Preface

Many physical phenomena are described by equations involving nondifferentiable functions, e.g., generic trajectories of quantum mechanics (Feynman and Hibbs 1965). Several different approaches to deal with nondifferentiable functions are proposed in the literature of variational calculus. We can mention the time scale approach, which typically deal with delta or nabla differentiable functions (Ferreira and Torres 2008; Malinowska and Torres 2009; Martins and Torres 2009); the fractional approach, allowing to consider functions that have no first order derivative but have fractional derivatives of all orders less than one (Almeida et al. 2010; Frederico and Torres 2008; Malinowska and Torres 2012); and the quantum approach, which is the subject of this book and is particularly useful to model physical and economical systems (Bangerezako 2004; Cresson et al. 2009; Malinowska and Torres 2010).

Quantum difference operators are receiving an increase of interest, mainly due to their applications—see, e.g., (Almeida and Torres 2009a; Annaby et al. 2012; Bangerezako 2004; Bangerezako 2005; Cresson et al. 2009; Ernst 2008; Kac and Cheung 2002). In 1992, Nottale introduced the theory of scale-relativity without the hypothesis of space–time differentiability (Nottale 1992; Nottale 1999). A rigorous mathematical foundation to Nottale's scale-relativity theory is nowadays given by means of a quantum calculus (Almeida and Torres 2009a; Almeida and Torres 2010; Cresson et al. 2009; Kac and Cheung 2002). Roughly speaking, we substitute the classical derivative by a difference operator, which allows us to deal with sets of nondifferentiable curves. For a deeper discussion of the motivation to study a nondifferentiable quantum calculus and its leading role in the understanding of complex physical systems, we refer the reader to (Almeida and Torres 2009a; Cresson et al. 2009; Kac and Cheung 2002; Nottale 1992).

Quantum calculus has several different dialects (Brito da Cruz et al. 2012; Brito da Cruz et al. 2013b, c; Ernst 2008; Kac and Cheung 2002). The most common one is based on Jackson's q-operators, where q stands for quantum (Annaby and Mansour 2012; Jackson 1908; Jackson 1910; Kac and Cheung 2002). The Jackson q-difference operator is defined by

$$D_q f(t) = \frac{f(qt) - f(t)}{t(q-1)}, \quad t \neq 0,$$

where q is a fixed number, normally taken from (0, 1). Here f is supposed to be defined on a q-geometric set A, i.e., A is a subset of $\mathbb{R}$ (or $\mathbb{C}$) for which $qt \in A$ whenever $t \in A$. The derivative at zero is defined to be $f'(0)$, provided that $f'(0)$ exists (Abu Risha et al. 2007; Andrews et al. 1999; Carmichael 1911; Carmichael 1913; Ismail 2005; Jackson 1908). Jackson also introduced the q-integral

$$\int_0^a f(t)d_qt = a(1-q)\sum_{k=0}^{\infty} q^k f(aq^k),$$

provided that the series converges, and in this case he defined

$$\int_a^b f(t)d_qt = \int_0^b f(t)d_qt - \int_0^a f(t)d_qt$$

(Al-Salam 1966; Jackson 1908; Jackson 1910; Kac and Cheung 2002). In 1949, Hahn introduced the quantum difference operator

$$D_{q,\omega}[f](t) = \frac{f(qt+\omega)-f(t)}{(q-1)t+\omega}, \quad t \neq \omega_0,$$

where $\omega_0 := \frac{\omega}{1-q}$, f is a real function defined on an interval I containing ω_0, and $q \in (0, 1)$ and $\omega \geq 0$ are real fixed numbers (Hahn 1949). The Hahn operator unifies (in the limit) the two most well known and used quantum difference operators: the Jackson q-difference derivative D_q, where $q \in (0, 1)$ (Gasper and Rahman 2004; Jackson 1951; Kac and Cheung 2002); and the forward difference Δ_ω, where $\omega > 0$ (Bird 1936; Jagerman 2000; Jordan 1965). The Hahn difference operator is a successful tool for constructing families of orthogonal polynomials and investigating some approximation problems—see, e.g., (Alvarez-Nodarse 2006; Costas-Santos and Marcellán 2007; Dobrogowska and Odzijewicz 2006; Kwon et al. 1998; Petronilho 2007). However, during 60 years, the construction of the proper inverse of Hahn's difference operator $\mathrm{D}_{q,\omega}$ remained an open question. Eventually, the problem was solved in 2009 by Aldwoah, who developed the associated integral calculus (Aldwoah 2009)—see also (Aldwoah and Hamza 2011; Annaby et al. 2012). A different approach would be to reduce the Hahn analysis to the Jackson q-analysis (Odzijewicz et al. 2001, Appendix A).

In this book, we develop the variational Hahn calculus. More precisely, we investigate problems of the calculus of variations using Hahn's difference operator and the Jackson–Nörlund integral. The calculus of variations is a classical area of mathematics with many applications in geometry, physics, economics, biology, engineering, dynamical systems, and control theory (Leizarowitz 1985; Leizarowitz 1989; Weinstock 1974). Although being an old theory, it is very much alive and still evolving—see, e.g., (Almeida et al. 2010; Almeida and Torres 2009b; Leizarowitz and Zaslavski 2003; Malinowska and Torres 2012; Martins and Torres 2009). The basic problem of calculus of variations can be formulated as follows: among all differentiable functions $y : [a, b] \to \mathbb{R}$ such that $y(a) = \alpha$ and $y(b) = \beta$, where α, β are fixed real numbers, find the one that minimize (or maximize) the functional

$$\mathcal{L}[y] = \int_a^b L(t, y(t), y'(t))dt.$$

It can be proved that the candidates to be minimizers (resp. maximizers) to this problem must satisfy the ordinary differential equation

$$\frac{d}{dt}\partial_3 L(t, y(t), y'(t)) = \partial_2 L(t, y(t), y'(t)),$$

called the Euler–Lagrange equation (by $\partial_i L$ we denote the partial derivative of L with respect to its ith argument). If the boundary condition $y(a) = \alpha$ is not present in the problem, then to find the candidates for extremizers one has to add another necessary condition: $\partial_3 L(a, y(a), y'(a)) = 0$; if $y(b) = \beta$ is not present, then $\partial_3 L(b, y(b), y'(b)) = 0$. These two conditions are known as natural boundary conditions or transversality conditions. Since many important physical phenomena are described by nondifferentiable functions, to develop a calculus of variations based on the Hahn quantum operator is an important issue. This is precisely what we do in this book. We discuss the fundamental concepts of a variational calculus, such as the Euler–Lagrange equations for the basic and isoperimetric problems, as well as Lagrange and optimal control problems. As particular cases, we obtain the classical discrete-time calculus of variations (Kelley and Peterson 2001, Chap. 8), the variational q-calculus (Bangerezako 2004; Bangerezako 2005), and the calculus of variations applied to Nörlund's sum (Fort 1937; Fort 1948). Variational functionals that depend on higher-order quantum derivatives are considered as well. Such problems arise in a natural way in applications of engineering, physics, and economics. As an example, we can consider the equilibrium of an elastic bending beam. Let us denote by $y(x)$ the deflection of the point x of the beam, $E(x)$ the elastic stiffness of the material, that can vary with x, and $\xi(x)$ the load that bends the beam. One may assume that, due to some constraints of physical nature, the dynamics does not depend on the usual derivative $y'(x)$ but on some quantum derivative $D_{q,\omega}[y](x)$. In this condition, the equilibrium of the beam correspond to the solution of the following higher-order Hahn's quantum variational problem:

$$\int_0^L \left[\frac{1}{2}\left(E(x)D^2_{q,\omega}[y](x)\right)^2 - \xi(x)y\left(q^2x + q\omega + \omega\right)\right]dx \to \min.$$

Note that we recover the classical problem of the equilibrium of the elastic bending beam when $(\omega, q) \to (0, 1)$. This problem is a particular case of problem (P) investigated in Sect. 2.7. Our higher-order Hahn's quantum Euler–Lagrange equation (Theorem 2.45) gives the main tool to solve such problems. As particular cases, we obtain the q-calculus Euler–Lagrange equation (Bangerezako 2004) and the h-calculus Euler–Lagrange equation (Bastos et al. 2011; Kelley and Peterson 2001).

Another generalization of the q-calculus considered in this book includes the quantum calculus that results from the n-power difference operator

$$D_n f(t) = \begin{cases} \frac{f(t^n)-f(t)}{t^n-t} & \text{if } t \in \mathbb{R}\backslash\{-1,0,1\}, \\ f'(t) & \text{if } t \in \{-1,0,1\}, \end{cases}$$

where n is a fixed odd positive integer (Aldwoah 2009). For that we develop a calculus based on the new and more general proposed operator $D_{n,q}$ (see Definition 3.2). The class of quantum systems thus obtained has two parameters and is wider than the standard class of quantum dynamical systems studied in the literature. We claim that the n,q-calculus offers a better mathematical modeling technique to deal with quantum physical systems of time-varying graininess. We trust that our n,q-quantum calculus will become a useful tool to investigate nonconservative dynamical systems in physics (Bartosiewicz and Torres 2008; El-Nabulsi and Torres 2007; El-Nabulsi and Torres 2008; Frederico and Torres 2007).

The subject of this short book is recent and is still evolving. The Hahn quantum variational calculus was started only in 2010 with the work (Malinowska and Torres 2010). Quantum variational problems involving Hahn's derivatives of higher-order were first investigated in (Brito da Cruz et al. 2012). Several quantum variational problems have been recently posed and studied (Aldwoah et al. 2012; Almeida and Torres 2009; Almeida and Torres 2011; Bangerezako 2004; Bangerezako 2005; Brito da Cruz et al. 2013a; Cresson 2005; Cresson et al. 2009; Frederico and Torres 2013; Martins and Torres 2012). The main purpose of this book is to present optimality conditions for generalized quantum variational problems in an unified and a coherent way, and call attention to a promising research area with possible applications in optimal control, physics, and economics (Cruz et al. 2010; Malinowska and Martins 2013; Sengupta 1997). The results presented in the book allow to deal with economical problems with a dynamic nature that does not depend on the usual derivative or the forward difference operator, but on the Hahn quantum difference operator $D_{q,\omega}$. This is connected with a moot question: what kind of "time" (continuous or discrete) should be used in the construction of dynamic models in economics? Although individual economic decisions are generally made at discrete time intervals, it is difficult to believe that they are perfectly synchronized as postulated by discrete models. The usual assumption that the economic activity takes place continuously is a convenient abstraction in many applications. In others, such as the ones studied in financial market equilibrium, the assumption of continuous trading corresponds closely to reality. We believe that our Hahn's approach helps to bridge the gap between two families of models: continuous and discrete.

This short book gives a gentle but solid introduction to the *Quantum Variational Calculus*. The audience is primarily advanced undergraduate and graduate students of mathematics, physics, engineering, and economics. However, the book provides also an opportunity for an introduction to the quantum variational calculus even for experienced researchers. Our aim is to introduce the theory of the quantum calculus of variations in a way suitable for self-study, and at the same time to give the reader the state of the art of a very active and promising research area. We will be extremely happy if the present book will motivate and encourage some readers to follow a research activity in the area, and to take part in the exploration of this exciting subject.

Keywords: Hahn's difference operator; Jackson–Norlünd's integral; Quantum calculus; *q*-differences; Calculus of variations and optimal control; Quantum variational problems; Necessary optimality conditions; Euler–Lagrange equations; Generalized natural boundary conditions; Isoperimetric problems; Leitmann's principle; Ramsey model; *n,q*-power difference operator; Generalized Nörlund sum; Generalized Jackson integral; *n,q*-difference equations.

Bialystok and Aveiro, July 2013 Agnieszka B. Malinowska
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