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The Interplay Between Time-dependent Speed of Propagation and Dissipation in Wave Models

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Abstract. The aim of this paper is to understand the influence of the interplay between a time-dependent increasing speed of propagation and a time-dependent coefficient in the dissipation on qualitative properties of solutions to the wave model

$$u_{tt} - a^2(t)\Delta u + b(t)u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x). \tag{0.1}$$

Our considerations are focused to energy estimates. The main difficulty is to find a good description of *non-effective* and *effective* dissipations depending on a given speed of propagation. The obtained energy estimates are optimal as special examples will show. At the end we will sketch very briefly how to get *scattering* and *over-damping* results. So, we propose a classification of different damping terms which is motivated by the thesis of J. Wirth for the case $a \equiv 1$.

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1. Introduction

The papers [12] and [13] are devoted to the study of the Cauchy problem for the wave equation with time-dependent dissipation

$$u_{tt} - \Delta u + b(t)u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x). \tag{1.1}$$

A description of the influence of the coefficient $b = b(t)$ on the qualitative behavior of solutions is given due to the following classification:

- *Scattering:* If $b(t)$ has a very weak influence, then there is a relation to the free wave equation. Such relations are described by so-called scattering results.

- *Non-effective:* If $b(t)$ has a weak influence, then the classical energy decays to 0 and corresponding L^p-L^q decay estimates are the classical Strichartz decay estimates with an additional term as a time-dependent coefficient coming from the decay of the energy itself. Such weak dissipations will be called non-effective.
- *Effective:* If $b(t)$ has a stronger influence, then L^p-L^q decay estimates are similar to those ones for the classical damped wave equation but with an additional decay function related to the dissipation itself. Such dissipations will be called effective.
- *Over-damping:* If $b(t)$ has a "very strong influence", then in general we can not expect any decay estimate of the classical wave type energy.

In both cases, *scattering* or *over-damping*, we have in general no energy decay. Roughly speaking, energy decay only appears for dissipations "between" the conditions $b \notin L_1(\mathbb{R}^+)$ and $1/b \notin L_1(\mathbb{R}^+)$ in (1.1). But we have to be more precise. This leads to distinguish between *non-effective* and *effective* dissipation. Correspondingly, we only cite here two results from [11]. Assuming the coefficient function $b = b(t)$ is a positive, smooth and monotone function of t, which satisfies

$$|b^{(k)}(t)| \leq C_k b(t) \left(\frac{1}{1+t}\right)^k$$

for all $k \in \mathbb{N}_0$.

Result 1.1. *Assume* $\limsup_{t\to\infty} tb(t) < 1$. *Then the solution* $u = u(t,x)$ *of* (1.1) *satisfies the* L^p-L^q *decay estimate*

$$\|(\partial_t, \nabla)u(t,\cdot)\|_{L^q} \leq C\frac{1}{\lambda(t)}(1+t)^{-\frac{n-1}{2}\left(\frac{1}{p}-\frac{1}{q}\right)} \left(\|u_1\|_{W_p^{N_p+1}} + \|u_2\|_{W_p^{N_p}}\right) \tag{1.2}$$

for $p \in (1,2]$, q *is the corresponding dual index,* $N_p = n\left(\frac{1}{p} - \frac{1}{q}\right)$ *and* $\lambda(t)$ *is an auxiliary function which is defined by*

$$\lambda(t) := \exp\left(\frac{1}{2}\int_0^t b(\tau)d\tau\right).$$

Result 1.2. *Assume* $tb(t) \to \infty$ *as* $t \to \infty$. *Then the solution* $u = u(t,x)$ *of* (1.1) *satisfies the* L^p-L^q *decay estimate*

$$\|(\partial_t, \nabla)u(t,\cdot)\|_{L^q} \leq C\left(1 + \int_0^t \frac{d\tau}{b(\tau)}\right)^{-\frac{n}{2}\left(\frac{1}{p}-\frac{1}{q}\right)-\frac{1}{2}} \left(\|u_1\|_{W_p^{N_p+1}} + \|u_2\|_{W_p^{N_p}}\right) \tag{1.3}$$

for $p \in (1,2]$, q *is the corresponding dual index and* $N_p = n\left(\frac{1}{p} - \frac{1}{q}\right)$.

What about wave models in (1.1) without any dissipation? In a series of papers (see [7], [8], [9] or [5]) the authors have obtained results about decay estimates for solutions to the Cauchy problem

$$u_{tt} - a^2(t)\Delta u = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x). \tag{1.4}$$

Therein $a(t)$ is chosen as $a^2(t) = \lambda^2(t)b^2(t)$, where $\lambda(t)$ is a monotonously increasing function and $b(t)$ is an oscillating function. In this paper we shall treat only the case of an increasing propagation speed and we are not interested in special oscillating parts in the coefficient $a(t)$. We recall that some results from [10] are obtained under the following assumptions to the coefficient $a = a(t)$:

(A1) $a(t) > 0$, $a'(t) > 0$, for $t \in [0, \infty)$,

(A2) $a_0 \dfrac{a(t)}{A(t)} \le \dfrac{a'(t)}{a(t)} \le a_1 \dfrac{a(t)}{A(t)}$, $a_0, a_1 > 0$,

(A3) $|a''(t)| \le a_2 a(t) \Big(\dfrac{a(t)}{A(t)} \Big)^2$, $a_2 \ge 0$,

(A4) $t + \dfrac{C}{\sqrt{a(t)}}$ is strictly increasing with a positive constant C and for large t.

Here $A(t) = 1 + \int_0^t a(s)ds$ is a primitive of $a(t)$. So a combination of the goals of [10] with the goals of [12] and [13] seems to be reasonable. For this reason we devote to the wave model

$$u_{tt} - a^2(t)\Delta u + b(t)u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x) \tag{1.5}$$

with time-dependent increasing speed of propagation and dissipation. An interesting issue is to introduce *precise descriptions* for *non-effective* and *effective dissipations* in model (1.5). Such a classification we shall propose in Sections 2 and 3. We derive energy estimates in both cases. Some examples show in Section 3.7 that our estimates are optimal. At the end of the paper we sketch very briefly scattering and over-damping results. Most of the results are part of the thesis of Mr. Bui Tang Bao Ngoc [1].

2. Non-effective dissipation

Let us devote to the wave model

$$u_{tt} - a^2(t)\Delta u + b(t)u_t = 0. \tag{2.1}$$

Our question is the following:

> *Under which assumptions to the coefficient $b = b(t)$ for a given time-dependent speed of propagation $a = a(t)$ can we call b a non-effective dissipation?*

Here non-effective means, that on the one hand we have really a dissipation effect (scattering to wave models is excluded), but on the other hand the wave model itself is hyperbolic like from the point of view of decay estimates for the wave type energy. Motivated by the considerations from [12] we assume:

(B1) $b(t) > 0$, $b(t) = \mu(t)\dfrac{a(t)}{A(t)}$, $b \notin L^1(\mathbb{R}_+)$,

(B2) $|\mu'(t)| \le C_\mu \mu(t)\dfrac{a(t)}{A(t)}$,

(B3) $\limsup_{t\to\infty} \mu(t) < 1$.

Instead of the assumption (B3) we assume sometimes

(B3)$'$ $\liminf_{t\to\infty} \mu(t) > 1$.

Finally, we suppose

(C) $\limsup_{t\to\infty} \mu(t) + \alpha(t) < 2$,

where $\alpha = \alpha(t)$ is defined by

$$\frac{a'(t)}{a(t)} =: \alpha(t)\frac{a(t)}{A(t)}.$$

Theorem 2.1. *Let us consider the Cauchy problem* (2.1) *under the assumptions* (A1) *to* (A3), (B1), (B2), (B3) *or* (B3)$'$ *and* (C). *Then we have the following estimates for the kinetic and elastic energy:*

$$\|u_t(t,\cdot)\|_{L^2} \le C\frac{\sqrt{a(t)}}{\lambda(t)}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big),$$

$$\|a(t)\nabla u(t,\cdot)\|_{L^2} \le C\frac{\sqrt{a(t)}}{\lambda(t)}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big).$$

Here $\lambda = \lambda(t)$ is defined by

$$\lambda(t) := \exp\Big(\frac{1}{2}\int_0^t b(\tau)d\tau\Big). \tag{2.2}$$

Proof. Applying partial Fourier transformation we have $\hat{u}_{tt} + a^2(t)|\xi|^2\hat{u} + b(t)\hat{u}_t = 0$. Later we will derive estimates for the fundamental solution $E = E(t,s,\xi)$ of an equivalent system of first order by different ways in different parts of the extended phase space $(0,\infty)\times\mathbb{R}^n$: in the dissipative zone and the hyperbolic zone. These zones are defined by

- $Z_{\text{hyp}}(N) := \{(t,\xi) : t \ge t_\xi\}$,
- $Z_{\text{diss}}(N) := \{(t,\xi) : 0 \le t \le t_\xi\}$,

where t_ξ satisfies $A(t_\xi)|\xi| = N$.

2.1. Considerations in the dissipative zone

Let us define the micro-energy $U = (N\delta(t)\hat{u}, D_t\hat{u})^T$, where we denote $\delta(t) := \dfrac{a(t)}{A(t)}$. Then the transformed equation can be written in the form of a system of first order (in D_t)

$$D_tU = A(t,\xi)U,\ A(t,\xi) = \begin{pmatrix} -i\dfrac{d_t\delta(t)}{\delta(t)} & N\delta(t) \\ \dfrac{a^2(t)|\xi|^2}{N\delta(t)} & ib(t) \end{pmatrix}.$$

Thus the solution $U = U(t,\xi)$ can be represented as $U(t,\xi) = E(t,s,\xi)U(s,\xi)$, where $E(t,s,\xi)$ is the fundamental solution, that is, the solution to the system

$$D_tE(t,s,\xi) = A(t,\xi)E(t,s,\xi),\ E(s,s,\xi) = I,\ 0 \le s \le t \le t_\xi.$$

In the further calculations we use the following statement:

Lemma 2.2.

1. *The assumption* (B3) *implies for the auxiliary function* $\lambda = \lambda(t)$ *the properties*
$$\int_0^t \frac{a(\tau)}{\lambda^2(\tau)} d\tau \lesssim \frac{A(t)}{\lambda^2(t)},$$
and $\dfrac{A(t)}{\lambda^2(t)}$ *is monotonously increasing if* t *tends to infinity.*
2. *The assumption* (B3)′ *implies* $\dfrac{a(t)}{\lambda^2(t)} \in L^1(\mathbb{R}^+)$ *with*
$$\int_t^\infty \frac{a(\tau)}{\lambda^2(\tau)} d\tau \lesssim \frac{A(t)}{\lambda^2(t)}.$$
Furthermore, $\dfrac{A(t)}{\lambda^2(t)}$ *is monotonously decreasing for large* t.

Proof. To prove these statements we only use integration by parts and straightforward calculations. □

Lemma 2.3. *Assume that* $a = a(t)$ *satisfies* (A1) *and* (A2), *and the function* $\mu = \mu(t)$ *satisfies the condition* (B3). *Then there exists a constant* $\delta \in (0,1)$ *such that*
$$a(t)^\delta \int_0^t \frac{a(\tau)^{1-\delta}}{\lambda^2(\tau)} d\tau \lesssim \frac{A(t)}{\lambda^2(t)}. \tag{2.3}$$

Proof. The statement follows directly after integration of the following inequality:
$$\frac{a(t)^{1-\delta}}{\lambda^2(t)} \lesssim \left(\frac{A(t)}{a(t)^\delta \lambda^2(t)}\right)' = \frac{a(t)^{1-\delta}}{\lambda^2(t)} - \delta \frac{A(t)a'(t)}{a(t)^{1+\delta}\lambda^2(t)} - \frac{A(t)\mu(t)\dfrac{a(t)}{A(t)}}{a(t)^\delta \lambda^2(t)},$$
where we use that the right-hand side can be estimated to below by
$$(1-c)\frac{a(t)^{1-\delta}}{\lambda^2(t)} - \delta \frac{A(t)a'(t)}{a(t)^{1+\delta}\lambda^2(t)} \text{ for large } t,$$
and $c < 1$ due to condition (B3). The desired estimate
$$\frac{a(t)^{1-\delta}}{\lambda^2(t)} \leq C\Big((1-c)\frac{a(t)^{1-\delta}}{\lambda^2(t)} - \delta \frac{A(t)a'(t)}{a(t)^{1+\delta}\lambda^2(t)}\Big)$$
is true if it exits a constant $C > \dfrac{1}{1-c}$ such that
$$A(t)a'(t) \leq (1 - c - C^{-1})\delta^{-1}a^2(t).$$
From that we can choose any δ satisfying $\delta < (\limsup_t A(t)a'(t)/a^2(t))^{-1}$. This supremum is finite by condition (A2). □

Denoting by $E^{(jk)}$ the entries of E we get for $k=1,2$ the system

$$D_tE^{(1k)} = -i\frac{d_t\delta(t)}{\delta(t)}E^{(1k)} + N\delta(t)E^{(2k)},$$
$$D_tE^{(2k)} = \frac{a^2(t)|\xi|^2}{N\delta(t)}E^{(1k)} + ib(t)E^{(2k)},\ E^{(jk)}(s,s,\xi)=\delta_{jk}.$$

Integration yields

$$E^{(1k)}(t,s,\xi) = \frac{\delta(t)}{\delta(s)}E^{(1k)}(s,s,\xi) + iN\delta(t)\int_s^t E^{(2k)}(\tau,s,\xi)d\tau, \tag{2.4}$$
$$E^{(2k)}(t,s,\xi) = \frac{\lambda^2(s)}{\lambda^2(t)}E^{(2k)}(s,s,\xi) + \frac{i|\xi|^2}{N\lambda^2(t)}\int_s^t \frac{a^2(\tau)}{\delta(\tau)}\lambda^2(\tau)E^{(1k)}(\tau,s,\xi)d\tau.$$

We are going to prove the following lemma:

Lemma 2.4. *Let us assume* (A1) *to* (A3) *and* (B3). *Then we have the following estimates for the entries $E^{(kl)}(t,0,\xi)$ of the fundamental solution $E(t,0,\xi)$ in the dissipative zone:*

$$(|E(t,0,\xi)|) := \begin{pmatrix} |E^{(11)}(t,0,\xi)| & |E^{(12)}(t,0,\xi)| \\ |E^{(21)}(t,0,\xi)| & |E^{(22)}(t,0,\xi)| \end{pmatrix} \lesssim \begin{pmatrix} \dfrac{a(t)}{A(t)} & \dfrac{a(t)^{1-\delta}}{\lambda^2(t)} \\ \dfrac{|\xi|^2K(t)}{\lambda^2(t)} & \dfrac{a(t)^{1-\delta}}{\lambda^2(t)} \end{pmatrix} \tag{2.5}$$

with $K(t) := \int_0^t a^2(\tau)\lambda^2(\tau)d\tau \le \lambda^2(t)a(t)A(t)$.

Proof. Let us consider

$$\begin{aligned} E^{(21)}(t,0,\xi) &= \frac{i|\xi|^2}{N\lambda^2(t)}\int_0^t \frac{a^2(\tau)}{\delta(\tau)}\lambda^2(\tau)E^{(11)}(\tau,0,\xi)d\tau \\ &= \frac{i|\xi|^2}{N\lambda^2(t)}\Big(\int_0^t \frac{a^2(\tau)}{\delta(0)}\lambda^2(\tau)d\tau + \int_0^t a^2(\tau)\lambda^2(\tau)iNd\tau\int_0^\tau E^{(21)}(\theta,0,\xi)d\theta d\tau\Big) \\ &= \frac{i|\xi|^2}{N\delta(0)\lambda^2(t)}K(t) - \frac{|\xi|^2}{\lambda^2(t)}\int_0^t a^2(\tau)\lambda^2(\tau)\int_0^\tau E^{(21)}(\theta,0,\xi)d\theta d\tau \\ &= \frac{i|\xi|^2}{C_N\lambda^2(t)}K(t) - \frac{|\xi|^2}{\lambda^2(t)}\int_0^t\Big(\int_\theta^t a^2(\tau)\lambda^2(\tau)d\tau\Big)E^{(21)}(\theta,0,\xi)d\theta. \end{aligned}$$

Rewriting the integral equation gives

$$\frac{C_N\lambda^2(t)E^{(21)}(t,0,\xi)}{|\xi|^2K(t)} = i + \int_0^t k_1(t,\theta,\xi)\frac{C_N\lambda^2(\theta)E^{(21)}(\theta,0,\xi)}{|\xi|^2K(\theta)}d\theta \tag{2.6}$$

with kernel

$$k_1(t,\theta,\xi) = -|\xi|^2\frac{K(\theta)}{K(t)\lambda^2(\theta)}\int_\theta^t a^2(\tau)\lambda^2(\tau)d\tau,\ \theta\in[0,t]. \tag{2.7}$$

Now we estimate

$$\int_0^t \sup_{\theta\le\tilde{t}\le t}|k_1(\tilde{t},\theta,\xi)|d\theta \lesssim |\xi|^2\int_0^{t_\xi}\sup_{\tilde{t}}\frac{K(\theta)}{\lambda^2(\theta)K(\tilde{t})}\big(K(\tilde{t})-K(\theta)\big)d\theta$$
$$\le |\xi|^2\int_0^{t_\xi}\frac{K(\theta)}{\lambda^2(\theta)}d\theta \lesssim |\xi|^2\int_0^{t_\xi}a(\theta)A(\theta)d\theta = \frac{1}{2}|\xi|^2A^2(t_\xi)\lesssim 1$$

uniformly in $Z_{\rm diss}(N)$. Therefore, we obtained

$$|E^{(21)}(t,0,\xi)| \lesssim \frac{|\xi|^2K(t)}{\lambda^2(t)}. \tag{2.8}$$

Substituting this estimate into the first integral equation implies

$$|E^{(11)}(t,0,\xi)| \le \frac{\delta(t)}{\delta(0)} + N\delta(t)\int_0^t\frac{|\xi|^2K(\tau)}{\lambda^2(\tau)}d\tau \lesssim \delta(t)+|\xi|^2\delta(t)A^2(t)\lesssim \delta(t) = \frac{a(t)}{A(t)}.$$

Next, we consider

$$E^{(22)}(t,0,\xi) = \frac{\lambda^2(0)}{\lambda^2(t)} + \frac{i|\xi|^2}{N\lambda^2(t)}\int_0^t\frac{a^2(\tau)}{\delta(\tau)}\lambda^2(\tau)E^{(12)}(\tau,0,\xi)d\tau$$
$$= \frac{\lambda^2(0)}{\lambda^2(t)} - \frac{|\xi|^2}{\lambda^2(t)}\int_0^t a^2(\tau)\lambda^2(\tau)\int_0^\tau E^{(22)}(\theta,0,\xi)d\theta d\tau,$$
$$\lambda^2(t)E^{(22)}(t,0,\xi) = 1-|\xi|^2\int_0^t\Big(\int_\theta^t a^2(\tau)\lambda^2(\tau)d\tau\Big)E^{(22)}(\theta,0,\xi)d\theta,$$

respectively. Our goal is to show that $|E^{(22)}(t,0,\xi)| \lesssim \dfrac{a(t)^{1-\delta}}{\lambda^2(t)}$. Therefore, we rewrite the integral equation as

$$\frac{\lambda^2(t)E^{(22)}(t,0,\xi)}{a(t)^{1-\delta}} = \frac{1}{a(t)^{1-\delta}} + \int_0^t k_2(t,\theta,\xi)\frac{\lambda^2(\theta)E^{(22)}(\theta,0,\xi)}{a(\theta)^{1-\delta}}d\theta \tag{2.9}$$

with the kernel

$$k_2(t,\theta,\xi) = -|\xi|^2\frac{a(\theta)^{1-\delta}}{a(t)^{1-\delta}\lambda^2(\theta)}\int_\theta^t a^2(\tau)\lambda^2(\tau)d\tau,\ \ \theta\in[0,t]. \tag{2.10}$$

The following integral over the kernel satisfies the desired estimate. It holds

$$\int_0^t\sup_{\theta\le\tilde{t}\le t}|k_2(\tilde{t},\theta,\xi)|d\theta \lesssim |\xi|^2\int_0^{t_\xi}\sup_{\tilde{t}}\frac{(a(\theta))^{1-\delta}}{(a(\tilde{t})^{1-\delta})\lambda^2(\theta)}\big(K(\tilde{t})-K(\theta)\big)d\theta$$
$$\le |\xi|^2\int_0^{t_\xi}\sup_{\tilde{t}}\frac{(a(\theta))^{1-\delta}K(\tilde{t})}{(a(\tilde{t}))^{1-\delta}\lambda^2(\theta)}d\theta \le |\xi|^2\lambda^2(t_\xi)A(t_\xi)(a(t_\xi))^\delta\int_0^{t_\xi}\frac{(a(\theta))^{1-\delta}}{\lambda^2(\theta)}d\theta$$
$$\lesssim |\xi|^2\lambda^2(t_\xi)A(t_\xi)\frac{A(t_\xi)}{\lambda^2(t_\xi)} \le |\xi|^2A^2(t_\xi)\lesssim 1.$$

Here we have used Lemma 2.3 and, therefore,

$$|E^{(22)}(t,0,\xi)| \lesssim \frac{a(t)^{1-\delta}}{\lambda^2(t)}. \tag{2.11}$$

After plugging this estimate into the first integral equation and using Lemma 2.3 again we have

$$|E^{(12)}(t,0,\xi)| \lesssim \delta(t)\int_0^t \frac{a(\tau)^{1-\delta}}{\lambda^2(\tau)}d\tau \lesssim \frac{a(t)^{1-\delta}}{A(t)}\frac{A(t)}{\lambda^2(t)} \lesssim \frac{a(t)^{1-\delta}}{\lambda^2(t)}. \tag{2.12}$$

This completes the proof. □

Now let us come back to

$$U(t,\xi) = E(t,0,\xi)U(0,\xi) \text{ for all } 0 \le t \le t_\xi. \tag{2.13}$$

Because of $a(t)|\xi||\hat{u}(t,\xi)| \le N\dfrac{a(t)}{A(t)}|\hat{u}(t,\xi)|$ in $Z_{\text{diss}}(N)$ the following statement can be concluded from (2.13) and Lemma 2.4:

Corollary 2.5. *We have in the dissipative zone $Z_{\text{diss}}(N)$ the following estimates for all $0 \le t \le t_\xi$:*

$$a(t)|\xi||\hat{u}(t,\xi)| \le C_N\frac{a(t)}{A(t)}|\hat{u}(0,\xi)| + C_N\frac{a(t)^{1-\delta}}{\lambda^2(t)}|D_t\hat{u}(0,\xi)|,$$
$$|D_t\hat{u}(t,\xi)| \le C_N\frac{|\xi|^2K(t)}{\lambda^2(t)}|\hat{u}(0,\xi)| + C_N\frac{a(t)^{1-\delta}}{\lambda^2(t)}|D_t\hat{u}(0,\xi)|.$$

Lemma 2.6. *Let us assume* (A1) *to* (A3) *and* (B3)$'$. *Then we have the following estimates for the entries $E^{(kl)}(t,0,\xi)$ of the fundamental solution $E(t,0,\xi)$:*

$$(|E(t,0,\xi)|) := \begin{pmatrix} |E^{(11)}(t,0,\xi)| & |E^{(12)}(t,0,\xi)| \\ |E^{(21)}(t,0,\xi)| & |E^{(22)}(t,0,\xi)| \end{pmatrix} \lesssim \begin{pmatrix} \dfrac{a(t)}{A(t)} & \dfrac{a(t)}{A(t)} \\ \dfrac{a(t)}{A(t)} & \dfrac{a(t)}{A(t)} \end{pmatrix}. \tag{2.14}$$

Proof. We start by estimating the first column. Plugging the representation for $E^{(21)}(t,s,\xi)$ into the integral equation for $E^{(11)}(t,s,\xi)$ gives

$$\frac{\delta(0)}{\delta(t)}E^{(11)}(t,0,\xi) = 1 - |\xi|^2\int_0^t\int_0^\tau \frac{\lambda^2(\theta)}{\lambda^2(\tau)}a^2(\theta)\frac{\delta(0)}{\delta(\theta)}E^{(11)}(\theta,0,\xi)d\theta d\tau,$$
$$\frac{1}{\delta(t)}|E^{(11)}(t,0,\xi)| \lesssim 1 + |\xi|^2\int_0^t\int_0^\tau \underbrace{\frac{\lambda^2(\theta)}{\lambda^2(\tau)}}_{\le 1}a^2(\theta)\frac{1}{\delta(\theta)}|E^{(11)}(\theta,0,\xi)|d\theta d\tau,$$
$$\frac{1}{\delta(t)}|E^{(11)}(t,0,\xi)| \lesssim \exp\left(|\xi|^2\int_0^t\int_0^\tau a^2(\theta)d\theta d\tau\right) \lesssim \exp\left(|\xi|^2A^2(t)\right) \lesssim 1,$$
$$|E^{(11)}(t,0,\xi)| \lesssim \delta(t) = \frac{a(t)}{A(t)}.$$

Here we have used the definition of dissipative zone and assumption (A2) for $a(t)$. Let us consider $E^{(21)}(t,0,\xi)$. We have

$$\begin{aligned}|E^{(21)}(t,0,\xi)| &\lesssim \frac{|\xi|^2}{\lambda^2(t)}\int_0^t \frac{a^2(\tau)}{\delta(\tau)}\lambda^2(\tau)|E^{(11)}(\tau,0,\xi)|d\tau \\ &\lesssim |\xi|^2\int_0^t \frac{a^2(\tau)}{\delta(\tau)}\underbrace{\frac{\lambda^2(\tau)}{\lambda^2(t)}}_{\leq 1}\delta(\tau)d\tau \lesssim |\xi|^2 A(t)a(t) \leq C_N\frac{a(t)}{A(t)}.\end{aligned}$$

Now we will estimate the entries of the second column. We get

$$\frac{1}{\delta(t)}E^{(12)}(t,0,\xi) = iN\lambda^2(0)\int_0^t \frac{d\tau}{\lambda^2(\tau)} - |\xi|^2\int_0^t\int_0^\tau \underbrace{\frac{\lambda^2(\theta)}{\lambda^2(\tau)}}_{\leq 1}\frac{a^2(\theta)}{\delta(\theta)}E^{(12)}(\theta,0,\xi)d\theta d\tau.$$

Because the first integral is uniformly bounded by the second statement from Lemma 2.2 we can obtain by the above reasoning together with assumption (A1) the desired estimate for $E^{(12)}$. For $E^{(22)}$ we have

$$\begin{aligned}\frac{1}{\delta(t)}|E^{(22)}(t,0,\xi)| &\lesssim \frac{A(t)}{a(t)\lambda^2(t)} + \frac{|\xi|^2 A(t)}{a(t)\lambda^2(t)}\int_0^t a^2(\tau)\lambda^2(\tau)d\tau \\ &\lesssim \frac{A(t)}{a(t)\lambda^2(t)} + \underbrace{\frac{|\xi|^2 A(t)}{a(t)}\int_0^t a^2(\tau)d\tau}_{\leq C_N}.\end{aligned}$$

If we notice the second statement of Lemma 2.2 the function $\dfrac{A(t)}{\lambda^2(t)}$ is decreasing. Taking account of the increasing behavior of $a(t)$ this implies that $\dfrac{A(t)}{a(t)\lambda^2(t)}$ is uniformly bounded for large t. This completes the proof. $\square$

2.2. Considerations in the hyperbolic zone

Here we use the hyperbolic micro-energy $U = (a(t)|\xi|\hat{u}, D_t\hat{u})^T$. Then U satisfies

$$D_tU = A(t,\xi)U := \begin{pmatrix} \dfrac{D_t a}{a} & a(t)|\xi| \\ a(t)|\xi| & ib(t)\end{pmatrix}U. \tag{2.15}$$

Let us carry out the first step of diagonalization. For this reason we introduce

$$M = \begin{pmatrix} 1 & -1 \\ 1 & 1\end{pmatrix},\quad M^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1\end{pmatrix} \text{ and } U^{(0)} := M^{-1}U.$$

So $D_tU^{(0)} = \mathcal{D}(t,\xi)U^{(0)} + \mathcal{R}(t)U^{(0)}$, where

$$\mathcal{D}(t,\xi) = \begin{pmatrix} \tau_1(t,\xi) & 0 \\ 0 & \tau_2(t,\xi) \end{pmatrix} := \begin{pmatrix} a(t)|\xi| & 0 \\ 0 & -a(t)|\xi| \end{pmatrix},$$

$$\mathcal{R}(t) = (R_{kl}(t)) := \frac{1}{2}\begin{pmatrix} \frac{D_ta}{a} + ib(t) & -\frac{D_ta}{a} + ib(t) \\ -\frac{D_ta}{a} + ib(t) & \frac{D_ta}{a} + ib(t) \end{pmatrix}.$$

Let $F_0(t)$ be the diagonal part of $\mathcal{R}(t)$. Now we carry out the second step of diagonalization procedure. Therefore we introduce the matrices

$$N^{(1)} = \begin{pmatrix} 0 & \dfrac{R_{12}}{\tau_1 - \tau_2} \\ \dfrac{R_{21}}{\tau_2 - \tau_1} & 0 \end{pmatrix} = \begin{pmatrix} 0 & i\dfrac{\delta_1(t)}{4a(t)|\xi|} \\ -i\dfrac{\delta_1(t)}{4a(t)|\xi|} & 0 \end{pmatrix}, \quad N_1 = I + N^{(1)}.$$

Here $\delta_1 := \dfrac{a'}{a} + b$. We have

$$\left(\frac{\frac{a'}{a}(t)}{4a(t)|\xi|}\right)^2 \lesssim \left(\frac{\frac{a}{A}(t)}{4a(t)|\xi|}\right)^2 \lesssim \Big(\frac{1}{A(t)|\xi|}\Big)^2 \leqslant \frac{C}{N^2}.$$

If we use the ansatz $b(t) = \mu(t)\dfrac{a(t)}{A(t)}$ and the assumptions (B3) or (B3)$'$ and (C), i.e., we have $\limsup_{t\to\infty}\mu(t) \lesssim 1$, then

$$\Big(\frac{b(t)}{4a(t)|\xi|}\Big)^2 = \Big(\frac{\mu(t)a(t)}{4a(t)A(t)|\xi|}\Big)^2 \lesssim \Big(\frac{1}{A(t)|\xi|}\Big)^2 \leq \frac{C}{N^2}.$$

Thus we can choose a sufficiently large N such that the determinant of N_1 is $\det N_1 = 1 - \Big(\dfrac{\delta_1(t)}{4a(t)|\xi|}\Big)^2 < 1/2$. Hence, the matrix N_1 is invertible. Setting

$$B^{(1)} = D_tN^{(1)} - (\mathcal{R} - F_0)N^{(1)} = \begin{pmatrix} -\dfrac{\delta_1^2(t)}{8a(t)|\xi|} & \partial_t\dfrac{\delta_1(t)}{4a(t)|\xi|} \\ -\partial_t\dfrac{\delta_1(t)}{4a(t)|\xi|} & \dfrac{\delta_1^2(t)}{8a(t)|\xi|} \end{pmatrix},$$

and $\mathcal{R}_1(t,\xi) = -N_1^{-1}B^{(1)}(t,\xi)$. We can conclude that

$$\big(D_t - \mathcal{D}(t,\xi) - \mathcal{R}(t)\big)N_1(t,\xi)U^{(1)} = N_1(t,\xi)\big(D_t - \mathcal{D}(t,\xi) - F_0(t) - \mathcal{R}_1(t,\xi)\big)U^{(1)}.$$

Now we shall find the solution $U^{(0)}(t,\xi) := N_1(t,\xi)U^{(1)}(t,\xi)$, where $U^{(1)}(t,\xi)$ is the solution to the system

$$\big(D_t - \mathcal{D}(t,\xi) - F_0(t) - \mathcal{R}_1(t,\xi)\big)U^{(1)}(t,\xi) = 0.$$

We can write $U^{(1)}(t,\xi) = E(t,t_\xi,\xi)U^{(1)}(t_\xi,\xi)$, where $E(t,s,\xi)$ is the fundamental solution, that is, the solution of the system

$$D_tE(t,s,\xi) = \big(D_t - \mathcal{D}(t,\xi) - F_0(t) - \mathcal{R}_1(t,\xi)\big)E(t,s,\xi), E(s,s,\xi) = I, t \geq s \geq t_\xi.$$

The solution $E_0 = E_0(t,s,\xi)$ of the "principal diagonal part" of the last system fulfils

$$D_t E_0(t,s,\xi) = (\mathcal{D}(t,\xi) + F_0(t))E_0(t,s,\xi), \quad E_0(s,s,\xi) = I, \ t \geq s \geq t_\xi.$$

Thus

$$E_0(t,s,\xi) = \frac{\sqrt{a(t)}}{\sqrt{a(s)}}\frac{\lambda(s)}{\lambda(t)}\begin{pmatrix} \exp\Big(\int_s^t ia(\tau)|\xi|d\tau\Big) & 0 \\ 0 & \exp\Big(-\int_s^t ia(\tau)|\xi|d\tau\Big)\end{pmatrix}.$$

Let us set

$$\mathcal{R}_2(t,s,\xi) = E_0(t,s,\xi)^{-1}\mathcal{R}_1(t,\xi)E_0(t,s,\xi),$$

$$Q(t,s,\xi) = I + \sum_{k=1}^\infty \int_s^t \mathcal{R}_2(t_1,s,\xi)\int_s^{t_1}\mathcal{R}_2(t_2,s,\xi)\dots\int_s^{t_{k-1}}\mathcal{R}_2(t_k,s,\xi)dt_k\dots dt_2dt_1.$$

Then $Q(t,s,\xi)$ solves the Cauchy problem

$$D_tQ(t,s,\xi) = \mathcal{R}_2(t,s,\xi)Q(t,s,\xi), \quad Q(s,s,\xi) = I, \ t \geq s \geq t_\xi.$$

The fundamental solution $E = E(t,s,\xi)$ is representable in the form $E(t,s,\xi) = E_0(t,s,\xi)Q(t,s,\xi)$. Taking into consideration the above representation for $Q(t,s,\xi)$ we are able to prove the following estimate:

$$|Q(t,s,\xi)| \leq \exp\Big(\int_s^t |\mathcal{R}_1(\tau,\xi)|d\tau\Big) \leq \exp\Big(\frac{1}{|\xi|}\Big(\frac{1}{A(\tau)}\Big)\Big|_s^t\Big) \leq C_N.$$

The backward transformation yields

$$U(t,\xi) = MN_1(t,\xi)E_0(t,s,\xi)Q(t,s,\xi)N_1^{-1}(s,\xi)M^{-1}U(s,\xi),$$

therefore, we may conclude

$$\left|\begin{pmatrix} a(t)|\xi|\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix}\right| \leq \frac{\sqrt{a(t)}}{\sqrt{a(s)}}\frac{\lambda(s)}{\lambda(t)}\left|\begin{pmatrix} a(s)|\xi|\hat{u}(s,\xi) \\ D_t\hat{u}(s,\xi)\end{pmatrix}\right| \quad \text{for all } t \geq s \geq t_\xi.$$

Corollary 2.7. *We have in the hyperbolic zone $Z_{\mathrm{hyp}}(N)$ the estimate*

$$\left|\begin{pmatrix} a(t)|\xi|\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix}\right| \leq C\frac{\sqrt{a(t)}}{\sqrt{a(t_\xi)}}\frac{\lambda(t_\xi)}{\lambda(t)}\left|\begin{pmatrix} a(t_\xi)|\xi|\hat{u}(t_\xi,\xi) \\ D_t\hat{u}(t_\xi,\xi)\end{pmatrix}\right|$$

for all $t \geq t_\xi$.

2.3. Gluing procedure and L^2-L^2 estimates

Before gluing the estimates in the dissipative zone and the hyperbolic zone we state the following lemma:

Lemma 2.8. *Assume that the functions $\mu = \mu(t)$ and $\alpha = \alpha(t)$ satisfy the assumption*

$$\limsup_{t\to\infty}\big(\mu(t) + \alpha(t)\big) < 2.$$

Then the following inequality holds:

$$\frac{\lambda(t)\sqrt{a(t)}}{A(t)} \leq C.$$

Proof. We have from the definition of λ and α

$$\frac{\lambda(t)\sqrt{a(t)}}{A(t)} \lesssim \frac{\exp\Big(\frac{1}{2}\int_0^t \mu(s)\frac{a(s)}{A(s)}ds\Big)\exp\Big(\frac{1}{2}\int_0^t \alpha(s)\frac{a(s)}{A(s)}ds\Big)}{\exp\Big(\int_0^t \frac{a(s)}{A(s)}ds\Big)}$$
$$= \exp\Big(\frac{1}{2}\int_0^t \big(\mu(s)+\alpha(s)-2\big)\frac{a(s)}{A(s)}ds\Big).$$

According to the assumption it holds $\mu(t)+\alpha(t)-2 \leq 0$ for $t \geq t_0$ with a suitable t_0. From that we may conclude

$$\frac{\lambda(t)\sqrt{a(t)}}{A(t)} \lesssim \exp\Big(\int_0^{t_0} \big(\mu(s)+\alpha(s)-2\big)\frac{a(s)}{A(s)}ds\Big) \leq C(t_0). \tag{2.16}$$

This completes the proof. □

From the statements of Corollaries 2.5 and 2.7 we derive the statement of our theorem.

Case 1: $\{|\xi| \geq N\}$. Then the statement of Corollary 2.7 implies immediately

$$\left|\begin{pmatrix} a(t)|\xi|\hat{u}(t,\xi) \\ D_t\hat{u}(t,\xi)\end{pmatrix}\right| \leq C\frac{\sqrt{a(t)}}{\lambda(t)}\left|\begin{pmatrix} |\xi|\hat{u}(0,\xi) \\ D_t\hat{u}(0,\xi)\end{pmatrix}\right|$$

for all $t \geq 0$.

Case 2: $\{|\xi| \leq N\}$ *and* $\{t \geq t_\xi\}$. Then the statements of Corollary 2.7 imply immediately

$$a(t)|\xi||\hat{u}(t,\xi)| + |D_t\hat{u}(t,\xi)| \leq C\frac{\sqrt{a(t)}}{\sqrt{a(t_\xi)}}\frac{\lambda(t_\xi)}{\lambda(t)}\Big(a(t_\xi)|\xi||\hat{u}(t_\xi,\xi)| + |D_t\hat{u}(t_\xi,\xi)|\Big)$$
$$\leq C_N\frac{\sqrt{a(t)}}{\lambda(t)}\Big(\frac{\lambda(t_\xi)}{\sqrt{a(t_\xi)}}a(t_\xi)|\xi||\hat{u}(t_\xi,\xi)| + \frac{\lambda(t_\xi)}{\sqrt{a(t_\xi)}}|D_t\hat{u}(t_\xi,\xi)|\Big).$$

From Corollary 2.5 we have for $t = t_\xi$

$$a(t_\xi)|\xi||\hat{u}(t,\xi)| + |D_t\hat{u}(t_\xi,\xi)| \leq C\frac{a(t_\xi)}{A(t_\xi)}|\hat{u}(0,\xi)| + C\frac{a(t_\xi)^{1-\delta}}{\lambda^2(t_\xi)}|D_t\hat{u}(0,\xi)|.$$

Summarizing we get

$$a(t)|\xi||\hat{u}(t,\xi)| + |D_t\hat{u}(t_\xi,\xi)| \leq C\frac{\sqrt{a(t)}}{\lambda(t)}\left(\frac{\sqrt{a(t_\xi)}\lambda(t_\xi)}{A(t_\xi)}|\hat{u}(0,\xi)| + \frac{a(t_\xi)^{\frac{1}{2}-\delta}}{\lambda(t_\xi)}|D_t\hat{u}(0,\xi)|\right)$$

for all admissible (t,ξ). If we choose $\delta \geq \dfrac{1}{2}$ and apply Lemma 2.8, then we may conclude

$$a(t)|\xi||\hat{u}(t,\xi)|+|D_t\hat{u}(t,\xi)| \leq C_N \frac{\sqrt{a(t)}}{\lambda(t)}\big(|\hat{u}(0,\xi)|+|D_t\hat{u}(0,\xi)|\big) \text{ for all admissible } (t,\xi).$$

Case 3: $\{|\xi| \leq N\}$ and $\{t \leq t_\xi\}$. Then the statements of Corollary 2.5 imply immediately

$$a(t)|\xi||\hat{u}(t,\xi)| + |D_t\hat{u}(t,\xi)| \leq C_N \frac{a(t)}{A(t)}|\hat{u}(0,\xi)| + C_N \frac{a(t)^{1-\delta}}{\lambda^2(t)}|D_t\hat{u}(0,\xi)|.$$

If we choose $\delta \geq \dfrac{1}{2}$ and apply Lemma 2.8, then we may conclude

$$a(t)|\xi||\hat{u}(t,\xi)|+|D_t\hat{u}(t,\xi)| \leq C_N \frac{\sqrt{a(t)}}{\lambda(t)}\big(|\hat{u}(0,\xi)|+|D_t\hat{u}(0,\xi)|\big) \text{ for all admissible } (t,\xi).$$

This completes the proof to Theorem 2.1. $\square$

Remark 2.9. If we choose in Theorem 2.1 the coefficient $a(t) \equiv 1$, then the obtained estimates coincide with the estimates from Result 1.1 for $p = q = 2$.

Example. Let $\mu \in (0,1)$ or $\mu \in \big(1, 2 - \frac{l}{l+1}\big)$. We choose with $l > 0$

$$a(t) = (1+t)^l,\ A(t) = \frac{1}{l+1}(1+t)^{l+1},\ b(t) = \frac{\mu(l+1)}{1+t}.$$

These coefficients satisfy the assumptions of Theorem 2.1. Taking into consideration $\lambda(t) = (1+t)^{\frac{\mu(l+1)}{2}}$ we may conclude

$$\big\|\big((1+t)^l\nabla u(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^2} \lesssim (1+t)^{\frac{l}{2}-\frac{\mu(l+1)}{2}}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big).$$

Example. Let $\mu \in (0,1)$. We choose

$$a(t) = e^t,\ A(t) = e^t,\ b(t) = \mu.$$

These coefficients satisfy the assumptions of Theorem 2.1. We have $\lambda(t) = e^{\frac{\mu}{2}t}$. So, we may conclude

$$\big\|\big(e^t\nabla u(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^2} \lesssim e^{\frac{t}{2}-\frac{\mu}{2}t}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big).$$

Example. Let $\mu > 0$ and $m \geq 1$. We choose

$$a(t) = (e^{[m]}+t)^l,\ \ A(t) = \frac{1}{l+1}(e^{[m]}+t)^{l+1},$$
$$\text{and}\ \ \mu(t) = \frac{\mu}{(l+1)\log(e^{[m]}+t)\cdots\log^{[m]}(e^{[m]}+t)}.$$

These coefficients satisfy the assumptions of Theorem 2.1.

We have $\lambda(t) = \big(\log^{[m]}(e^{[m]}+t)\big)^{\frac{\mu}{2}}$. So, we may conclude

$$\big\|\big((e^{[m]}+t)^l\nabla u(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^2} \lesssim \frac{(e^{[m]}+t)^{\frac{l}{2}}}{\big(\log^{[m]}(e^{[m]}+t)\big)^{\frac{\mu}{2}}}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big).$$

3. Effective dissipation

We consider the following Cauchy problem

$$u_{tt} - a^2(t)\Delta u + b(t)u_t = 0, \ \ u(0,x) = u_1(x), \ \ u_t(0,x) = u_2(x). \tag{3.1}$$

In the previous section we have concerned with the influence of the dissipation term $b(t)u_t$ for a given speed of propagation $a(t)$ such that the equation (3.1) is from the point of view of long time behavior of solutions and its energies in some sense close to the *wave equation with increasing speed of propagation*. Now our question is the following:

Under which assumptions to the coefficient $b = b(t)$ for a given time-dependent speed of propagation $a = a(t)$ can we call b an effective dissipation?

Here effective means, that on the one hand we have really a dissipation effect (overdamping is excluded), but on the other hand the model is parabolic like from the point of view of decay estimates for the wave type energy.

We will apply a transformation of the damped wave equation from (3.1) to a wave equation with time-dependent speed of propagation and potential. Thus, we define the new function

$$v(t,x) := \exp\Big(\frac{1}{2}\int_0^t b(\tau)d\tau\Big)u(t,x).$$

After some calculations we get

$$v_{tt} - a^2(t)\Delta v - \Big(\frac{1}{4}b^2(t) + \frac{1}{2}b'(t)\Big)v = 0.$$

Applying partial Fourier transformation we have

$$\hat{v}_{tt} + m(t,\xi)\hat{v} = 0, \tag{3.2}$$

here

$$m(t,\xi) = a^2(t)|\xi|^2 - \frac{1}{4}b^2(t) - \frac{1}{2}b'(t). \tag{3.3}$$

To study the interacting between $a(t)$ and $b(t)$ we assume:

(B$'$1) $b(t) > 0$, $b(t) = \mu(t)\dfrac{a(t)}{A(t)}$, $\dfrac{a^2(t)}{b(t)} \notin L^1(\mathbb{R}^+)$,

(B$'$2) $\big|d_t^k\mu(t)\big| \le C_k\mu(t)\left(\dfrac{a(t)}{A(t)}\right)^k$ for $k = 1, 2$,

(B$'$3) $\mu(t) \to \infty$ as $t \to \infty$, $\dfrac{\mu(t)}{A(t)}$ is monotonous.

Using assumption (B$'$1) we can rewrite the formula (3.3) by

$$m(t,\xi) = a^2(t)|\xi|^2 - \frac{1}{4}\mu^2(t)\frac{a^2(t)}{A^2(t)} - \frac{1}{2}\Big(\mu(t)\frac{a(t)}{A(t)}\Big)'.$$

Assumptions (B$'$2) and (B'3) show that $b'(t)$ is a negligible term in comparison with $b^2(t)$, this means $|b'(t)| = o(b^2(t))$ as $t \to \infty$.

We introduce the auxiliary symbol

$$\langle\xi\rangle_{b(t)} := \sqrt{\Big|a^2(t)|\xi|^2 - \frac{b^2(t)}{4}\Big|} = \sqrt{\Big|a^2(t)|\xi|^2 - \frac{\mu^2(t)}{4}\frac{a^2(t)}{A^2(t)}\Big|}. \tag{3.4}$$

The main result of this section is the following statement:

Theorem 3.1. *Let us assume the conditions* (A1) *to* (A3) *and* (B′1) *to* (B′3). *Then we have the following* L^2-L^2 *estimates:*
For the kinetic energy we have

$$\|u_t(t,\cdot)\|_{L^2} \lesssim a(t)\left(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)^{-\frac12}\big(\|u_1\|_{H^1}+\|u_2\|_{L^2}\big).$$

For the"elastic" energy we have

$$\|a(t)\nabla u(t,\cdot)\|_{L^2} \lesssim a(t)\left(1+\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)^{-\frac12}\big(\|u_1\|_{H^1}+\|u_2\|_{L^2}\big).$$

3.1. Regions and zones

We define the separating curve $t_\xi = t(|\xi|)$ by

$$\Gamma = \Big\{(t,\xi) : |\xi| = \frac12\frac{\mu(t)}{A(t)}\Big\},$$

and introduce the following regions in the extended phase space $(0,\infty)\times\mathbb{R}^n_\xi$:

the hyperbolic region: $\Pi_{\rm hyp} = \Big\{(t,\xi) : |\xi| > \frac12\frac{\mu(t)}{A(t)}\Big\}$,

the elliptic region: $\Pi_{\rm ell} = \Big\{(t,\xi) : |\xi| < \frac12\frac{\mu(t)}{A(t)}\Big\}$.

The auxiliary symbol $\langle\xi\rangle_{b(t)}$ is differentiable in these regions and satisfies

$$\partial_t\langle\xi\rangle_{b(t)} = \pm\frac{a'(t)a(t)|\xi|^2 - \frac{\mu(t)a(t)}{2A(t)}\Big(\frac{\mu(t)a(t)}{2A(t)}\Big)'}{\langle\xi\rangle_{b(t)}},\quad \partial_{|\xi|}\langle\xi\rangle_{b(t)} = \pm\frac{a^2(t)|\xi|}{\langle\xi\rangle_{b(t)}}, \tag{3.5}$$

where the upper sign is taken in the hyperbolic region.

We will also divide both regions of the extended phase space into zones. For this reason we define
the hyperbolic zone:

$$Z_{\rm hyp}(N) = \Big\{(t,\xi) : \langle\xi\rangle_{b(t)} \ge N\mu(t)\frac{a(t)}{2A(t)}\Big\}\cap\Pi_{\rm hyp},$$

the pseudo-differential zone:

$$Z_{\rm pd}(N,\varepsilon) = \Big\{(t,\xi) : \varepsilon\mu(t)\frac{a(t)}{2A(t)} \le \langle\xi\rangle_{b(t)} \le N\mu(t)\frac{a(t)}{2A(t)}\Big\}\cap\Pi_{\rm hyp},$$

the dissipative zone:

$$Z_{\rm diss}(c_0) = \Big\{(t,\xi) : |\xi| \le c_0\frac{1}{A(t)}\Big\}\cap\Pi_{\rm ell},$$

the elliptic zone:

$$Z_{\mathrm{ell}}(c_0,\varepsilon)=\Big\{(t,\xi):|\xi|\geq c_0\frac{1}{A(t)}\Big\}\cap\Big\{\langle\xi\rangle_{b(t)}\geq\varepsilon\mu(t)\frac{a(t)}{2A(t)}\Big\}\cap\Pi_{\mathrm{ell}},$$

the reduced zone:

$$Z_{\mathrm{red}}(\varepsilon)=\Big\{(t,\xi):\langle\xi\rangle_{b(t)}\leq\varepsilon\mu(t)\frac{a(t)}{2A(t)}\Big\}.$$

Remark 3.2. The dissipative zone can be skipped if we assume the further assumption

$$\text{(S1)}\qquad \frac{a^2(t)}{b(t)A^2(t)}=\frac{a(t)}{\mu(t)A(t)}\in L^1(\mathbb{R}^+).$$

Under this additional assumption we define $Z_{\mathrm{ell}}(\varepsilon):=Z_{\mathrm{ell}}(0,\varepsilon)$.

3.2. The hyperbolic region

3.2.1. Symbols in Π_{hyp}.

Definition 3.3. Let us define the following classes of symbols in the hyperbolic zone:

$$S_l\{m_1,m_2,m_3\}=\Big\{c(t,\xi):|D_\xi^\alpha D_t^k c(t,\xi)|\leq C_{\alpha,k}\langle\xi\rangle_{b(t)}^{m_1-|\alpha|}a(t)^{m_2+|\alpha|}\Big(\frac{a(t)}{A(t)}\Big)^{m_3+k}\quad\text{for all } (t,\xi)\in Z_{\mathrm{hyp}}(N),\ \alpha,\ \text{ and } k\leq l\Big\}.$$

Lemma 3.4. *The family of symbol classes $S_l\{m_1,m_2,m_3\}$ generates a hierarchy having the following properties:*

- $S_l\{m_1,m_2,m_3\}$ *is a vector space,*
- $S_l\{m_1,m_2,m_3\}S_l\{m_1',m_2',m_3'\}\subset S_l\{m_1+m_1',m_2+m_2',m_3+m_3'\}$,
- $D_t^kD_\xi^\alpha S_l\{m_1,m_2,m_3\}\subset S_{l-k}\{m_1-|\alpha|,m_2+|\alpha|,m_3+k\}$,
- $S_0\{-1,0,2\}\subset L_\xi^\infty L_t^1(Z_{\mathrm{hyp}}(N))$.

Proof. We only verify the fourth property. Indeed, if $c=c(t,\xi)\in S_0\{-1,0,2\}$, then

$$\int_{t_\xi}^\infty|c(\tau,\xi)|d\tau\lesssim\int_{t_\xi}^\infty\frac{a^2(\tau)d\tau}{\langle\xi\rangle_{b(\tau)}A^2(\tau)}\sim\int_{t_\xi}^\infty\frac{a(\tau)d\tau}{|\xi|A^2(\tau)}\leq\frac{C}{A(t_\xi)|\xi|}\leq\frac{C}{N\mu(t_\xi)}<\infty$$

due to the definition of the hyperbolic zone and assumption (B′3). Remark, that here we used

$$\langle\xi\rangle_{b(t)}\sim a(t)|\xi|\ \text{ uniformly on } Z_{\mathrm{hyp}}(N)\tag{3.6}$$

to conclude what we wanted to have. □

3.2.2. Consideration in the hyperbolic zone. Now we consider the micro-energy

$$V=(\langle\xi\rangle_{b(t)}\hat{v},D_t\hat{v})^T. \tag{3.7}$$

Then it holds

$$D_tV=\begin{pmatrix}0 & \langle\xi\rangle_{b(t)}\\ \langle\xi\rangle_{b(t)} & 0\end{pmatrix}V+\begin{pmatrix}\frac{D_t\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(t)}} & 0\\ -\frac{\left(\mu(t)\frac{a(t)}{A(t)}\right)'}{2\langle\xi\rangle_{b(t)}} & 0\end{pmatrix}V. \tag{3.8}$$

Lemma 3.5. *Let us assume the conditions* (B′1)*,* (B′2) *and* (B′3)*. Then the following estimate holds for the fundamental solution $E_V(t,s,\xi)$ with $(s,\xi),t,\xi)\in Z_{\text{hyp}}(N)$, $s\le t$:*

$$|E_V(t,s,\xi)|\lesssim\frac{\sqrt{a(t)}}{\sqrt{a(s)}}.$$

Proof. Let us carry out the first step of diagonalization. For this reason we set

$$M=\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix},\quad M^{-1}=\frac{1}{2}\begin{pmatrix}1 & 1\\ -1 & 1\end{pmatrix},\quad \text{and } V^{(0)}:=M^{-1}V.$$

Hence,

$$D_tV^{(0)}=\mathcal{D}(t,\xi)V^{(0)}+\mathcal{R}(t,\xi)V^{(0)}, \tag{3.9}$$

where

$$\mathcal{D}(t,\xi)=\begin{pmatrix}\langle\xi\rangle_{b(t)} & 0\\ 0 & -\langle\xi\rangle_{b(t)}\end{pmatrix}\in S_2\{1,0,0\}, \tag{3.10}$$

$$\mathcal{R}(t,\xi)=\begin{pmatrix}\frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}-\frac{b'(t)}{4\langle\xi\rangle_{b(t)}} & -\frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}+\frac{b'(t)}{4\langle\xi\rangle_{b(t)}}\\ -\frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}-\frac{b'(t)}{4\langle\xi\rangle_{b(t)}} & \frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}+\frac{b'(t)}{4\langle\xi\rangle_{b(t)}}\end{pmatrix}\in S_1\{0,0,1\}. \tag{3.11}$$

Let $F_0(t,\xi)$ be the diagonal part of $\mathcal{R}(t,\xi)$. Now we carry out the second step of diagonalization procedure. Therefore we introduce the matrices

$$\begin{aligned}N^{(1)}&=\begin{pmatrix}0 & \frac{\mathcal{R}_{12}}{\tau_1-\tau_2}\\ \frac{\mathcal{R}_{21}}{\tau_2-\tau_1} & 0\end{pmatrix}\\ &=\begin{pmatrix}0 & -\frac{D_t\langle\xi\rangle_{b(t)}}{4\langle\xi\rangle^2_{b(t)}}-\frac{b'(t)}{8\langle\xi\rangle^2_{b(t)}}\\ -\frac{D_t\langle\xi\rangle_{b(t)}}{4\langle\xi\rangle^2_{b(t)}}+\frac{b'(t)}{8\langle\xi\rangle^2_{b(t)}} & 0\end{pmatrix}\in S_1\{-1,0,1\},\end{aligned}$$

$N_1(t,\xi)=I+N^{(1)}(t,\xi)\in S_1\{0,0,0\}$. For sufficiently large time $t_0=t_0(\varepsilon)$ the matrix $N_1(t,\xi)$ is invertible with uniformly bounded inverse $N_1^{-1}(t,\xi)$ for $t\ge t_0$ in $Z_{\text{hyp}}(N)$ (see Remark 3.6). Now we can follow the usual procedure to diagonalize. Let

$$\begin{aligned}B^{(1)}(t,\xi)&=D_tN^{(1)}(t,\xi)-\big(\mathcal{R}(t,\xi)-F_0(t,\xi)\big)N^{(1)}(t,\xi)\in S_0\{-1,0,2\},\\ \mathcal{R}_1(t,\xi)&=-N_1^{-1}(t,\xi)B^{(1)}(t,\xi)\in S_0\{-1,0,2\}.\end{aligned}$$

Then we can conclude

$$\big(D_t-\mathcal{D}(t,\xi)-\mathcal{R}(t,\xi)\big)N_1(t,\xi)V^{(1)}=N_1(t,\xi)\big(D_t-\mathcal{D}(t,\xi)-F_0(t,\xi)-\mathcal{R}_1(t,\xi)\big)V^{(1)}.$$

Now we shall find the solution $V^{(0)}(t,\xi)=:N_1(t,\xi)V^{(1)}(t,\xi)$, where $V^{(1)}(t,\xi)$ is the solution to the system

$$\big(D_t-\mathcal{D}(t,\xi)-F_0(t,\xi)-\mathcal{R}_1(t,\xi)\big)V^{(1)}(t,\xi)=0.$$

We can write $V^{(1)}(t,\xi)=E_V(t,t_\xi,\xi)V^{(1)}(t_\xi,\xi)$. Here $E_V(t,s,\xi)$ is the fundamental solution to the following system:

$$\big(D_t-\mathcal{D}(t,\xi)-F_0(t,\xi)-\mathcal{R}_1(t,\xi)\big)E_V(t,s,\xi)=0,\quad E_V(s,s,\xi)=I,\ t\geq s\geq t_\xi.$$

The solution $E_0=E_0(t,s,\xi)$ of the "principal diagonal part" of this system fulfils

$$D_tE_0(t,s,\xi)=\big(\mathcal{D}(t,\xi)+F_0(t,\xi)\big)E_0(t,s,\xi),\quad E_0(s,s,\xi)=I,\ t\geq s\geq t_\xi.$$

Thus

$$E_0(t,s,\xi)=\exp\left(i\int_s^t\big(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi)\big)d\tau\right),$$

and we can get

$$|E_0(t,s,\xi)|\lesssim\exp\left(\int_s^t\frac{\partial_t\langle\xi\rangle_{b(\tau)}}{2\langle\xi\rangle_{b(\tau)}}d\tau\right)=\frac{\sqrt{\langle\xi\rangle_{b(t)}}}{\sqrt{\langle\xi\rangle_{b(s)}}}\sim\frac{\sqrt{a(t)|\xi|}}{\sqrt{a(s)|\xi|}}\sim\frac{\sqrt{a(t)}}{\sqrt{a(s)}}.$$

Let us set

$$\mathcal{R}_2(t,s,\xi)=E_0^{-1}(t,s,\xi)\mathcal{R}_1(t,\xi)E_0(t,s,\xi),$$

$$Q(t,s,\xi)=I+\sum_{k=1}^{\infty}\int_s^t\mathcal{R}_2(t_1,s,\xi)\int_s^{t_1}\mathcal{R}_2(t_2,s,\xi)\cdots\int_s^{t_{k-1}}\mathcal{R}_2(t_k,s,\xi)dt_k\cdots dt_2dt_1.$$

Then $Q=Q(t,s,\xi)$ solves the Cauchy problem

$$D_tQ(t,s,\xi)=\mathcal{R}_2(t,s,\xi)Q(t,s,\xi),\quad Q(s,s,\xi)=I,\ t\geq s\geq t_\xi.$$

The fundamental solution $E_V(t,s,\xi)$ is representable in the form $E_V(t,s,\xi)=E_0(t,s,\xi)Q(t,s,\xi)$. Furthermore, applying the fourth property from Lemma 3.4 to $\mathcal{R}_1\in S_0\{-1,0,2\}\subset L^\infty_\xi L^1_t(Z_{\text{hyp}})$ we see that

$$|Q(t,s,\xi)|\leq\exp\Big(\int_s^t|\mathcal{R}_1(\tau,\xi)|d\tau\Big)\leq C_N.$$

This completes the proof. □

Remark 3.6. The large constant N guarantees the invertibility in the whole hyperbolic zone. The remaining problem consists in the understanding of invertibility in the pseudo-differential zone. For $t\geq t_0(\varepsilon)$ this zone can be skipped after the choice $N=\varepsilon$. The other set $\{t\in(0,t_0(\varepsilon)]:(t,\xi)\in Z_{\text{pd}}(N,\varepsilon)\}$ is compact.

3.3. The elliptic region

3.3.1. Symbols in Π_{ell}. The symbols in the elliptic zone are constructed in a similar manner as in the hyperbolic zone with a little change for the auxiliary symbol $\langle\xi\rangle_{b(t)}$ which can be estimated by

$$\langle\xi\rangle_{b(t)} \sim \frac{b(t)}{2} \sim \mu(t)\frac{a(t)}{2A(t)} \qquad \text{uniformly on } Z_{\mathrm{ell}}(c_0,\varepsilon). \tag{3.12}$$

Definition 3.7. Let us define the following classes of symbols in the elliptic zone:

$$S_l\{m_1,m_2,m_3\} = \Big\{c = c(t,\xi) : |D_\xi^\alpha D_t^k c(t,\xi)| \le C_{\alpha,k}\langle\xi\rangle_{b(t)}^{m_1-|\alpha|} a(t)^{m_2+|\alpha|} \left(\frac{a(t)}{A(t)}\right)^{m_3+k} \quad \text{for all } (t,\xi)\in Z_{\mathrm{ell}}(c_0,\varepsilon),\ \alpha \text{ and } k\le l\Big\}.$$

Lemma 3.8. *The family of symbol classes $S_l\{m_1,m_2,m_3\}$ generates a hierarchy having the following properties:*

- $S_l\{m_1,m_2,m_3\}$ *is a vector space,*
- $S_l\{m_1,m_2,m_3\}S_l\{m_1',m_2',m_3'\} \subset S_l\{m_1+m_1',m_2+m_2',m_3+m_3'\}$,
- $D_t^kD_\xi^\alpha S_l\{m_1,m_2,m_3\} \subset S_{l-k}\{m_1-|\alpha|,m_2+|\alpha|,m_3+k\}$,
- $S_0\{-1,0,2\} \subset L_\xi^\infty L_t^1(Z_{\mathrm{ell}}(c_0,\varepsilon))$.

Proof. We only verify the fourth property. Indeed, if $c\in S_0\{-1,0,2\}$, then

$$\int_{t_{\xi_1}}^{t_{\xi_2}} |c(\tau,\xi)|d\tau \lesssim \int_{t_{\xi_1}}^{t_{\xi_2}} \frac{a^2(\tau)}{\langle\xi\rangle_{b(t)}A^2(\tau)}d\tau \sim \int_{t_{\xi_1}}^{t_{\xi_2}} \frac{a(\tau)}{\mu(\tau)A(\tau)}d\tau$$
$$\lesssim \int_{t_{\xi_1}}^{t_{\xi_2}} \frac{\sqrt{1-\varepsilon^2}a(\tau)}{|\xi|A^2(\tau)}d\tau \lesssim \frac{1}{|\xi|A(t_{\xi_1})} \lesssim C(\varepsilon,c_0),$$

where t_{ξ_1}, t_{ξ_2} denotes the lower, upper boundary of the elliptic zone, respectively. From the definitions of the elliptic zone and dissipative zone we have $\mu(t) \ge \frac{2|\xi|A(t)}{\sqrt{1-\varepsilon^2}}$ for all $t\in[t_{\xi_1},t_{\xi_2}]$ and $|\xi|A(t_{\xi_1})\sim 1$. □

3.3.2. Consideration in the elliptic zone. In this region we introduce again the micro-energy

$$V = (\langle\xi\rangle_{b(t)}\hat v, D_t\hat v)^T.$$

Then we can get the system of differential equations

$$D_tV = \left(\begin{pmatrix} 0 & \langle\xi\rangle_{b(t)} \\ -\langle\xi\rangle_{b(t)} & 0 \end{pmatrix} + \begin{pmatrix} \frac{D_t\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(t)}} & 0 \\ -\frac{b'(t)}{2\langle\xi\rangle_{b(t)}} & 0 \end{pmatrix}\right)V. \tag{3.13}$$

In a first step we use the diagonalizer of the first matrix which is defined as follows:

$$M = \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}, \quad M^{-1} = \frac{1}{2}\begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix}, \quad \text{and } V^{(0)} := M^{-1}V.$$

Hence,

$$D_tV^{(0)} = \mathcal{D}(t,\xi)V^{(0)} + \mathcal{R}(t,\xi)V^{(0)}, \tag{3.14}$$

where

$$\mathcal{D}(t,\xi)=\begin{pmatrix} -i\langle\xi\rangle_{b(t)} & 0\\ 0 & i\langle\xi\rangle_{b(t)}\end{pmatrix}\in S_2\{1,0,0\}, \tag{3.15}$$

$$\mathcal{R}(t,\xi)=\frac{1}{2}\begin{pmatrix} \frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}-i\frac{b'(t)}{4\langle\xi\rangle_{b(t)}} & -\frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}+i\frac{b'(t)}{4\langle\xi\rangle_{b(t)}}\\ -\frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}-i\frac{b'(t)}{4\langle\xi\rangle_{b(t)}} & \frac{D_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}+i\frac{b'(t)}{4\langle\xi\rangle_{b(t)}}\end{pmatrix}\in S_1\{0,0,1\}. \tag{3.16}$$

Let $F_0=F_0(t,\xi)$ be the diagonal part of $\mathcal{R}=\mathcal{R}(t,\xi)$. Now we carry out the second step of diagonalization procedure. Thus, we consider the difference δ of the entries of $\mathcal{D}(t,\xi)+F_0(t,\xi)$. We have

$$i\delta(t,\xi)=2\langle\xi\rangle_{b(t)}+\frac{b'(t)}{2\langle\xi\rangle_{b(t)}}\sim 2\langle\xi\rangle_{b(t)}+\frac{b^2(t)}{2\langle\xi\rangle_{b(t)}}\sim\langle\xi\rangle_{b(t)}\in S_2(1,0,0) \tag{3.17}$$

for $t\geq t_0$ with a sufficiently large $t_0=t_0(\varepsilon)$ by using $|b'(t)|=o(b^2(t))$. Now we can follow the usual procedure of diagonalization. Therefore, we introduce the matrices

$$\begin{aligned} N^{(1)}&=\begin{pmatrix} 0 & -\frac{\mathcal{R}_{12}}{\delta}\\ \frac{\mathcal{R}_{21}}{\delta} & 0\end{pmatrix}\\ &\sim\begin{pmatrix} 0 & i\frac{D_t\langle\xi\rangle_{b(t)}}{4\langle\xi\rangle^2_{b(t)}}-\frac{b'(t)}{8\langle\xi\rangle^2_{b(t)}}\\ i\frac{D_t\langle\xi\rangle_{b(t)}}{4\langle\xi\rangle^2_{b(t)}}+\frac{b'(t)}{8\langle\xi\rangle^2_{b(t)}} & 0\end{pmatrix}\in S_1\{-1,0,1\},\end{aligned}$$

$N_1(t,\xi)=I+N^{(1)}(t,\xi)\in S_1\{0,0,0\}$. For a sufficiently large time $t\geq t_0$ the matrix $N_1(t,\xi)$ is invertible with uniformly bounded inverse $N_1^{-1}(t,\xi)$ in $Z_{\mathrm{ell}}(c_0,\varepsilon)$. Let

$$\begin{aligned} B^{(1)}(t,\xi)&=D_tN^{(1)}(t,\xi)-\big(\mathcal{R}(t,\xi)-F_0(t,\xi)\big)N^{(1)}(t,\xi)\in S_0\{-1,0,2\},\\ \mathcal{R}_1(t,\xi)&=-N_1^{-1}(t,\xi)B^{(1)}(t,\xi)\in S_0\{-1,0,2\}.\end{aligned}$$

We can conclude that

$$\big(D_t-\mathcal{D}(t,\xi)-\mathcal{R}(t,\xi)\big)N_1(t,\xi)V^{(1)}=N_1(t,\xi)\big(D_t-\mathcal{D}(t,\xi)-F_0(t,\xi)-\mathcal{R}_1(t,\xi)\big)V^{(1)}.$$

Now we shall find the solution $V^{(1)}(t,\xi):=N_1^{-1}(t,\xi)V^{(0)}(t,\xi)$, where $V^{(1)}(t,\xi)$ is the solution to the system

$$\big(D_t-\mathcal{D}(t,\xi)-F_0(t,\xi)-\mathcal{R}_1(t,\xi)\big)V^{(1)}(t,\xi)=0.$$

We can write $V^{(1)}(t,\xi)=E_{V,1}(t,s,\xi)V^{(1)}(s,\xi)$. Here $E_{V,1}(t,s,\xi)$ is the fundamental solution to the following system:

$$\big(D_t-\mathcal{D}(t,\xi)-F_0(t,\xi)-\mathcal{R}_1(t,\xi)\big)E_{V,1}(t,s,\xi)=0,\quad E_{V,1}(s,s,\xi)=I,\ t\geq s\geq t_\xi.$$

We transform the system for $E_{V,1}(t,s,\xi)$ to an integral equation for a new matrix-valued function $Q_{\mathrm{ell}}(t,s,\xi)$ by considering

$$\exp\left(-i\int_s^t\big(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi)\big)d\tau\right)E_{V,1}(t,s,\xi).$$

Using this ansatz we have after differentiation

$$\begin{aligned}
&D_t\left(\exp\left(-i\int_s^t \big(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi)\big)d\tau\right)E_{V,1}(t,s,\xi)\right)\\
&= -\big(\mathcal{D}(t,\xi)+F_0(t,\xi)\big)\exp\left(-i\int_s^t \big(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi)\big)d\tau\right)E_{V,1}(t,s,\xi)\\
&\quad+\exp\left(-i\int_s^t \big(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi)\big)d\tau\right)\big(\mathcal{D}(t,\xi)+F_0(t,\xi)+\mathcal{R}_1(t,\xi)\big)E_{V,1}(t,s,\xi)\\
&= \exp\left(-i\int_s^t \big(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi)\big)d\tau\right)\mathcal{R}_1(t,\xi)E_{V,1}(t,s,\xi).
\end{aligned}$$

Consequently,

$$\begin{aligned}
E_{V,1}(t,s,\xi) =\; &\exp\left(i\int_s^t \big(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi)\big)d\tau\right)E_{V,1}(s,s,\xi)\\
&+i\int_s^t \exp\left(i\int_\theta^t \big(\mathcal{D}(\tau,\xi)+F_0(\tau,\xi)\big)d\tau\right)\mathcal{R}_1(\theta,\xi)E_{V,1}(\theta,s,\xi)d\theta.
\end{aligned}$$

We introduce an unknown weight factor to represent $Q_{\mathrm{ell},1}$ in the following way:

$$Q_{\mathrm{ell},1}(t,s,\xi)=\exp\Big(-\int_s^t w(\tau,\xi)d\tau\Big)E_{V,1}(t,s,\xi).$$

Then we get

$$\begin{aligned}
Q_{\mathrm{ell},1}(t,s,\xi) &= \exp\left(\int_s^t \big(i\mathcal{D}(\tau,\xi)+iF_0(\tau,\xi)-w(\tau,\xi)I\big)d\tau\right)\\
&+\int_s^t \exp\left(\int_\theta^t \big(i\mathcal{D}(\tau,\xi)+iF_0(\tau,\xi)-w(\tau,\xi)I\big)d\tau\right)\mathcal{R}_1(\theta,\xi)Q_{\mathrm{ell},1}(\theta,s,\xi)d\theta.
\end{aligned}$$

The main entries of the diagonal matrix $i\mathcal{D}(t,\xi)+iF_0(t,\xi)$ are given by

$$\text{(I)}=\langle\xi\rangle_{b(t)}+\frac{\partial_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}+\frac{b'(t)}{4\langle\xi\rangle_{b(t)}},\quad \text{(II)}=-\langle\xi\rangle_{b(t)}+\frac{\partial_t\langle\xi\rangle_{b(t)}}{2\langle\xi\rangle_{b(t)}}-\frac{b'(t)}{4\langle\xi\rangle_{b(t)}}.$$

For the difference (II)−(I) we get

$$\text{(II)}-\text{(I)}=-2\langle\xi\rangle_{b(t)}-\frac{b'(t)}{2\langle\xi\rangle_{b(t)}}=-\frac{b^2(t)+b'(t)-4a^2(t)|\xi|^2}{2\langle\xi\rangle_{b(t)}}\le 0$$

in $Z_{\mathrm{ell}}(c_0,\varepsilon)$ for $t\ge t_0$ by using $|b'(t)|=o(b^2(t))$. We are choosing the weight $w(t,\xi)=(I)$. By this choice the matrix

$$\begin{aligned}
H(t,s,\xi) &= \exp\left(\int_s^t \big(i\mathcal{D}(\tau,\xi)+iF_0(\tau,\xi)-w(\tau,\xi)I\big)d\tau\right)\\
&= \operatorname{diag}\left(1,\exp\left(\int_s^t\Big(-2\langle\xi\rangle_{b(\tau)}-\frac{b'(\tau)}{2\langle\xi\rangle_{b(\tau)}}\Big)d\tau\right)\right)\to\begin{pmatrix}1&0\\0&0\end{pmatrix}
\end{aligned}$$

as $t \to \infty$ with a fixed s. Hence, the matrix $H(t,s,\xi)$ is uniformly bounded for (s,ξ), $(t,\xi) \in Z_{\text{ell}}(c_0,\varepsilon)$. Taking account of $\mathcal{R}_1 \in S_0\{-1,0,2\}$ is uniformly integrable over the elliptic zone and that the matrix which can be represented by

$$Q_{\text{ell},1}(t,s,\xi) = H(t,s,\xi) + \sum_{k=1}^{\infty} i^k \int_s^t H(t,t_1,\xi)\mathcal{R}_1(t_1,s,\xi)$$
$$\times \int_s^{t_1} H(t_1,t_2,\xi)\mathcal{R}_1(t_2,s,\xi) \cdots \int_s^{t_{k-1}} H(t_{k-1},t_k,\xi)\mathcal{R}_1(t_k,s,\xi)dt_k \cdots dt_2 dt_1$$

is uniformly bounded in $Z_{\text{ell}}(c_0,\epsilon)$ from the last considerations we may conclude

$$\begin{aligned} E_{V,1}(t,s,\xi) &= \exp\Big(\int_s^t w(\tau,\xi)d\tau\Big) Q_{\text{ell},1}(t,s,\xi) \\ &= \exp\Big(\int_s^t \Big(\langle\xi\rangle_{b(\tau)} + \frac{\partial\langle\xi\rangle_{b(\tau)}}{2\langle\xi\rangle_{b(\tau)}} + \frac{b'(\tau)}{4\langle\xi\rangle_{b(\tau)}}\Big)d\tau\Big) Q_{\text{ell},1}(t,s,\xi) \\ &\sim \exp\Big(\int_s^t \Big(\langle\xi\rangle_{b(\tau)} + \frac{\partial\langle\xi\rangle_{b(\tau)}}{2\langle\xi\rangle_{b(\tau)}} + \frac{b'(\tau)}{2b(\tau)}\Big)d\tau\Big) Q_{\text{ell},1}(t,s,\xi) \\ &\sim \frac{\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(s)}} \exp\Big(\int_s^t \langle\xi\rangle_{b(\tau)}d\tau\Big) Q_{\text{ell},1}(t,s,\xi). \end{aligned}$$

Summarizing the considerations of this section we have proved the following lemma:

Lemma 3.9. *Under the assumptions* (B′1), (B′2) *and* (B′3) *the fundamental solution* $E_V(t,s,\xi)$ *to the operator* $D_t - \mathcal{D}(t,\xi) - F_0(t,\xi) - \mathcal{R}_1(t,\xi)$, *with* $(t,\xi),(s,\xi) \in Z_{\text{ell}}(c_0,\varepsilon) \cap \{t \geq t_0(\varepsilon)\}, s \leq t$ *can be represented as*

$$E_{V,1}(t,s,\xi) \sim \frac{\langle\xi\rangle_{b(t)}}{\langle\xi\rangle_{b(s)}} \exp\Big(\int_s^t \langle\xi\rangle_{b(\tau)}d\tau\Big) Q_{\text{ell},1}(t,s,\xi).$$

3.4. The reduced zone

In this zone we can estimate $\langle\xi\rangle_{b(t)}$ by $\varepsilon\dfrac{b(t)}{2}$. Thus, we consider the micro-energy

$$V = \Big(\varepsilon\frac{b(t)}{2}\hat{v}, D_t\hat{v}\Big)^T. \tag{3.18}$$

We get the following system of first order

$$D_tV = \begin{pmatrix} \frac{D_tb(t)}{b(t)} & \varepsilon\frac{b(t)}{2} \\ \frac{a^2(t)|\xi|^2 - \frac14 b^2(t) - \frac12 b'(t)}{\varepsilon\frac12 b(t)} & \end{pmatrix} V. \tag{3.19}$$

To estimate the entries of this matrix we will use

- $|b'(t)| = o(b^2(t))$,
- $\langle\xi\rangle_{b(t)} \lesssim \varepsilon\frac{b(t)}{2}$,
- consequently, $\frac{a^2(t)|\xi|^2 - \frac14 b^2(t) - \frac12 b'(t)}{\varepsilon\frac12 b(t)} \lesssim \varepsilon\frac{b(t)}{2} - \frac{b'(t)}{\varepsilon b(t)} \lesssim \varepsilon b(t)$.

Thus, we can estimate the norm of the coefficient matrix by $\varepsilon b(t)$ for sufficiently large t.

Summarizing the following statement holds:

Lemma 3.10. *If we assume* (B′1) *to* (B′3)*, then the fundamental solution* $E_V(t,s,\xi)$ *to* (3.19) *can be estimated by*

$$|E_V(t,s,\xi)| \lesssim \exp\Big(\varepsilon \int_s^t b(\tau)d\tau\Big)$$

for $t_0 \le s \le t$ *with sufficiently large* $t_0 = t_0(\varepsilon)$ *and* $(t,\xi),(s,\xi) \in Z_{\mathrm{red}}(\varepsilon)$*.*

Remark 3.11*.* We can make the reduced zone as small as we want by the control of the constant ε.

3.4.1. The dissipative zone. By Remark 3.2 the dissipative zone can be skipped if we assume $(S1)$. Now, let us assume, that the assumption $(S1)$ does not hold, i.e., $\dfrac{a(t)}{\mu(t)A(t)} \notin L^1(\mathbb{R}^+)$. Thus, we introduced the dissipative zone to ensure integrability of $S_0\{-1,0,2\}$ over the elliptic zone $Z_{\mathrm{ell}}(c_0,\epsilon)$. In this case we can apply directly Lemma 2.6 (because of assumption $\lim_{t\to\infty}\mu(t)\to\infty$) to estimate the fundamental solution $E(t,s,\xi)$ related to the micro-energy $U = (\frac{a(t)}{A(t)}\hat{u}, D_t\hat{u})^T$ and relate this estimate to the fundamental solution $E_V(t,s,\xi)$ related to $V = (\langle\xi\rangle_{b(t)}\hat{v}, D_t\hat{v})^T$.

3.5. Estimates for the fundamental solution

We want to obtain estimates for the energy of the solution to our original Cauchy problem. For this reason we need to transform back to estimate the fundamental solution $E(t,s,\xi)$ which is related to the micro-energy $(a(t)|\xi|\hat{u}, D_t\hat{u})$.
Outside the reduced zone it holds

$$E(t,s,\xi) = T(t,\xi)E_V(t,s,\xi)T^{-1}(s,\xi), \tag{3.20}$$

where the matrix $T(t,\xi)$ is defined in the following way:

$$\begin{pmatrix} h(t,\xi)\hat{u} \\ D_t\hat{u} \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{h(t,\xi)}{\lambda(t)\langle\xi\rangle_{b(t)}} & 0 \\ i\frac{b(t)}{2\lambda(t)\langle\xi\rangle_{b(t)}} & \frac{1}{\lambda(t)} \end{pmatrix}}_{T(t,\xi)} \begin{pmatrix} \langle\xi\rangle_{b(t)}\hat{v} \\ D_t\hat{v} \end{pmatrix} \tag{3.21}$$

with inverse

$$T^{-1}(t,\xi) = \begin{pmatrix} \frac{\lambda(t)\langle\xi\rangle_{b(t)}}{h(t,\xi)} & 0 \\ -i\frac{b(t)\lambda(t)}{2h(t,\xi)} & \lambda(t) \end{pmatrix}. \tag{3.22}$$

Recall that outside of the dissipative zone we have $h(t,\xi) = a(t)|\xi|$ and, especially, in the dissipative zone we use $h(t,\xi) = \dfrac{a(t)}{A(t)}$. Inside the reduced zone we have estimated $\langle\xi\rangle_{b(t)}$ by $\varepsilon\frac{b(t)}{2}$. Therefore, we change the definition of the matrix $T(t,\xi)$ by

$$\begin{pmatrix} \frac{2h(t,\xi)}{\varepsilon\lambda(t)b(t)} & 0 \\ i\frac{1}{\lambda(t)} & \frac{1}{\lambda(t)} \end{pmatrix}, \qquad |T(t,\xi)| \sim \lambda^{-1}(t) \tag{3.23}$$

for all $(t,\xi) \in Z_{\mathrm{red}}(\varepsilon)$.

Remark 3.12. We may conclude that in the hyperbolic and reduced zones the fundamental solution to our original Cauchy problem in the extended phase space can be estimated by

$$(|E(t,s,\xi)|) \lesssim \frac{\lambda(s)}{\lambda(t)}(|E_V(t,s,\xi)|).$$

Some auxiliary estimates. We continue with some auxiliary estimates which are used to obtain energy estimates later on.

Lemma 3.13. *Let us suppose* (B$'$1) *to* (B$'$3) *and let* $\lambda(t) = \exp\left(\frac{1}{2}\int_0^t b(\tau)d\tau\right)$. *Then the following holds:*

1. *The definition of* $\langle\xi\rangle_{b(t)}$ *in the elliptic zone implies* $\langle\xi\rangle_{b(t)} - \dfrac{b(t)}{2} \leq -\dfrac{a^2(t)|\xi|^2}{b(t)}$.
2. $\dfrac{\lambda(s)}{\lambda(t)}\exp\left(\int_s^t \langle\xi\rangle_{b(\tau)}d\tau\right) \lesssim \exp\left(-|\xi|^2\int\limits_s^t \dfrac{a^2(\tau)}{b(\tau)}d\tau\right)$.
3. *With* $A(t_{\xi_1})|\xi| \sim 1$ (*separating line between dissipative and elliptic zone*) *it holds*
$$\exp\left(-|\xi|^2\int_0^{t_{\xi_1}}\frac{a^2(\tau)}{b(\tau)}d\tau\right) \sim 1.$$

Proof. The first statement is equivalent to the following inequality

$$\sqrt{\frac{b^2(t)}{4} - a^2(t)|\xi|^2} - \frac{b(t)}{2} \leq -\frac{a^2(t)|\xi|^2}{b(t)}.$$

The second statement follows directly from the first one together with the definition of $\lambda(t)$. The third statement can be directly obtained from the following estimate:

$$\begin{aligned}|\xi|^2\int_0^{t_{\xi_1}}\frac{a^2(\tau)}{b(\tau)}d\tau &= |\xi|^2\int_0^{t_{\xi_1}}\frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau \\ &\leq |\xi|^2\int_0^{t_{\xi_1}}\frac{a(\tau)A(\tau)}{\mu_0}d\tau \lesssim |\xi|^2A^2(t_{\xi_1}) \lesssim 1.\end{aligned}$$

The proof is complete. □

A refined estimate for the fundamental solution in the elliptic zone. Inside the elliptic zone we have

$$|E_V(t,s,\xi)| \lesssim \frac{b(t)}{b(s)}\exp\left(\int_s^t \langle\xi\rangle_{b(\tau)}d\tau\right).$$

This yields in combination with (3.20) the estimate

$$\begin{aligned}(|E(t,s,\xi)|) &\lesssim \begin{pmatrix}\frac{a(t)|\xi|}{b(t)} & 0\\ & b(t)\end{pmatrix}\exp\left(\int_s^t\left(\langle\xi\rangle_{b(\tau)} - \frac{b(\tau)}{2}\right)d\tau\right)\begin{pmatrix}\frac{1}{a(s)|\xi|} & 0\\ \frac{1}{a(s)|\xi|} & \frac{1}{b(s)}\end{pmatrix}\\ &\lesssim \exp\left(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\right)\begin{pmatrix}\frac{a(t)}{a(s)} & \frac{a(t)|\xi|}{b(s)}\\ \frac{b(t)}{a(s)|\xi|} & \frac{b(t)}{b(s)}\end{pmatrix}.\end{aligned} \tag{3.24}$$

Here we have used the first statement from Lemma 3.13. The estimate for the first row seems to be optimal while the estimate for the second row is not optimal, because at least for an increasing coefficient function $b(t)$ for fixed ξ some entries of the matrix from the right-hand side of (3.24) become unbounded for increasing t. This estimate contradicts to our a priori knowledge that the wave type energy itself is decreasing. For this reason we need a refined estimate which will be presented in the following steps. Let us assume that $\Phi_k(t,s,\xi)$, $k=1,2$, are solutions to the equation

$$\Phi_{tt} + a^2(t)|\xi|^2\Phi + b(t)\Phi_t = 0$$

with initial values

$$\Phi_k(s,s,\xi) = \delta_{1k}, \quad \partial_t\Phi_k(s,s,\xi) = \delta_{2k}.$$

Then we have

$$\begin{pmatrix} a(t)|\xi|v(t,\xi) \\ D_t v(t,\xi) \end{pmatrix} = \begin{pmatrix} \dfrac{a(t)}{a(s)}\Phi_1(t,s,\xi) & ia(t)|\xi|\Phi_2(t,s,\xi) \\ \dfrac{D_t\Phi_1(t,s,\xi)}{a(s)|\xi|} & iD_t\Phi_2(t,s,\xi) \end{pmatrix} \begin{pmatrix} a(s)|\xi|v(s,\xi) \\ D_t v(s,\xi) \end{pmatrix}.$$

Hence, if we compare with the estimate (3.24), then

$$|\Phi_1(t,s,\xi)| \lesssim \exp\left(-|\xi|^2\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\right), \tag{3.25}$$

$$|\Phi_2(t,s,\xi)| \lesssim \frac{1}{b(s)}\exp\left(-|\xi|^2\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\right), \tag{3.26}$$

$$|\partial_t\Phi_1(t,s,\xi)| \lesssim b(t)\exp\left(-|\xi|^2\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\right), \tag{3.27}$$

$$|\partial_t\Phi_1(t,s,\xi)| \lesssim \frac{b(t)}{b(s)}\exp\left(-|\xi|^2\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\right). \tag{3.28}$$

Let $\Psi_k(t,s,\xi) = \partial_t\Phi_k(t,s,\xi)$, $k=1,2$. Then we obtain the equations of first order

$$\partial_t\Psi_k + b(t)\Psi_k = -a^2(t)|\xi|^2\Phi_k(t,s,\xi), \qquad \Psi_k(s,s,\xi) = \delta_{k2}.$$

Applying Duhamel's principle we get

$$\begin{aligned}
\Psi_1(t,s,\xi) &= -|\xi|^2\int_s^t a^2(\tau)\frac{\lambda^2(\tau)}{\lambda^2(t)}\Phi_1(\tau,s,\xi)d\tau, \\
|\Psi_1(t,s,\xi)| &\lesssim \frac{|\xi|^2}{\lambda^2(t)}\int_s^t a^2(\tau)\lambda^2(\tau)\exp\left(-|\xi|^2\int_s^\tau \frac{a^2(\theta)}{b(\theta)}d\theta\right)d\tau \\
&\lesssim \frac{a^2(t)|\xi|^2}{\lambda^2(t)}\int_s^t \lambda^2(\tau)\exp\left(-|\xi|^2\int_s^\tau \frac{a^2(\theta)}{b(\theta)}d\theta\right)d\tau \\
&\lesssim \frac{a^2(t)|\xi|^2}{\lambda^2(t)}\int_s^t \underbrace{b(\tau)\lambda^2(\tau)}_{\partial_\tau\lambda^2(\tau)}\frac{1}{b(\tau)}\exp\left(-|\xi|^2\int_s^\tau \frac{a^2(\theta)}{b(\theta)}d\theta\right)d\tau
\end{aligned}$$

$$\lesssim \frac{a^2(t)|\xi|^2}{\lambda^2(t)}\Big(\lambda^2(\tau)\frac{1}{b(\tau)}\exp\Big(-|\xi|^2\int_s^\tau \frac{a^2(\theta)}{b(\theta)}d\theta\Big)\Big)\Big|_s^t$$
$$+\frac{a^2(t)|\xi|^2}{\lambda^2(t)}\int_s^t \lambda^2(\tau)\underbrace{\Big(\frac{|\xi|^2a^2(\tau)}{b(\tau)}+\frac{b'(\tau)}{b^2(\tau)}\Big)}_{\lesssim C(\tau)<1}\exp\Big(-|\xi|^2\int_s^\tau \frac{a^2(\theta)}{b(\theta)}d\theta\Big)d\tau$$
$$\lesssim \frac{a^2(t)|\xi|^2}{b(t)}\exp\Big(-|\xi|^2\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)-\frac{a^2(t)|\xi|^2}{b(s)}\frac{\lambda^2(s)}{\lambda^2(t)}.$$

Here we have used $a^2(t)|\xi|^2/b^2(t)\le 1/2$ from the definition of the elliptic zone and $\frac{b'(t)}{b^2(t)}=o(1)$. We see that the second summand is subordinate to the first one because

$$\frac{b(s)}{b(t)}\exp\Big(-|\xi|^2\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)\frac{\lambda^2(t)}{\lambda^2(s)}=\exp\Big(\int_s^t\underbrace{\Big(b(\tau)-\frac{a^2(\tau)|\xi|^2}{b(\tau)}-\frac{b'(\tau)}{b(\tau)}\Big)}_{>0,\ \text{if}\,\tau\ge t_0}d\tau\Big)$$

for $t_0\le s\le t$ with t_0 sufficiently large. Similarly, we can represent Ψ_2 in the following way:

$$\Psi_2(t,s,\xi)=\frac{\lambda^2(s)}{\lambda^2(t)}-|\xi|^2\int_s^t a^2(\tau)\frac{\lambda^2(\tau)}{\lambda^2(t)}\Phi_2(\tau,s,\xi)d\tau,$$
$$|\Psi_2(t,s,\xi)|\lesssim\frac{\lambda^2(s)}{\lambda^2(t)}+\frac{|\xi|^2}{\lambda^2(t)}\int_s^t a^2(\tau)\lambda^2(\tau)\frac{1}{b(s)}\exp\Big(-|\xi|^2\int_s^\tau \frac{a^2(\theta)}{b(\theta)}d\theta\Big)d\tau$$
$$\lesssim\frac{\lambda^2(s)}{\lambda^2(t)}+\frac{a^2(t)|\xi|^2}{b(t)b(s)}\exp\Big(-|\xi|^2\int_s^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big).$$

Thus, we have proved the following lemma:

Lemma 3.14. *Let $(s,\xi),(t,\xi)\in Z_{\mathrm{ell}}(c_0,\varepsilon)$ with $s\le t$. Then the fundamental solution $E(t,s,\xi)$ can be estimated in the following way:*

$$(|E(t,s,\xi)|)\lesssim\exp\Big(-|\xi|^2\int_s^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big)\begin{pmatrix}\frac{a(t)}{a(s)} & \frac{a(t)|\xi|}{b(s)}\\ \frac{a^2(t)|\xi|}{a(s)b(t)} & \frac{a^2(t)|\xi|^2}{b(s)b(t)}\end{pmatrix}+\frac{\lambda^2(s)}{\lambda^2(t)}\begin{pmatrix}0&0\\0&1\end{pmatrix}. \tag{3.29}$$

Remark 3.15. Let us choose a fixed s. Then the second summand in (3.29) is dominated by the first one. If we set $s=t_{\xi_2}$, then in the two cases $\Big(\frac{\mu(t)}{A(t)}$ is increasing or decreasing$\Big)$ we can use $a(t_{\xi_2})|\xi|\sim b(t_{\xi_2})$ to get the following estimate:

$$(|E(t,t_{\xi_2},\xi)|)\lesssim\exp\Big(-|\xi|^2\int_{t_{\xi_2}}^t\frac{a^2(\tau)}{b(\tau)}d\tau\Big)\begin{pmatrix}\frac{a(t)}{a(t_{\xi_2})} & \frac{a(t)}{a(t_{\xi_2})}\\ \frac{a^2(t)|\xi|}{a(t_{\xi_2})b(t)} & \frac{a^2(t)|\xi|}{a(t_{\xi_2})b(t)}\end{pmatrix}. \tag{3.30}$$

3.6. Gluing procedure

Case 1: *The function* $\dfrac{\mu(t)}{A(t)}$ *is monotonously decreasing.*

In the previous sections we have considered the fundamental solution in different zones. Now we have to glue the estimates from Lemmas 2.6, 3.14, 3.10 and 3.5. We obtain for the part of the hyperbolic zone which contains large frequencies $\{\xi : |\xi| \geq c > 0\}$ the following estimate for the fundamental solution:

$$(|E(t,0,\xi)|) \lesssim \sqrt{a(t)} \exp\Big(-\frac{1}{2}\int_0^t b(\tau)d\tau\Big)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

to our original problem in the extended phase space, cf. Lemma 3.5 and Remark 3.12. It remains to consider the influence of the dissipative zone, the elliptic zone, the reduced zone and the hyperbolic zone for small frequencies. We denote by $t_{\xi_k}, k = 1,2,3$, the separating lines between the dissipative zone and the elliptic zone ($k=1$), between the elliptic zone and the reduced zone ($k=2$) and between the reduced zone and the hyperbolic zone ($k=3$).

Case 1.1: $t \leq t_{\xi_1}$.

In this case we follow directly Lemma 2.6.

Case 1.2: $t_{\xi_1} \leq t \leq t_{\xi_2}$.

Now we have to glue the estimates from Lemmas 2.6 and 3.10. We have the following corollary:

Corollary 3.16. *The following estimates hold for all* $t \in [t_{\xi_1}, t_{\xi_2}]$:

$$\begin{aligned}(|E(t,0,\xi)|) \lesssim{}& \exp\Big(-|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)\begin{pmatrix} a(t)|\xi| & a(t)|\xi| \\ \frac{a^2(t)|\xi|^2}{b(t)} & \frac{a^2(t)|\xi|^2}{b(t)} \end{pmatrix}\\ &+ \exp\Big(-\int_{t_{\xi_1}}^t b(\tau)d\tau\Big)\, a(t_{\xi_1})|\xi| \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.\end{aligned}$$

Proof. The fundamental solution $E(t,0,\xi)$ can be represented as

$$E(t,t_{\xi_1},\xi)E(t_{\xi_1},0,\xi).$$

This yields for all $(t,\xi) \in Z_{\text{ell}}(c_0,\varepsilon)$

$$\begin{aligned}(|E(t,0,\xi)|) &\lesssim (|E(t,t_{\xi_1},\xi)|)(|E(t_{\xi_1},0,\xi)|)\\ &\lesssim \Bigg(\exp\Big(-|\xi|^2\int_{t_{\xi_1}}^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)\begin{pmatrix} \frac{a(t)}{a(t_{\xi_1})} & \frac{a(t)|\xi|}{b(t_{\xi_1})} \\ \frac{a^2(t)|\xi|}{a(t_{\xi_1})b(t)} & \frac{a^2(t)|\xi|^2}{b(t_{\xi_1})b(t)} \end{pmatrix}\\ &\qquad + \frac{\lambda^2(t_{\xi_1})}{\lambda^2(t)}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\Bigg) \times \frac{a(t_{\xi_1})}{A(t_{\xi_1})}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\end{aligned}$$

$$\lesssim \exp\Big(-|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)\begin{pmatrix} a(t)|\xi| & a(t)|\xi| \\ \frac{a^2(t)|\xi|^2}{b(t)} & \frac{a^2(t)|\xi|^2}{b(t)} \end{pmatrix}$$
$$+\exp\Big(-\int_{t_{\xi_1}}^t b(\tau)d\tau\Big)a(t_{\xi_1})|\xi|\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}.$$

Here we used $a(t_{\xi_1})|\xi| \lesssim b(t_{\xi_1})$, $|\xi| \sim \dfrac{c_0}{A(t_{\xi_1})}$ together with the third statement from Lemma 3.13 to extend the above integral. This completes the proof. □

Case 1.3: $t_{\xi_2} \le t \le t_{\xi_3}$.

Now we will glue the estimates from Lemma 3.10 and Corollary 3.16.

Corollary 3.17. *The following estimate holds for all $t \in [t_{\xi_2}, t_{\xi_3}]$:*

$$(|E(t,0,\xi)|) \lesssim \exp\Big(-|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\Big)a(t)|\xi|\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Proof. From Lemma 3.10 and Remark 3.12 we have the following estimate:

$$(|E(t,t_{\xi_2},\xi)|) \lesssim \frac{\lambda(t_{\xi_2})}{\lambda(t)}\exp\Big(\varepsilon\int_{t_{\xi_2}}^t b(\tau)d\tau\Big)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.$$

Taking account of the representation of the fundamental solution $E(t,0,\xi)$ as $E(t,t_{\xi_2},\xi)E(t_{\xi_2},0,\xi)$ gives after application of Corollary 3.16 the following estimate:

$$\begin{aligned}
&(|E(t,0,\xi)|)\lesssim(|E(t,t_{\xi_2},\xi)|)(|E(t_{\xi_2},0,\xi)|)\\
&\lesssim\exp\Big(\Big(\varepsilon-\frac{1}{2}\Big)\int_{t_{\xi_2}}^t b(\tau)d\tau\Big)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\Big(\exp\Big(-\int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)d\tau\Big)a(t_{\xi_1})|\xi|\begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\\
&\quad+\exp\Big(-|\xi|^2\int_0^{t_{\xi_2}}\frac{a^2(\tau)}{b(\tau)}d\tau\Big)\begin{pmatrix} a(t_{\xi_2})|\xi| & a(t_{\xi_2})|\xi| \\ \frac{a^2(t_{\xi_2})|\xi|^2}{b(t_{\xi_2})} & \frac{a^2(t_{\xi_2})|\xi|^2}{b(t_{\xi_2})} \end{pmatrix}\Big)\\
&\lesssim\Big(\exp\Big(\Big(\varepsilon-\frac{1}{2}\Big)\int_{t_{\xi_2}}^t b(\tau)d\tau\Big)\exp\Big(-|\xi|^2\int_0^{t_{\xi_2}}\frac{a^2(\tau)}{b(\tau)}d\tau\Big)\Big(a(t_{\xi_2})|\xi|+\frac{a^2(t_{\xi_2})|\xi|^2}{b(t_{\xi_2})}\Big)\\
&\quad+\exp\Big(\Big(\varepsilon-\frac{1}{2}\Big)\int_{t_{\xi_2}}^t b(\tau)d\tau\Big)\exp\Big(-\int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)d\tau\Big)a(t_{\xi_1})|\xi|\Big)\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}.
\end{aligned}$$

From the definition of $Z_{\mathrm{red}}(\varepsilon)$ with a sufficiently small ε we have

$$a^2(t)|\xi|^2 \le \Big(\frac{1}{2}-\varepsilon\Big)b^2(t).$$

For $t \le t_{\xi_2}$ we use

$$a(t)|\xi| \lesssim b(t).$$

Hence, the integral

$$\exp\Big(-\int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)d\tau\Big)$$

can be estimated by

$$\exp\Big(-|\xi|^2\int_{t_{\xi_1}}^{t_{\xi_2}} \frac{a^2(\tau)}{b(\tau)}d\tau\Big),$$

and the last integral can be extended up to $t=0$. Using $t \geq t_{\xi_2}$ and the increasing behavior of a we conclude from the last estimates the desired statement. $\square$

Case 1.4: $t_{\xi_3} \leq t < \infty$.

From Lemma 3.5 and Remark 3.12 we obtain the following statement:

Corollary 3.18. *The following estimate holds for all* $t \in [t_{\xi_3}, \infty)$:

$$(|E(t,t_{\xi_3},\xi)|) \lesssim \frac{\sqrt{a(t)}}{\sqrt{a(t_{\xi_3})}} \exp\left(-\frac{1}{2}\int_{t_{\xi_3}}^{t} b(\tau)d\tau\right)\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}.$$

Finally, we have to glue the estimates from Corollaries 3.17 and 3.18.

Corollary 3.19. *The following estimate holds for all* $t \in [t_{\xi_3}, \infty)$:

$$\begin{aligned}(|E(t,0,\xi)|) \lesssim \exp\left(-|\xi|^2\int_0^{t_{\xi_3}} \frac{a^2(\tau)}{b(\tau)}d\tau\right)\exp\left(-\frac{1}{2}\int_{t_{\xi_3}}^{t} b(\tau)d\tau\right) \\ \times \sqrt{a(t)}\sqrt{a(t_{\xi_3})}|\xi|\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}.\end{aligned}$$

Case 2: *The function* $\dfrac{\mu(t)}{A(t)}$ *is monotonously increasing.*

The elliptic region lies in this case on the top of the hyperbolic region. For small frequencies the set $\{\xi : |\xi| \leq c_0\}$ lies completely inside the elliptic zone. For this reason we can use the estimates from the elliptic zone and obtain immediately

$$(|E(t,0,\xi)|) \lesssim \exp\left(-|\xi|^2\int_0^{t} \frac{a^2(\tau)}{b(\tau)}d\tau\right)\begin{pmatrix} \frac{a(t)}{a(0)} & \frac{a(t)}{a(0)} \\ \frac{a^2(t)|\xi|}{a(0)b(t)} & \frac{a^2(t)|\xi|}{a(0)b(t)}\end{pmatrix}. \tag{3.31}$$

It remains to consider the influence of the elliptic zone, the reduced zone and the hyperbolic zone for large frequencies. We denote by $t_{\xi_k}, k=1,2$, the separating lines between the hyperbolic zone and the reduced zone ($k=1$) and between the reduced zone and the elliptic zone ($k=2$).

Case 2.1: $t \leq t_{\xi_1}$.

In this case we follow directly Lemma 3.5 and Remark 3.12 to obtain

$$(|E(t,0,\xi)|) \lesssim \frac{\sqrt{a(t)}}{\sqrt{a(0)}} \exp\left(-\frac{1}{2}\int_0^{t} b(\tau)d\tau\right)\begin{pmatrix} 1 & 1 \\ 1 & 1\end{pmatrix}.$$

Case 2.2: $t_{\xi_1} \leq t \leq t_{\xi_2}$.

In this case we need to glue the estimates in the hyperbolic zone and in the reduced zone. We have

$$|E(t,0,\xi)| \lesssim \exp\left(\left(\varepsilon - \frac{1}{2}\right)\int_{t_{\xi_1}}^t b(\tau)d\tau - \frac{1}{2}\int_0^{t_{\xi_1}} b(\tau)d\tau\right)\frac{\sqrt{a(t_{\xi_1})}}{\sqrt{a(0)}}.$$

Case 2.3: $t_{\xi_2} \leq t$.

In this case we need to glue the estimates in the elliptic zone and in the reduced zone. Summarizing yields the following corollary:

Corollary 3.20. *The following estimate holds for all $t \in [t_{\xi_2}, \infty)$:*

$$(|E(t,0,\xi)|) \lesssim \exp\left(-|\xi|^2\int_{t_{\xi_2}}^t \frac{a^2(\tau)}{b(\tau)}d\tau + \left(\varepsilon - \frac{1}{2}\right)\int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)d\tau - \frac{1}{2}\int_0^{t_{\xi_1}} b(\tau)d\tau\right)$$
$$\times \frac{\sqrt{a(t_{\xi_1})}}{\sqrt{a(0)}}\begin{pmatrix} \frac{a(t)}{a(t_{\xi_2})} & \frac{a(t)}{a(t_{\xi_2})} \\ \frac{a^2(t)|\xi|}{a(t_{\xi_2})b(t)} & \frac{a^2(t)|\xi|}{a(t_{\xi_2})b(t)} \end{pmatrix}.$$

3.6.1. L^2-L^2 estimates – end of the proof.

Case 1: *The function $\dfrac{\mu(t)}{A(t)}$ is monotonously decreasing.*

In the case $t \in [0, t_{\xi_1}]$ we have from Lemma 2.6 the estimate

$$|E(t,0,\xi)| \lesssim \frac{a(t)}{A(t)} \lesssim \frac{a(t)}{\sqrt{1 + \int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau}}.$$

This follows directly from

$$\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau = \int_0^t \frac{a(\tau)A(\tau)}{\mu(\tau)}d\tau \lesssim \int_0^t a(\tau)A(\tau)d\tau \lesssim A^2(t)$$

for large t.

In the case $t \in [t_{\xi_1}, t_{\xi_2}]$ we will estimate separately each row in the estimate from Corollary 3.16. Let us consider the first row. It holds

$$a(t)|\xi|\exp\left(-|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right) \lesssim a(t)\left(1 + \int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right)^{-\frac{1}{2}},$$

therefore, we get the desired decay estimate. Using the monotonicity of a for the second row we can estimate by the first one

$$\frac{a^2(t)}{b(t)}|\xi|^2 = a(t)|\xi|\frac{a(t)|\xi|}{b(t)} \lesssim a(t)|\xi|,$$
$$a(t_{\xi_1})\exp\left(-\int_0^t b(\tau)d\tau\right) \lesssim a(t)\exp\left(-|\xi|^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right).$$

In the case $t \in [t_{\xi_2}, t_{\xi_3}]$ by using Corollary 3.17 we can estimate like in the case $t \in [t_{\xi_1}, t_{\xi_2}]$.

To derive the corresponding estimates from Corollary 3.19 we have in the case $t \in [t_{\xi_3}, \infty)$ to estimate the term

$$S(t, |\xi|) := |\xi| \exp\left(-|\xi|^2 \int_0^{t_{\xi_3}} \frac{a^2(\tau)}{b(\tau)} d\tau\right) \exp\left(-\frac{1}{2} \int_{t_{\xi_3}}^t b(\tau) d\tau\right).$$

Lemma 3.21. *The maximum of the function $S(t, |\xi|)$ is taken at a point $|\xi|$ independent of $t \geq t_{\xi_3}$ and*

$$S(t, |\xi|) \leq \max_{\xi \in \mathbb{R}^n} \left\{ |\xi| \exp\left(-|\xi|^2 \int_0^t \frac{a^2(\tau)}{b(\tau)} d\tau\right) \right\}.$$

Proof. To estimate the function $S(t, |\xi|)$ it is important that we will prove that the first partial derivative $\partial_{|\xi|} S(t, |\xi|)$ is negative for $|\xi|$ small. This follows from

$$\begin{aligned}
&\partial_{|\xi|} S(t, |\xi|) \\
&\quad = S(t, |\xi|) \left(\frac{1}{|\xi|} - 2|\xi| \int_0^{t_{\xi_3}} \frac{a^2(\tau)}{b(\tau)} d\tau - \frac{a^2(t_{\xi_3})|\xi|^2}{b(t_{\xi_3})} d_{|\xi|} t_{\xi_3} + \frac{b(t_{\xi_3})}{2} d_{|\xi|} t_{\xi_3} \right) \\
&\quad < S(t, |\xi|) \left(\frac{1}{|\xi|} + \Big(\frac{b(t_{\xi_3})}{2} - \frac{a^2(t_{\xi_3})|\xi|^2}{b(t_{\xi_3})} \Big) d_{|\xi|} t_{\xi_3} \right) \\
&\quad < S(t, |\xi|) \left(\frac{1}{|\xi|} + \frac{(1-\varepsilon^2) b(t_{\xi_3})}{4} d_{|\xi|} t_{\xi_3} \right).
\end{aligned}$$

Here we have used

$$\frac{a^2(t_{\xi_3})|\xi|^2}{b(t_{\xi_3})} = \frac{(1+\varepsilon^2) b(t_{\xi_3})}{4}.$$

Hence, a sufficiently small ε guarantees $\frac{(1-\varepsilon^2)b(t_{\xi_3})}{4} > 0$. Taking account of $d_{|\xi|} t_{\xi_3} < 0$, $|d_{|\xi|} t_{\xi_3}| \geq \frac{\mu(t_{\xi_3})}{|\xi| b(t_{\xi_3})}$ and $\mu(t_{\xi_3}) \to \infty$ for $|\xi| \to 0$ we have the desired decreasing behavior of the function $S(t, |\xi|)$ in $|\xi|$. Now let us fix $t > 0$. Then the function $S(t, |\xi|)$ takes its maximum for $|\tilde{\xi}|$ satisfying $t = t_{\tilde{\xi}_3}$, that is, the second integral vanishes in $S(t, |\xi|)$. This completes the proof. $\square$

Corollary 3.19 and Lemma 3.21 yield the following result:

$$|E(t, 0, \xi)| \lesssim a(t) \left(1 + \int_0^t \frac{a^2(\tau)}{b(\tau)} d\tau\right)^{-\frac{1}{2}} \text{ for } t \in [t_{\xi_3}, \infty).$$

In this way all statements are proved.

Case 2: *The function* $\dfrac{\mu(t)}{A(t)}$ *is monotonously increasing.*

For small frequencies $\{\xi : |\xi| \leq c_0\}$ we can apply the estimate in (3.31). Here we use that $\dfrac{A(t)}{\mu(t)}$ is monotonously decreasing. For large frequencies $\{\xi : |\xi| \geq c_0\}$

we consider the estimates from Corollary 3.20, that is, we have

$$\exp\left(-|\xi|^2\int_{t_{\xi_2}}^t \frac{a^2(\tau)}{b(\tau)}d\tau + \Big(\varepsilon-\frac{1}{2}\Big)\int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)d\tau - \frac{1}{2}\int_0^{t_{\xi_1}} b(\tau)d\tau\right)$$
$$\lesssim \exp\left(-c_0^2\int_0^t \frac{a^2(\tau)}{b(\tau)}d\tau\right).$$

Here we use for ε sufficiently small the inequality

$$\Big(\frac{1}{2}-\varepsilon\Big)\int_{t_{\xi_1}}^{t_{\xi_2}} b(\tau)d\tau \geq |\xi|^2\int_{t_{\xi_1}}^{t_{\xi_2}} \frac{a^2(\tau)}{b(\tau)}d\tau.$$

Moreover, the following estimate holds for $c_0 < \dfrac{1}{\sqrt{2}}\dfrac{\mu(0)}{A(0)}$:

$$\frac{b(t)}{2} \geq c_0^2\frac{a^2(t)}{b(t)} \quad \text{iff} \quad \frac{b^2(t)}{2} \geq c_0^2 a^2(t) \quad \text{iff} \quad \frac{1}{2}\frac{\mu^2(t)}{A^2(t)} \geq c_0^2.$$

We can see that the first row in the estimate from Corollary 3.20 has its maximum for large t inside of $\{\xi : |\xi| \leq c_0\}$. From that the theorem is completely proved.

Remark 3.22. If we choose in Theorem 3.1 the coefficient $a(t) \equiv 1$, then the obtained estimates coincide with the estimates from Result 1.2 for $p = q = 2$.

Examples. We will give some examples for special coefficients.

Example. Let $a(t) = (1+t)^l$, $b(t) = (1+t)^k$, $k \in (-1, 2l+1]$ with $l > 0$. Then we have

$$\big\|\big((1+t)^l\nabla u(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^2} \lesssim (1+t)^{\frac{k-1}{2}}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big).$$

Example. Let $a(t) = e^t$, $b(t) = e^{\beta t}$, $\beta \in (0, 2]$. Then we have

$$\big\|\big(e^t\nabla u(t,\cdot), u_t(t,\cdot)\big)\big\|_{L^2} \lesssim e^{\frac{\beta}{2}t}\big(\|u_1\|_{H^1} + \|u_2\|_{L^2}\big).$$

3.7. Comparison with known results

In [1] some results for scale-invariant models are proved by applying direct calculations and the theory of special functions. Now let us compare these results with the main results Theorem 2.1 for non-effective dissipations from Section 2 and Theorem 3.1 for effective dissipations from Section 3 to see that our estimates are optimal.

3.7.1. Speed of potential order. We start with the Cauchy problem

$$u_{tt} - (1+t)^{2l}\Delta u + \frac{\mu(l+1)}{(1+t)}u_t = 0, \; u(0,x) = u_1(x), \; u_t(0,x) = u_2(x) \tag{3.32}$$

with $l > 0$. Then the following result can be found in [1]:

Proposition 3.23. *We have the following estimate for the energies of solutions to* (3.32)*:*

$$\|u_t(t,\cdot),(1+t)^l\nabla u(t,\cdot)\|_{L^2}\lesssim(1+t)^{l+(l+1)\max\{\rho-\frac{1}{2},-1\}}\left(\|u_1\|_{H^1}+\|u_2\|_{L^2}\right) \quad (3.33)$$

with $\rho=\frac{1}{2}(1-\mu-\frac{l}{l+1})$.

Case 1: *Non-effective dissipation* $\left(\max\{\rho-\frac{1}{2},-1\}=\rho-\frac{1}{2}\right)$.

With $\mu\neq 1$ we can see that $a(t)=(1+t)^{1+l}$, $b(t)=\dfrac{\mu(l+1)}{(1+t)}$ satisfy all assumptions from Theorem 2.1. Otherwise, from the definition of ρ and the condition $\left(\max\{\rho-\frac{1}{2},-1\}=\rho-\frac{1}{2}\right)$ we obtain $\mu+\frac{l}{l+1}<2$, i.e., this condition satisfies the condition (C): $\limsup_{t\to\infty}\mu(t)+\alpha(t)<2$.

Applying Theorem 2.1 in the case of non-effective dissipation the asymptotic profile for the "kinetic energy" $\|u_t(t,\cdot)\|_{L^2}$ and for the "elastic energy" $\|(1+t)^l\nabla u(t,\cdot)\|_{L^2}$ is determined by

$$\frac{\sqrt{a(t)}}{\lambda(t)}=\frac{(1+t)^{\frac{l}{2}}}{e^{\frac{1}{2}\int_0^t\frac{\mu(l+1)}{1+s}ds}}=(1+t)^{\frac{l}{2}-\frac{\mu(l+1)}{2}}.$$

This profile coincides with the profiles from the estimates in Proposition 3.23.

Case 2: *Effective dissipation* $\left(\max\{\rho-\frac{1}{2},-1\}=-1\right)$.

From the definition of ρ we can see that the above condition implies $\mu+\frac{l}{l+1}\geq 2$. Applying Theorem 3.1 for the case of effective dissipation the asymptotic profile of the "kinetic energy" $\|u_t(t,\cdot)\|_{L^2}$ and for the "elastic energy" $\|(1+t)^l\nabla u(t,\cdot)\|_{L^2}$ is determined by

$$a(t)\left(1+\int_0^t\frac{a^2(\tau)}{b(\tau)}d\tau\right)^{-\frac{1}{2}}=\frac{(1+t)^l}{\sqrt{1+\int_0^t\frac{(1+\tau)^{2l+1}}{\mu(l+1)}d\tau}}\sim\frac{1}{1+t}.$$

Due to assumption (B$'$3) it is not allowed to apply Theorem 3.1 directly to the Cauchy problem (3.32). But, if we formally do it for $\mu\geq 2-\frac{l}{l+1}$, then this profile coincides with the profiles from the estimates of Proposition 3.23. For the case $\mu=0$ some L^p-L^q estimates on the conjugate line are proposed in [6].

3.7.2. Speed of exponential order. Now we consider another model case to compare with the general results Theorem 2.1 for non-effective dissipations and Theorem 3.1 for effective dissipations. We devote the Cauchy problem

$$u_{tt}-e^{2t}\Delta u+\mu u_t=0,\ u(0,x)=u_1(x),\ u_t(0,x)=u_2(x). \quad (3.34)$$

Then the following result can be found in [1]:

Proposition 3.24. *We have the following estimates for the solutions to* (3.34)*:*

$$\|u_t(t,\cdot),(e^t\nabla u(t,\cdot)\|_{L^2}\lesssim e^{t+t\max\{\rho-\frac{1}{2},-1\}}\left(\|u_1\|_{H^1}+\|u_2\|_{L^2}\right) \quad (3.35)$$

with $\rho=-\frac{\mu}{2}$.

Case 1: *Non-effective dissipation* $\left(\max\{\rho - \frac{1}{2}, -1\} = \rho - \frac{1}{2}\right)$.

The assumptions from Theorem 2.1 are satisfied for $\mu \neq 1$. Keep in mind that $\rho - \frac{1}{2} > -1 \Leftrightarrow -\frac{\mu}{2} - \frac{1}{2} > -1$, this condition implies $\mu + 1 < 2$, i.e., it satisfies the condition $(C) : \limsup_{t\to\infty} \mu(t) + \alpha(t) < 2$.

Applying Theorem 2.1 in the case of non-effective dissipations the asymptotic profile for the "kinetic energy" $\|u_t(t,\cdot)\|_{L^2}$ and for the "elastic energy" $\|e^t \nabla u(t,\cdot)\|_{L^2}$ is determined by

$$\frac{\sqrt{a(t)}}{\lambda(t)} = \frac{e^{\frac{t}{2}}}{e^{\frac{1}{2}\int_0^t \mu ds}} = e^{\frac{t}{2} - \frac{\mu t}{2}}.$$

This profile coincides with the profiles from the estimates from Proposition 3.24.

Case 2: *Effective dissipation* $\left(\max\{\rho - \frac{1}{2}, -1\} = -1\right)$.

From the definition of ρ we can see that the above condition implies $\mu + 1 \geq 2$. Hence, this condition does not satisfy the condition (C). Applying Theorem 3.1 in the case of effective dissipations the asymptotic profile of the "kinetic energy" $\|u_t(t,\cdot)\|_{L^2}$ and of the "elastic energy" $\|e^t \nabla u(t,\cdot)\|_{L^2}$ is determined by

$$a(t)\left(1 + \int_0^t \frac{a^2(s)}{b(s)} ds\right)^{-\frac{1}{2}} = \frac{e^t}{\sqrt{1 + \int_0^t \frac{e^{2s}}{\mu} ds}} \sim C.$$

Due to assumption (B$'$3) it is not allowed to apply Theorem 3.1 to the Cauchy problem (3.35). But if we formally do it for $\mu \geq 1$, then this profile coincides with the profiles from the estimates of Proposition 3.24. For the case $\mu = 0$ some L^p-L^q estimates on the conjugate line are proposed in [4].

4. Scattering and over-damping

From the thesis of J. Wirth [11] we expect scattering and over-damping results, too.

4.1. Scattering

We will concern with conditions for $b = b(t)$ that the solutions $u = u(t,x)$ of

$$u_{tt} - a^2(t)\Delta u + b(t)u_t = 0,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x) \tag{4.1}$$

behave asymptotically equal to the solution of the corresponding wave equation with strictly increasing speed of propagation

$$v_{tt} - a^2(t)\Delta v = 0,\ v(0,x) = v_1(x),\ v_t(0,x) = v_2(x) \tag{4.2}$$

with some suitable Cauchy data (v_1, v_2).

Here we introduce the energy space $E(\mathbb{R}^n) = \dot{H}^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ and we assume $(u_1, u_2) \in E(\mathbb{R}^n)$, this means, $(|D|u_1, u_2) \in L^2(\mathbb{R}^n)$.

Result 4.1. *We assume that the coefficient $b = b(t)$ satisfies $b \in L^1(\mathbb{R}^+)$. Then there exists the Møller wave operator $W_+ : E \to E$ mapping the Cauchy data $(a(0)u_1, u_2) \in E$ from* (4.1) *to Cauchy data $(a(0)v_1, v_2)$ from* (4.2) *by*

$$(a(0)v_1, v_2)^T = W_+(a(0)u_1, u_2)^T$$

such that the asymptotic equivalence of solutions of the problems (4.1) *and* (4.2) *holds in the following way:*

$$\frac{1}{\sqrt{a(t)}} \|(a(t)u, D_t u) - (a(t)v, D_t v)\|_E \to 0 \tag{4.3}$$

while $t \to \infty$. Moreover, we have the decay estimate

$$\frac{1}{\sqrt{a(t)}} \|(a(t)u, D_t u) - (a(t)v, D_t v)\|_E \lesssim \|(u_1, u_2)\|_E \int_t^\infty b(\tau)d\tau \tag{4.4}$$

with the convergence rate $\int_t^\infty b(\tau)d\tau$ to 0 *as $t \to \infty$.*

4.2. Over-damping

We consider now "large" coefficients $b = b(t)$ in the damping term. For this reason we may assume

$$\text{(OD)} \qquad \int_0^\infty \frac{a^2(\tau)}{b(\tau)} d\tau < \infty.$$

Then the formal application of Theorem 3.1 implies among other things

$$\|\nabla u(t, \cdot)\|_{L^2} \leq C(\|u_1\|_{H^1} + \|u_2\|_{L^2}).$$

The following result shows that no more can be expected in this case of so-called over-damping.

Result 4.2. *Assume* (A1) *to* (A3), (B′1) *to* (B′3) *and* (OD). *Then for $(u_1, u_2) \in L^2(\mathbb{R}^n) \times H^{-1}(\mathbb{R}^n)$ the limit*

$$u(\infty, x) = \lim_{t\to\infty} u(t, x)$$

exists in $L^2(\mathbb{R}^n)$ and is different from zero for non-zero data. Furthermore, under the regularity assumption $(u_1, u_2) \in H^2(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ it holds

$$\|u(t, \cdot) - u(\infty, \cdot)\|_{L^2} = \mathcal{O}\left(\int_t^\infty \frac{a^2(\tau)}{b(\tau)} d\tau \right).$$

5. Concluding remarks

Remark 5.1. There are several papers which are devoted to the Cauchy problem for the following non-linear wave equations

$$u_{tt} - a(t)^2\Delta u = u_t^2 - a(t)^2|\nabla u|^2,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x). \quad (5.1)$$

In particular, in two papers [14] and [15] it is explained how the above class of special semi-linear Cauchy problems can be reduced by Nirenberg's transformation to a linear model with constrain condition. The above papers and the paper [2] concern with the problem of global existence (in time) for small data solutions to the semi-linear Cauchy problem

$$u_{tt} - a(t)^2\Delta u = u_t^2 - a(t)^2|\nabla u|^2,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x). \quad (5.2)$$

It would be a challenge to apply this approach to the case of *non-effective* dissipations to the following semi-linear Cauchy problem

$$u_{tt} - a(t)^2\Delta u + b(t)u_t = u_t^2 - a(t)^2|\nabla u|^2,\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x). \quad (5.3)$$

Remark 5.2. Another interesting application to the case of *effective* dissipations is the question for global small data solutions to the following semi-linear model

$$u_{tt} - a(t)^2\Delta u + b(t)u_t = f(u),\ u(0,x) = u_1(x),\ u_t(0,x) = u_2(x), \quad (5.4)$$

where $f(u) \approx |u|^p$. In a recent paper [3] the authors have constructed counter-examples which provide a nonexistence result for weak solutions to (5.4).

Both problems are attacked in forthcoming papers.

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