Chapter 2
A Topography-Fitted Coordinate System

This chapter is devoted to the presentation of the curvilinear coordinates that we use for the description of a shallow mass flow down arbitrary topography. In the first section basic concepts on the geometry and kinematics of a surface are provided. They will be smoothly applied in the next section, which deals with the parameterization of the topographic surface. With these prerequisites topography-fitted coordinates are then introduced. These coordinates have been used to describe thin flows by Dressler [1], De Toni and Scotton [2], Bouchut and Westdickenberg [3], Ionescu [4], to name only a few. We follow the approach initiated by Bouchut and Westdickenberg [3], by which the results are expressed in matrix form. However, unlike [3], we use vectors and tensors and their contravariant components to derive the main results, which makes the computations geometrically more intuitive, and allows an easy extension of the flow modelling from ideal fluids to arbitrary materials, from slightly curved topographic surfaces to those with arbitrary curvature, and from rigid to moving material and erodible topographies.

2.1 Basics of the Geometry and Kinematics of a Surface

We present some elementary concepts and rules from the geometry and kinematics of a surface. These concepts/rules are probably already familiar to the reader, but providing them here we offer careful definitions, notations and a collection of properties which will enable us to model the basal topography. Besides, the material serves to keep the book self-determined as much as possible.

2.1.1 Basics of the Geometry of a Surface

Consider a regular surface $S$ in the three-dimensional Euclidean space $\mathcal{E}$, given by the parametrization
\[ x_1 = x_1(\Delta^1, \Delta^2), \quad x_2 = x_2(\Delta^1, \Delta^2), \quad x_3 = x_3(\Delta^1, \Delta^2) \quad (2.1) \]

with respect to an orthogonal Cartesian coordinate system \( Ox_1x_2x_3 \). Here, \((\Delta^1, \Delta^2)\) belongs to some open set \( D \subset \mathbb{R}^2 \), the functions \( x_1, x_2, x_3 \) are in \( C^2(D) \) (i.e., \( x_1, x_2, x_3 \) are twice continuously differentiable on \( D \)), and

\[ \text{rank} \left( \frac{\partial x_i}{\partial \Delta^\alpha} \right) = 2. \quad (2.2) \]

A point \( Q \in S \) has the Cartesian coordinates determined by (2.1) for some \((\Delta^1, \Delta^2) \in D\), and the position vector \( r \) given by

\[ r(\Delta^1, \Delta^2) = x_1(\Delta^1, \Delta^2) e_1 + x_2(\Delta^1, \Delta^2) e_2 + x_3(\Delta^1, \Delta^2) e_3, \]

where \( \{e_1, e_2, e_3\} \) is the standard basis of the translation vector space \( V_3 \) of \( E \) and associated to the Cartesian system \( Ox_1x_2x_3 \), see Fig. 2.1a. Condition (2.2) guarantees the linear independence of the vectors

\[ \tau_\alpha \equiv \frac{\partial r}{\partial \Delta^\alpha}, \quad \alpha \in \{1, 2\}, \quad (2.3) \]

at \( Q \), and so \( \tau_1 \) and \( \tau_2 \) build a basis, called the natural basis, for the vector space which they generate (called, the tangent space to \( S \) at \( Q \)). Let \( \{\tau^1, \tau^2\} \) be the reciprocal basis corresponding to the natural basis \( \{\tau_1, \tau_2\} \) (see (1.2) for the concept of reciprocal basis). With the following definitions of the scalars \( \phi_{\alpha\beta}, \phi^{\alpha\beta}, \alpha, \beta \in \{1, 2\}, \)

\[ \phi_{\alpha\beta} \equiv \tau_\alpha \cdot \tau_\beta, \quad \phi^{\alpha\beta} \equiv \tau^\alpha \cdot \tau^\beta, \quad (2.4) \]

we immediately obtain the relations

\[ \tau^\beta = \phi_{\alpha\beta} \tau^\alpha, \quad \tau_\beta = \phi^{\alpha\beta} \tau_\alpha, \quad (\phi^{\alpha\beta}) = (\phi_{\alpha\beta})^{-1}, \quad (2.5) \]

Fig. 2.1 a The Cartesian coordinate system \( Ox_1x_2x_3 \), the position vector \( r \) of a point \( Q \) on the surface \( S \), and the bases \( \{e_1, e_2, e_3\}, \{\tau_1, \tau_2, n\} \) of \( V_3 \); b a unit normal vector field on \( S \).
2.1 Basics of the Geometry and Kinematics of a Surface

see also (1.3). According to (2.4), it is clear that the matrices \( (\phi_{\alpha\beta}) \) and \( (\phi^{\alpha\beta}) \) are symmetric and positive definite; the scalars \( \phi_{\alpha\beta} \) are called the coefficients of the first fundamental form of \( S \) at \( Q \) (corresponding to the parametrization (2.1) of the surface \( S \)).

In the next considerations we assume that we are given a unit normal vector field \( n \) to \( S \), varying continuously on \( S \), see Fig. 2.1b. We take such a vector field as

\[
n = \frac{\tau_1 \times \tau_2}{\|\tau_1 \times \tau_2\} ;
\]

this \( n \) is shown in Fig. 2.1a. For further use we note that \( \tau_1, \tau_2, n \), evaluated at \( Q \in S \), form a basis of the vector space \( V_3 \).

Now, since \( n \cdot n = 1 \), the derivatives of \( n \) with respect to \( \Delta^1 \) and \( \Delta^2 \) are vectors in the tangent space to \( S \) at \( Q \), so that we have the representation

\[
\frac{\partial n}{\partial \Delta^\beta} = -b_{\alpha\beta} \tau^\alpha ,
\]

which defines the coefficients \( b_{\alpha\beta} \) of the second fundamental form of \( S \) (corresponding to the parametrization (2.1) and to the normal vector field \( n \)). With the relation \( \tau_\alpha \cdot n = 0 \) it can be shown that \( (b_{\alpha\beta}) \) is a symmetric matrix. Indeed,

\[
b_{\alpha\beta} = -\tau_\alpha \cdot \frac{\partial n}{\partial \Delta^\beta} = \frac{\partial \tau_\alpha}{\partial \Delta^\beta} \cdot n = \frac{\partial}{\partial \Delta^\alpha} \left( \frac{\partial r}{\partial \Delta^\alpha} \right) \cdot n
\]

The curvature tensor of the surface \( S \) (to which the unit vector field \( n \) has been associated) is defined by

\[
\mathcal{H} \equiv b_{\alpha\beta} \tau^\alpha \otimes \tau^\beta .
\]

Relation (2.5)\(^2\) can be used to represent \( \mathcal{H} \) as

\[
\mathcal{H} = b_{\gamma\beta} \tau^\gamma \otimes \tau^\beta = \phi^{\alpha\gamma} b_{\gamma\beta} \tau_\alpha \otimes \tau^\beta = W^\alpha_{\beta} \tau_\alpha \otimes \tau^\beta ,
\]

where

\[
W^\alpha_{\beta} = \phi^{\alpha\gamma} b_{\gamma\beta} .
\]

The matrix \( W \equiv (W^\alpha_{\beta}) \) is called the Weingarten curvature matrix, and the scalar

\[
\Omega \equiv \frac{1}{2} \text{tr} \mathcal{H} = \frac{1}{2} \text{tr} W
\]

\(^1\)The pair \((S, n)\) is called an oriented surface. Of course, \((S, -n)\) is another oriented surface.

\(^2\)\(n \cdot n = 1 \implies \frac{\partial}{\partial \Delta^\alpha} (n \cdot n) = 0 \implies \frac{\partial n}{\partial \Delta^\alpha} \cdot n = 0\).
defines the mean curvature of $S$. Note that the following matrix relation

$$W = (\phi^{\alpha\gamma})(b_{\gamma\beta})$$

(2.12)

holds. We also mention that relation (2.7) can be further written as

$$\frac{\partial n}{\partial \Delta^\beta} = -b_{\gamma\beta} \tau^\gamma = -\phi^{\alpha\gamma} b_{\gamma\beta} \tau_\alpha = -W^\alpha_{\beta} \tau_\alpha .$$

(2.13)

Let us now change the parametric representation of $S$, i.e., consider the change of variables

$$\Delta^\alpha = \Delta^\alpha(\xi^1, \xi^2), \quad (\xi^1, \xi^2) \in D_0, \quad \alpha \in \{1, 2\} ,$$

(2.14)

with $D_0$ an open subset of $\mathbb{R}^2$. Thus, according to (2.1), another parameterization of $S$ is expressible as

$$x_k = x_k(\Delta^1(\xi^1, \xi^2), \Delta^2(\xi^1, \xi^2)) \equiv \tilde{x}_k(\xi^1, \xi^2), \quad k \in \{1, 2, 3\} ,$$

(2.15)

and the position vector $r$ can be written as

$$r = \tilde{r}(\xi^1, \xi^2) = \tilde{x}_1(\xi^1, \xi^2)e_1 + \tilde{x}_2(\xi^1, \xi^2)e_2 + \tilde{x}_3(\xi^1, \xi^2)e_3 .$$

On $S$ we consider the normal vector field $n$ defined in (2.6). We denote by $F$ the Jacobian matrix of the transformation (2.14),

$$F \equiv \left( \frac{\partial \Delta^\alpha}{\partial \xi^\beta} \right) , \quad \text{det } F \neq 0 ,$$

(2.16)

and by $\tilde{\phi}_{\alpha\beta}, \tilde{b}_{\alpha\beta}, \tilde{W}$ the quantities corresponding to the parametrization (2.15), similar to $\phi_{\alpha\beta}, b_{\alpha\beta}$ and $W$. It can be checked that the following relations,

$$(\tilde{\phi}_{\alpha\beta}) = F^T (\phi_{\alpha\beta}) F , \quad (\tilde{b}_{\alpha\beta}) = F^T (b_{\alpha\beta}) F , \quad \tilde{W} = F^{-1} W F ,$$

(2.17)

hold (see Exercise 2.1). Notice that, if $n$ is given by (2.6), as we have agreed, $n$ can be computed by a similar formula when dealing with the parameterization (2.15), that is,

$$n = \frac{\tilde{\tau}_1 \times \tilde{\tau}_2}{\| \tilde{\tau}_1 \times \tilde{\tau}_2 \|} , \quad \text{with } \tilde{\tau}_\alpha \equiv \frac{\partial \tilde{r}}{\partial \xi^\alpha} , \quad \alpha \in \{1, 2\} ,$$

(2.18)

if and only if $\text{det } F > 0$ (see Exercise 2.2).
2.1.2 Basics of a Moving Surface

Just to be not so abstract and to motivate the introduction of the concept of a moving surface, we mention that the topographic surface corresponding to a topographic bed which is erodible, or on which sediment may be deposited, is described as a moving surface; its speed of propagation (see the definition below) is the erosion/deposition rate (see the forthcoming Sect. 2.2.2).

We consider a moving surface (with respect to the reference system $Ox_1x_2x_3$), that is, a one-parameter family $S_t \equiv \{S_t\}_{t \in I}$ ($I \subset \mathbb{R}$—open interval) of regular surfaces $S_t$ given as

$$x_1 = x_1(\Delta^1, \Delta^2, t), \quad x_2 = x_2(\Delta^1, \Delta^2, t), \quad x_3 = x_3(\Delta^1, \Delta^2, t), \quad (2.19)$$

where $(\Delta^1, \Delta^2) \in \mathcal{D} \subset \mathbb{R}^2$, $\mathcal{D}$ is an open subset, and the functions $x_1$, $x_2$, $x_3$ are of class $C^2$ on $\mathcal{D} \times I$.

Clearly, for each instant $t$ we are given a parameterized surface $S_t$ as in Sect. 2.1.1. By choosing a continuous unit normal vector field to $S_t$ according to (2.6), everything that has been shown in Sect. 2.1.1 is valid for $S_t$. Of course, the geometric quantities, e.g. the coefficients of the fundamental forms, $\phi_{\alpha\beta}, b_{\alpha\beta}$, are generally time dependent.

Now we are interested in the time dependence of $x_1$, $x_2$, $x_3$. With the aid of the position vector of the surface point $(\Delta^1, \Delta^2)$ at time $t$,

$$r = r(\Delta^1, \Delta^2, t) = x_1(\Delta^1, \Delta^2, t) e_1 + x_2(\Delta^1, \Delta^2, t) e_2 + x_3(\Delta^1, \Delta^2, t) e_3,$$

we define the velocity $u_S$ of $(\Delta^1, \Delta^2)$ at the instant $t$ by

$$u_S \equiv \frac{\partial r}{\partial t}. \quad (2.20)$$

With respect to the basis $\{\tau_1, \tau_2, n\}$ of $\mathcal{V}_3$, $u_S$ has the representation

$$u_S = U^\beta \tau_\beta + Un. \quad (2.21)$$

The normal component $U$ of $u_S$ satisfies the relation

$$U = u_S \cdot n, \quad (2.22)$$

and is a quantity intrinsic to $S$, that is, independent of the parameterization of $S$ (see the forthcoming (2.26)$_2$). It is called the speed of displacement (propagation) of the surface $S$.

If the moving surface (2.19) is implicitly described by

$$F(x_1, x_2, x_3, t) = 0, \quad (2.23)$$
a unit normal vector \( n \) to \( \mathcal{S} \) can be determined as

\[
n = \nabla F / \| \nabla F \| \quad \text{with} \quad \nabla F \equiv \frac{\partial F}{\partial x_k} e_k.
\]  

(2.24)

We suppose that \( F \) is chosen in such a way that \( n \) computed via (2.24) is the same as the prescribed unit normal vector (2.6). Inserting (2.19) into (2.23) and differentiating the emerging relation with respect to \( t \), we obtain

\[
\frac{\partial F}{\partial t} + \nabla F \cdot \mathbf{u}_S = 0,
\]

(2.25)

which is called the evolution equation for \( \mathcal{S} \) or the kinematic equation of \( \mathcal{S} \). Substituting \( \nabla F = \| \nabla F \| n \) into (2.25) and using (2.22), we deduce

\[
\frac{\partial F}{\partial t} + \| \nabla F \| \mathbf{U} = 0 \quad \iff \quad \mathbf{U} = -\frac{\partial F}{\partial t} / \| \nabla F \|.
\]

(2.26)

If the speed of displacement \( \mathbf{U} \) is known, however \( F \) in (2.23) is unknown, relation (2.26)\(_1\) stands for the determination of \( F \) as a function of space and time. This is the case in this book when modelling an erosion/deposition process: the erosion/deposition rate \( \mathbf{U} \) is postulated, and the evolution of the topographic profile is determined via the kinematic equation. Relation (2.26)\(_2\) indicates the independence of \( \mathbf{U} \) of the parameterization of \( \mathcal{S} \).

### 2.2 Mathematical Description of the Topographic Surface

Nowadays, remote sensing technology, such as Light Detection and Ranging (LIDAR) or Interferometric Synthetic Aperture Radar (InSAR), are widely applied in the modern Geographic Information Systems (GIS), where the topographic surface is generally represented by a set of altitudes of terrain locations over a regular horizontal grid, i.e., the DTM. These profile data can be interpolated for an explicit representation in Cartesian coordinates, which explains the mathematical description which we give below for the topographic surface.

If the topographic bed is deformable\(^3\) or moving or is associated with erosion/deposition processes, it will be called active. If not active, the topographic bed will be designated as stationary or rigid or stagnant. In Sect. 2.2.1 we consider a rigid bed, so that the corresponding topographic surface is modelled as a stationary surface. In Sect. 2.2.2 we switch to an active topographic bed, and model its topographic surface as a moving surface.

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\(^3\)For slow motion of a large (ice) mass on a deforming lithosphere, this case is important.
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2.2.1 Topographic Surface as a Stationary Surface

In this section we assume a rigid topographic bed, and describe the topographic surface as a surface $S_b$ given explicitly by

$$x_3 = b(x_1, x_2), \quad (x_1, x_2) \in \mathcal{D} \subset \mathbb{R}^2,$$

where $\mathcal{D}$ is an open set, and $b \in C^2(\mathcal{D})$. Physically, $x_1$ and $x_2$ lie on the horizontal plane, whilst $x_3$ points upwards against the direction of gravity; $b$ gives the bed elevation. Our derivations use the implicit representation of $S_b$,

$$F_b(x_1, x_2, x_3) = 0, \quad \text{with} \quad F_b(x_1, x_2, x_3) \equiv x_3 - b(x_1, x_2), \quad (2.27)$$

and the parametric representation

$$x_1 = x, \quad x_2 = y, \quad x_3 = b(x, y) \quad \text{for} \quad (x, y) \in \mathcal{D}, \quad (2.28)$$

which gives the position vector $r_b$ of a point $Q$ on $S_b$ as

$$r_b = r_b(x, y) = x e_1 + y e_2 + b(x, y) e_3. \quad (2.29)$$

We may think of the representation (2.28) as corresponding to the choice $\Delta^1 \equiv x$, $\Delta^2 \equiv y$ in the general presentation from Sect. 2.1.1. We take the continuous unit normal vector field $n_b$ on $S_b$ to point into the flowing material. Figure 2.1, in which the surface $S$ was designed to represent the topographic surface $S_b$, can be used to have a geometric/physical representation on $S_b$.

Considering parameterization (2.28) of $S_b$ and the above mentioned unit vector $n_b$, we determine the matrices $(\phi_{\alpha\beta})$, $(\phi^{\alpha\beta})$ and the Weingarten matrix $W$, which have been introduced in Sect. 2.1.1.

First, the two vectors of the natural basis of the tangent space to $S_b$ are given by

$$\tau_1 \equiv \frac{\partial r_b}{\partial x} = e_1 + \frac{\partial b}{\partial x} e_3, \quad \tau_2 \equiv \frac{\partial r_b}{\partial y} = e_2 + \frac{\partial b}{\partial y} e_3, \quad (2.30)$$

implying the following expressions for the coefficients of the first fundamental form,

$$\phi_{11} \equiv \tau_1 \cdot \tau_1 = 1 + \left(\frac{\partial b}{\partial x}\right)^2, \quad \phi_{22} \equiv \tau_2 \cdot \tau_2 = 1 + \left(\frac{\partial b}{\partial y}\right)^2, \quad \phi_{12} \equiv \tau_1 \cdot \tau_2 = \frac{\partial b}{\partial x} \frac{\partial b}{\partial y}.$$ 

With these expressions it can be easily checked that the matrix $(\phi_{\alpha\beta})$ emerges as

$$(\phi_{\alpha\beta}) = I + \text{grad } b \otimes \text{grad } b, \quad (2.31)$$
which yields, see Exercise 2.4,

\[(\phi^\alpha\beta) = (\phi_{\alpha\beta})^{-1} = \mathbf{I} - \frac{1}{1 + \nabla b \cdot \nabla b} \nabla b \otimes \nabla b. \tag{2.32}\]

With the notations

\[c \equiv (1 + \nabla b \cdot \nabla b)^{-1/2}, \quad s \equiv c \nabla b, \tag{2.33}\]

the matrices \((\phi_{\alpha\beta}), (\phi^{\alpha\beta})\) can be written in a more compact form as

\[(\phi_{\alpha\beta}) = \mathbf{I} + \frac{1}{c^2} s \otimes s, \quad (\phi^{\alpha\beta}) = \mathbf{I} - s \otimes s. \tag{2.34}\]

We explicitly record the relation

\[
\left(\mathbf{I} + \frac{1}{c^2} s \otimes s\right)^{-1} = \mathbf{I} - s \otimes s, \tag{2.35}
\]

to which we will refer in some derivations.

The unit normal vector \(n_b\) can be determined by using representation (2.27) of \(S_b\). One obtains

\[n_b = \frac{\nabla F_b}{\|\nabla F_b\|} = -s_1 e_1 - s_2 e_2 + c e_3, \tag{2.36}\]

where \(s_1, s_2\) are the components of \(s, (s_1, s_2)^T \equiv s\). Now we are able to compute the coefficients of the second fundamental form,

\[b_{11} = -\tau_1 \cdot \frac{\partial n_b}{\partial x} = -\left(e_1 + \frac{\partial b}{\partial x} e_3\right) \cdot \left(-\frac{\partial s_1}{\partial x} e_1 - \frac{\partial s_2}{\partial x} e_2 + \frac{\partial c}{\partial x} e_3\right) = \frac{\partial s_1}{\partial x} - \frac{\partial b}{\partial x} \frac{\partial c}{\partial x} = c \frac{\partial^2 b}{\partial x^2}, \tag{2.37}\]

\[b_{12} = -\tau_1 \cdot \frac{\partial n_b}{\partial y} = -\left(e_1 + \frac{\partial b}{\partial x} e_3\right) \cdot \left(-\frac{\partial s_1}{\partial y} e_1 - \frac{\partial s_2}{\partial y} e_2 + \frac{\partial c}{\partial y} e_3\right) = \frac{\partial s_1}{\partial y} - \frac{\partial b}{\partial x} \frac{\partial c}{\partial y} = \frac{\partial^2 b}{\partial x \partial y} = b_{21}, \]

\[b_{22} = -\tau_2 \cdot \frac{\partial n_b}{\partial y} = -\left(e_2 + \frac{\partial b}{\partial y} e_3\right) \cdot \left(-\frac{\partial s_1}{\partial y} e_1 - \frac{\partial s_2}{\partial y} e_2 + \frac{\partial c}{\partial y} e_3\right) = \frac{\partial s_2}{\partial y} - \frac{\partial b}{\partial y} \frac{\partial c}{\partial y} = \frac{\partial^2 b}{\partial y^2}.\]
which implies

\[(b_{\alpha \beta}) = c \mathbf{H}, \quad \text{with} \quad \mathbf{H} \equiv \text{grad} \, (\text{grad} \, b) = \begin{pmatrix}
\frac{\partial^2 b}{\partial x^2} & \frac{\partial^2 b}{\partial x \partial y} \\
\frac{\partial^2 b}{\partial x \partial y} & \frac{\partial^2 b}{\partial y^2}
\end{pmatrix} \tag{2.38}
\]

With (2.12), (2.34) and (2.38) we immediately deduce that the Weingarten matrix \(\mathbf{W}\) is given by

\[\mathbf{W} = (\phi^{\alpha \beta}) (b_{\beta \gamma}) = c \, (\mathbf{I} - \mathbf{s} \otimes \mathbf{s}) \mathbf{H}. \tag{2.39}\]

We note the following relations,

\[\mathbf{s} \cdot \mathbf{s} + c^2 = 1, \tag{2.40}\]

\[\text{grad} \, c = -\frac{1}{c} \, (\text{grad} \, \mathbf{s})^T \mathbf{s} = -c^2 \mathbf{H} \mathbf{s}, \quad \text{grad} \, \mathbf{s} = \mathbf{W}, \tag{2.41}\]

which we will need in some computations. Relation (2.40) expresses the fact that \(\|\mathbf{n}\| = 1\), and for the proof of (2.41) we refer the interested reader to Exercise 2.3.

For numerical computations it is sometimes more convenient to use surface parameters on \(S_b\) different from the Cartesian parameters \(x, y\) as in (2.28). We denote these parameters by \(\xi^1, \xi^2\). Thus, the change of variables

\[x = x(\xi^1, \xi^2), \quad y = y(\xi^1, \xi^2), \quad (\xi^1, \xi^2) \in \mathcal{D}_0, \tag{2.42}\]

with functions \(x, y\) of class \(C^2\) on the computational domain \(\mathcal{D}_0\), yields the parameterization

\[x_1 = x(\xi^1, \xi^2), \quad x_2 = y(\xi^1, \xi^2), \quad x_3 = b(x(\xi^1, \xi^2), y(\xi^1, \xi^2)) \equiv \tilde{b}(\xi^1, \xi^2), \tag{2.43}\]

and the position vector \(\mathbf{r}_b\) of a point on \(S_b\), see (2.29), as

\[\mathbf{r}_b(x, y) = x(\xi^1, \xi^2)\mathbf{e}_1 + y(\xi^1, \xi^2)\mathbf{e}_2 + \tilde{b}(\xi^1, \xi^2)\mathbf{e}_3 \equiv \tilde{\mathbf{r}}_b(\xi^1, \xi^2). \tag{2.44}\]

We denote by \(\mathbf{F}\) the Jacobian matrix of the transformation (2.42),

\[\mathbf{F} = \begin{pmatrix}
\frac{\partial x}{\partial \xi^1} & \frac{\partial x}{\partial \xi^2} \\
\frac{\partial y}{\partial \xi^1} & \frac{\partial y}{\partial \xi^2}
\end{pmatrix}, \quad \det \mathbf{F} \neq 0. \tag{2.45}\]

If we assume

\[\det \mathbf{F} > 0, \tag{2.46}\]
the prescribed normal vector field on \( S_b \) can be determined by formula (2.18), where now \( \tilde{\tau}_1, \tilde{\tau}_2 \) correspond to (2.43), see Exercise 2.1. However, in the derivations employed in this book we will never express \( n_b \) by this formula, so that assumption (2.46) is not important, only \( \det F \neq 0 \) counts.

From now on, we will no longer use the natural basis \( \{\tau_1, \tau_2\} \) corresponding to the Cartesian parameterization, see (2.30). That is why the vectors \( \tilde{\tau}_1, \tilde{\tau}_2 \) will be simply denoted by \( \tau_1, \tau_2 \), and they should not be confused with \( \tau_1, \tau_2 \) from (2.30). In short, \( \tau_1, \tau_2 \) refer to the parameterization (2.43), and, therefore, with (2.44) we have

\[
\tau_1 = \frac{\partial x}{\partial \xi^1} e_1 + \frac{\partial y}{\partial \xi^1} e_2 + \partial \tilde{b} / \partial \xi^1 e_3, \quad \tau_2 = \frac{\partial x}{\partial \xi^2} e_1 + \frac{\partial y}{\partial \xi^2} e_2 + \partial \tilde{b} / \partial \xi^2 e_3 .
\] (2.47)

However, we indicate by the tilde symbol the coefficients of the first and second fundamental forms referring to the parameterization (2.43). We further introduce the notations

\[
M_0 \equiv (\tilde{\phi}^{\alpha \beta}), \quad \tilde{H} \equiv (\tilde{b}_{\alpha \beta}),
\] (2.48)

and use them, together with (2.34), (2.38), (2.39) and (2.41)\(_1\), to rewrite formulae (2.17), now relating quantities corresponding to the parameterizations (2.28) and (2.43). That is,

\[
(\tilde{\phi}^{\alpha \beta}) = F^T (\phi^{\alpha \beta}) F \iff M_0^{-1} = F^T \left( I + \frac{1}{c^2} s \otimes s \right) F , \\
(\tilde{b}_{\alpha \beta}) = F^T (b_{\alpha \beta}) F \iff \tilde{H} = c F^T H F, \\
\tilde{W} = F^{-1} W F \iff \tilde{W} = M_0 \tilde{H} .
\] (2.49)

In (2.49)\(_3\) we have used the relation

\[
M_0 = F^{-1} (I - s \otimes s) F^{-T} ,
\] (2.50)
as deduced from (2.49)\(_1\) and (2.35).

We close this section by the following two remarks. First, recalling definition (1.12) for Grad \( f \), Grad \( v \), where \( f \) is a scalar field and \( v \) is a 2-column matrix field depending on \( \xi^1 \) and \( \xi^2 \), we have

Grad \( c = F^T \) grad \( c \), \quad Grad \( s = \) (grad \( s \))\( F \).

When combined with (2.41), (2.49)\(_2\) and \( W F = F \tilde{W} \), this yields the formulae

Grad \( c = -c \tilde{H} F^{-1} s \), \quad Grad \( s = F \tilde{W} \). (2.51)

Secondly, we will need the change of basis matrix \( P \) from the basis \( \{e_1, e_2, e_3\} \) to the basis \( \{\tau_1, \tau_2, n_b\} \) of \( \mathcal{V}_3 \), as well as the inverse matrix \( P^{-1} \). Inspection of (2.36) and (2.47) shows that
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\[
P = \begin{pmatrix}
\frac{\partial x}{\partial \xi^1} & \frac{\partial x}{\partial \xi^2} & -s_1 \\
\frac{\partial y}{\partial \xi^1} & \frac{\partial y}{\partial \xi^2} & -s_2 \\
\frac{\partial b}{\partial x} \frac{\partial x}{\partial \xi^1} + \frac{\partial b}{\partial y} \frac{\partial y}{\partial \xi^1} & \frac{\partial b}{\partial x} \frac{\partial x}{\partial \xi^2} + \frac{\partial b}{\partial y} \frac{\partial y}{\partial \xi^2} & c
\end{pmatrix}
\]

(2.52)

\[
P = \begin{pmatrix}
F & -s \\
(F^T \text{grad } b)^T & c
\end{pmatrix}
\]

To deduce \( P^{-1} \) we decompose \( P \) as

\[
P = \begin{pmatrix}
I & -s \\
\frac{1}{c} s^T & c
\end{pmatrix}
\begin{pmatrix}
F & 0 \\
0 & 1
\end{pmatrix}
\]

(2.53)

which immediately allows the derivation in block-form of \( P^{-1} \),

\[
P^{-1} = \begin{pmatrix}
F & 0 \\
0 & 1
\end{pmatrix}^{-1} \begin{pmatrix}
I & -s \\
\frac{1}{c} s^T & c
\end{pmatrix}^{-1} = \begin{pmatrix}
F^{-1} & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
I - s \otimes s & c s \\
-s^T & c
\end{pmatrix}
\]

(2.54)

\[
P^{-1} = \begin{pmatrix}
F^{-1} & 0 \\
0 & 1
\end{pmatrix} \begin{pmatrix}
I - s \otimes s & c F^{-1} s \\
-s^T & c
\end{pmatrix}
\]

SUMMARY of notations and relations

\[
S_b : r_b(x, y) = x e_1 + ye_2 + b(x, y)e_3
\]

\[
\tilde{r}_b(\xi^1, \xi^2) = x(\xi^1, \xi^2)e_1 + y(\xi^1, \xi^2)e_2 + \tilde{b}(\xi^1, \xi^2)e_3
\]

\[
\tilde{b}(\xi^1, \xi^2) \equiv b(x(\xi^1, \xi^2), y(\xi^1, \xi^2))
\]

\[
F \equiv \begin{pmatrix}
\frac{\partial x}{\partial \xi^1} & \frac{\partial x}{\partial \xi^2} \\
\frac{\partial y}{\partial \xi^1} & \frac{\partial y}{\partial \xi^2} \\
\frac{\partial b}{\partial \xi^1} & \frac{\partial b}{\partial \xi^2}
\end{pmatrix}
\]

\[
\tau_\alpha \equiv \frac{\partial \tilde{r}_b}{\partial \xi_\alpha} = \frac{\partial x}{\partial \xi_\alpha} e_1 + \frac{\partial y}{\partial \xi_\alpha} e_2 + \frac{\partial \tilde{b}}{\partial \xi_\alpha} e_3, \quad \alpha \in \{1, 2\}
\]

\[\text{Bouchut and Westdickenberg [3] noticed this useful decomposition.}\]
2 A Topography-Fitted Coordinate System

\[ c \equiv (1 + \text{grad} b \cdot \text{grad} b)^{-1/2}, \quad s \equiv c \text{ grad} b, \quad s \cdot s + c^2 = 1 \]

\[ n_b = \frac{\tau_1 \times \tau_2}{\| \tau_1 \times \tau_2 \|} = -s_1 e_1 - s_2 e_2 + ce_3, \quad (s_1, s_2)^T \equiv s \]

\[ \left( I + \frac{1}{c^2} s \otimes s \right)^{-1} = I - s \otimes s \tilde{\phi}_{\alpha\beta} \equiv \tau_\alpha \cdot \tau_\beta, \quad \tilde{\phi}_{\alpha\beta} \equiv \tau^\alpha \cdot \tau^\beta \]

\[ M_0 \equiv (\tilde{\phi}_{\alpha\beta}) = F^{-1}(I - s \otimes s)F^{-T} \]

\[ M_0^{-1} = (\tilde{\phi}_{\alpha\beta}) = F^T \left( I + \frac{1}{c^2} s \otimes s \right) F \]

\[ \frac{\partial n_b}{\partial \xi^\beta} = -\tilde{b}_{\alpha\beta} \tau^\alpha \]

\[ H \equiv \text{grad} (\text{grad} b), \quad \tilde{H} \equiv (\tilde{b}_{\alpha\beta}) = c F^T HF \]

\[ W = c (I - s \otimes s) H, \quad \tilde{W} = F^{-1}WF = M_0 \tilde{H} \]

\[ \Omega = \frac{1}{2} \text{tr} W = \frac{1}{2} \text{tr} \tilde{W}, \]

\[ \text{Grad} c = -c H F^{-1}s, \quad \text{Grad} s = FW. \]

### 2.2.2 Topographic Surface as a Moving Surface

Now we assume an active topographic bed. The topographic surface varies in time, and its mathematical model is a moving surface \( S_b \equiv \{ S_t \}_{t \in I} \), see Sect. 2.1.2, given explicitly by

\[ x_3 = b(x_1, x_2, t), \quad (x_1, x_2) \in D \subset \mathbb{R}^2, \quad t \in I \subset \mathbb{R}, \]

where \( D \) is open and \( b \in C^2(D \times I) \). Analogously to the case of a rigid topography, we use the implicit representation of \( S_b \),

\[ F_b(x_1, x_2, x_3, t) = 0, \quad \text{with} \quad F_b(x_1, x_2, x_3, t) \equiv x_3 - b(x_1, x_2, t), \quad (2.55) \]

and the parametric representation

\[ x_1 = x, \quad x_2 = y, \quad x_3 = b(x, y, t), \quad (2.56) \]

which gives the position vector \( r_b \) of a point on \( S_t \) as

\[ r_b = r_b(x, y, t) = xe_1 + ye_2 + b(x, y, t)e_3. \quad (2.57) \]
Furthermore, we consider the transformation
\[ x = x(\xi^1, \xi^2, t), \quad y = y(\xi^1, \xi^2, t), \]  
where the functions \( x, y \) are of class \( C^2 \) on \( D_0 \times I \) and such that \( \det \mathbf{F} \neq 0 \), with \( \mathbf{F} \) defined as in (2.45). This implies another parameterization of \( S_b \),
\[ x_1 = x(\xi^1, \xi^2, t), \quad x_2 = y(\xi^1, \xi^2, t), \quad x_3 = \tilde{b}(\xi^1, \xi^2, t), \]  
(2.59)
where
\[ \tilde{b}(\xi^1, \xi^2, t) \equiv b(x(\xi^1, \xi^2, t), y(\xi^1, \xi^2, t), t). \]  
(2.60)
The position vector \( \mathbf{r}_b \) of a point on \( S_t \), see (2.57), emerges as
\[ \mathbf{r}_b(x, y, t) = x(\xi^1, \xi^2, t)e_1 + y(\xi^1, \xi^2, t)e_2 + \tilde{b}(\xi^1, \xi^2, t)e_3 \equiv \tilde{r}_b(\xi^1, \xi^2, t). \]  
(2.61)
Clearly, the description of a topographic bed which is eroded or on which sediments are deposited is similar to that of a rigid topographic bed. What is different is the time dependence of the surface elevation \( b \) and of the change of parameters (2.58). Consequently, the geometric properties deduced for a rigid topography in Sect. 2.2.1 hold also for an active topography, and next we only refer to properties implied by the above mentioned time dependence.

Let \( \mathbf{u}_S \) denote the velocity of some surface point \((\xi^1, \xi^2)\) at the instant \( t \), that is,
\[ \mathbf{u}_S = \frac{\partial \tilde{r}_b}{\partial t} = \frac{\partial x}{\partial t} e_1 + \frac{\partial y}{\partial t} e_2 + \frac{\partial \tilde{b}}{\partial t} e_3. \]  
(2.62)
As pointed out in Sect. 2.1.2, its normal component \( \mathcal{U} = \mathbf{u}_S \cdot \mathbf{n}_b \) is the speed of propagation of \( S_b \), and can be deduced by formula (2.26) with \( \mathbf{F} = F_b \), where \( F_b \) is given by (2.55). We obtain
\[ \mathcal{U} = c \frac{\partial b}{\partial t} \quad \text{with} \quad c \equiv \left[ 1 + \left( \frac{\partial b}{\partial x} \right)^2 + \left( \frac{\partial b}{\partial y} \right)^2 \right]^{-1/2}, \]  
(2.63)
where definition (2.33)\(_1\) of \( c \) is repeated for convenience. In sediment transport context, the speed of propagation \( \mathcal{U} \) is called erosion/deposition rate, see Sect. 4.1. If \( \mathcal{U} > 0 \), sediments are deposited, and erosion occurs for \( \mathcal{U} < 0 \).

For later use we are interested in the determination of the tangential components \( \mathcal{U}^1 \) and \( \mathcal{U}^2 \) of the velocity \( \mathbf{u}_S \) in the representation
\[ \mathbf{u}_S = \mathcal{U}^\alpha \mathbf{\tau}_\alpha + \mathcal{U} \mathbf{n}_b, \]  
(2.64)
in terms of the components of \( \mathbf{u}_S \) with respect to the Cartesian basis. In (2.64) the vectors \( \mathbf{\tau}_1 \) and \( \mathbf{\tau}_2 \) tangent to \( S_t \) refer to parameterization (2.59), that is,
\[ \tau_1 = \frac{\partial x}{\partial \xi_1} e_1 + \frac{\partial y}{\partial \xi_1} e_2 + \frac{\partial \tilde{b}}{\partial \xi_1} e_3, \quad \tau_2 = \frac{\partial x}{\partial \xi_2} e_1 + \frac{\partial y}{\partial \xi_2} e_2 + \frac{\partial \tilde{b}}{\partial \xi_2} e_3, \] (2.65)

see also (2.47) for the case of a rigid topography. With the notations

\[ v_S \equiv \begin{pmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \end{pmatrix}, \quad u_S \equiv \begin{pmatrix} U^1 \\ U^2 \end{pmatrix}, \] (2.66)

formula (1.1) and representations (2.62) and (2.64) of \( u_S \) yield

\[ \begin{pmatrix} u_S \\ \mathcal{U} \end{pmatrix} = P^{-1} \begin{pmatrix} v_S \\ \frac{\partial \tilde{b}}{\partial t} \end{pmatrix}. \] (2.67)

In (2.67), \( P \) is the change of basis matrix from \( \{ e_1, e_2, e_3 \} \) to \( \{ \tau_1, \tau_2, n_b \} \), see (2.52), and its inverse \( P^{-1} \) is shown in (2.54). Insertion of \( P^{-1} \) and of \( \frac{\partial \tilde{b}}{\partial t} \) as given by

\[ \frac{\partial \tilde{b}}{\partial t} = \frac{\partial b}{\partial t} + \text{grad} \cdot v_S = \frac{\partial b}{\partial t} + \frac{1}{c} s \cdot v_S = \frac{1}{c} (\mathcal{U} + s \cdot v_S), \] (2.68)

into (2.67), yields

\[ \begin{pmatrix} u_S \\ \mathcal{U} \end{pmatrix} = \begin{pmatrix} F^{-1} (v_S + \mathcal{U}s) \\ \mathcal{U} \end{pmatrix} \iff u_S = F^{-1} (v_S + \mathcal{U}s), \] (2.69)

which we wanted to show.

**Remark** Assuming that the erosion/deposition rate \( \mathcal{U} \) is known, relation (2.63), written in the form

\[ \frac{\partial b}{\partial t} = \frac{1}{c} \mathcal{U}, \] (2.70)

stands as an evolution equation for the bed elevation \( b \). If the \( (\xi^1, \xi^2) \) coordinates are used, the equation for the bed elevation reads

\[ \frac{\partial \tilde{b}}{\partial t} = \frac{1}{c} (\mathcal{U} + s \cdot v_S), \] (2.71)

see (2.68). A law for the erosion/deposition rate \( \mathcal{U} \) will be given in Sect. 5.3.
2.3 Topography-Fitted Coordinates

In this section we introduce curvilinear coordinates for the points lying in the vicinity of the basal surface $S_b$. Two of such coordinates are $\xi^1, \xi^2$ from the parametric representation of the ground surface $S_b$. The third coordinate, $\zeta$, measures the distance from a point near $S_b$ to $S_b$, and adequately serves to apply the shallowness approximations, even in steep topographies. We divide the content of this section into two parts, depending on whether the erosion/deposition process is taken into account or not.

2.3.1 Coordinates Fitted to a Stationary Topographic Surface

In this section the topographic bed is assumed rigid, and so the bed surface is described as the stationary topographic surface given by (2.43), to which the assigned unit normal vector field $n_b$ points into the flowing mass. We denote the Cartesian coordinates of a point $P \in \mathcal{E}$ near the surface $S_b$ by $(x_1, x_2, x_3)$, and use $(\xi^1, \xi^2, \xi^3 \equiv \zeta)$ for the topography-fitted curvilinear coordinates of $P$, introduced as follows.

Let $\mathbf{r}$ be the position vector of $P$,

$$
\mathbf{r}(x_1, x_2, x_3) = x_1 \mathbf{e}_1 + x_2 \mathbf{e}_2 + x_3 \mathbf{e}_3,
$$

and let $\mathbf{r}_b$ be the position vector of the point $Q \in S_b$ representing the orthogonal projection of $P$ onto $S_b$,

$$
\mathbf{r}_b = \tilde{\mathbf{r}}_b(\xi^1, \xi^2) = x(\xi^1, \xi^2) \mathbf{e}_1 + y(\xi^1, \xi^2) \mathbf{e}_2 + \tilde{b}(\xi^1, \xi^2) \mathbf{e}_3, \quad \text{(2.72)}
$$
The vector relation
\[
\mathbf{r}(x_1, x_2, x_3) = \mathbf{r}_b(x_1, x_2) + \zeta \mathbf{n}_b(x_1, x_2)
\]
defines the change of coordinates
\[
(x_1, x_2, x_3) \leftrightarrow (\xi^1, \xi^2, \zeta)
\]
near the basal surface \(S_b\), see (2.44). The relation, see Fig. 2.2,

\[
\mathbf{r} = \mathbf{r}_b + \zeta \mathbf{n}_b ,
\]
more specifically,

\[
\mathbf{r}(x_1, x_2, x_3) = \tilde{\mathbf{r}}_b(\xi^1, \xi^2) + \zeta \tilde{\mathbf{n}}_b(\xi^1, \xi^2) \equiv \tilde{\mathbf{r}}(\xi^1, \xi^2, \zeta) ,
\]
defines the change of coordinates
\[
(x_1, x_2, x_3) \leftrightarrow (\xi^1, \xi^2, \zeta) ,
\]
on the condition that the Jacobian \(J\) of the transformation (2.74) is not zero,

\[
J \equiv \det A^{-1} \neq 0 , \quad \text{with} \quad A^{-1} \equiv \left( \frac{\partial x_i}{\partial \xi^j} \right)_{i,j \in \{1,2,3\}} .
\]

The geometric interpretation of the curvilinear coordinates \((\xi^1, \xi^2, \zeta)\) of \(P\) is clear, namely, \(\xi^1, \xi^2\) are the surface coordinates of the orthogonal projection \(Q\) of \(P\) onto \(S_b\), and \(|\zeta|\) is the distance from \(P\) to \(S_b\).

To use this change of coordinates in analytical derivations we need to know the matrices \(A^{-1}\) and \(A\). This will also allow an explicit expression of restriction (2.75)_1, which we henceforth assume to be fulfilled. In order to obtain \(A^{-1}\) we introduce the vectors

\[
g_k \equiv \frac{\partial \tilde{\mathbf{r}}}{\partial \xi^k} = A^{-1}_{jk} \mathbf{e}_j , \quad k \in \{1, 2, 3\} ,
\]
which build the natural basis of \(V_3\) at the point \(P\), associated to the coordinates \((\xi^1, \xi^2, \zeta)\). Relation (2.76) shows that the Jacobian matrix \(A^{-1}\) defined in (2.75) represents the change of basis matrix from \(\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}\) to \(\{\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3\}\). With definitions (2.47) and (2.76), the position vectors (2.72), (2.73), and relation (2.13), we have

---

5 Denoting the Jacobian matrix of (2.74) by the inverse of a matrix \(A\) we preserve the notation used by Bouchut and Westdickenberg [3] for the same quantity.
\[ g_1 = \frac{\partial \tilde{r}_b}{\partial \xi^1} + \zeta \frac{\partial \tilde{n}_b}{\partial \xi^1} = (1 - \zeta \tilde{W}_1^1) \tau_1 - \zeta \tilde{W}_1^2 \tau_2, \]
\[ g_2 = \frac{\partial \tilde{r}_b}{\partial \xi^2} + \zeta \frac{\partial \tilde{n}_b}{\partial \xi^2} = -\zeta \tilde{W}_2^1 \tau_1 + (1 - \zeta \tilde{W}_2^2) \tau_2, \]
\[ g_3 = n_b, \]

or, with a shorter notation,
\[ g_\alpha = (\delta_\beta^\alpha - \zeta \tilde{W}_\beta^\alpha) \tau_\beta, \quad g_3 = n_b. \]

This indicates that \( A^{-1} \) is the outcome of two successive transformations,
\[ A^{-1} = P_1 P, \]

where \( P, P_1 \) are the following change of basis matrices,
\[ P \quad \text{from} \{e_1, e_2, e_3\} \to \{\tau_1, \tau_2, n_b\}, \]
\[ P_1 \quad \text{from} \{\tau_1, \tau_2, n_b\} \to \{g_1, g_2, g_3\}. \]

The matrix \( P \) is given by (2.52), and \( P_1 \) can immediately be deduced from (2.77),
\[ P_1 = \begin{pmatrix} 1 - \zeta \tilde{W}_1^1 & -\zeta \tilde{W}_1^2 & 0 \\ -\zeta \tilde{W}_2^1 & 1 - \zeta \tilde{W}_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} I - \zeta \tilde{W} & 0 \\ 0 & 1 \end{pmatrix}, \]

so that
\[ A^{-1} = P_1 P = \begin{pmatrix} F(I - \zeta \tilde{W}) & -s \\ \frac{1}{c} (I - \zeta \tilde{W})^T F^T s^T c \end{pmatrix}. \]

Introducing the notation
\[ B \equiv F(I - \zeta \tilde{W}) = (I - \zeta W) F, \]

where \( F\tilde{W} = WF \) has been used, \( A^{-1} \) is expressible as
\[ A^{-1} \equiv \left( \frac{\partial x_i}{\partial \xi^j} \right) = \begin{pmatrix} B & -s \\ \frac{1}{c} (B^T s)^T c \end{pmatrix}. \]

Now, similarly to (2.53), we decompose \( A^{-1} \) as
\[ A^{-1} = \begin{pmatrix} I & -s \\ \frac{1}{c} s^T c \end{pmatrix} \begin{pmatrix} B & 0 \\ 0 & 1 \end{pmatrix}. \]
which immediately allows the derivation in block form of the matrix $A$,

$$
A = \begin{pmatrix}
    B & 0 \\
    0 & 1
\end{pmatrix}^{-1} \begin{pmatrix}
    I & -s \\
    \frac{1}{c}s^T & c
\end{pmatrix}^{-1} \begin{pmatrix}
    B^{-1} & 0 \\
    B^{-1}s & cs
\end{pmatrix} = \begin{pmatrix}
    B^{-1} & 0 \\
    0 & 1
\end{pmatrix} \begin{pmatrix}
    I - s \otimes s & c \\
    -s^T & c
\end{pmatrix},
$$

and therefore

$$
A = \left( \frac{\partial \xi^i}{\partial x^j} \right) = \begin{pmatrix}
    B^{-1}(I - s \otimes s) & cB^{-1}s \\
    -s^T & c
\end{pmatrix}.
$$

(2.81)

With $A^{-1}$ given in (2.80), we can compute the Jacobian determinant $J$,

$$
J \equiv \det A^{-1} = \frac{1}{c} \det B = \frac{1}{c} \det F \det(I - \tilde{\zeta}W),
$$

(2.82)

and explicit condition (2.75)$_1$. Since $\det F \neq 0$, (2.75)$_1$ reads

$$
\det(I - \tilde{\zeta}W) = \det(I - \zeta W) \neq 0,
$$

(2.83)

where (2.17)$_3$ has been used. This restricts the use of the change of coordinates (2.74) to realistic situations: (2.74) can be applied if and only if the basal topography plus the domain occupied by the flowing mass satisfy (2.83). We record expression (2.82) for the Jacobian $J$ in the form

$$
J = J_0 \det(I - \tilde{\zeta}W), \quad J_0 \equiv \frac{1}{c} \det F.
$$

(2.84)

We return to the basis $\{g_1, g_2, g_3\}$ in (2.76) and introduce its reciprocal basis $\{g^1, g^2, g^3\}$. The combination of (2.76) and (2.81) implies the following formulae for the contravariant coefficients $g^{ij} \equiv g^i \cdot g^j$ and the covariant coefficients $g_{ij} \equiv g_i \cdot g_j$ of the metric tensor,

$$
(g^{ij}) = AA^T = \begin{pmatrix}
    M & 0 \\
    0 & 1
\end{pmatrix}, \quad (g_{ij}) = (AA^T)^{-1} = \begin{pmatrix}
    M^{-1} & 0 \\
    0 & 1
\end{pmatrix},
$$

(2.85)

where

$$
M \equiv B^{-1}(I - s \otimes s)B^{-T}.
$$

(2.86)

Since $\{g^1, g^2, g^3 = n_b\}$ is a basis of $V_3$, the matrix $(g^{ij})$ is positive definite, therefore $M$ is equally a positive definite matrix. We notice that

$$
M|_{\zeta=0} = M_0,
$$

(2.87)
which yields (2.86) as

\[ M = (I - \zeta \tilde{W})^{-1} M_0 (I - \zeta \tilde{W})^{-T}, \]  

(2.88)

see expressions (2.50) and (2.79) of \( M_0 \) and \( B \), respectively.

**SUMMARY of notations and relations**

\[ r = r_b + \zeta n_b \iff r(x_1, x_2, x_3) = \tilde{r}_b(\xi_1, \xi_2) + \zeta \tilde{n}_b(\xi_1, \xi_2) \]

\[ \equiv \tilde{r}(\xi_1, \xi_2, \zeta), \quad \text{with} \quad \xi_3 \equiv \zeta \]

\[ g_k = \frac{\partial \tilde{r}}{\partial \xi^k}, \quad k \in \{1, 2, 3\} \]

\[ g_\alpha = (\delta^\beta_\alpha - \zeta \tilde{W}_\alpha^\beta) \tau_\beta, \quad \alpha \in \{1, 2\}, \quad g_3 = n_b \]

\[ B \equiv F(I - \zeta \tilde{W}) = (I - \zeta W)F \]

\[ M \equiv B^{-1}(I - s \otimes s)B^T = (I - \zeta \tilde{W})^{-1} M_0 (I - \zeta \tilde{W})^{-T} \]

\[ M|_{\zeta=0} = M_0 \]

\[ A^{-1} = \left( \frac{\partial x_i}{\partial \xi^j} \right) = \begin{pmatrix} B & -s \\ \frac{1}{c} (B^T s)^T & c \end{pmatrix} \]

\[ g_k = \frac{\partial \tilde{r}}{\partial \xi^k} = A^{-1}_{jk} e_j, \quad k \in \{1, 2, 3\} \]

\[ A = \left( \frac{\partial \xi^i}{\partial x_j} \right) = \begin{pmatrix} B^{-1}(I - s \otimes s) & c B^{-1} s \\ -s^T & c \end{pmatrix} \]

\[ J_0 \equiv \frac{1}{c} \det F, \quad J \equiv \det A^{-1} = J_0 \det(I - \zeta \tilde{W}) \]

\[ g^{ij} \equiv g^i \cdot g^j, \quad g_{ij} \equiv g_i \cdot g_j \]

\[ (g^{ij}) = AA^T = \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix}, \quad (g_{ij}) = (AA^T)^{-1} = \begin{pmatrix} M^{-1} & 0 \\ 0 & 1 \end{pmatrix}, \]
2.3.2 On the Components of Vectors and Tensors

We use the basis vectors $g_1, g_2, g_3$ to represent vectors in $\mathcal{V}_3$ and second order tensors on $\mathcal{V}_3$,

$$v = v^i g_i, \quad \sigma = T^{ij} g_i \otimes g_j.$$  \hfill (2.89)

Expressions (2.77) of $g_1, g_2, g_3$ show that $g_1, g_2$ are parallel to the surface $S_b$, and $g_3$ is perpendicular to $S_b$. Due to these geometric properties it is convenient to decompose a vector $v \in \mathcal{V}_3$ into a tangential component $v_\tau$ and a normal component $v_n$,

$$v = v^1 g_1 + v^2 g_2 + v^3 g_3 \quad \text{with} \quad v_\tau \equiv v^1 g_1 + v^2 g_2, \quad v_n \equiv v^3 g_3.$$  \hfill (2.90)

This decomposition is independent of the choice of the surface coordinates $\xi^1, \xi^2$, a fact which can also be seen from the relations

$$v_n = (v \cdot n_b) n_b, \quad v_\tau = v - (v \cdot n_b) n_b,$$

which are satisfied by $v_\tau$ and $v_n$. We collect the components of $v_\tau$ with respect to $g_1, g_2$ in a 2-column matrix, and use the notation $v$ for the normal component $v^3 = v \cdot n_b$ of $v$, i.e.,

$$v \equiv \begin{pmatrix} v^1 \\ v^2 \end{pmatrix}, \quad v \equiv v^3.$$

Similarly, we collect the components $T^{ij}$ of a symmetric second order tensor $\sigma$, see (2.89), as follows,

$$T \equiv \begin{pmatrix} T^{11} & T^{12} \\ T^{12} & T^{22} \end{pmatrix} = T^T, \quad t \equiv \begin{pmatrix} T^{13} \\ T^{23} \end{pmatrix} = \begin{pmatrix} T^{31} \\ T^{32} \end{pmatrix}.$$

We, therefore, use the following block matrices for the contravariant components of vectors and symmetric second order tensors,

$$\begin{pmatrix} v^1 \\ v^2 \\ v^3 \end{pmatrix} = \begin{pmatrix} v \\ v \end{pmatrix}, \quad (T^{ij}) = \begin{pmatrix} T & t \\ t^T & T^{33} \end{pmatrix}.$$  \hfill (2.91)

Note that, using (2.78)\textsubscript{1} in (2.90)\textsubscript{2}, $v_\tau$ can be written as

$$v_\tau = (\delta^\alpha_\beta - \zeta \tilde{W}^\alpha_\beta) v^3 \tau_\alpha,$$  \hfill (2.92)

which is the representation of $v_\tau$ with respect to $\tau_1, \tau_2$. Moreover, with (2.85)\textsubscript{2} we have
\|v_\tau\|^2 = (v^\alpha g_\alpha) \cdot (v^\beta g_\beta) = (g_\alpha \cdot g_\beta) v^\alpha v^\beta = M^{-1} v \cdot v . \quad (2.93)

We will also use the Cartesian components of vectors and the mixed components of a symmetric second order tensor, that is

\[ v = v_i e_i , \quad \sigma = \Sigma^{ij} e_i \otimes g_j . \quad (2.94) \]

For them we introduce the notations

\[ \mathcal{V} \equiv \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} , \quad \nu \equiv v_3 , \]

and

\[ \mathcal{S} \equiv \begin{pmatrix} \Sigma^{11} & \Sigma^{12} \\ \Sigma^{21} & \Sigma^{22} \end{pmatrix} , \quad s \equiv \begin{pmatrix} \Sigma^{13} \\ \Sigma^{23} \end{pmatrix} , \quad t \equiv \begin{pmatrix} \Sigma^{31} \\ \Sigma^{32} \end{pmatrix} , \quad \sigma \equiv \Sigma^{33} , \]

so that

\[ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} \mathcal{V} \\ \nu \end{pmatrix} , \quad (\Sigma^{ij}) = \begin{pmatrix} \mathcal{S} \\ s \end{pmatrix} T^T \sigma . \quad (2.95) \]

Note that \( \mathcal{S} \) is not a symmetric matrix as \( T \) in (2.91). Recalling that the change of basis matrix from \( \{ e_1, e_2, e_3 \} \) to \( \{ g_1, g_2, g_3 \} \) is \( A^{-1} \), see (2.75), formula (1.1) yields the relation between the contravariant components \( v^i \) and the Cartesian components \( v_i \) of \( v \), see (2.89) and (2.94),

\[ \begin{pmatrix} \mathcal{V} \\ \nu \end{pmatrix} = A^{-1} \begin{pmatrix} v \\ v \end{pmatrix} , \]

which gives, owing to (2.80),

\[ \mathcal{V} = B v - s \nu , \quad \nu = \frac{1}{c} s \cdot B v + c \nu . \quad (2.96) \]

In particular, the relation between the Cartesian coordinates, \( \mathcal{V}_\tau , \nu_\tau \), and the curvilinear coordinates, \( \nu , 0 \), of the tangential component \( \mathcal{V}_\tau \) of the vector \( \nu \), see (2.90), is

\[ \begin{pmatrix} \mathcal{V}_\tau \\ \nu_\tau \end{pmatrix} = A^{-1} \begin{pmatrix} v \\ 0 \end{pmatrix} , \]

that is,
\[ \mathbf{v}_{\tau} = \mathbf{Bv}, \quad v_{\tau} = \frac{1}{c} \mathbf{s} \cdot \mathbf{Bv} \quad (2.97) \]

Then, inserting \( g_i = A_{ki}^{-1} e_k \) into \((2.89)_2\) and comparing the emerging representation of the symmetric tensor \( \mathbf{\sigma} \) with \((2.94)_2\), we obtain the relation between the contravariant components \( T^{ij} \) and the mixed components \( \Sigma^{ij} \) of \( \mathbf{\sigma} \),

\[ (\Sigma^{ij}) = A^{-1} (T^{kl}) . \]

This is equivalent to

\[ \mathbf{S} = \mathbf{BT} - \mathbf{s} \otimes \mathbf{t}, \quad \mathbf{s} = \mathbf{Bt} - T^{33} \mathbf{s}, \]
\[ \mathbf{t} = \frac{1}{c} \mathbf{TB}^T \mathbf{s} + c \mathbf{t}, \quad \sigma = \frac{1}{c} \mathbf{s} \cdot \mathbf{Bt} + c T^{33} . \quad (2.98) \]

### 2.3.3 Coordinates Fitted to a Moving Topographic Surface

Now we consider a moving topographic surface \( S_b \), described by \((2.59)\), so that the position vector of a point on \( S_b \) is given by \((2.61)\), which we rewrite here for convenience,

\[ \mathbf{r}_b = \mathbf{r}_b(\xi^1, \xi^2, t) = x(\xi^1, \xi^2, t)e_1 + y(\xi^1, \xi^2, t)e_2 + \tilde{b}(\xi^1, \xi^2, t)e_3 . \quad (2.99) \]

**Fig. 2.3** The coordinates \((\xi^1, \xi^2, \zeta)\) and \((\hat{\xi}^1, \hat{\xi}^2, \hat{\zeta})\) of the point \( P \) at times \( t \) and \( \hat{t} \), respectively.
The change of coordinates which we introduce for points in a neighbourhood of the surface $S_b$ emerges from the same vector relation as that from Sect. 2.3.1, i.e., $r = r_b + \zeta n_b$. Explicitly,

$$r(x_1, x_2, x_3) = \tilde{r}_b(\xi^1, \xi^2, t) + \zeta \tilde{n}_b(\xi^1, \xi^2, t) \equiv \tilde{r}(\xi^1, \xi^2, \zeta, t). \tag{2.100}$$

However, the coordinates $\xi^1, \xi^2, \zeta$ of $P \in E$ are now time-dependent,

$$\xi^1(x_1, x_2, x_3, t), \quad \xi^2(x_1, x_2, x_3, t), \quad \zeta(x_1, x_2, x_3, t), \tag{2.101}$$

which is geometrically obvious: since the surface is moving, the orthogonal projection of $P$ onto $S_b$ and the distance of $P$ to the surface $S_b$ are changing in time, see Fig. 2.3, where the coordinates $(\hat{\xi}^1, \hat{\xi}^2, \hat{\zeta})$ of the point $P$ at the instant $\hat{t}$ are

$$\hat{\xi}^1 \equiv \xi^1(x_1, x_2, x_3, \hat{t}), \quad \hat{\xi}^2 \equiv \xi^2(x_1, x_2, x_3, \hat{t}), \quad \hat{\zeta} \equiv \zeta(x_1, x_2, x_3, \hat{t}).$$

As a consequence, as soon as the dependence of $\xi^1, \xi^2, \zeta$ on $x_1, x_2, x_3$ is envisaged, we simply take over the results from Sect. 2.3.1, e.g. formula (2.81) for the matrix $A$, which gives the partial derivatives of (2.101) with respect to the Cartesian coordinates. Additionally, we are interested in obtaining the partial derivatives with respect to $t$ of the coordinates (2.101),

$$\dot{\xi}^1 \equiv \frac{\partial \xi^1}{\partial t}, \quad \dot{\xi}^2 \equiv \frac{\partial \xi^2}{\partial t}, \quad \dot{\zeta} \equiv \frac{\partial \zeta}{\partial t}. \tag{2.102}$$

They are needed for the derivation of some rules of differentiation when the change of coordinates (2.100) is used, see Sect. 3.1.3. We deduce the coordinate velocities $\dot{\xi}^1, \dot{\xi}^2, \dot{\zeta}$ by using the identity, see (2.100) and (2.101),

$$r(x_1, x_2, x_3) = \tilde{r}(\xi^1(x_1, x_2, x_3, t), \xi^2(x_1, x_2, x_3, t), \xi^3(x_1, x_2, x_3, t), t), \tag{2.103}$$

which we differentiate with respect to $t$ to obtain

$$\theta = \frac{\partial \tilde{r}}{\partial t} + \dot{\xi}^k \frac{\partial \tilde{r}}{\partial \xi^k} \iff w = -\dot{\xi}^k g_k, \tag{2.104}$$

with

$$w \equiv \frac{\partial \tilde{r}}{\partial t}, \tag{2.105}$$

and where definition (2.76) of $g_k$ has been used. In the context of computational methods, $w$ is called the mesh velocity. Relation (2.102) indicates that the coordinate velocities $\dot{\xi}^1, \dot{\xi}^2, \dot{\zeta}$ are the contravariant components of the vector $-w$ with respect to $g_1, g_2, g_3 = n_b$. That is, with
we have
\[
\left( \begin{array}{c} \dot{\xi}_1 \\ \dot{\xi}_2 \end{array} \right) = -w, \quad \dot{\zeta} = -w.
\] (2.105)

It is therefore necessary to determine \( w \) and \( \tilde{w} \). To this end we use definitions (2.100) and (2.64) of \( \tilde{r} \) and \( u_S \), respectively, which yields
\[
w = \frac{\partial \tilde{r}_b}{\partial t} + \zeta \frac{\partial \tilde{n}_b}{\partial t} = u_S + \zeta \frac{\partial \tilde{n}_b}{\partial t}.
\] (2.106)

First, let us obtain the representation of \( u_S \) with respect to \( g_1, g_2, n_b \). From (2.78) we deduce that \( \tau_1, \tau_2 \) are written in terms of \( g_1, g_2 \) as
\[
\tau_\beta = W^\alpha_\beta g_\alpha, \quad \tilde{W}^\alpha_\beta = (I - \zeta \tilde{W})^{-1},
\] and hence representation (2.64) emerges as
\[
u_S = U^\beta \tau_\beta + U n_b = W^\alpha_\beta U^\gamma U^\alpha g_\alpha + U n_b.
\] (2.107)

Here \( u_S \equiv (U^1, U^2)^T \) and \( U \) are given by (2.69) and (2.63), respectively.

Next consider \( \frac{\partial \tilde{n}_b}{\partial t} \). The relation \( n_b \cdot n_b = 1 \) implies \( \tilde{n}_b \cdot \frac{\partial \tilde{n}_b}{\partial t} = 0 \), so that \( \frac{\partial \tilde{n}_b}{\partial t} \) is a tangent vector to \( S_t \), i.e.,
\[
\frac{\partial \tilde{n}_b}{\partial t} = a^\alpha g_\alpha,
\] (2.108)

with
\[
a^\alpha = \frac{\partial \tilde{n}_b}{\partial t} \cdot g^\alpha = g^\beta a^\alpha \frac{\partial \tilde{n}_b}{\partial t} \cdot g_\beta \overset{(2.78)}{=} g^\beta (\delta^\gamma_\beta - \zeta \tilde{W}^\gamma_\beta) \frac{\partial \tilde{n}_b}{\partial t} \cdot \tau_\gamma.
\]

Moreover, we compute
\[
\frac{\partial \tilde{n}_b}{\partial t} \cdot \tau_\gamma = \tilde{n}_b \cdot \frac{\partial \tau_\gamma}{\partial t} = \tilde{n}_b \cdot \frac{\partial}{\partial t} \left( \frac{\partial \tilde{r}_b}{\partial \xi^\gamma} \right) = \tilde{n}_b \cdot \frac{\partial}{\partial \xi^\gamma} \left( \frac{\partial \tilde{r}_b}{\partial t} \right) = \tilde{n}_b \cdot \frac{\partial}{\partial \xi^\gamma} \left( U^\beta \tau_\beta + U n_b \right) = \tilde{n}_b \cdot \frac{\partial}{\partial \xi^\gamma} \left( U^\beta \frac{\partial \tilde{n}_b}{\partial \xi^\gamma} \cdot \tau_\beta - \frac{\partial U}{\partial \xi^\gamma} \right) = \tilde{b}_\omega \gamma U^\omega \frac{\partial U}{\partial \xi^\gamma}.
\]

Therefore, we obtain
\[
\frac{\partial \tilde{n}_b}{\partial t} = -g^\beta (\delta^\gamma_\beta - \zeta \tilde{W}^\gamma_\beta) \left( \tilde{b}_\omega \gamma U^\omega + \frac{\partial U}{\partial \xi^\gamma} \right) g_\alpha.
\] (2.109)
Finally, substituting (2.107) and (2.109) into (2.106), we deduce

\[
 w = \left\{ W_\alpha^\beta \mathcal{U}^\beta - \zeta g^{\beta\alpha} (\delta^\gamma_\beta - \zeta \bar{W}^\gamma_\beta) \left( \bar{b}_{\omega\gamma} \mathcal{U}^\omega + \frac{\partial \mathcal{U}}{\partial \xi^\gamma} \right) \right\} g_\alpha + \mathcal{U} n_b ,
\]

which, after a routine calculus using (2.48), (2.49), (2.85), and (2.88), yields the components \( w, w \) (see (2.104)) of \( w \) as

\[
 w = \mathbf{u}_S - \zeta (I - \zeta \bar{W})^{-1} M_0 \text{Grad} \mathcal{U} , \quad w = \mathcal{U} .
\] (2.110)

Relation (2.110) is essential when selecting the parameterization of the moving topographic surface as described in the next section.

### SUMMARY of notations and relations

- \( r = r_b + \zeta n \iff r (x_1, x_2, x_3) = \bar{r}_b (\xi^1, \xi^2, t) + \zeta \bar{n}_b (\xi^1, \xi^2, t) \)
- \( \equiv \tilde{r} (\xi^1, \xi^2, \zeta, t) \)
- \( w \equiv \frac{\partial \tilde{r}}{\partial t} = -\dot{\zeta}^k g_k \)
- \( w = w^\alpha g_\alpha + wn_b , \quad w = \begin{pmatrix} w^1 \\ w^2 \end{pmatrix} \)
- \( w = \mathbf{u}_S - \zeta (I - \zeta \bar{W})^{-1} M_0 \text{Grad} \mathcal{U} , \quad w = \mathcal{U} . \)

To conclude, by (2.73)/(2.100) in this section we have introduced the curvilinear coordinates \( \xi^1, \xi^2, \zeta \) replacing the Cartesian coordinates \( x_1, x_2, x_3 \) of points lying in that vicinity of the topographic surface defined by condition (2.83). Using these curvilinear coordinates when deducing the modelling equations for shallow flows in the following chapter, the computational mesh results in a body-fitted grid that follows the geometry of the topography. For active topographic beds, at all times the topographic surface corresponds to \( \zeta = 0 \) in the computational \( (\xi^1, \xi^2, \zeta) \)-domain. Note that the change of parameters on the topographic surface, see (2.42) or (2.58), is left arbitrary. It can be freely chosen to the benefit of the numerical method. Most authors use the arc lengths on the coordinate lines \( y = \text{constant}, x = \text{constant} \) on \( \mathcal{S}_b \) as parameters \( \xi^1 \) and \( \xi^2 \), respectively. Another choice for the time-dependent transformation (2.58) is discussed in the next section. Both these parameterizations of the basal surface are used in the example given in Sect. 5.4.
2.4 The Topography-Fitted Coordinates in the Context of the Unified Coordinates (UC) Approach

It is generally non-trivial to determine the change of parameters (2.58) for a complex terrain surface when erosion/deposition processes occur. An idea of how to face this problem was proposed by Tai and Kuo [5], and was suggested by the unified coordinates (UC) approach (see e.g. Hui [6, 7], Hui and Xu [8]). In this section we shortly describe this method and show how it inspired the choice for the change of parameters (2.58) in [5].

The UC method is essentially an approach to search for an optimum coordinate system \((\xi^1, \xi^2, \xi^3)\) in Computational Fluid Dynamics, in the sense that, (a) the system of partial differential equations, when expressed in terms of \((\xi^1, \xi^2, \xi^3)\), is in conservative form, (b) contact discontinuities are sharply resolved, and (c) a body-fitted mesh, commonly adopted in computing flows of complex geometries, can be automatically generated. With \(x_1, x_2, x_3\)—the Cartesian coordinates, the new coordinate system is searched for as defined by a space-time change of coordinates

\[
(x_1, x_2, x_3, t) \iff (\xi^1, \xi^2, \xi^3, \lambda) \quad (2.111)
\]
of the form

\[
\begin{align*}
    dx_1 &= A \, d\xi^1 + L \, d\xi^2 + P \, d\xi^3 + U \, d\lambda, \\
    dx_2 &= B \, d\xi^1 + M \, d\xi^2 + Q \, d\xi^3 + V \, d\lambda, \\
    dx_3 &= C \, d\xi^1 + N \, d\xi^2 + R \, d\xi^3 + W \, d\lambda, \\
    dt &= d\lambda,
\end{align*}
\]

or, equivalently,

\[
\begin{pmatrix}
    \frac{\partial x_1}{\partial \xi^1} & \frac{\partial x_1}{\partial \xi^2} & \frac{\partial x_1}{\partial \xi^3} & \frac{\partial x_1}{\partial \lambda} \\
    \frac{\partial x_2}{\partial \xi^1} & \frac{\partial x_2}{\partial \xi^2} & \frac{\partial x_2}{\partial \xi^3} & \frac{\partial x_2}{\partial \lambda} \\
    \frac{\partial x_3}{\partial \xi^1} & \frac{\partial x_3}{\partial \xi^2} & \frac{\partial x_3}{\partial \xi^3} & \frac{\partial x_3}{\partial \lambda} \\
    \frac{\partial t}{\partial \xi^1} & \frac{\partial t}{\partial \xi^2} & \frac{\partial t}{\partial \xi^3} & \frac{\partial t}{\partial \lambda}
\end{pmatrix}
= \begin{pmatrix}
    A & L & P & U \\
    B & M & Q & V \\
    C & N & R & W \\
    0 & 0 & 0 & 1
\end{pmatrix}. \quad (2.113)
\]

The mesh velocity components \(U, V, W\) are assigned to fulfill the above mentioned requirements (b), (c), and the unknown functions \(A, B, \ldots, R\) in (2.112) are determined by the rules of differentiations which they satisfy.\(^6\) These rules emerge as additional equations which are solved in conjunction with the physical equations.

\(^6\)For instance, \(A = \frac{\partial x_1}{\partial \xi^1}, \ U = \frac{\partial x_1}{\partial \lambda}, \) so that, \(\frac{\partial A}{\partial \lambda} = \frac{\partial U}{\partial x_1}, \) and once \(U\) is known, this represents an evolution equation for \(A.\)
Thus, the curvilinear coordinates \((\xi^1, \xi^2, \xi^3)\) are generated while advancing in time the numerical procedure. For several prescriptions of the grid velocity \((U, V, W)\) and for more details on the UC approach see Hui and Xu [8].

When erosion/deposition processes are taken into account, the time-dependent change of coordinates \((2.101)\) can be conceived as a space-time change of coordinates \((2.111)\), where

\[
\begin{pmatrix}
\frac{\partial \xi^1}{\partial x_1} & \frac{\partial \xi^1}{\partial x_2} & \frac{\partial \xi^1}{\partial x_3} & \frac{\partial \xi^1}{\partial t} \\
\frac{\partial \xi^2}{\partial x_1} & \frac{\partial \xi^2}{\partial x_2} & \frac{\partial \xi^2}{\partial x_3} & \frac{\partial \xi^2}{\partial t} \\
\frac{\partial \xi^3}{\partial x_1} & \frac{\partial \xi^3}{\partial x_2} & \frac{\partial \xi^3}{\partial x_3} & \frac{\partial \xi^3}{\partial t} \\
\frac{\partial \lambda}{\partial x_1} & \frac{\partial \lambda}{\partial x_2} & \frac{\partial \lambda}{\partial x_3} & \frac{\partial \lambda}{\partial t}
\end{pmatrix}
= \begin{pmatrix}
A & 0 & -w \\
0 & A & -w \\
0 & 0 & 1
\end{pmatrix}, \quad (2.114)
\]

see \((2.81)\) and \((2.105)\). This gives,

\[
\begin{pmatrix}
\frac{\partial x_1}{\partial \xi^1} & \frac{\partial x_1}{\partial \xi^2} & \frac{\partial x_1}{\partial \xi^3} & \frac{\partial x_1}{\partial \lambda} \\
\frac{\partial x_2}{\partial \xi^1} & \frac{\partial x_2}{\partial \xi^2} & \frac{\partial x_2}{\partial \xi^3} & \frac{\partial x_2}{\partial \lambda} \\
\frac{\partial x_3}{\partial \xi^1} & \frac{\partial x_3}{\partial \xi^2} & \frac{\partial x_3}{\partial \xi^3} & \frac{\partial x_3}{\partial \lambda} \\
\frac{\partial t}{\partial \xi^1} & \frac{\partial t}{\partial \xi^2} & \frac{\partial t}{\partial \xi^3} & \frac{\partial t}{\partial \lambda}
\end{pmatrix}
= \begin{pmatrix}
A^{-1} & A^{-1}(w) \\
0 & 1
\end{pmatrix}, \quad (2.115)
\]

see Exercise 2.5, which indicates that the transformation \((2.111)\) corresponding to the topography-fitted coordinates has the form \((2.113)\), with

\[
\begin{pmatrix}
A & L & P \\
B & M & Q \\
C & N & R
\end{pmatrix} = A^{-1}, \quad \begin{pmatrix}
U \\
V \\
W
\end{pmatrix} = A^{-1} \begin{pmatrix}
w \\
w
\end{pmatrix}. \quad (2.116)
\]

Based on this remark we show how the change of parameters \((2.58)\) can be chosen. Note that in expression \((2.110)\), \(u_S\) is a term which refers solely to the transformation \((2.58)\) that has not yet been prescribed. One effect is that the mesh velocity \((2.116)\) bears some degree of freedom, which resembles the UC and other moving mesh approaches. Thus, for simplicity, the mesh velocity \((2.116)\) can be taken such that

\[
u_S = 0. \quad (2.117)
\]

According to \((2.66)\), this choice for \(u_S\) reads as
\[ \mathbf{v}_S = -\mathbf{U}s, \quad (2.118) \]

or, componentwise,

\[ \frac{\partial x}{\partial t} = -c \mathbf{U} \frac{\partial b}{\partial x}, \quad \frac{\partial y}{\partial t} = -c \mathbf{U} \frac{\partial b}{\partial y}, \quad (2.119) \]

see (2.69). Therefore, assumption (2.117) prescribes the time derivatives of \( x(\xi^1, \xi^2, t) \) and \( y(\xi^1, \xi^2, t) \), and it only remains to define the initial values \( x(\xi^1, \xi^2, 0), y(\xi^1, \xi^2, 0) \) to obtain the functions \( x, y \) by forward integration. These initial values can be determined e.g., by considering the parameters \( \xi^1, \xi^2 \) as being the arc lengths on the coordinate lines \( y = \text{constant} \) and \( x = \text{constant} \) on the topographic surface at time \( t = 0 \), see also the forthcoming Sect. 5.4. With this choice, \( \det \mathbf{F}(\xi^1, \xi^2, 0) > 0 \), and so, for sufficiently smooth right-hand sides in (2.119), \( \det \mathbf{F} > 0 \) for at least a short time interval \([0, T)\). That is, conditions (2.119) do indeed define a (time-dependent) change of coordinates \( (x, y) \leftrightarrow (\xi^1, \xi^2) \) on \([0, T)\). As a matter of fact, the case \( \det \mathbf{F} = 0 \) can be avoided in a numerical algorithm. In the numerical computations by Y.C. Tai, C.Y. Kuo and collaborators [5, 9–11], \( \det \mathbf{F} \) emerging from (2.119) never happened to vanish.

In view of (2.110), assumption (2.117) is equivalent to the requirement

\[ \mathbf{w}|_{\xi=0} = \mathbf{U} \mathbf{n}_b, \quad (2.120) \]

which has been used by Tai and Kuo [5]. Thus, (2.117) states that, at the basal topography the computational mesh moves along the normal direction to the topographic surface with the velocity \( \mathbf{U} \).

The requirements (a) and (b) of the UC method are also taken into account in [5], resulting in a distinct approach, in this book called non-conventional method (see the forthcoming Sect. 4.6), to formulate depth-averaged models for thin flows.

**Exercises to Chap. 2**

**Ex 2.1** Prove (2.17).

**Ex 2.2** Show that \( \tilde{\tau}_1 \times \tilde{\tau}_2 = (\det \mathbf{F}) \tau_1 \times \tau_2 \), with \( \tau_1, \tau_2 \) as defined by (2.3), \( \mathbf{F} \) given by (2.16), and for \( \tilde{\tau}_1, \tilde{\tau}_2 \) see (2.18). With the aid of this relation justify the remark following (2.17).

**Ex 2.3** Prove (2.41).

**Ex 2.4** Show that, if \( \mathbf{a} \equiv (a_1, a_2)^T \) is a 2-column matrix, then \( \mathbf{I} + \mathbf{a} \otimes \mathbf{a} \) is invertible and
(I + a ⊗ a)^{-1} = I - \frac{1}{a \cdot a} a \otimes a.

**Ex 2.5** Let A be an invertible $3 \times 3$ matrix and let $b$ be a 3-column matrix. Check that
\[
\begin{pmatrix}
A & b \\
0 & 1
\end{pmatrix}^{-1} = \begin{pmatrix}
A^{-1} & -A^{-1}b \\
0 & 1
\end{pmatrix}.
\]

**Ex 2.6** Prove formula (3.11).

**Ex 2.7** Prove formula (3.70).

**Ex 2.8** Use relations (2.36) and (2.109) to show that
\[
\frac{\partial c}{\partial t} = -cF^{-1}s \cdot (\tilde{H}u_S + \text{Grad } \mathcal{U}), \quad \frac{\partial s}{\partial t} = F\tilde{W}u_S + FM_0\text{Grad } \mathcal{U},
\]
where $c$ and $s$ are defined in (2.33) (considered with $b = b(x, y, t)$), and are understood as functions of $(\xi^1, \xi^2, t)$ via the transformation (2.58).

**References**

Shallow Geophysical Mass Flows down Arbitrary Topography
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