provide in this section a series of definitions which will be useful in the sequel.

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DOI 10.1007/978-3-319-02827-9_2,

Definition 2.6 (Convex set) A set $S \subset \mathbb{R}^n$ is *convex* if it holds that, $\forall x_1 \in S$ and $\forall x_2 \in S$,

$$\alpha x_1 + (1-\alpha)x_2 \in S, \quad \forall \alpha \in [0, 1]$$

The point

$$x = \alpha x_1 + (1-\alpha)x_2$$

where $0 \le \alpha \le 1$ is called a *convex combination* of the pair $\{x_1, x_2\}$. The set of all such points is the line segment connecting x_1 and x_2. Obviously, a set S is convex if a segment between any two points in S lies in S.

Definition 2.7 (Convex function) A function $f : S \to \mathbb{R}$ with a convex set $S \subseteq \mathbb{R}^n$ is *convex* if and only if, $\forall x_1 \in S$, $\forall x_2 \in S$ and $\forall \alpha \in [0, 1]$,

$$f\big(\alpha x_1 + (1-\alpha)x_2\big) \le \alpha f(x_1) + (1-\alpha) f(x_2)$$

Definition 2.8 (C-set) A set $S \subset \mathbb{R}^n$ is a C-set if it is a *convex* and *compact* set, containing the origin in its interior.

Definition 2.9 (Convex hull) The *convex hull* of a set $S \subset \mathbb{R}^n$ is the smallest convex set containing S.

It is well known [133] that for any finite set $S = \{s_1, s_2, \dots, s_r\}$, where $s_i \in \mathbb{R}^n$, $i = 1, 2, \dots, r$, the convex hull of S is given by

$$\text{Convex Hull}(S) = \left\{ s \in \mathbb{R}^n : s = \sum_{i=1}^{r} \alpha_i s_i : \forall s_i \in S \right\}$$

where $\sum_{i=1}^{r} \alpha_i = 1$ and $\alpha_i \ge 0$, $i = 1, 2, \dots, r$.

2.2.2 Ellipsoidal Set

Ellipsoidal sets or ellipsoids are one of the famous classes of convex sets. Ellipsoids represent a large category used in the study of dynamical systems due to their simple numerical representation [32, 75]. Next we provide a formal definition for ellipsoids and a few properties.

Definition 2.10 (Ellipsoidal set) An ellipsoid $E(P, x_0) \subset \mathbb{R}^n$ with center x_0 and shape matrix P is a set of the form,

$$E(P, x_0) = \left\{ x \in \mathbb{R}^n : (x - x_0)^T P^{-1} (x - x_0) \le 1 \right\} \tag{2.1}$$

where $P \in \mathbb{R}^{n \times n}$ is a positive definite matrix.

If $x_0 = 0$ then it is possible to write,

$$E(P) = \left\{x \in \mathbb{R}^n : x^T P^{-1} x \leq 1\right\} \tag{2.2}$$

Define $Q = \sqrt{P}$ as the Cholesky factor of matrix P, which satisfies

$$Q^T Q = Q Q^T = P$$

With matrix Q, it is possible to show an alternative dual representation of ellipsoid (2.1)

$$D(Q, x_0) = \left\{x \in \mathbb{R}^n : x = x_0 + Qz\right\}$$

where $z \in \mathbb{R}^n$ such that $z^T z \leq 1$.

Ellipsoids are probably the most commonly used in the control field since they are associated with powerful tools such as Linear Matrix Inequalities (LMI) [32, 112]. When using ellipsoids, almost all the control optimization problems can be reduced to the optimization of a linear function under LMI constraints. This optimization problem is convex and is by now a powerful design tool in many control applications.

A linear matrix inequality is a condition of the type [32, 112],

$$F(x) \succeq 0$$

where $x \in \mathbb{R}^n$ is a vector variable and

$$F(x) = F_0 + \sum_{i=1}^{n} F_i x_i$$

with symmetric matrices $F_i \in \mathbb{R}^{m \times m}$.

LMIs can either be understood as feasibility conditions or constraints for optimization problems. Optimization of a linear function over LMI constraints is called semi-definite programming, which is considered as an extension of linear programming. Nowadays, a major benefit in using LMIs is that for solving an LMI problem, several polynomial time algorithms were developed and implemented in free available software packages, such as LMI Lab [43], YALMIP [87], CVX [49], SEDUMI [121], etc.

The Schur complements are a very useful tool for manipulating matrix inequalities. The Schur complements state that the nonlinear conditions of the special forms,

$$\begin{cases} P(x) \succ 0 \\ P(x) - Q(x)^T R(x)^{-1} Q(x) \succ 0 \end{cases} \tag{2.3}$$

or

$$\begin{cases} R(x) \succ 0 \\ R(x) - Q(x) P(x)^{-1} Q(x)^T \succ 0 \end{cases} \tag{2.4}$$

can be equivalently written in the LMI form,

$$\begin{bmatrix} P(x) & Q(x)^T \\ Q(x) & R(x) \end{bmatrix} \succ 0 \tag{2.5}$$

The Schur complements allows one to convert certain nonlinear matrix inequalities into a higher dimensional LMI. For example, it is well known [75] that the support function of the ellipsoid $E(P)$, evaluated at the vector $f \in \mathbb{R}^n$ is,

$$\phi_{E(P)}(z) = \sqrt{f^T P f} \tag{2.6}$$

then clearly, $E(P)$ is a subset of the polyhedral set[1] $\mathscr{P}(f, 1)$, where

$$\mathscr{P}(f, 1) = \left\{x \in \mathbb{R}^n : \left|f^T x\right| \leq 1\right\}$$

if and only if

$$f^T P f \leq 1$$

or by using the Schur complements this condition can be rewritten as [32, 55],

$$\begin{bmatrix} 1 & f^T P \\ Pf & P \end{bmatrix} \succeq 0 \tag{2.7}$$

An ellipsoid $E(P, x_0) \subset \mathbb{R}^n$ is uniquely defined by its matrix P and by its center x_0. Since matrix P is symmetric, the complexity of the representation (2.1) is

$$\frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}$$

The main drawback of ellipsoids is however that having a fixed and symmetrical structure they may be too conservative and this conservativeness is increased by the related operations. It is well known [75] that[2]

- The convex hull of of a set of ellipsoids, in general, is not an ellipsoid.
- The sum of two ellipsoids is not, in general, an ellipsoid.
- The difference of two ellipsoids is not, in general, an ellipsoid.
- The intersection of two ellipsoids is not, in general, an ellipsoid.

2.2.3 Polyhedral Set

Polyhedral sets provide a useful geometrical representation for linear constraints that appear in diverse fields such as control and optimization. In a convex setting,

[1] A rigorous definition of polyhedral sets will be given in Sect. 2.2.3.

[2] The reader is referred to [75] for the definitions of operations with ellipsoids.

they provide a good compromise between complexity and flexibility. Due to their linear and convex nature, the basic set operations are relatively easy to implement [76, 129]. Principally, this is related to their dual (half-spaces/vertices) representation [33, 93] which allows choosing which formulation is best suited for a particular problem. This section is started by recalling some theoretical concepts.

Definition 2.11 (Hyperplane) A hyperplane $H(f, g)$ is a set of the form,

$$H(f, g) = \left\{x \in \mathbb{R}^n : f^T x = g\right\} \tag{2.8}$$

where $f \in \mathbb{R}^n$, $g \in \mathbb{R}$.

Definition 2.12 (Half-space) A closed half-space $\mathscr{H}(f, g)$ is a set of the form,

$$\mathscr{H}(f, g) = \left\{x \in \mathbb{R}^n : f^T x \leq g\right\} \tag{2.9}$$

where $f \in \mathbb{R}^n$, $g \in \mathbb{R}$.

Definition 2.13 (Polyhedral set) A convex polyhedral set $P(F, g)$ is a set of the form,

$$P(F, g) = \left\{x \in \mathbb{R}^n : F_i^T x \leq g_i,\ i = 1, 2, \ldots, n_1\right\} \tag{2.10}$$

where $F_i^T \in \mathbb{R}^n$ denotes the i-th row of the matrix $F \in \mathbb{R}^{n_1 \times n}$ and g_i is the i-th component of the column vector $g \in \mathbb{R}^{n_1}$.

A polyhedral set contains the origin if and only if $g \geq 0$, and includes the origin in its interior if and only if $g > 0$.

Definition 2.14 (Polytope) A polytope is a *bounded* polyhedral set.

Definition 2.15 (Dimension of polytope) A polytope $P(F, g) \subset \mathbb{R}^n$ is of dimension $d \leq n$, if there exists a d-dimension ball with radius $\varepsilon > 0$ contained in $P(F, g)$ and there exists no $(d + 1)$-dimension ball with radius $\varepsilon > 0$ contained in $P(F, g)$. A polytope is full dimensional if $d = n$.

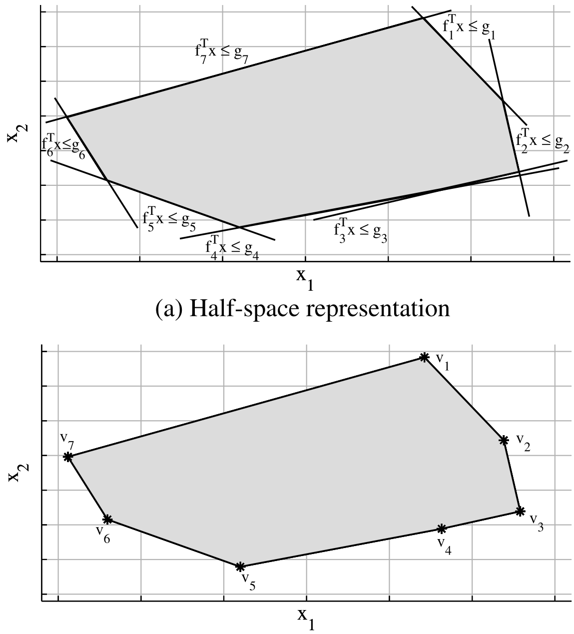
Definition 2.16 (Face, facet, vertex, edge) An $(n - 1)$-dimensional face F_{ai} of polytope $P(F, g) \subset \mathbb{R}^n$ is defined as,

$$F_{ai} = \left\{x \in P : F_i^T x = g_i\right\} \tag{2.11}$$

and can be interpreted as the intersection between P and a *non-redundant supporting hyperplane*

$$F_{ai} = P \cap \left\{x \in \mathbb{R}^n : F_i^T x = g_i\right\} \tag{2.12}$$

The non-empty intersection of two faces of dimension $(n - r)$ with $r = 0, 1, \ldots, n - 1$ leads to the description of $(n - r - 1)$-dimensional face. The faces of $P(F, g)$ with dimension 0, 1 and $(n - 1)$ are called *vertices*, *edges* and *facets*, respectively.



(a) Half-space representation

(b) Vertex representation

Fig. 2.1 Exemplification for the equivalent of half-space and vertex representations of polytopes

One of the fundamental properties of polytope is that it can be presented in half-space representation as in Definition 2.13 or in vertex representation as

$$P(V) = \left\{ x \in \mathbb{R}^n : x = \sum_{i=1}^{r} \alpha_i v_i \right\}$$

where $v_i \in \mathbb{R}^n$ is the i-column of matrix $V \in \mathbb{R}^{n \times r}$, $\sum_{i=1}^{r} \alpha_i = 1$ and $\alpha_i \geq 0$, $i = 1, 2, \ldots, r$, see Fig. 2.1.

Note that the transformation from half-space (vertex) representation to vertex (half-space) representation may be time-consuming with several well-known algorithms: Fourier-Motzkin elimination [39], CDD [41], Equality Set Projection [64].

Recall that the expression $x = \sum_{i=1}^{r} \alpha_i v_i$ with a given set of vectors $\{v_1, v_2, \ldots, v_r\}$ and

$$\sum_{i=1}^{r} \alpha_i = 1, \quad \alpha_i \geq 0$$

is called *the convex hull* of the set $\{v_1, v_2, \ldots, v_r\}$ and will be denoted as

$$x = \mathrm{Conv}\{v_1, v_2, \ldots, v_r\}$$

Definition 2.17 (Simplex) A simplex $S \subset \mathbb{R}^n$ is an n-dimensional polytope, which is the convex hull of $n + 1$ vertices.

For example, a $2D$-simplex is a triangle, a $3D$-simplex is a tetrahedron, and a $4D$-simplex is a pentachoron.

Definition 2.18 (Redundant half-space) For a given polytope $P(F,g)$, the polyhedral set $P(\overline{F},\overline{g})$ is defined by removing the i-th half-space F_i^T from matrix F and the corresponding component g_i from vector g. The facet (F_i^T, g_i) is *redundant* if and only if

$$\overline{g_i} < g_i \tag{2.13}$$

where

$$\overline{g_i} = \max_x \{F_i^T x\} \quad \text{s.t. } x \in P(\overline{F},\overline{g})$$

Definition 2.19 (Redundant vertex) For a given polytope $P(V)$, the polyhedral set $P(\overline{V})$ is defined by removing the i-th vertex v_i from the matrix V. The vertex v_i is *redundant* if and only if

$$p_i < 1 \tag{2.14}$$

where

$$p_i = \min_p \{1^T p\} \quad \text{s.t. } \begin{cases} \overline{V} p = v_i, \\ p \geq 0 \end{cases}$$

Definition 2.20 (Minimal representation) A half-space or vertex representation of polytope P is *minimal* if and only if the removal of any facet or any vertex would change P, i.e. there are no redundant facets or redundant vertices.

Clearly, a minimal representation of a polytope can be achieved by removing from the half-space (vertex) representation all the redundant facets (vertices).

Definition 2.21 (Normalized representation) A polytope

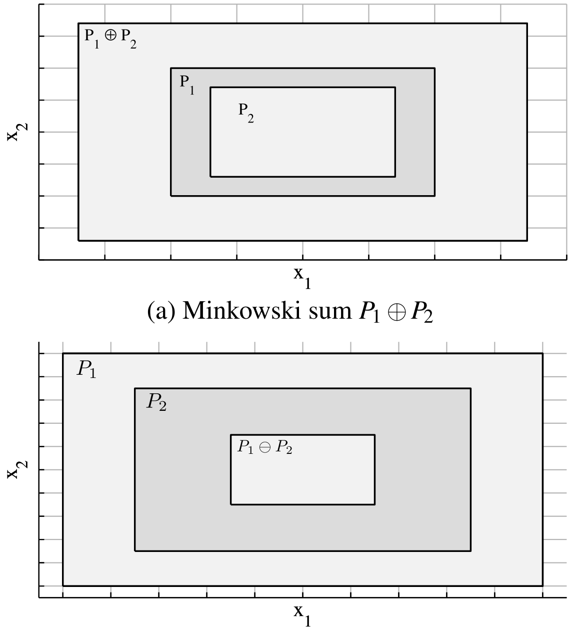
$$P(F,g) = \{x \in \mathbb{R}^n : F_i^T x \leq g_i,\ i = 1,2,\dots,n_1\}$$

is in a normalized representation if it has the following property

$$F_i^T F_i = 1, \quad \forall i = 1,2,\dots,n_1$$

A normalized full dimensional polytope has a *unique* minimal representation. This fact is very useful in practice, since normalized full dimensional polytopes in minimal representation allow us to avoid any ambiguity when comparing them.

Next, some basic operations on polytopes will be briefly reviewed. Note that although the focus lies on polytopes, most of the operations described here are directly or with minor changes applicable to polyhedral sets. Additional details on polytope computation can be found in [42, 51, 133].



(a) Minkowski sum $P_1 \oplus P_2$

(b) Pontryagin difference $P_1 \ominus P_2$.

Fig. 2.2 Minkowski sum and Pontryagin difference of polytopes

Definition 2.22 (Intersection) The intersection of two polytopes $P_1 \subset \mathbb{R}^n$, $P_2 \subset \mathbb{R}^n$ is a polytope,

$$P_1 \cap P_2 = \left\{x \in \mathbb{R}^n : x \in P_1, x \in P_2\right\}$$

Definition 2.23 (Minkowski sum) The Minkowski sum of two polytopes $P_1 \subset \mathbb{R}^n$, $P_2 \subset \mathbb{R}^n$ is a polytope, see Fig. 2.2(a),

$$P_1 \oplus P_2 = \{x_1 + x_2 : x_1 \in P_1, x_2 \in P_2\}$$

It is well known [133] that if P_1 and P_2 are given in vertex representation, i.e.

$$\begin{aligned} P_1 &= \mathrm{Conv}\{v_{11}, v_{12}, \ldots, v_{1p}\}, \\ P_2 &= \mathrm{Conv}\{v_{21}, v_{22}, \ldots, v_{2q}\} \end{aligned}$$

then their Minkowski sum can be computed as,

$$P_1 \oplus P_2 = \mathrm{Conv}\{v_{1i} + v_{2j}\}, \quad \forall i = 1, 2, \ldots, p, \ \forall j = 1, 2, \ldots, q$$

Definition 2.24 (Pontryagin difference) The Pontryagin difference of two polytopes $P_1 \subset \mathbb{R}^n$, $P_2 \subset \mathbb{R}^n$ is the polytope, see Fig. 2.2(b),

$$P_1 \ominus P_2 = \{x_1 \in P_1 : x_1 + x_2 \in P_1, \forall x_2 \in P_2\}$$

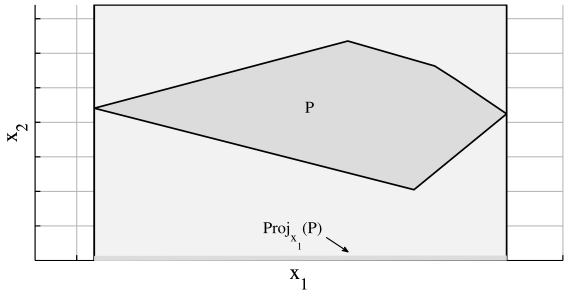


Fig. 2.3 Projection of a 2-dimensional polytope P onto a line x_1

Note that the Pontryagin difference is not the complement of the Minkowski sum. For two polytopes P_1 and P_2, it holds only that $(P_1 \ominus P_2) \oplus P_2 \subseteq P_1$.

Definition 2.25 (Projection) Given a polytope $P \subset \mathbb{R}^{n_1+n_2}$, the orthogonal projection of P onto the x_1-space $\mathbb{R}^{n_1}$ is defined as, see Fig. 2.3,

$$\mathrm{Proj}_{x_1}(P) = \left\{x_1 \in \mathbb{R}^{n_1} : \exists x_2 \in \mathbb{R}^{n_2} \text{ such that } \left[x_1^T \; x_2^T\right]^T \in P\right\}$$

It is well known [133] that the Minkowski sum operation on polytopes in their half-space representation is complexity-wise equivalent to a projection. Current projection methods for polytopes that can operate in general dimensions can be grouped into four classes: Fourier elimination [66], block elimination [12], vertex based approaches and wrapping-based techniques [64].

Clearly, the complexity of the representation of polytopes is not a function of the space dimension only, but it may be arbitrarily big. For the half-space (vertex) representation, the complexity of the polytopes is a linear function of the number of rows of the matrix F (the number of columns of the matrix V). As far as the complexity issue concerns, it is worth to be mentioned that none of these representations can be regarded as more convenient. Apparently, one can define an arbitrary polytope with relatively few vertices, however this may nevertheless have a surprisingly large number of facets. This happens, for example when some vertices contribute to many facets. And equally, one can define an arbitrary polytope with relatively few facets, however this may have relatively many more vertices. This happens, for example when some facets have many vertices [42].

The main advantage of the polytopes is their flexibility. It is well known [33] that any convex body can be approximated arbitrarily close by a polytope. Particularly, for a given bounded, convex and closed set S and for a given scalar $0 < \varepsilon < 1$, then there exists a polytope P such that,

$$(1-\varepsilon)S \subseteq P \subseteq S$$

for an inner ε-approximation of the set S and

$$S \subseteq P \subseteq (1+\varepsilon)S$$

for an outer ε-approximation of the set S.

2.3 Set Invariance Theory

2.3.1 Problem Formulation

Consider the following uncertain and/or time-varying linear discrete-time system,

$$x(k+1) = A(k)x(k) + B(k)u(k) + Dw(k) \tag{2.15}$$

where $x(k) \in \mathbb{R}^n$, $u(k) \in \mathbb{R}^m$, $w(k) \in \mathbb{R}^d$ are, respectively the state, input and disturbance vectors. The matrices $A(k) \in \mathbb{R}^{n\times n}$, $B(k) \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{n\times d}$. $A(k)$ and $B(k)$ satisfy,

$$\begin{cases} A(k) = \sum_{i=1}^{q} \alpha_i(k) A_i, \qquad B(k) = \sum_{i=1}^{q} \alpha_i(k) B_i \\ \sum_{i=1}^{q} \alpha_i(k) = 1, \quad \alpha_i(k) \geq 0 \end{cases} \tag{2.16}$$

where the matrices A_i, B_i, $i = 1, 2, \ldots, q$ are the extreme realizations of $A(k)$ and $B(k)$.

Theorem 2.1 *$A(k)$, $B(k)$ given as,*

$$\begin{cases} A(k) = \sum_{i=1}^{q_1} \alpha_i(k) A_i, \qquad B(k) = \sum_{j=1}^{q_2} \beta_j(k) B_j, \\ \sum_{i=1}^{q_1} \alpha_i(k) = 1, \quad \alpha_i(k) \geq 0, \ \forall i = 1, 2, \ldots, q_1, \\ \sum_{j=1}^{q_2} \beta_j(k) = 1, \quad \beta_j(k) \geq 0, \ \forall j = 1, 2, \ldots, q_2 \end{cases} \tag{2.17}$$

can be transformed into the form of (2.16).

Proof For simplicity, the case $D = 0$ is considered. One has,

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{q_1} \alpha_i(k) A_i x(k) + \sum_{j=1}^{q_2} \beta_j(k) B_j u(k) \\ &= \sum_{i=1}^{q_1} \alpha_i(k) A_i x(k) + \sum_{i=1}^{q_1} \alpha_i(k) \sum_{j=1}^{q_2} \beta_j(k) B_j u(k) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{q_1} \alpha_i(k) \left\{ A_i x(k) + \sum_{j=1}^{q_2} \beta_j(k) B_j u(k) \right\} \\
&= \sum_{i=1}^{q_1} \alpha_i(k) \left\{ \sum_{j=1}^{q_2} \beta_j(k) A_i x(k) + \sum_{j=1}^{q_2} \beta_j(k) B_j u(k) \right\} \\
&= \sum_{i=1}^{q_1} \alpha_i(k) \sum_{j=1}^{q_2} \beta_j(k) \left\{ A_i x(k) + B_j u(k) \right\} \\
&= \sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \alpha_i(k) \beta_j(k) \left\{ A_i x(k) + B_j u(k) \right\}
\end{aligned}
$$

Consider the polytope Q_c whose vertices are obtained by taking all possible combinations of $\{A_i, B_j\}$ with $i = 1, 2, \ldots, q_1$ and $j = 1, 2, \ldots, q_2$. Since

$$
\sum_{i=1}^{q_1} \sum_{j=1}^{q_2} \alpha_i(k) \beta_j(k) = \sum_{i=1}^{q_1} \alpha_i(k) \sum_{j=1}^{q_2} \beta_j(k) = 1
$$

it follows that $\{A(k), B(k)\}$ can be expressed as a convex combination of the vertices of Q_c. □

The state, the control and the disturbance are subject to the following polytopic constraints,

$$
\begin{cases}
x(k) \in X, & X = \left\{ x \in \mathbb{R}^n : F_x x \le g_x \right\} \\
u(k) \in U, & U = \left\{ u \in \mathbb{R}^m : F_u u \le g_u \right\} \\
w(k) \in W, & W = \left\{ w \in \mathbb{R}^d : F_w w \le g_w \right\}
\end{cases}
\tag{2.18}
$$

where the matrices F_x, F_u, F_w and the vectors g_x, g_u, g_w are assumed to be constant with $g_x > 0$, $g_u > 0$, $g_w > 0$ such that the origin is contained in the interior of X, U and W. Here the inequalities are element-wise.

The aim of this section is to briefly review the set invariance theory, whose definitions are reported in the next subsection.

2.3.2 Basic Definitions

The relationship between the dynamic (2.15) and constraints (2.18) leads to the introduction of invariance/viability concepts [9, 28]. First we consider the case when no inputs are present,

$$
x(k+1) = A(k)x(k) + Dw(k) \tag{2.19}
$$

Definition 2.26 (Robustly positively invariant set) [24, 67] The set $\Omega \subseteq X$ is robustly positively invariant for system (2.19) if and only if, $\forall x(k) \in \Omega$, $\forall w(k) \in W$,

$$x(k+1) = A(k)x(k) + Dw(k) \in \Omega$$

Hence if the state $x(k)$ reaches Ω, it will remain inside Ω in spite of disturbance $w(k)$. The term *positively* refers to the fact that only forward evolutions of system (2.19) are considered and will be omitted in future sections for brevity.

The maximal robustly invariant set $\Omega_{\max} \subseteq X$ is a robustly invariant set, that contains all the robustly invariant sets contained in X.

A concept similar to invariance, but with possibly stronger requirements, is the concept of contractivity introduced in the following definition.

Definition 2.27 (Robustly contractive set) [24, 67] For a given $0 \le \lambda \le 1$, the set $\Omega \subseteq X$ is robustly λ-contractive for system (2.19) if and only if, $\forall x(k) \in \Omega$, $\forall w(k) \in W$,

$$x(k+1) = A(k)x(k) + Dw(k) \in \lambda\Omega$$

Other useful definitions which will be used in the sequence are reported next.

Definition 2.28 (Robustly controlled invariant set and admissible control) [24, 67] The set $C \subseteq X$ is robustly controlled invariant for the system (2.15) if for all $x(k) \in C$, there exists a control value $u(k) \in U$ such that, $\forall w(k) \in W$,

$$x(k+1) = A(k)x(k) + B(k)u(k) + Dw(k) \in C$$

Such a control action is called *admissible*.

The maximal robustly controlled invariant set $C_{\max} \subseteq X$ is a robustly controlled invariant set and contains all the robust controlled invariant sets contained in X.

Definition 2.29 (Robustly controlled contractive set) [24, 67] For a given $0 \le \lambda \le 1$ the set $C \subseteq X$ is robustly controlled contractive for the system (2.15) if for all $x(k) \in C$, there exists a control value $u(k) \in U$ such that, $\forall w(k) \in W$,

$$x(k+1) = A(k)x(k) + B(k)u(k) + Dw(k) \in \lambda C$$

Note that in Definition 2.27 and Definition 2.29 if $\lambda = 1$ we will, respectively retrieve the robustly invariance and robustly controlled invariance.

2.3.3 Ellipsoidal Invariant Sets

In this subsection, ellipsoids will be used for set invariance description. For simplicity, the case of vanishing disturbances is considered. In other words, the system under consideration is,

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{2.20}$$

It is assumed that the state and control constraints are symmetric,

$$\begin{cases} x(k) \in X, X = \{x : |F_i^T x| \le 1\}, & \forall i = 1, 2, \dots, n_1 \\ u(k) \in U, U = \{u : |u_j| \le u_{j\max}\}, & \forall j = 1, 2, \dots, m \end{cases} \tag{2.21}$$

where $u_{j\max}$ is the j-component of vector $u_{\max} \in \mathbb{R}^m$.

Consider now the problem of checking robustly controlled invariance. The set $E(P) = \{x \in \mathbb{R}^n : x^T P^{-1} x \le 1\}$ is controlled invariant if and only if for all $x \in E(P)$ there exists an input $u = \Phi(x) \in U$ such that,

$$\big(A_i x + B_i \Phi(x)\big)^T P^{-1} \big(A_i x + B_i \Phi(x)\big) \le 1, \quad \forall i = 1, 2, \dots, q \tag{2.22}$$

One possible choice for $u = \Phi(x)$ is a linear controller $u = Kx$. By defining $A_{ci} = A_i + B_i K$ with $i = 1, 2, \dots, q$, condition (2.22) is equivalent to,

$$x^T A_{ci}^T P^{-1} A_{ci} x \le 1, \quad \forall i = 1, 2, \dots, q \tag{2.23}$$

It is well known [29] that for the linear system (2.20), it is sufficient to check condition (2.23) for all x on the boundary of $E(P)$, i.e. for all x such that $x^T P^{-1} x = 1$. Therefore (2.23) can be transformed into,

$$x^T A_{ci}^T P^{-1} A_{ci} x \le x^T P^{-1} x, \quad \forall i = 1, 2, \dots, q$$

or equivalently,

$$A_{ci}^T P^{-1} A_{ci} \preceq P^{-1}, \quad \forall i = 1, 2, \dots, q$$

By using the Schur complements, this condition can be rewritten as,

$$\begin{bmatrix} P^{-1} & A_{ci}^T \\ A_{ci} & P \end{bmatrix} \succeq 0, \quad \forall i = 1, 2, \dots, q$$

The condition provided here is not linear in P. By using the Schur complements again, one gets,

$$P - A_{ci} P A_{ci}^T \succeq 0, \quad \forall i = 1, 2, \dots, q$$

or

$$\begin{bmatrix} P & A_{ci} P \\ P A_{ci}^T & P \end{bmatrix} \succeq 0, \quad \forall i = 1, 2, \dots, q$$

By substituting $A_{ci} = A_i + B_i K$, $i = 1, 2, \dots, q$, one obtains,

$$\begin{bmatrix} P & A_i P + B_i K P \\ P A_i^T + P K^T B_i^T & P \end{bmatrix} \succeq 0, \quad \forall i = 1, 2, \dots, q \tag{2.24}$$

Though this condition is nonlinear, since P and K are the unknowns. Still it can be re-parameterized into a linear condition by setting $Y = KP$. Condition (2.24) becomes,

$$\begin{bmatrix} P & A_i P + B_i Y \\ P A_i^T + Y^T B_i^T & P \end{bmatrix} \succeq 0, \quad \forall i = 1, 2, \ldots, q \tag{2.25}$$

Condition (2.25) is necessary and sufficient for ellipsoid $E(P)$ with the linear controller $u = Kx$ to be robustly invariant. Concerning the constraints (2.21), using equation (2.7) it follows that,

- The state constraints are satisfied if $E(P) \subseteq X$, or,

$$\begin{bmatrix} 1 & F_i^T P \\ P F_i & P \end{bmatrix} \succeq 0, \quad \forall i = 1, 2, \ldots, n_1 \tag{2.26}$$

- The input constraints are satisfied if $E(P) \subseteq X_u$ where,

$$X_u = \left\{ x \in \mathbb{R}^n : \left| K_j x \right| \le u_{j\max} \right\}, \quad j = 1, 2, \ldots, m$$

and K_j is the j-row of the matrix $K \in \mathbb{R}^{m \times n}$, hence,

$$\begin{bmatrix} u_{j\max}^2 & K_j P \\ P K_j^T & P \end{bmatrix} \succeq 0,$$

Since $Y_j = K_j P$ where Y_j is the j-row of the matrix $Y \in \mathbf{R}^{m \times n}$, it follows that,

$$\begin{bmatrix} u_{j\max}^2 & Y_j \\ Y_j^T & P \end{bmatrix} \succeq 0 \tag{2.27}$$

Define a vector $T_j \in \mathbb{R}^m$ as,

$$T_j = [0 \quad 0 \quad \ldots \quad 0 \quad \underbrace{1}_{j\text{-th position}} \quad 0 \quad \ldots \quad 0 \quad 0]$$

Since $Y_j = T_j Y$, equation (2.27) can be transformed into,

$$\begin{bmatrix} u_{j\max}^2 & T_j Y \\ Y^T T_j^T & P \end{bmatrix} \succeq 0 \tag{2.28}$$

It is generally desirable to have the *largest* ellipsoid among the ones satisfying conditions (2.25), (2.26), (2.28). In the literature [55, 125], the size of ellipsoid $E(P)$ is usually measured by the determinant or the trace of matrix P. Here the trace of matrix P is chosen due to its linearity. The trace of a square matrix is defined to be the sum of the elements on the main diagonal of the matrix. Maximization of the trace of matrices corresponds to the search for the maximal sum of eigenvalues of matrices. With the trace of matrix as the objective function, the problem of choosing the *largest* robustly invariant ellipsoid can be formulated as,

$$J = \max_{P,Y} \{\text{trace}(P)\} \tag{2.29}$$

subject to

- Invariance condition (2.25).
- Constraints satisfaction (2.26), (2.28).

It is clear that the solution P, Y of problem (2.29) may lead to the controller $K = YP^{-1}$ such that the closed loop system with matrix $A_c(k) = A(k) + B(k)K$ is at the stability margin. In other words, the ellipsoid $E(P)$ thus obtained might not be contractive (although being invariant). Indeed, the system trajectories might not converge to the origin. In order to ensure $x(k) \to 0$ as $k \to \infty$, it is required that for all $x \in E(P)$, to have

$$\big(A_i x + B_i \Phi(x)\big)^T P^{-1} \big(A_i x + B_i \Phi(x)\big) < 1, \quad \forall i = 1, 2, \ldots, q$$

With the same argument as above, one can conclude that an ellipsoid $E(P)$ with a linear controller $u = Kx$ is robustly contractive if the following set of LMI conditions is satisfied,

$$\begin{bmatrix} P & A_i P + B_i Y \\ P A_i^T + Y^T B_i^T & P \end{bmatrix} \succ 0, \quad \forall i = 1, 2, \ldots, q \tag{2.30}$$

where $Y = KP$.

It should be noted that condition (2.30) is the same as (2.25), except that condition (2.30) requires the left hand side to be a strictly positive matrix.

2.3.4 Polyhedral Invariant Sets

The problem of set invariance description using polyhedral sets is addressed in this subsection. With linear constraints on the state and the control vectors, polyhedral invariant sets are preferred to ellipsoidal invariant sets, since they offer a better approximation of the domain of attraction [22, 38, 54]. To begin, let us consider the case, when the control input is of the form $u(k) = Kx(k)$. Then the system (2.15) becomes,

$$x(k+1) = A_c(k)x(k) + Dw(k) \tag{2.31}$$

where

$$A_c(k) = A(k) + B(k)K = \text{Conv}\{A_{ci}\} \tag{2.32}$$

with $A_{ci} = A_i + B_i K$, $i = 1, 2, \ldots, q$.

The state constraints of the system (2.31) are,

$$x \in X_c, \quad X_c = \left\{x \in \mathbb{R}^n : F_c x \le g_c\right\} \tag{2.33}$$

where

$$F_c = \begin{bmatrix} F_x \\ F_u K \end{bmatrix}, \qquad g_c = \begin{bmatrix} g_x \\ g_u \end{bmatrix}$$

The following definition plays an important role in computing robustly invariant sets for system (2.31) with constraints (2.33).

Definition 2.30 (Pre-image set) For the system (2.31), the one step admissible *pre-image set* of the set X_c is a set $X_c^{(1)} \subseteq X_c$ such that for all $x \in X_c^{(1)}$, it holds that, $\forall w \in W$,

$$A_{ci}x + Dw \in X_c, \quad \forall i = 1, 2, \dots, q$$

The pre-image set $X_c^{(1)}$ can be computed as [23, 27],

$$X_c^{(1)} = \left\{x \in X_c : F_c A_{ci} x \le g_c - \max_{w \in W}\{F_c D w\},\ i = 1, 2, \dots, q\right\} \tag{2.34}$$

Example 2.1 Consider the following uncertain and time-varying system,

$$x(k+1) = A(k)x(k) + Bu(k) + Dw(k)$$

where

$$A(k) = \alpha(k)A_1 + \big(1 - \alpha(k)\big)A_2, \quad 0 \le \alpha(k) \le 1,$$
$$A_1 = \begin{bmatrix} 1.1 & 1 \\ 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0.6 & 1 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

The constraints (2.18) have the particular realization given by the matrices,

$$F_x = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad g_x = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}, \qquad F_w = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad g_w = \begin{bmatrix} 0.2 \\ 0.2 \\ 0.2 \\ 0.2 \end{bmatrix},$$
$$F_u = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad g_u = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

The controller is chosen as,

$$K = [-0.3856 \quad -1.0024]$$

The closed loop matrices are,

$$A_{c1} = \begin{bmatrix} 1.1000 & 1.0000 \\ -0.3856 & -0.0024 \end{bmatrix}, \qquad A_{c2} = \begin{bmatrix} 0.6000 & 1.0000 \\ -0.3856 & -0.0024 \end{bmatrix}$$

The set $X_c = \{x \in \mathbb{R}^2 : F_c x \le g_c\}$ is,

$$F_c = \begin{bmatrix} 1.0000 & 0 \\ 0 & 1.0000 \\ -1.0000 & 0 \\ 0 & -1.0000 \\ -0.3856 & -1.0024 \\ 0.3856 & 1.0024 \end{bmatrix}, \qquad g_c = \begin{bmatrix} 3.0000 \\ 3.0000 \\ 3.0000 \\ 3.0000 \\ 2.0000 \\ 2.0000 \end{bmatrix}$$

By solving the LP problem (2.13), it follows that the second and the fourth inequalities of X_c, i.e. $[0\ 1]x \le 3$ and $[0\ -1]x \le 3$, are redundant. After eliminating the redundant inequalities and normalizing the half-space representation, the set X_c is given as,

$$X_c = \left\{x \in \mathbb{R}^2 : \widehat{F}_c x \le \widehat{g}_c\right\}$$

where

$$\widehat{F}_c = \begin{bmatrix} 1.0000 & 0 \\ -1.0000 & 0 \\ -0.3590 & -0.9333 \\ 0.3590 & 0.9333 \end{bmatrix}, \qquad \widehat{g}_c = \begin{bmatrix} 3.0000 \\ 3.0000 \\ 0.9311 \\ 0.9311 \end{bmatrix}$$

Using (2.34), the one step admissible pre-image set X_c^1 of X_c is defined as,

$$X_c^{(1)} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} \widehat{F}_c \\ \widehat{F}_c A_1 \\ \widehat{F}_c A_2 \end{bmatrix} x \le \begin{bmatrix} \widehat{g}_c \\ \widehat{g}_c - \max_{w \in W}\{\widehat{F}_c w\} \\ \widehat{g}_c - \max_{w \in W}\{\widehat{F}_c w\} \end{bmatrix} \right\} \tag{2.35}$$

After removing redundant inequalities, the set $X_c^{(1)}$ is presented in minimal normalized half-space representation as,

$$X_c^{(1)} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 1.0000 & 0 \\ -1.0000 & 0 \\ -0.3590 & -0.9333 \\ 0.3590 & 0.9333 \\ 0.7399 & 0.6727 \\ -0.7399 & -0.6727 \\ 0.3753 & -0.9269 \\ -0.3753 & 0.9269 \end{bmatrix} x \le \begin{bmatrix} 3.0000 \\ 3.0000 \\ 0.9311 \\ 0.9311 \\ 1.8835 \\ 1.8835 \\ 1.7474 \\ 1.7474 \end{bmatrix} \right\}$$

The sets X, X_c and $X_c^{(1)}$ are depicted in Fig. 2.4.

Procedure 2.1 Robustly invariant set computation [46, 68]

- **Input:** The matrices $A_{c1}, A_{c2}, \ldots, A_{cq}, D$, the sets X_c in (2.33) and W.
- **Output:** The robustly invariant set Ω.

1. Set $i=0$, $F_0=F_c$, $g_0=g_c$ and $X_0=\{x\in\mathbb{R}^n: F_0x\le g_0\}$.
2. Set $X_1=X_0$.
3. Eliminate redundant inequalities of the following polytope,

$$P=\left\{x\in\mathbb{R}^n:\begin{bmatrix}F_0\\ F_0A_{c1}\\ F_0A_{c2}\\ \vdots\\ F_0A_{cq}\end{bmatrix}x\le\begin{bmatrix}g_0\\ g_0-\max_{w\in W}\{F_0Dw\}\\ g_0-\max_{w\in W}\{F_0Dw\}\\ \vdots\\ g_0-\max_{w\in W}\{F_0Dw\}\end{bmatrix}\right\}$$

4. Set $X_0=P$ and update consequently the matrices F_0 and g_0.
5. If $X_0=X_1$ then stop and set $\Omega=X_0$. Else continue.
6. Set $i=i+1$ and go to step 2.

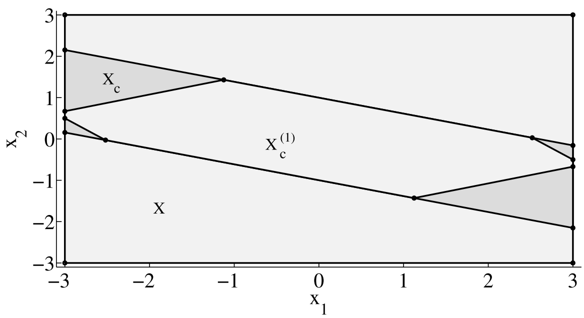


Fig. 2.4 One step pre-image set for Example 2.1

Clearly, $\Omega\subseteq X_c$ is robustly invariant if it equals to its one step admissible pre-image set, i.e., $\forall x\in\Omega$, $\forall w\in W$,

$$A_ix+Dw\in\Omega,\quad \forall i=1,2,\ldots,q$$

Using this observation, Procedure 2.1 can be used for computing the set Ω for system (2.31) with constraints (2.33).

A natural question for Procedure 2.1 is that if there exists a finite index i such that $X_0=X_1$, or equivalently if Procedure 2.1 terminates after a finite number of iterations.

In the absence of disturbances, the following theorem holds [25].

Theorem 2.2 [25] *Assume that the system (2.31) is robustly asymptotically stable. Then there exists a finite index $i=i_{\max}$, such that $X_0=X_1$ in Procedure 2.1.*

Procedure 2.2 Robustly invariant set computation

- **Input:** The matrices $A_{c1}, A_{c2}, \dots, A_{cq}$, D, the sets X_c and W.
- **Output:** The robustly invariant set Ω.

1. Set $i = 0$, $F_0 = F_c$, $g_0 = g_c$ and $X_0 = \{x \in \mathbb{R}^n : F_0 x \le g_0\}$.
2. Consider the following polytope

$$P = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} F_0 \\ F_0 A_{c1} \\ F_0 A_{c2} \\ \vdots \\ F_0 A_{cq} \end{bmatrix} x \le \begin{bmatrix} g_0 \\ g_0 - \max_{w \in W}\{F_0 D w\} \\ g_0 - \max_{w \in W}\{F_0 D w\} \\ \vdots \\ g_0 - \max_{w \in W}\{F_0 D w\} \end{bmatrix} \right\}$$

   and iteratively check the redundancy of the following subset of inequalities

$$\left\{ x \in \mathbb{R}^n : F_0 A_{cj} x \le g_0 - \max_{w \in W}\{F_0 D w\} \right\}$$

   with $j = 1, 2, \dots, q$.
3. If all of the inequalities are redundant with respect to X_0, then stop and set $\Omega = X_0$. Else continue.
4. Set $X_0 = P$.
5. Set $i = i + 1$ and go to step 2.

Remark 2.1 In the presence of disturbances, the necessary and sufficient condition for the existence of a finite index i is that the minimal robustly invariant set[3] [71, 97, 105] is a subset of X_c.

Note that checking equality of two polytopes X_0 and X_1 in step 5 is computationally demanding, i.e. one has to check $X_0 \subseteq X_1$ and $X_1 \subseteq X_0$. Note also that if the set Ω is invariant at the iteration i then the following set of inequalities,

$$\begin{bmatrix} F_0 A_{c1} \\ F_0 A_{c2} \\ \vdots \\ F_0 A_{cq} \end{bmatrix} x \le \begin{bmatrix} g_0 - \max_{w \in W}\{F_0 D_1 w\} \\ g_0 - \max_{w \in W}\{F_0 D_2 w\} \\ \vdots \\ g_0 - \max_{w \in W}\{F_0 D_q w\} \end{bmatrix}$$

is redundant with respect to $\Omega = \{x \in \mathbb{R}^n : F_0 x \le g_0\}$. Hence Procedure 2.1 can be made more efficient as in Procedure 2.2.

[3]The set $\Omega \subseteq X_c$ is minimal robustly invariant if it is a robustly invariant set and is a subset of any robustly invariant set contained in X_c.

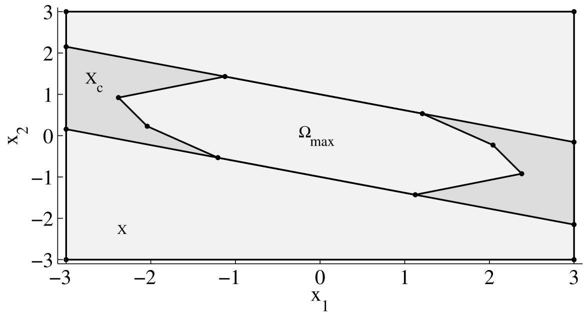


Fig. 2.5 Maximal robustly invariant set $\Omega_{\max}$ for Example 2.2

It is well known [29, 46, 71] that the set Ω resulting from Procedure 2.1 or Procedure 2.2, is actually the maximal robustly invariant set for system (2.31) and constraints (2.33), that is $\Omega = \Omega_{\max}$.

Example 2.2 Consider the uncertain system in Example 2.1 with the same state, control and disturbance constraints. Using Procedure 2.2, the set $\Omega_{\max}$ is found after 5 iterations as,

$$\Omega_{\max} = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} -0.3590 & -0.9333 \\ 0.3590 & 0.9333 \\ 0.6739 & 0.7388 \\ -0.6739 & -0.7388 \\ 0.8979 & 0.4401 \\ -0.8979 & -0.4401 \\ 0.3753 & -0.9269 \\ -0.3753 & 0.9269 \end{bmatrix} x \le \begin{bmatrix} 0.9311 \\ 0.9311 \\ 1.2075 \\ 1.2075 \\ 1.7334 \\ 1.7334 \\ 1.7474 \\ 1.7474 \end{bmatrix} \right\}$$

The sets X, X_c and $\Omega_{\max}$ are depicted in Fig. 2.5.

Definition 2.31 (One-step robustly controlled set) Given system (2.15), the one-step robustly controlled set, denoted as C_1 of the set $C_0 = \{x \in \mathbb{R}^n : F_0 x \le g_0\}$ is given by all states that can be steered in one step into C_0 when a suitable control action is applied. The set C_1 can be shown to be [23, 27],

$$C_1 = \left\{ x \in \mathbb{R}^n : \exists u \in U : F_0(A_i x + B_i u) \le g_0 - \max_{w \in W}\{F_0 D w\} \right\}, \quad i = 1, 2, \ldots, q \tag{2.36}$$

Remark 2.2 If C_0 is robustly invariant, then $C_0 \subseteq C_1$. Hence C_1 is a robustly controlled invariant set.

Recall that $\Omega_{\max}$ is the maximal robustly invariant set with respect to a predefined control law $u(k) = Kx(k)$. Define C_N as the set of all states, that can be steered into $\Omega_{\max}$ in no more than N steps along an admissible trajectory, i.e. a trajectory satisfy-

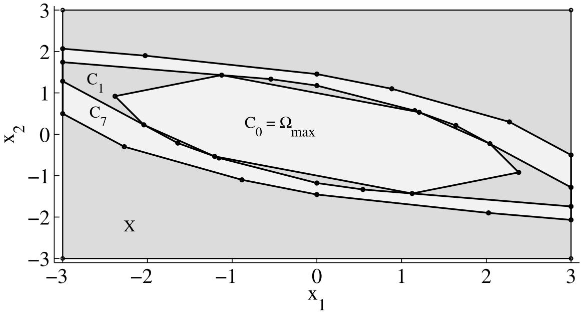


Fig. 2.6 Robustly controlled invariant sets for Example 2.3

ing control, state and disturbance constraints. This set can be computed recursively by Procedure 2.3.

Since $\Omega_{\max}$ is a robustly invariant set, it follows that $C_{i-1} \subseteq C_i$ and therefore C_i is a robustly controlled invariant set and a sequence of nested polytopes. Note that the complexity of C_N does not have an analytic dependence on N and may increase without bound, thus placing a practical limitation on the choice of N.

Example 2.3 Consider the uncertain system in Example 2.1 with the same state, input and disturbance constraints. Using Procedure 2.3, the robustly controlled invariant sets C_N with $N = 1$ and $N = 7$ are obtained and shown in Fig. 2.6. Note that $C_7 = C_8$ is the maximal robustly controlled invariant set.

The set C_7 is presented in minimal normalized half-space representation as,

$$C_7 = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} 0.3731 & 0.9278 \\ -0.3731 & -0.9278 \\ 0.4992 & 0.8665 \\ -0.4992 & -0.8665 \\ 0.1696 & 0.9855 \\ -0.1696 & -0.9855 \\ 0.2142 & 0.9768 \\ -0.2142 & -0.9768 \\ 0.7399 & 0.6727 \\ -0.7399 & -0.6727 \\ 1.0000 & 0 \\ -1.0000 & 0 \end{bmatrix} x \le \begin{bmatrix} 1.3505 \\ 1.3505 \\ 1.3946 \\ 1.3946 \\ 1.5289 \\ 1.5289 \\ 1.4218 \\ 1.4218 \\ 1.8835 \\ 1.8835 \\ 3.0000 \\ 3.0000 \end{bmatrix} \right\}$$

2.4 On the Domain of Attraction

In this section we study the problem of estimating the domain of attraction for uncertain and/or time-varying linear discrete-time systems in closed-loop with a *saturated* controller and state constraints.

Procedure 2.3 Robustly N-step controlled invariant set computation

- **Input:** The matrices $A_1, A_2, \dots, A_q, D$ and the sets X, U, W and $\Omega_{\max}$.
- **Output:** The N-step robustly controlled invariant set C_N.

1. Set $i = 0$ and $C_0 = \Omega_{\max}$ and let the matrices F_0, g_0 be the half-space representation of C_0, i.e. $C_0 = \{x \in \mathbb{R}^n : F_0 x \le g_0\}$
2. Compute the expanded set $P_i \subset \mathbb{R}^{n+m}$

$$P_i = \left\{ (x,u) \in \mathbb{R}^{n+m} : \begin{bmatrix} F_i(A_1 x + B_1 u) \\ F_i(A_2 x + B_2 u) \\ \vdots \\ F_i(A_q x + B_q u) \end{bmatrix} \le \begin{bmatrix} g_i - \max_{w \in W}\{F_i D w\} \\ g_i - \max_{w \in W}\{F_i D w\} \\ \vdots \\ g_i - \max_{w \in W}\{F_i D w\} \end{bmatrix} \right\}$$

3. Compute the projection $P_i^{(n)}$ of P_i on $\mathbb{R}^n$

$$P_i^{(n)} = \left\{ x \in \mathbb{R}^n : \exists u \in U \text{ such that } (x,u) \in P_i \right\}$$

4. Set

$$C_{i+1} = P_i^{(n)} \cap X$$

and let F_{i+1}, g_{i+1} be the half-space representation of C_{i+1}, i.e.

$$C_{i+1} = \left\{ x \in \mathbb{R}^n : F_{i+1} x \le g_{i+1} \right\}$$

5. If $C_{i+1} = C_i$, then stop and set $C_N = C_i$. Else continue.
6. If $i = N$, then stop else continue.
7. Set $i = i + 1$ and go to step 2.

2.4.1 Problem Formulation

Consider the following uncertain and/or time-varying linear discrete-time system,

$$x(k+1) = A(k)x(k) + B(k)u(k) \tag{2.37}$$

where

$$\begin{cases} A(k) = \sum_{i=1}^{q} \alpha_i(k) A_i, \qquad B(k) = \sum_{i=1}^{q} \alpha_i(k) B_i \\ \sum_{i=1}^{q} \alpha_i(k) = 1, \quad \alpha_i(k) \ge 0 \end{cases} \tag{2.38}$$

with given matrices $A_i \in \mathbb{R}^{n \times n}$ and $B_i \in \mathbb{R}^{n \times m}$, $i = 1, 2, \dots, q$.

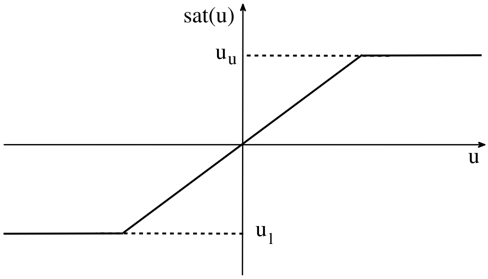


Fig. 2.7 Saturation function

Both the state vector $x(k)$ and the control vector $u(k)$ are subject to the constraints,

$$\begin{cases} x(k) \in X, & X = \{x \in \mathbb{R}^n : F_i^T x \le g_i\}, \quad \forall i = 1, 2, \ldots, n_1 \\ u(k) \in U, & U = \{u \in \mathbb{R}^m : u_{jl} \le u_j \le u_{ju}\}, \quad \forall j = 1, 2, \ldots, m \end{cases} \tag{2.39}$$

where $F_i^T \in \mathbb{R}^n$ is the i-th row of the matrix $F_x \in \mathbb{R}^{n_1 \times n}$, g_i is the i-th component of the vector $g_x \in \mathbb{R}^{n_1}$, u_{il} and u_{iu} are respectively, the i-th component of the vectors u_l and u_u, which are the lower and upper bounds of input u. It is assumed that the matrix F_x and the vectors g_x, u_l and u_u are constant with $g_x > 0$, $u_l < 0$ and $u_u > 0$.

Our aim is to estimate the domain of attraction for the system,

$$x(k+1) = A(k)x(k) + B(k)\,\mathrm{sat}\big(Kx(k)\big) \tag{2.40}$$

subject to the constraints (2.39), where

$$u(k) = Kx(k) \tag{2.41}$$

is a given controller that robustly stabilizes system (2.37).

2.4.2 Saturation Nonlinearity Modeling—A Linear Differential Inclusion Approach

A linear differential inclusion approach used for modeling the saturation function is briefly reviewed in this subsection. This modeling framework was first proposed by Hu et al. [55, 57, 58]. Then its generalization was developed by Alamo et al. [5, 6]. The main idea of the linear differential inclusion approach is to use an auxiliary vector variable $v \in \mathbb{R}^m$, and to compose the output of the saturation function as a convex combination of u and v.

The saturation function is defined as

$$\mathrm{sat}(u) = \big[\mathrm{sat}(u_1) \quad \mathrm{sat}(u_2) \quad \ldots \quad \mathrm{sat}(u_m)\big]^T \tag{2.42}$$

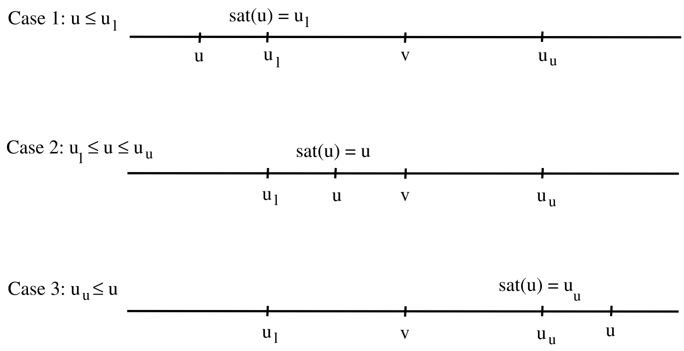


Fig. 2.8 Linear differential inclusion approach

where, see Fig. 2.7,

$$\text{sat}(u_i) = \begin{cases} u_{il}, & \text{if } u_i \leq u_{il} \\ u_i, & \text{if } u_{il} \leq u_i \leq u_{iu} \\ u_{iu}, & \text{if } u_{iu} \leq u_i \end{cases} \quad i = 1, 2, \ldots, m \tag{2.43}$$

To underline the details of the approach, let us first consider the case u and consequently v are scalars. Clearly, for any u, there exist $u_l \leq v \leq u_u$ and $0 \leq \beta \leq 1$ such that,

$$\text{sat}(u) = \beta u + (1 - \beta)v \tag{2.44}$$

or, equivalently

$$\text{sat}(u) \in \text{Conv}\{u, v\} \tag{2.45}$$

Figure 2.8 illustrates this fact.

Analogously, for $m = 2$ and v such that

$$\begin{cases} u_{1l} \leq v_1 \leq u_{1u} \\ u_{2l} \leq v_2 \leq u_{2u} \end{cases} \tag{2.46}$$

the saturation function can be expressed as,

$$\text{sat}(u) = \beta_1 \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \beta_2 \begin{bmatrix} u_1 \\ v_2 \end{bmatrix} + \beta_3 \begin{bmatrix} v_1 \\ u_2 \end{bmatrix} + \beta_4 \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \tag{2.47}$$

with $\sum_{j=1}^4 \beta_j = 1$, $\beta_j \geq 0$. Or, equivalently

$$\text{sat}(u) \in \text{Conv}\left\{ \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} u_1 \\ v_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ u_2 \end{bmatrix}, \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right\} \tag{2.48}$$

Define D_m as the set of $m \times m$ diagonal matrices whose diagonal elements are either 0 or 1. For example, if $m = 2$ then

$$D_2 = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$$

There are 2^m elements in D_m. Denote each element of D_m as E_j, $j = 1, 2, \ldots, 2^m$ and define $E_j^- = I - E_j$. For example, if

$$E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

then

$$E_1^- = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Clearly, if $E_j \in D_m$, then E_j^- is also in D_m. The generalization of the results (2.45) (2.48) is reported by the following lemma [55, 57, 58],

Lemma 2.1 [57] *Consider two vectors $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^m$ such that $u_{il} \le v_i \le u_{iu}$ for all $i = 1, 2, \ldots, m$, then it holds that*

$$\operatorname{sat}(u) \in \operatorname{Conv}\{E_j u + E_j^- v\}, \quad j = 1, 2, \ldots, 2^m \tag{2.49}$$

Consequently, there exist $\beta_j \ge 0$ and $\sum_{j=1}^{2^m} \beta_j = 1$ such that,

$$\operatorname{sat}(u) = \sum_{j=1}^{2^m} \beta_j \left(E_j u + E_j^- v\right)$$

2.4.3 *The Ellipsoidal Set Approach*

The aim of this subsection is twofold. First, we provide an invariance condition of ellipsoidal sets for uncertain and/or time-varying linear discrete-time systems with a saturated input and state constraints. This invariance condition is an extended version of the previously published results in [57] for the robust case. Secondly, we propose a method for computing a *saturated* controller $u(k) = \operatorname{sat}(Kx(k))$ that makes a given invariant ellipsoid *contractive* with the maximal contraction factor. For simplicity, the case of bounds equal to $u_{\max}$ is considered, namely

$$-u_l = u_u = u_{\max}$$

and let us assume that the set X in (2.39) is symmetric and $g_i = 1, \forall i = 1, 2, \ldots, n_1$. Clearly, the latter assumption is nonrestrictive as long as, $\forall g_i > 0$

$$F_i^T x \le g_i \quad \Leftrightarrow \quad \frac{F_i^T}{g_i} x \le 1$$

For a given matrix $H \in \mathbb{R}^{m\times n}$, define X_c as the intersection between X and the polyhedral set $F(H, u_{\max}) = \{x \in \mathbb{R}^n : |Hx| \le u_{\max}\}$, i.e.

$$X_c = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} F_x \\ H \\ -H \end{bmatrix} x \le \begin{bmatrix} 1 \\ u_{\max} \\ u_{\max} \end{bmatrix} \right\} \tag{2.50}$$

We are now ready to state the main result of this subsection,

Theorem 2.3 *If there exist a positive definite matrix $P \in \mathbb{R}^{n\times n}$ and a matrix $H \in \mathbb{R}^{m\times n}$ such that, $\forall i = 1, 2, \ldots, q$, $\forall j = 1, 2, \ldots, 2^m$,*

$$\begin{bmatrix} P & \{A_i + B_i(E_j K + E_j^- H)\}P \\ P\{A_i + B_i(E_j K + E_j^- H)\}^T & P \end{bmatrix} \succeq 0, \tag{2.51}$$

and $E(P) \subset X_c$, then the ellipsoid $E(P)$ is a robustly invariant set for system (2.40) *with constraints* (2.39).

Proof Assume that there exist P and H such that condition (2.51) is satisfied. Using Lemma 2.1 and by choosing $v = Hx$, it follows that,

$$\operatorname{sat}\big(Kx(k)\big) = \sum_{j=1}^{2^m} \beta_j(k)\big(E_j K x(k) + E_j^- H x(k)\big)$$

for all $x(k)$ such that $|Hx(k)| \le u_{\max}$. Subsequently,

$$\begin{aligned} x(k+1) &= \sum_{i=1}^{q} \alpha_i(k) \left\{ A_i + B_i \sum_{j=1}^{2^m} \beta_j(k)\big(E_j K + E_j^- H\big) \right\} x(k) \\ &= \sum_{i=1}^{q} \alpha_i(k) \left\{ \sum_{j=1}^{2^m} \beta_j(k) A_i + B_i \sum_{j=1}^{2^m} \beta_j(k)\big(E_j K + E_j^- H\big) \right\} x(k) \\ &= \sum_{i=1}^{q} \alpha_i(k) \sum_{j=1}^{2^m} \beta_j(k) \big\{A_i + B_i\big(E_j K + E_j^- H\big)\big\} x(k) \\ &= \sum_{i=1}^{q} \sum_{j=1}^{2^m} \alpha_i(k)\beta_j(k) \big\{A_i + B_i\big(E_j K + E_j^- H\big)\big\} x(k) = A_c(k) x(k) \end{aligned}$$

where

$$A_c(k) = \sum_{i=1}^{q} \sum_{j=1}^{2^m} \alpha_i(k)\beta_j(k)\big\{A_i + B_i\big(E_j K + E_j^- H\big)\big\}$$

Since

$$\sum_{i=1}^{q} \sum_{j=1}^{2^m} \alpha_i(k)\beta_j(k) = \sum_{i=1}^{q} \alpha_i(k) \left\{ \sum_{j=1}^{2^m} \beta_j(k) \right\} = 1$$

if follows that $A_c(k)$ belongs to the polytope P_c, whose vertices are obtained by taking all possible combinations of $A_i + B_i(E_j K + E_j^- H)$ with $i = 1, 2, \ldots, q$ and $j = 1, 2, \ldots, 2^m$.

The set $E(P) = \{x \in \mathbb{R}^n : x^T P^{-1} x \le 1\}$ is invariant, if and only if

$$x^T A_c(k)^T P^{-1} A_c(k) x \le 1 \tag{2.52}$$

for all $x \in E(P)$. With the same argument as in Sect. 2.3.3, condition (2.52) can be transformed to,

$$\begin{bmatrix} P & A_c(k)P \\ P A_c(k)^T & P \end{bmatrix} \succeq 0 \tag{2.53}$$

Since $A_c(k)$ belongs to the polytope P_c, it follows that one should check (2.53) at the vertices of P_c. So the set of LMI conditions to be satisfied is the following, $\forall i = 1, 2, \ldots, q$, $\forall j = 1, 2, \ldots, 2^m$,

$$\begin{bmatrix} P & \{A_i + B_i(E_j K + E_j^- H)\}P \\ P\{A_i + B_i(E_j K + E_j^- H)\}^T & P \end{bmatrix} \succeq 0 \qquad \square$$

Note that condition (2.51) involves the multiplication between two unknown parameters H and P. By defining $Y = HP$, condition (2.51) can be rewritten as, $\forall i = 1, 2, \ldots, q$, $\forall j = 1, 2, \ldots, 2^m$,

$$\begin{bmatrix} P & (A_i P + B_i E_j K P + B_i E_j^- Y) \\ (P A_i^T + P K^T E_j B_i^T + Y^T E_j^- B_i^T) & P \end{bmatrix} \succeq 0, \tag{2.54}$$

Thus the unknown matrices P and Y enter linearly in (2.54).

As in Sect. 2.3.3, in general one would like to have the largest invariant ellipsoid for system (2.37) under *saturated* controller $u(k) = \mathrm{sat}(Kx(k))$ with respect to constraints (2.39). This can be achieved by solving the following LMI problem,

$$J = \max_{P,Y} \big\{\mathrm{trace}(P)\big\} \tag{2.55}$$

subject to

- Invariance condition (2.54).
- Constraint satisfaction,
  - On state
$$\begin{bmatrix} 1 & F_i^T P \\ P F_i & P \end{bmatrix} \succeq 0, \quad \forall i = 1, 2, \dots, n_1$$
  - On input
$$\begin{bmatrix} u_{i\max}^2 & Y_i \\ Y_i^T & P \end{bmatrix} \succeq 0, \quad \forall i = 1, 2, \dots, m$$
  where Y_i is the i-th row of the matrix Y.

Example 2.4 Consider the following linear uncertain, time-varying discrete-time system,

$$x(k+1) = A(k)x(k) + B(k)u(k)$$

with

$$A(k) = \alpha(k)A_1 + \big(1 - \alpha(k)\big)A_2, \qquad B(k) = \alpha(k)B_1 + \big(1 - \alpha(k)\big)B_2$$

and

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ 0 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 1 & 0.2 \\ 0 & 1 \end{bmatrix}, \qquad B_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1.5 \end{bmatrix}$$

At each sampling time $\alpha(k) \in [0,\ 1]$ is an uniformly distributed pseudo-random number. The constraints are,

$$-10 \le x_1 \le 10, \qquad -10 \le x_2 \le 10, \qquad -1 \le u \le 1$$

The controller is chosen as,

$$K = [-1.8112 \quad -0.8092]$$

By solving the LMI problem (2.55), the matrices P and Y are obtained,

$$P = \begin{bmatrix} 5.0494 & -8.9640 \\ -8.9640 & 28.4285 \end{bmatrix}, \qquad Y = [0.4365 \quad -4.2452]$$

Hence

$$H = YP^{-1} = [-0.4058 \quad -0.2773]$$

Solving the LMI problem (2.30), the ellipsoid $E(P_1)$ is obtained with

$$P_1 = \begin{bmatrix} 1.1490 & -3.1747 \\ -3.1747 & 9.9824 \end{bmatrix}$$

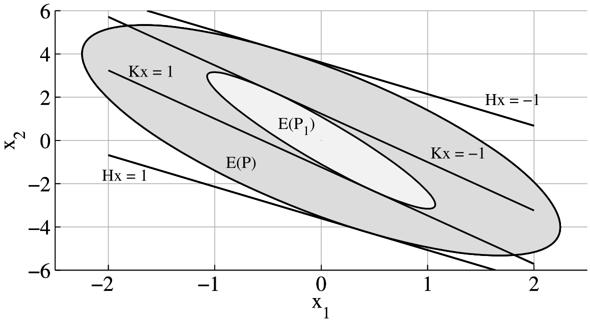


Fig. 2.9 Invariant sets with different control laws for Example 2.4. The set $E(P)$ is obtained for $u(k) = \mathrm{sat}(Kx(k))$ and the set $E(P_1)$ is obtained for $u(k) = Kx(k)$

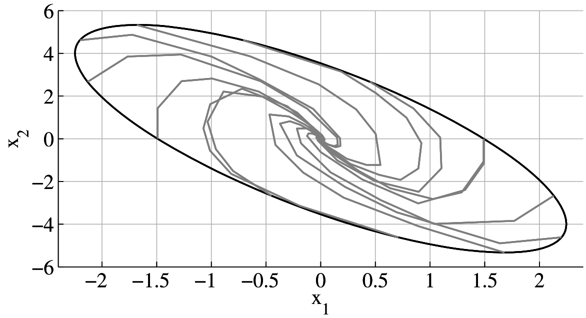


Fig. 2.10 State trajectories of the closed loop system for Example 2.4

under the linear feedback $u(k) = Kx(k)$. Figure 2.9 presents two invariant ellipsoids with different control laws. $E(P)$ is obtained for $u(k) = \mathrm{sat}(Kx(k))$ and $E(P_1)$ is obtained for $u(k) = Kx(k)$.

Figure 2.10 shows state trajectories of the closed loop system with the controller $u(k) = \mathrm{sat}(Kx(k))$ for different initial conditions and realizations of $\alpha(k)$.

In the first part of this subsection, Theorem 2.3 was exploited in the following manner: if $E(P)$ is robustly invariant for the system,

$$x(k+1) = A(k)x(k) + B(k)\,\mathrm{sat}\big(Kx(k)\big)$$

then there exists a stabilizing linear controller $u(k) = Hx(k)$, such that $E(P)$ is robustly invariant for system,

$$x(k+1) = A(k)x(k) + B(k)Hx(k)$$

with $H \in \mathbb{R}^{m\times n}$ obtained by solving the LMI problem (2.55).

Theorem 2.3 now will be exploited in a different manner. We would like to design a *saturated* controller $u(k) = \mathrm{sat}(Kx(k))$ that makes a *given* invariant ellipsoid $E(P)$ contractive with the maximal contraction factor. This invariant ellipsoid $E(P)$ can be inherited for example together with a linear controller $u(k) = Hx(k)$ from the optimization of some convex objective function $J(P)$,[4] for example trace(P). In the second stage, using H and $E(P)$, a saturated controller $u(k) = \mathrm{sat}(Kx(k))$ which maximizes some contraction factor $1 - g$ is computed.

It is worth noticing that the invariance condition (2.29) corresponds to the one in condition (2.54) with $E_j = 0$ and $E_j^- = I - E_j = I$. Following the proof of Theorem 2.3, it can be shown that for the system,

$$x(k+1) = A(k)x(k) + B(k)\,\mathrm{sat}\big(Kx(k)\big)$$

the set $E(P)$ is contractive with the contraction factor $1 - g$ if

$$\big\{A_i + B_i\big(E_jK + E_j^-H\big)\big\}^T P^{-1}\big\{A_i + B_i\big(E_jK + E_j^-H\big)\big\} - P^{-1} \preceq -gP^{-1} \quad (2.56)$$

$\forall i = 1, 2, \ldots, q$, $\forall j = 1, 2, \ldots, 2^m$ such that $E_j \neq 0$. Using the Schur complements, (2.56) becomes,

$$\begin{bmatrix} (1-g)P^{-1} & (A_i + B_i(E_jK + E_j^-H))^T \\ (A_i + B_i(E_jK + E_j^-H)) & P \end{bmatrix} \succeq 0 \quad (2.57)$$

$\forall i = 1, 2, \ldots, p$, $\forall j = 1, 2, \ldots, 2^m$ with $E_j \neq 0$.

Hence, the problem of computing a saturated controller that makes a given invariant ellipsoid contractive with the maximal contraction factor can be formulated as,

$$J = \max_{g,K}\{g\} \quad (2.58)$$

subject to (2.57).

Recall that here the only unknown parameters are the matrix $K \in \mathbb{R}^{m \times n}$ and the scalar g, the matrices P and H being given in the first stage.

Remark 2.3 The proposed two-stage control design presented here benefits from global uniqueness properties of the solution. This is due to the one-way dependence of the two (prioritized) objectives: the trace maximization precedes the associated contraction factor.

Example 2.5 Consider the uncertain system in Example 2.4 with the same state and input constraints. In the first stage, by solving (2.29), the matrices P and Y are obtained,

$$P = \begin{bmatrix} 100.0000 & -43.1051 \\ -43.1051 & 100.0000 \end{bmatrix}, \qquad Y = [-3.5691 \quad -6.5121]$$

[4] Practically, the design of the invariant ellipsoid $E(P)$ and the controller $u(k) = Hx(k)$ can be done by solving the LMI problem (2.29).

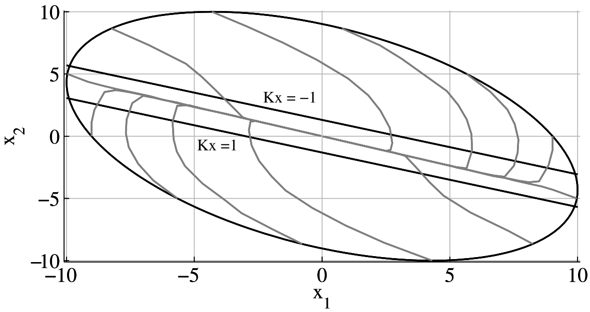


Fig. 2.11 Invariant ellipsoid and state trajectories of the closed loop system for Example 2.5

Hence $H = YP^{-1} = [-0.0783\ \ -0.0989]$. In the second stage, by solving (2.58), the matrix K is obtained,

$$K = [-0.3342 \quad -0.7629]$$

Figure 2.11 shows the invariant ellipsoid $E(P)$. This figure also shows state trajectories of the closed loop system with the controller $u(k) = \mathrm{sat}(Kx(k))$ for different initial conditions and realizations of $\alpha(k)$.

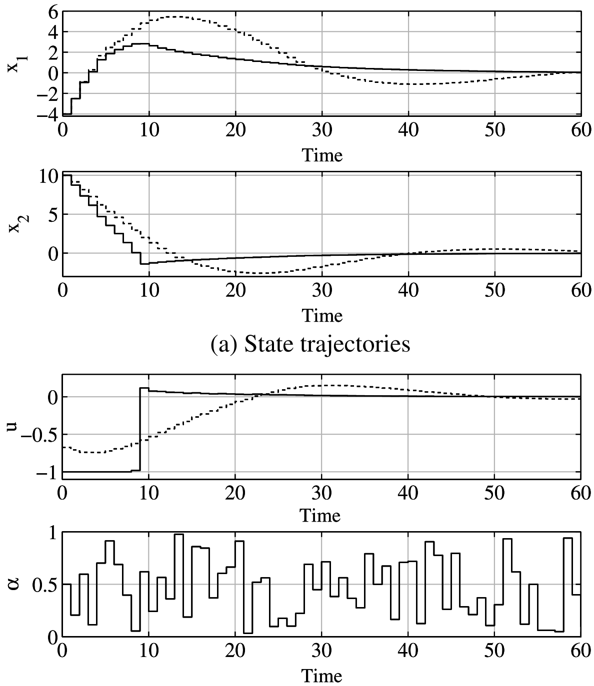
For the initial condition $x(0) = [-4\ 10]^T$, Fig. 2.12(a) presents state trajectories of the closed loop system as functions of time for the saturated controller $u(k) = \mathrm{sat}(Kx(k))$ (solid) and for the linear controller $u(k) = Hx(k)$ (dashed). It is worth noticing that the time to regulate the plant to the origin by using $u(k) = Hx(k)$ is longer than the time to regulate the plant to the origin by using $u(k) = \mathrm{sat}(Kx(k))$. The reason is that when using $u(k) = Hx(k)$, the control action is saturated only at some points of the boundary of $E(P)$, while using $u(k) = \mathrm{sat}(Kx(k))$, the control action is saturated not only on the boundary of $E(P)$, the saturation being active also inside $E(P)$. This phenomena can be observed in Fig. 2.12(b). The same figure presents the realization of $\alpha(k)$.

2.4.4 The Polyhedral Set Approach

The problem of estimating the domain of attraction is addressed by using polyhedral sets in this subsection. For a given linear controller $u(k) = Kx(k)$, it is clear that the largest polyhedral invariant set is the maximal robustly invariant set $\Omega_{\max}$ which can be found using Procedure 2.1 or Procedure 2.2. From this point on, it is assumed that $\Omega_{\max}$ is known.

The aim is to find the *largest* polyhedral invariant set $\Omega_s \subseteq X$ characterizing an estimation of the domain of attraction for system (2.37) under $u(k) = \mathrm{sat}(Kx(k))$. To this aim, recall that from Lemma 2.1, the saturation function can be expressed as,

$$\mathrm{sat}\big(Kx(k)\big) = \sum_{j=1}^{2^m} \beta_j(k)\big(E_j Kx + E_j^- v\big), \qquad \sum_{j=1}^{2^m} \beta_j(k) = 1, \quad \beta_j \geq 0 \qquad (2.59)$$



(a) State trajectories

(b) Input trajectories and α realization

Fig. 2.12 State and input trajectories for Example 2.5 for the controller $u = \mathrm{sat}(Kx)$ (*solid*), and for the controller $u = Hx$ (*dashed*) in the figures for x_1, x_2 and u

where $u_l \le v \le u_u$ and E_j is an element of D_m and $E_j^- = I - E_j$.

Using (2.59), the closed loop system can be rewritten as,

$$
\begin{aligned}
x(k+1) &= \sum_{i=1}^{q} \alpha_i(k) \left\{ A_i x(k) + B_i \sum_{j=1}^{2^m} \beta_j(k) \big(E_j K x(k) + E_j^- v\big) \right\} \\
&= \sum_{i=1}^{q} \alpha_i(k) \left\{ \sum_{j=1}^{2^m} \beta_j(k) A_i x(k) + B_i \sum_{j=1}^{2^m} \beta_j(k) \big(E_j K x(k) + E_j^- v\big) \right\} \\
&= \sum_{i=1}^{q} \alpha_i(k) \sum_{j=1}^{2^m} \beta_j(k) \big\{ A_i x(k) + B_i \big(E_j K x(k) + E_j^- v\big) \big\}
\end{aligned}
$$

or

$$
x(k+1) = \sum_{i=1}^{q} \alpha_i(k) \sum_{j=1}^{2^m} \beta_j(k) \big\{ (A_i + B_i E_j K) x(k) + B_i E_j^- v \big\} \tag{2.60}
$$

The variable $v \in \mathbb{R}^m$ can be considered as an external controlled input for the system (2.60). Hence, the problem of finding Ω_s for the system (2.40) boils down to the problem of computing the largest controlled invariant set for the system (2.60).

Procedure 2.4 Invariant set computation

- **Input:** The matrices $A_1, \dots, A_q$, $B_1, \dots, B_q$, the matrix K and the sets X, U and $\Omega_{\max}$
- **Output:** An invariant approximation of the invariant set Ω_s for the closed loop system (2.40).

1. Set $i = 0$ and $C_0 = \Omega_{\max}$ and let the matrices F_0, g_0 be the half space representation of C_0, i.e. $C_0 = \{x \in \mathbb{R}^n : F_0 x \le g_0\}$
2. Compute the expanded set $P_j \subset \mathbb{R}^{n+m}$, $\forall j = 1, 2, \dots, 2^m$

$$P_j = \left\{ (x, v) \in \mathbb{R}^{n+m} : \begin{bmatrix} F_i\{(A_1 + B_1 E_j K)x + B_1 E_j^- v\} \\ F_i\{(A_2 + B_2 E_j K)x + B_2 E_j^- v\} \\ \vdots \\ F_i\{(A_q + B_q E_j K)x + B_q E_j^- v\} \end{bmatrix} \le \begin{bmatrix} g_i \\ g_i \\ \vdots \\ g_i \end{bmatrix} \right\}$$

3. Compute the projection $P_j^{(n)}$ of P_j on $\mathbb{R}^n$,

$$P_j^{(n)} = \left\{ x \in \mathbb{R}^n : \exists v \in U \text{ such that } (x, v) \in P_j \right\}, \quad \forall j = 1, 2, \dots, 2^m$$

4. Set

$$C_{i+1} = X \bigcap_{j=1}^{2^m} P_j^{(n)}$$

and let the matrices F_{i+1}, g_{i+1} be the half space representation of C_{i+1}, i.e.

$$C_{i+1} = \left\{ x \in \mathbb{R}^n : F_{i+1} x \le g_{i+1} \right\}$$

5. If $C_{i+1} = C_i$, then stop and set $\Omega_s = C_i$. Else continue.
6. Set $i = i + 1$ and go to step 2.

The system (2.60) can be considered as an uncertain system with respect to the parameters α_i and β_j. Hence using the results in Sect. 2.3.4, Procedure 2.4 can be used to obtain Ω_s.

Since $\Omega_{\max}$ is robustly invariant, it follows that $C_{i-1} \subseteq C_i$. Hence C_i is a robustly invariant set. The set sequence $\{C_0, C_1, \dots\}$ converges to Ω_s, which is the largest polyhedral invariant set.

Remark 2.4 Each polytope C_i represents an inner invariant approximation of the domain of attraction for the system (2.37) under the controller $u(k) = \operatorname{sat}(Kx(k))$. That means Procedure 2.4 can be stopped at any time before converging to the true largest invariant set Ω_s and obtain an inner invariant approximation of the domain of attraction.

Procedure 2.5 Invariant set computation

- **Input:** The matrices $A_1, A_2, \ldots, A_q$ and the sets X_H and $\Omega_{\max}$.
- **Output:** The invariant set Ω_s^H.

1. Set $i = 0$ and $C_0 = \Omega_{\max}$ and let the matrices F_0, g_0 be the half-space representation of the set C_0, i.e. $C_0 = \{x \in \mathbb{R}^n : F_0 x \le g_0\}$
2. Compute the set $P_j \subset \mathbb{R}^n$

$$P_j = \left\{ x \in \mathbb{R}^n : \begin{bmatrix} F_i(A_1 + B_1 E_j K + B_1 E_j^- H)x \\ F_i(A_2 + B_2 E_j K + B_2 E_j^- H)x \\ \vdots \\ F_i(A_q + B_q E_j K + +B_q E_j^- H)x \end{bmatrix} \le \begin{bmatrix} g_i \\ g_i \\ \vdots \\ g_i \end{bmatrix} \right\}$$

3. Set

$$C_{i+1} = X_H \bigcap_{j=1}^{2^m} P_j$$

and let the matrices F_{i+1}, g_{i+1} be the half-space representation of C_{i+1}, i.e.

$$C_{i+1} = \left\{ x \in \mathbb{R}^n : F_{i+1} x \le g_{i+1} \right\}$$

4. If $C_{i+1} = C_i$, then stop and set $\Omega_s = C_i$. Else continue.
5. Set $i = i + 1$ and go to step 2.

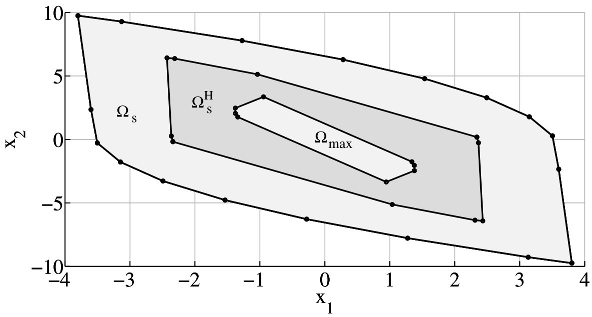


Fig. 2.13 Invariant sets for different control laws and different methods for Example 2.6. The set Ω_s is obtained for $u(k) = \mathrm{sat}(Kx(k))$ using Procedure 2.4. The set Ω_s^H is obtained for $u(k) = \mathrm{sat}(Kx(k))$ using Procedure 2.5. The set $\Omega_{\max}$ is obtained for $u(k) = Kx$ using Procedure 2.2

It is worth noticing that the matrix $H \in \mathbb{R}^{m \times n}$ resulting from the LMI problem (2.55) can also be used for computing an inner polyhedral invariant approximation Ω_s^H of the domain of attraction. Clearly, Ω_s^H is a subset of Ω_s, since v is now in a restricted form $v(k) = Hx(k)$. In this case, using (2.60) one obtains,

$$x(k+1) = \sum_{i=1}^{q} \alpha_i(k) \sum_{j=1}^{2^m} \beta_j(k) \left\{ \left(A_i + B_i E_j K + B_i E_j^- H\right) x(k) \right\} \tag{2.61}$$

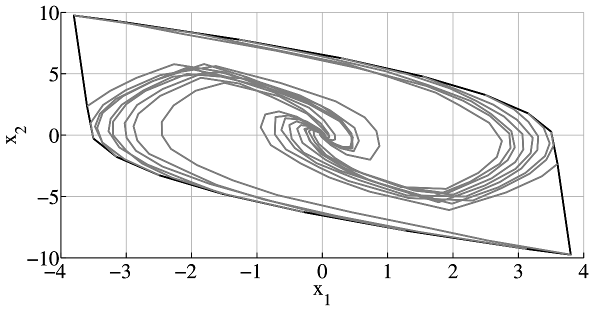


Fig. 2.14 State trajectories of the closed loop system with $u(k) = \mathrm{sat}(Kx(k))$ for Example 2.6

Define the set X_H as,

$$X_H = \left\{x \in \mathbb{R}^n : F_H x \le g_H\right\} \tag{2.62}$$

where

$$F_H = \begin{bmatrix} F_x \\ H \\ -H \end{bmatrix}, \qquad g_H = \begin{bmatrix} g_x \\ u_u \\ u_l \end{bmatrix}$$

Procedure 2.5 can be used for computing Ω_s^H.

Since the matrix $\sum_{i=1}^{q} \alpha_i(k) \sum_{j=1}^{2^m} \beta_j(k)\{(A_i + B_i E_j K + B_i E_j^- H)\}$ is asymptotically stable, Procedure 2.5 terminates in finite time [29]. In other words, there exists a finite index $i = i_{\max}$ such that $C_{i_{\max}} = C_{i_{\max}+1}$.

Example 2.6 Consider Example 2.4 with the same state and control constraints. The controller is $K = [-1.8112\ -0.8092]$.

Using Procedure 2.4, the set Ω_s is obtained after 121 iterations and depicted in Fig. 2.13. This figure also shows the set Ω_s^H obtained by using Procedure 2.5 with the auxiliary matrix $H = [-0.4058\ -0.2773]$, and the set $\Omega_{\max}$ obtained with the controller $u(k) = Kx$ using Procedure 2.2.

Ω_s^H and Ω_s are presented in minimal normalized half-space representation as,

$$\Omega_s^H = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} -0.8256 & -0.5642 \\ 0.8256 & 0.5642 \\ 0.9999 & 0.0108 \\ -0.9999 & -0.0108 \\ 0.9986 & 0.0532 \\ -0.9986 & -0.0532 \\ -0.6981 & -0.7160 \\ 0.6981 & 0.7160 \\ 0.9791 & 0.2033 \\ -0.9791 & -0.2033 \\ -0.4254 & -0.9050 \\ 0.4254 & 0.9050 \end{bmatrix} x \le \begin{bmatrix} 2.0346 \\ 2.0346 \\ 2.3612 \\ 2.3612 \\ 2.3467 \\ 2.3467 \\ 2.9453 \\ 2.9453 \\ 2.3273 \\ 2.3273 \\ 4.7785 \\ 4.7785 \end{bmatrix} \right\}$$

$$\Omega_s = \left\{ x \in \mathbb{R}^2 : \begin{bmatrix} -0.9996 & -0.0273 \\ 0.9996 & 0.0273 \\ -0.9993 & -0.0369 \\ 0.9993 & 0.0369 \\ -0.9731 & -0.2305 \\ 0.9731 & 0.2305 \\ 0.9164 & 0.4004 \\ -0.9164 & -0.4004 \\ 0.8434 & 0.5372 \\ -0.8434 & -0.5372 \\ 0.7669 & 0.6418 \\ -0.7669 & -0.6418 \\ 0.6942 & 0.7198 \\ -0.6942 & -0.7198 \\ 0.6287 & 0.7776 \\ -0.6287 & -0.7776 \\ 0.5712 & 0.8208 \\ -0.5712 & -0.8208 \end{bmatrix} x \le \begin{bmatrix} 3.5340 \\ 3.5340 \\ 3.5104 \\ 3.5104 \\ 3.4720 \\ 3.4720 \\ 3.5953 \\ 3.5953 \\ 3.8621 \\ 3.8621 \\ 4.2441 \\ 4.2441 \\ 4.7132 \\ 4.7132 \\ 5.2465 \\ 5.2465 \\ 5.8267 \\ 5.8267 \end{bmatrix} \right\}$$

Figure 2.14 presents state trajectories of the closed loop system with $u(k) = \text{sat}(Kx(k))$ for different initial conditions and realizations of $\alpha(k)$.

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