Chapter 1
Hamiltonian Representation of Magnetic Field

In this chapter we present the Hamiltonian formulation of the equations for magnetic field lines. We specifically consider the magnetic field corresponding the toroidal plasma configuration. Using the action principle of the classical mechanics the Hamiltonian equation for the magnetic field lines will be derived and its different forms will be presented. A particular emphasize will be given to the equations in flux coordinates, in which the field lines are the straight lines. We conclude the chapter introducing the simple model of the magnetic field and study the properties of its magnetic field lines using the methods of classical mechanics.

Let $\mathbf{B} = (B_X, B_Y, B_Z)$ is a static magnetic field in the three-dimensional (3D) space (X, Y, Z). It is divergent-free field satisfying the condition,

$$\nabla \cdot \mathbf{B} = 0. \tag{1.1}$$

The magnetic field lines , $\mathbf{r}(\tau) = (X(\tau), Y(\tau), Z(\tau))$, are three- dimensional curves tangent to the magnetic field $\mathbf{B}$ defined by the equation

$$\frac{d\mathbf{r}}{d\tau} = \mathbf{B}, \tag{1.2}$$

where τ is independent parameter related to the length element of curve $ds = (dX^2 + dY^2 + dZ^2)^{1/2}$: $d\tau = |\mathbf{B}|^{-1} ds$.

The magnetic field lines in the 3D-space may behave in a various way. Depending on the initial conditions they may be confined in a finite domain or may extend to infinity; they may lie on the surfaces or fully cover certain finite 3D domains. Notably that the magnetic confinement of high temperature plasmas for the thermonuclear fusion is based on the creation of the magnetic field whose field lines cover nested toroidal surfaces also known as *magnetic surfaces* which illustrated in Fig. 1.1.

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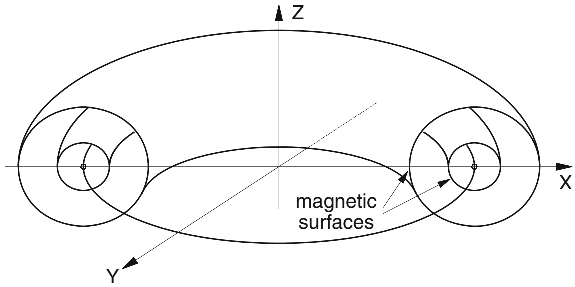

Fig. 1.1 Field lines lie on the nested toroidal surfaces $\psi(X, Y, Z) = \text{const}$

1.1 Hamiltonian Equations for Magnetic Field Lines

The most powerful mathematical tool to study the magnetic field lines is the methods of Hamiltonian dynamics which is the most convenient to describe the regular and chaotic field lines. Below, we formulate the Eq. (1.2) for magnetic field lines in a Hamiltonian form.

The Hamiltonian formulation is based on the presentation of the magnetic field by the vector potential $\mathbf{A}\,(\mathbf{r})$ related with $\mathbf{B}$ as $\mathbf{B} = \nabla \times \mathbf{A}$. The magnetic field is invariant with respect to the gauge transformation of the vector potential $\mathbf{A} \to \mathbf{A} + \nabla g$, where $g(\mathbf{r})$ is the arbitrary scalar (gauge) function.

The field lines can be derived from a variational principle , that is the action

$$S[\mathbf{r}(\tau)] = \int_{\mathbf{r_1}}^{\mathbf{r_2}} \mathbf{A}\,(\mathbf{r}(\tau)) \cdot \frac{d\mathbf{r}}{d\tau} d\tau, \tag{1.3}$$

defined as the integral over the vector potential $\mathbf{A}\,(\mathbf{r})$ along the field line $\mathbf{r}(\tau)$ connecting the endpoints $\mathbf{r_1}$ and $\mathbf{r_2}$ should be minimal [Cary and Littlejohn (1983)]. The action in the classical mechanics is given by

$$S[\mathbf{r}(\tau)] = \int_{\mathbf{r_1}}^{\mathbf{r_2}} (\mathbf{p} \cdot d\mathbf{r} - H dt)\,, \tag{1.4}$$

where $(\mathbf{r}, \mathbf{p})$ are canonical variables, H is the Hamiltonian function, and t is a time. The Hamiltonian equations derived from the variation of the action (1.4) are

$$\frac{d\mathbf{r}}{dt} = \frac{\partial H}{\partial \mathbf{p}}, \qquad \frac{d\mathbf{p}}{dt} = -\frac{\partial H}{\partial \mathbf{r}}. \tag{1.5}$$

Let (u, v, w) be the orthogonal coordinate system, and (A_u, A_v, A_w) be the corresponding components of the vector potential $\mathbf{A}$. By choosing the appropriate gauge function g that $A_w = -\partial g/\partial w$, one can remove the component A_w. Then introducing the momenta $P_u = A_u(U, V, W)$, $P_v = A_v(U, V, W)$, and the Hamiltonian function $H = H(U, V, P_u, P_v) = 0$ the action (1.3) can be presented as

$$S = \int_{\mathbf{r_1}}^{\mathbf{r_2}} (P_u dU + P_v dV - H d\tau) \, . \tag{1.6}$$

The corresponding field lines are described by the *autonomous* Hamiltonian equation,

$$\begin{aligned} \frac{dU}{d\tau} &= \frac{\partial H}{\partial P_u}, \qquad \frac{dV}{d\tau} = \frac{\partial H}{\partial P_v}, \\ \frac{dP_u}{d\tau} &= -\frac{\partial H}{\partial U}, \qquad \frac{dP_v}{d\tau} = -\frac{\partial H}{\partial V}. \end{aligned} \tag{1.7}$$

The Hamiltonian $H(U, V, P_u, P_v) = 0$ can be chosen as

$$H(U, V, P_u, P_v) = P_u - A_u \left(U, V, W(P_v, U, V)\right) = 0, \tag{1.8}$$

by expressing the variable W via (P_v, U, V) $[P_v = A_v(U, V, W)]$ or

$$H(U, V, P_u, P_v) = P_v - A_v \left(U, V, W(P_u, U, V)\right) = 0, \tag{1.9}$$

by W via (P_u, U, V) $[P_u = A_u(U, V, W)]$. The choice of the Hamiltonian depends on whether the variable U (or V) can be considered as a time-like independent variable which does not change its direction. Then field lines can be described by the *non-autonomous* Hamiltonian system. In the first case $U = \tau$ the Hamiltonian equations is given by

$$\frac{dV}{dU} = \frac{\partial H_u}{\partial P_v}, \quad \frac{dP_v}{dU} = -\frac{\partial H_u}{\partial V}, \tag{1.10}$$

where $H_u = -A_u \left(U, V, W(P_v, U, V)\right)$. The similar equation we have in the case $V = \tau$. The two-dimensional system (1.10) is known as *one and half degrees of freedom Hamiltonian system*. This term has a following meaning: the canonically conjugated variables (V, P_v) correspond to "one degree" and the time-like independent variable U corresponds to a half degree since the time runs only in one direction.

Specifically we consider the Hamiltonian equations for field lines in a toroidal system. Let (R, Z, φ) be cylindrical coordinate system , where R is directed along the major radius, Z is the vertical coordinate, and φ is the toroidal angle. They are also related to the quasitoroidal coordinates (r, θ, φ):

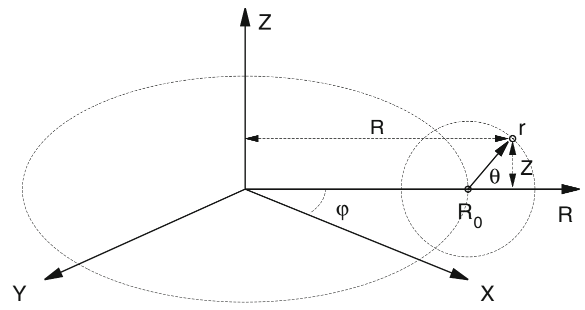

Fig. 1.2 Geometrical coordinates of a toroidal system: (r, θ, φ) is the quasitoroidal coordinates; (R, Z, φ) is the cylindrical coordinate system

$$R = R_0 + r\cos\theta, \qquad Z = r\sin\theta,$$
$$r = \sqrt{(R - R_0)^2 + Z^2}, \quad \theta = \arctan\frac{Z}{R - R_0},$$

shown in Fig. 1.2. Here R_0 is the major radius of the torus center. The ratio $\varepsilon = r/R_0$ is known an inverse aspect ratio (the ratio R/r is called an *aspect ratio*).

Let $\left(A_R, A_Z, A_\varphi\right)$ be the components of the vector potential $\mathbf{A}$ in the cylindrical coordinate system. By choosing $W \equiv R$ we have

$$P_z = A_Z(R, Z, \varphi), \qquad P_\varphi = RA_\varphi, \qquad u = \varphi, \qquad v = Z,$$
$$H = H(Z, \varphi, P_z, P_\varphi) = P_\varphi - R(Z, P_z)A_\varphi(R(Z, P_z), Z, \varphi) = 0. \quad (1.11)$$

The corresponding Hamiltonian equations are

$$\frac{dZ}{d\tau} = \frac{\partial H}{\partial P_z}, \qquad \frac{dP_z}{d\tau} = -\frac{\partial H}{\partial Z},$$
$$\frac{d\varphi}{d\tau} = \frac{\partial H}{\partial P_\varphi}, \qquad \frac{dP_\varphi}{d\tau} = -\frac{\partial H}{\partial \varphi}. \quad (1.12)$$

The tokamak plasmas the toroidal angle φ can be chosen as an independent time-like variable τ. Then field line equations have the following form

$$\frac{dZ}{d\varphi} = \frac{\partial H_\varphi}{\partial P_Z}, \qquad \frac{dP_Z}{d\varphi} = -\frac{\partial H_\varphi}{\partial Z}, \quad (1.13)$$

where $H_\varphi = -RA_\varphi$.

1.1.1 Equations in Normalized Variables

Introducing the normalized variables,

$$x = \frac{R}{R_0}, \quad z = \frac{Z}{R_0}, \qquad p_z = \frac{P_Z}{B_0 R_0},$$
$$\psi_\varphi = \frac{H_\varphi}{B_0 R_0^2} = -\frac{R A_\varphi}{B_0 R_0^2}, \quad \psi_z = \frac{A_z}{B_0 R_0}, \tag{1.14}$$

Equations (1.12) and (1.13) can be presented in the form

$$\frac{dz}{d\tau} = \frac{\partial h}{\partial p_z}, \qquad \frac{dp_z}{d\tau} = -\frac{\partial h}{\partial z},$$
$$\frac{d\varphi}{d\tau} = \frac{\partial h}{\partial P_\varphi}, \qquad \frac{dp_\varphi}{d\tau} = -\frac{\partial h}{\partial \varphi},$$
$$h = H/B_0 R_0^2, \tag{1.15}$$

and

$$\frac{dz}{d\varphi} = \frac{\partial \psi_\varphi}{\partial p_z}, \quad \frac{dp_z}{d\varphi} = -\frac{\partial \psi_\varphi}{\partial z}. \tag{1.16}$$

The relation between the canonical variables (z, p_z) and to the geometrical coordinates (R, Z, φ) is given by $p_z = A_Z(R, Z, \varphi)/B_0 R_0$. Particularly, when the effect of diamagnetic currents[1] to the toroidal component of magnetic field, B_φ, is negligible, and thus $B_\varphi = B_0 R_0/R$, $A_Z(R, Z, \varphi) = B_0 R_0 \ln(R/R_0)$ one has

$$p_z = \ln \frac{R}{R_0}, \qquad R = R_0 e^{p_z}, \tag{1.17}$$

where B_0 is the magnitude of B_φ at $R = R_0$.

The poloidal magnetic field components, B_R, B_z, and the toroidal field, B_φ, are determined by

$$B_R = -\frac{\partial A_\varphi}{\partial Z} = \frac{B_0 R_0}{R} \frac{\partial \psi_\varphi}{\partial z},$$
$$B_z = \frac{1}{R} \frac{\partial R A_\varphi}{\partial R} = -B_0 \frac{\partial \psi_\varphi}{\partial p_z},$$
$$B_\varphi = -\frac{\partial A_Z}{\partial R} = -B_0 \frac{\partial \psi_z}{\partial x}. \tag{1.18}$$

In the cylindrical coordinate system (R, Z, φ) the equations of field lines (1.2) can be presented as

[1] A diamagnetic current is created due to a circular motion of charged particles in an induced magnetic field. It produces a magnetic field which opposes the external magnetic field and thus the total magnetic field is reduced. Therefore a plasma possesses diamagnetic properties. Typically, the magnetic field due to a diamagnetic current is much smaller than the strong toroidal field B_0.

$$\frac{dR}{d\varphi} = \frac{RB_R}{B_\varphi}, \quad \frac{dZ}{d\varphi} = \frac{RB_z}{B_\varphi}. \tag{1.19}$$

1.2 Magnetic Flux Coordinates

The magnetic field can be conveniently presented in the so-called magnetic flux coordinates consisting of one flux coordinate, ψ, and two angle-like coordinates, ϑ, φ. In these coordinates a magnetic field in a equilibrium plasma lie on the closed surface $\Psi(\mathbf{r}) =$ constant, and field lines are given by a straight line, i.e., $\vartheta = \iota(\psi)\varphi + \vartheta_0$, where $\iota(\psi)$ is a function of the flux ψ. These coordinates can be introduced as action-angle variables of the Hamiltonian system (1.12) for magnetic field lines. Below we follow this approach.

1.2.1 Action-Angle Variables

The powerful method to study Hamiltonian system is based on the action-angle variables. The action-angle variables are set of canonical variables useful to study integrable systems. They are also useful in classical perturbation theory of Hamiltonian systems, particularly in the formulation of the Kolmogorov's theorem on conservation of the conditionally-periodic motion under small perturbations, known as the Kolmogorov–Arnold–Moser (KAM) theory (see Sect. 7.1). Below we give a brief description of these variables for a magnetic system.

Consider the equilibrium plasma with nested toroidal magnetic surfaces $\psi(\mathbf{r}) =$ constant as shown in Fig. 1.1. On the torus there are two basic irreducible closed contours C_θ and C_φ that wound torus along the poloidal angle ϑ (the short way around the torus) and the toroidal angle φ (the long way around the torus) as illustrated in Fig. 1.3. We consider these angles, (ϑ, φ), as canonical coordinates of Hamiltonian system. Let ψ_θ and ψ_φ be canonical momenta conjugated to the canonical coordinates (ϑ, φ), respectively, so that the Hamiltonian function H depends only on these variables, $H = H(\psi_\theta, \psi_\varphi)$. The Hamiltonian system (1.12) takes the form

$$\begin{aligned} \frac{d\vartheta}{d\tau} &= \frac{\partial H}{\partial \psi_\theta} = \omega_\theta, \quad \frac{d\psi_\theta}{d\tau} = -\frac{\partial H}{\partial \psi_\varphi} = 0, \\ \frac{d\varphi}{d\tau} &= \frac{\partial H}{\partial \psi_\varphi} = \omega_\varphi, \quad \frac{d\psi_\varphi}{d\tau} = -\frac{\partial H}{\partial \varphi} = 0, \end{aligned} \tag{1.20}$$

with the Hamiltonian function

$$H = H(\vartheta, \varphi, \psi_\theta, \psi_\varphi) = \psi_\varphi - RA_\varphi(R, Z, \varphi) = 0, \tag{1.21}$$

where the cylindrical coordinates (R, Z) are the functions of (ψ_θ, ϑ): $R = R(\psi_\theta, \vartheta)$ and $Z = Z(\psi_\theta, \vartheta)$. The quantities ω_θ, ω_φ are called frequencies of the system.

Formally, the action-angle variables are introduced by the canonical change of variables $(z, \varphi, p_z, p_\varphi) \to (\vartheta, \varphi, \psi_\vartheta, \psi_\varphi)$,

$$\begin{aligned} \psi_\theta &= \frac{1}{2\pi}\oint_{C_\vartheta} p_z(z; \psi_\theta)dz, & \psi_\varphi &= \frac{1}{2\pi}\oint_{C_\varphi} p_\varphi(z; \psi_\varphi)d\varphi, \\ \vartheta_\vartheta &= \frac{\partial F(\psi_\vartheta, \psi_\varphi; z, \varphi)}{\partial \psi_\vartheta} & \vartheta_\varphi &= \frac{\partial F(\psi_\vartheta, \psi_\varphi; z, \varphi)}{\partial \psi_\varphi}, \end{aligned} \tag{1.22}$$

given by the generating function

$$F(\psi_\vartheta, \psi_\varphi; z, \varphi) = \int^z p_z(z'; \psi_\theta)dz' + \int^\varphi p_\varphi(\varphi'; \psi_\varphi)d\varphi'. \tag{1.23}$$

In (1.22) the integration is taken along the closed contour C_θ formed by crossing the magnetic surface with the poloidal plane $\varphi = \text{const}$ (see Fig. 1.3). Using Eq. (1.14) and the calibration $A_R = 0$, one can easily see that ψ_θ equals to the normalized magnetic flux of the toroidal field B_φ through the surface area S_θ enclosed by the contour C_θ,

$$\begin{aligned} \psi_\theta &= \frac{1}{2\pi}\int\limits_{S_\theta} dp_z dz = \frac{1}{2\pi B_0 R_0^2}\int\limits_{S_\theta} \frac{\partial A_z}{\partial R} dR dZ \\ &= \int\limits_{S_\theta} \nabla \times \mathbf{A} \cdot \mathbf{e}_\varphi dR dZ = \frac{1}{2\pi B_0 R_0^2}\int\limits_{S_\theta} B_\varphi dR dZ. \end{aligned} \tag{1.24}$$

Similarly, one can show that ψ_φ is a normalized magnetic flux of the poloidal field B_z through area S_φ enclosed by the closed toroidal contour C_φ (see Fig. 1.3), i.e., $\psi_p = -RA_\varphi / B_0 R_0^2$,

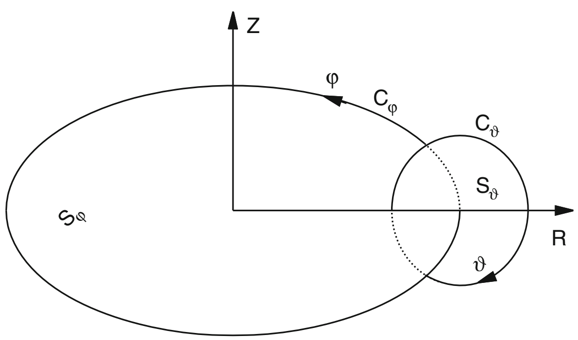

Fig. 1.3 Two basic irreducible closed contours C_θ and C_φ that wound torus along the poloidal angle ϑ (the short way around the torus) and the toroidal angle φ (the long way around the torus)

$$\psi_\varphi = -\frac{1}{2\pi}\oint_{C_\varphi}\frac{A_\varphi R d\varphi}{B_0 R_0^2} = \frac{1}{2\pi B_0 R_0^2}\int_{S_\varphi}\nabla\times\mathbf{A}\cdot\mathbf{e}_z R dR d\varphi$$
$$= -\frac{1}{2\pi B_0 R_0^2}\int_{S_\varphi} B_z dS. \tag{1.25}$$

The actions ψ_ϑ and ψ_φ are known as a *toroidal* and *poloidal* fluxes, respectively.[2]

The Eq. (1.22) also determine the relation between the flux coordinates (ψ_ϑ, ψ_φ, ϑ, φ) and the quasitoroidal coordinates (r, θ, φ) (or the cylindrical coordinates (R, Z, φ)). Particularly, in equilibrium plasmas these geometrical coordinates are periodic functions of the angular variable ϑ,

$$R = \sum_m R_m(\psi)e^{im\vartheta}, \quad Z = \sum_m Z_m(\psi)e^{im\vartheta},$$
$$r = \sum_m r_m(\psi)e^{im\vartheta}, \quad \theta = \vartheta + \sum_m \alpha_m(\psi)e^{im\vartheta}, \tag{1.26}$$

where the Fourier coefficients $R_m(\psi)$, $Z_m(\psi)$, $r_m(\psi)$, and $\alpha_m(\psi)$ are functions of the flux coordinate ψ, (ψ_ϑ or ψ_φ). Typical dependencies of the poloidal angle ϑ on the geometrical angle θ is shown in Fig. 2.2.

Furthermore, we will notate the poloidal flux ψ_φ as ψ, and the toroidal flux ψ_ϑ as ψ_t.

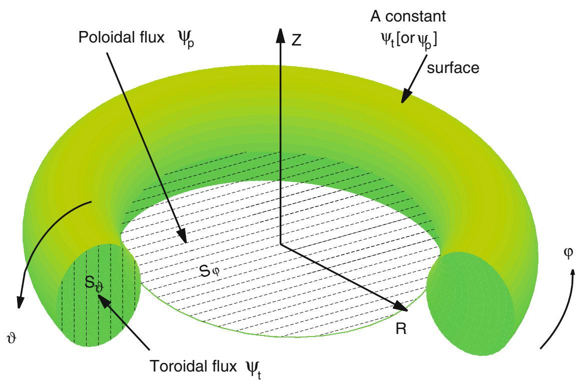

Fig. 1.4 Definitions of magnetic coordinates: the toroidal angle φ, the poloidal angle ϑ, the poloidal flux ψ_p, and the toroidal flux ψ_t

[2] The magnetic flux ψ cannot be arbitrary small. According to the quantization rule, $\oint p_z dz = hn$, ($n = 1, 2, \dots$), $p_z = eA/c$ one has $\psi = \Phi_0 n$, where $\Phi_0 = hc/e$ is a quant of magnetic flux [h is the Planck's constant]. In fusion plasmas the magnetic flux is so large, that its discreteness does not play any role.

1.2.2 The Clebsch Form

The new variables $(\psi_t, \psi, \vartheta, \varphi)$ are called *magnetic coordinates* (or *Boozer coordinates*) (see Fig. 1.4). The position of field lines on the given magnetic surface is uniquely given by a *poloidal angle*, ϑ, (the short way around the torus) and *toroidal angle*, φ (the long way around the torus). In terms of toroidal and poloidal fluxes, $\psi_t, \psi,$ poloidal and toroidal angles, ϑ, φ, the vector potential $\mathbf{A}$ and the divergence-free magnetic field $\mathbf{B}$ can be presented in the Clebsch form

$$\begin{aligned}\mathbf{A} &= B_0 R_0 \left(\nabla g + \psi_t \nabla\vartheta + \psi \nabla\varphi\right),\\ \mathbf{B} &= B_0 \left(\nabla\psi_t \times \nabla\vartheta - \nabla\varphi \times \nabla\psi\right).\end{aligned} \tag{1.27}$$

In find the equation of magnetic field lines we consider ψ as a function of ψ_t, ϑ, and φ, $\psi = \psi(\psi_t, \vartheta, \varphi)$. Then using

$$\nabla\psi = \frac{\partial\psi}{\partial\psi_\vartheta}\nabla\psi_\vartheta + \frac{\partial\psi}{\partial\vartheta}\nabla\vartheta + \frac{\partial\psi}{\partial\varphi}\nabla\varphi,$$

we obtain the magnetic field components in these coordinates are

$$\begin{aligned}B_\vartheta &= \mathbf{B}\cdot\nabla\vartheta = B_0 J \frac{\partial\psi}{\partial\psi_\vartheta},\\ B_\psi &= \mathbf{B}\cdot\nabla\psi_t = -B_0 J \frac{\partial\psi}{\partial\vartheta},\\ B_\varphi &= \mathbf{B}\cdot\nabla\varphi = J B_0,\end{aligned} \tag{1.28}$$

where $J = \nabla\psi_t \cdot (\nabla\vartheta \times \nabla\varphi)$ is the Jacobian of coordinate transformations. The equations for magnetic field lines take the Hamiltonian form

$$\frac{d\vartheta}{d\varphi} = \frac{B_\vartheta}{B_\varphi} = \frac{\partial\psi}{\partial\psi_t}, \qquad \frac{d\psi_t}{d\varphi} = \frac{B_\psi}{B_\varphi} = -\frac{\partial\psi}{\partial\vartheta}, \tag{1.29}$$

with ψ as a Hamiltonian function. This system is equivalent to the non-autonomous Hamiltonian system with 1+1/2-degree-of-freedom in which the toroidal angle φ plays a role of independent time-like variable, and (ϑ, ψ_t) as canonical-conjugated variables.

Field line equations can be also presented as the the autonomous two-degree-of-freedom Hamiltonian system similar to the system (1.12),

$$\begin{aligned}\frac{d\vartheta}{d\tau} &= \frac{\partial H}{\partial\psi_t}, \qquad \frac{d\psi_t}{d\tau} = -\frac{\partial H}{\partial\vartheta},\\ \frac{d\varphi}{d\tau} &= \frac{\partial H}{\partial\psi}, \qquad \frac{d\psi}{d\tau} = -\frac{\partial H}{\partial\varphi},\end{aligned} \tag{1.30}$$

with the Hamiltonian function

$$H \equiv H\left(\vartheta, \varphi, \psi_t, \psi\right) = \psi - \psi\left(\psi_t, \vartheta, \varphi\right) = 0, \tag{1.31}$$

and $(\vartheta, \varphi, \psi_t, \psi)$ as canonically- conjugated variables. The poloidal flux $\psi = \psi(\vartheta, \psi_t, \varphi)$ is a 2π—periodic function of ϑ, φ.

1.2.3 The Safety Factor

For the equilibrium magnetic field configuration with the nested magnetic surfaces, $\psi =$ constant, a field line on the magnetic surface $\psi =$ constant is a linear function of φ:

$$\vartheta = \frac{\varphi}{q(\psi)} + \vartheta_0 = \iota(\psi)\varphi + \vartheta_0, \tag{1.32}$$

where the quantities $q(\psi)$ and $\iota(\psi)$ are defined as

$$\frac{1}{q(\psi)} = \iota(\psi) = \frac{d\psi_p^{(0)}(\psi)}{d\psi_t}, \qquad \psi(\psi_t) = \int \frac{d\psi_t}{q(\psi)} = \int \iota(\psi) d\psi_t. \tag{1.33}$$

The $q(\psi)$ known a *safety factor* is used in a tokamak research, and $\iota(\psi)$ known a *rotational transform* (or a *winding number*) is used a stellarator research, respectively. The safety factor is equal increment of the toroidal angle, $\Delta\varphi$ of the magnetic surface per one full turn along the poloidal angle ϑ: $q(\psi) = \Delta\varphi/\Delta\vartheta = \Delta\varphi/2\pi$. The term a *safety factor* for $q(\psi)$ is introduced because its role in determining a stability of plasmas: higher values of q lead to greater stablity [see Wesson (2004)]. It also plays an important role in transport of heat and particles in a magnetically fusion plasmas.[3] The increment of the toroidal angle, $\Delta\varphi$ can be also calculated using Eq. (1.16). Then the safety factor is given by the integral

$$q(\psi) = \frac{\Delta\varphi}{2\pi} = \frac{1}{2\pi} \oint_c \frac{dz}{\partial\psi/\partial p_z}, \tag{1.34}$$

taken along the closed contour C of $\psi(z, p_z) = \text{const}$.

One should note that the poloidal angle ϑ and the toroidal angle φ are not uniquely defined. By the transformation of angles (ϑ, φ) into new ones $(\bar{\vartheta}, \bar{\varphi})$

$$\begin{aligned} \bar{\vartheta} &= \vartheta + \iota(\psi) G(\psi, \vartheta, \varphi), \\ \bar{\varphi} &= \varphi + G(\psi, \vartheta, \varphi), \end{aligned} \tag{1.35}$$

[3] Particularly, as will be discussed Chap. 11 the barriers to a particle transport caused by a small scale turbulent field may be formed near the low–order rational values of q.

where $G(\psi, \vartheta, \varphi)$ is an arbitrary periodic function of (ϑ, φ), one can show that the magnetic field conserves the Clebsch form (1.27) also in the variables $(\bar{\vartheta}, \bar{\varphi})$.

Beside of this the magnetic field $\mathbf{B}$ is not changed if we add to ϑ (or φ) any arbitrary function $f(\psi)$ of ψ,

$$\Theta = \vartheta + f(\psi). \tag{1.36}$$

We will use this property later to define the poloidal angle ϑ which would be convenient to treat the field lines in a tokamak plasma with a magnetic separatrix (see Sect. 2.31).

1.3 Model of Single-Null Divertor Tokamak

In this section we consider the model of magnetic field which is topologically equivalent to the tokamak magnetic field with the separatrix. In this example we demonstrate the action-angle formalism of Hamiltonian dynamics to study the structure of field lines.

The poloidal flux of the magnetic field of the model is given by

$$\psi(z, p) = -\frac{\alpha^2}{2} z^2 \left(1 - \frac{z^2}{2z_a^2}\right) + \frac{\beta^2}{2} p^2, \tag{1.37}$$

where α, β, z_a are dimensionless parameters. The Hamiltonian equations of field lines are

$$\begin{aligned} \frac{dz}{d\varphi} &= \frac{\partial \psi(z, p)}{\partial p} = \beta^2 p, \\ \frac{dp}{d\varphi} &= -\frac{\partial \psi(z, p)}{\partial z} = \alpha^2 z \left(1 - \frac{z^2}{z_a^2}\right). \end{aligned} \tag{1.38}$$

The fixed points (z_s, p_s) of the system are determined by

$$\frac{dz}{d\varphi} = \frac{\partial \psi(z, p)}{\partial p} = 0, \qquad \frac{dp}{d\varphi} = -\frac{\partial \psi(z, p)}{\partial z} = 0. \tag{1.39}$$

According to (1.18) they correspond to the nulls of poloidal field B_R, B_z: $B_R(z_s, p_s) = 0$, $B_z(z_s, p_s) = 0$. The system (1.38) has two fixed points: one is a hyperbolic fixed point at $(z_s = 0, p_s = 0)$ which usually called the X-point and one is an elliptic fixed point at $(z_s = z_a, p_s = 0)$ corresponding to the magnetic axis (also called O-point) . The magnetic flux at the magnetic axis, corresponding to the minimum of $\psi(z, p)$ is equal to $\psi_a = -z_a^2 \alpha^2 / 4$.

The magnetic field configuration of the system in the (z, p) plane is shown in Fig. 1.5a. In the plasma region the poloidal flux $\psi \equiv \psi(z, p) < 0$, and the field lines

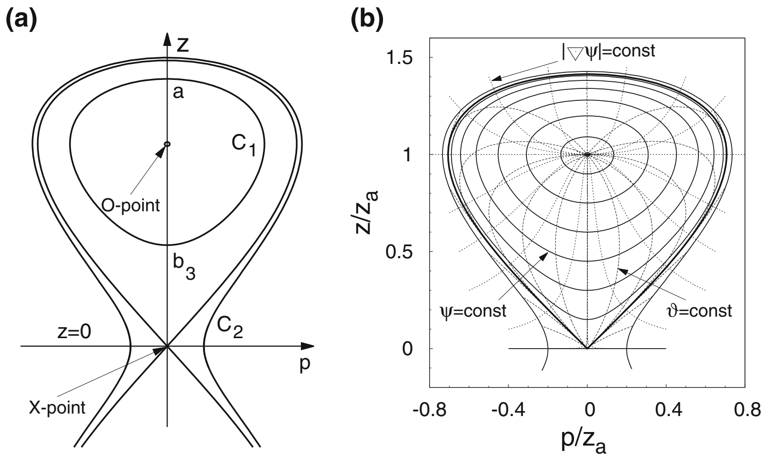

Fig. 1.5 **a** Magnetic configuration of the model in the (z, p) plane: *Curve* C_1 corresponds to field lines on the closed magnetic surface; *curve* C_2—an open field line; and *curve* 3—a magnetic separatrix. **b** Magnetic surfaces $\psi(z, p) = \text{const}$, isolines of $\vartheta(z, p) = \text{const}$, normals to magnetic surfaces $|\nabla\psi(z, p)| = \text{const}$, *thick curve* corresponds to the separatrix

lie on the magnetic surfaces $\psi(z, p) = \text{const}$ whose cross-section with the plane $\varphi = \text{const}$ is shown by the contour C_1. The open field lines lie on the magnetic surfaces with $\psi(z, p) > 0$ with the cross-section C_2. The magnetic separatrix is given by the surface $\psi(z, p) = 0$ (surface 3).

1.3.1 Field Lines in Action-Angle Variables

Below we determine the field lines using the action-angle variables (ψ_t, ϑ) which will be used later to study the effect of magnetic perturbations. Recall that in these variables field lines are straight lines, $\psi = \text{const}$, $\vartheta = \varphi/q(\psi) + \vartheta_0$.

1.3.1.1 Closed Field Lines

The action variable ψ_t for the closed field lines ($\psi < 0$) is defined as an integral along closed contour C_1 (see Fig. 1.5a),

$$\psi_t = \frac{1}{2\pi}\oint_{C_1} p(z;\psi)dz = \frac{1}{2\pi\beta}\oint_{C_1}\sqrt{2\left(\psi + \frac{\alpha^2 z^2}{2} - \frac{\alpha^2 z^4}{4z_a^2}\right)}dz$$
$$= \frac{\sqrt{2}\alpha}{2\pi z_a \beta}\int_b^a \sqrt{\left(a^2 - z^2\right)\left(z^2 - b^2\right)}dz,$$

where a and b are maximal and minimal vertical positions of field lines along the contour C_1, respectively,

$$(a, b) = z_a\sqrt{1 \mp \sqrt{\psi_N}}, \qquad \psi_N = 1 + \frac{\psi}{|\psi_a|}.$$

Here ψ_N is the normalized poloidal flux. Using the integral [Prudnikov et al. (1986)]

$$\int_b^a \sqrt{\left(a^2 - z^2\right)\left(z^2 - b^2\right)}dz = \frac{a}{3}\left[(a^2 + b^2)E(k) - 2b^2 K(k)\right],$$
$$k = \frac{\sqrt{a^2 - b^2}}{a},$$

where $K(k)$ and $E(k)$ are the complete elliptic integrals of the first kind and the second kind, respectively, with a module k, we obtain

$$\psi_t = \psi_t^{(s)}\frac{\sqrt{2(1+\sqrt{\psi_N})}}{\pi}\left[E(k) - \left(1 - \sqrt{\psi_N}\right)K(k)\right], \tag{1.40}$$

with

$$k = \frac{\sqrt{2} \quad \psi_N^{1/4}}{\sqrt{1 + \psi_N^{1/2}}}. \tag{1.41}$$

Here $\psi_t^{(s)} = \alpha z_a^2/3\beta$ is the value of toroidal flux at the separatrix.

In order to introduce the angle variable ϑ we set that $\vartheta = 0$ at the highest point $z = a$ of the contour C_1, and $\vartheta = \mp\pi$ at the lowest point $z = b$. Then for the closed field lines ($\psi < 0$) the angle ϑ is introduced as the integral.

$$\begin{aligned}
\vartheta(z) &= \frac{\partial}{\partial\psi_t}\int_z^a p(z';\psi)dz' = \frac{1}{\sqrt{2}\beta}\frac{d\psi}{d\psi_t}\int_z^a \frac{dz'}{\sqrt{\psi+\frac{\alpha^2 z^2}{2}-\frac{\alpha^2 z^4}{4z_a^2}}} \\
&= \frac{2z_a}{\sqrt{2}\alpha\beta}\frac{d\psi}{d\psi_t}\int_z^a \frac{dz'}{\sqrt{\left(a^2-z^2\right)\left(z^2-b^2\right)}} \\
&= \frac{\sqrt{2}z_a}{\alpha\beta a}\frac{d\psi}{d\psi_t}F\left(\arcsin\frac{\sqrt{a^2-z^2}}{\sqrt{2}a\psi_N^{1/4}},k\right),
\end{aligned} \tag{1.42}$$

where we have used the integral

$$\int_z^a \frac{dz}{\sqrt{\left(a^2-z^2\right)\left(z^2-b^2\right)}} = \frac{1}{a}F(\phi,k),$$

$$\phi = \arcsin\sqrt{\frac{a^2-z^2}{a^2-b^2}},$$

with the incomplete elliptic integral of the first kind $F(\phi,k)$. Using the condition $\vartheta(z=b)=\pi$ we find the derivative $d\psi/d\psi_t$ which is equal to the inverse safety factor,

$$q(\psi) = \left(\frac{d\psi}{d\psi_t}\right)^{-1} = \frac{\sqrt{2}}{\pi\alpha\beta\sqrt{1+\sqrt{\psi_N}}}K(k). \tag{1.43}$$

Inverting the relation (1.42) with respect to the coordinate z and using the relation $p=\beta^{-2}dz/d\varphi=q^{-1}\beta^{-2}dz/d\vartheta$ which follows from (1.38) we obtain field line coordinates $z(\vartheta)$, $p(\vartheta)$ as a function of the angle variable ϑ:

$$\begin{aligned}
z(\vartheta) &= a\sqrt{1-k^2\,\mathrm{sn}^2\left(K(k)\vartheta/\pi,k\right)} = a\,\mathrm{dn}\left(K(k)\vartheta/\pi,k\right), \\
p(\vartheta) &= -\frac{\sqrt{2}z_a\psi_N^{1/2}\alpha}{\beta}\,\mathrm{sn}\left(K(k)\vartheta/\pi,k\right)\,\mathrm{cn}\left(K(k)\vartheta/\pi,k\right),
\end{aligned} \tag{1.44}$$

where $\mathrm{sn}(u;k)$, $\mathrm{cn}(u;k)$, $\mathrm{dn}(u;k)$ are the Jacobi elliptic functions. The dependence of the field lines on the toroidal angle φ can be obtained by using the relation $\vartheta=\varphi/q(\psi)$. Field lines reach the lowest vertical point $z=b$ at $\varphi=\pm\pi q(\psi)$, and the highest vertical point $z=a$ at $\varphi=0$.

1.3.1.2 Open Field Lines

We define the toroidal flux ψ_t for the open field lines $\psi_N > 1$ by requiring that it should be a continuous function of the poloidal flux ψ at the separatrix $\psi = 0$. This can be done by choosing the integration contour C_2 along the field line starting and ending at the p-axis ($z = 0$) after one full turn as shown in (see Fig. 1.5a), i.e.,

$$\psi_t = \frac{1}{2\pi}\oint_{C_2} p(z;\psi)dz = \frac{1}{\pi\beta}\int_0^a \sqrt{2\left(\psi + \frac{\alpha^2 z^2}{2} - \frac{\alpha^2 z^4}{4z_a^2}\right)}dz$$
$$= \frac{\sqrt{2}\alpha}{2\pi z_a \beta}\int_0^a \sqrt{\left(a^2 - z^2\right)\left(z^2 + b^2\right)}dz,$$

where $b^2 = z_a^2(\sqrt{\psi_N} - 1)$. Then using the integral

$$\int_0^a \sqrt{\left(a^2 - z^2\right)\left(z^2 + b^2\right)}dz = \frac{\sqrt{a^2 + b^2}}{3}\left[(a^2 - b^2)E\left(k^{-1}\right) + b^2 K\left(k^{-1}\right)\right],$$

where the module k is given by (1.41), one obtains

$$\psi_t = \psi_t^{(s)}\frac{2\psi_N^{1/4}}{\pi}\left[E\left(k^{-1}\right) + \frac{1}{2}\left(\sqrt{\psi_N} - 1\right)K\left(k^{-1}\right)\right]. \tag{1.45}$$

The angle variable ϑ is defined similar to (1.42) with the settings $\vartheta(z = a) = 0$ and $\vartheta(z = 0) = \pm\pi$. Then, it is not difficult to obtain

$$\vartheta(z) = \frac{1}{\alpha\beta\psi_N^{1/4}}\frac{d\psi}{d\psi_t}F\left(\arccos\frac{z}{a}, k^{-1}\right). \tag{1.46}$$

From the condition $\vartheta(z = 0) = \pi$ one obtains the safety factor,

$$q(\psi) = \frac{1}{\pi\alpha\beta\psi_N^{1/4}}K\left(k^{-1}\right). \tag{1.47}$$

Field line coordinates $z(\vartheta)$, $p(\vartheta)$ as a function of the angle variables ϑ are given by

$$z(\vartheta) = a\ \mathrm{cn}\left(\frac{K\left(k^{-1}\right)\vartheta}{\pi}, k^{-1}\right),$$
$$p(\vartheta) = -\frac{a\alpha\psi_N^{1/4}}{\beta}\ \mathrm{sn}\left(K(k^{-1})\vartheta/\pi, k^{-1}\right)\ \mathrm{dn}\left(K(k)\vartheta/\pi, k^{-1}\right). \tag{1.48}$$

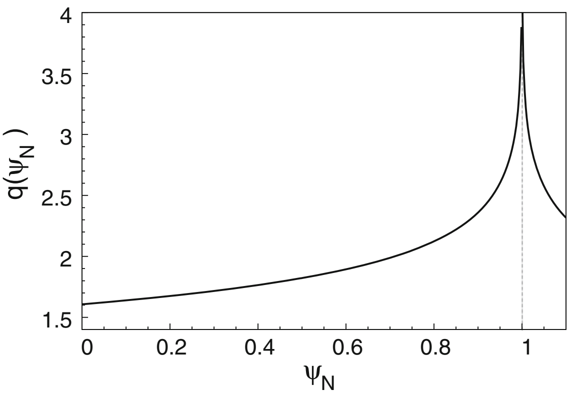

Fig. 1.6 Dependence of the safety factor $q(\psi)$ (1.43) and (1.47) on ψ_N. The values of the parameters $\gamma = 0.44$, $z_a = 1$, and $\alpha = 0.4553$. The value of q_{95} is 2.6

Figure 1.5b shows the isolines of $\vartheta(z, p) =$ const, normals to magnetic surfaces $\nabla\psi(z, p) =$ const, thick curve corresponds to the separatrix.

1.3.2 Magnetic Separatrix

The field line on the separatrix ($\psi = 0$) can be obtained from (1.44) or (1.48) by putting $\vartheta = \varphi/q(\psi)$ and taking the limit $\psi_N \to 1$, $k \to 1$ and using the relations $\mathrm{dn}(u, k = 1) = \mathrm{cn}(u, k = 1) = 1/\cosh(u)$, $\mathrm{sn}(u, k = 1) = \tanh(u)$, and $K(k)/\pi q(\psi) \to \gamma$. Therefore, we have

$$z_s(\varphi) = \frac{\sqrt{2}z_a}{\cosh(\gamma\varphi)}, \qquad p_s(\varphi) = -\frac{\sqrt{2}z_a\alpha\sinh(\gamma\varphi)}{\beta\cosh^2(\gamma\varphi)}. \tag{1.49}$$

The safety factor $q(\psi)$ (1.43) and (1.47) near the separatrix ($\psi \to \pm 0$, $\psi_N \to 1 \pm 0$) has the following asymptotics,

$$q(\psi) = \frac{1}{2\pi\gamma}\ln\frac{Q}{|\psi|} + O(\psi), \quad \psi \to \pm 0, \tag{1.50}$$

where $\gamma = \alpha\beta$ and $Q = 16z_a^2\alpha^2$. As we will see below, the asymptotical form (1.50) is a generic for the magnetic field configurations with the magnetic separatrix which has the so-called first order null of the poloidal magnetic field (see Sect. 2.4.1). Figure 1.6 shows the profile $q(\psi)$ for the values of the parameters $\gamma = 0.44$, $z_a = 1$, and $\alpha = 0.4553$. The number q_{95} is the value of the safety factor at the magnetic surface which has 95 % of the total poloidal flux through the cross section confined by the separatrix.

1.4 Bibliographic Comments

The Hamiltonian formulation of the equations of magnetic field lines has been already used in early studies of the problem of stability and destruction of magnetic surfaces in tokamaks and stellarators [Kerst (1962); Rosenbluth et al. (1966); Filonenko et al. (1967); Freis et al. (1973); Hamzeh (1974); Finn (1975), Matsuda and Yoshikawa (1975); Boozer and Rechester (1978)]. More general Hamiltonian formulation of magnetic field lines based on the variational principles of classical mechanics has been given by Cary and Littlejohn (1983). Boozer (1983) and White (2001) formulated the Hamiltonian equations of field lines in magnetic flux coordinates using the Clebsch form of magnetic field.

Historically, the Clebsch presentation of magnetic field has been first given by Kruskal and Kulsrud (1958); Hamada (1959, 1962); Greene and Johnson (1962) for equilibrium plasmas. The validity of this presentation for the arbitrary magnetic field was shown by Boozer (1983), and a rigorous mathematical proof that every solenoidal field in an arbitrary smooth simple toroidal domain can written in this form is given by Yoshida (1993). The detailed description of magnetic flux coordinates can be found, for example, in the books by Balescu (1988) and Dhaeseleer et al. (1991).

The formalism of action-angle variables is described in many textbooks on classical mechanics, for example, by Landau and Lifshits (1976) and Goldstein (1980). The general definition of action-angle variables in Hamiltonian systems with inseparable variables is given in a textbook by Arnold (1989).

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