Preface

The reader is invited to immerse himself in a "love story" which has been unfolding for 35 centuries: the love story between mathematicians and geometry. In addition to accompanying the reader up to the present state of the art, the purpose of this *Trilogy* is precisely to tell this story. The *Geometric Trilogy* will introduce the reader to the multiple complementary aspects of geometry, first paying tribute to the historical work on which it is based and then switching to a more contemporary treatment, making full use of modern logic, algebra and analysis. In this *Trilogy*, Geometry is definitely viewed as an autonomous discipline, never as a sub-product of algebra or analysis. The three volumes of the *Trilogy* have been written as three independent but complementary books, focusing respectively on the axiomatic, algebraic and differential approaches to geometry. They contain all the useful material for a wide range of possibly very different undergraduate geometry courses, depending on the choices made by the professor. They also provide the necessary geometrical background for researchers in other disciplines who need to master the geometric techniques.

The present book leads the reader on a walk through 35 centuries of geometry: from the papyrus of the Egyptian scribe *Ahmes*, 16 centuries before Christ, to Hilbert's famous axiomatization of geometry, 19 centuries after Christ. We discover step by step how all the ingredients of contemporary geometry have slowly acquired their final form.

It is a matter of fact: for three millennia, geometry has essentially been studied via "synthetic" methods, that is, from a given system of axioms. It was only during the 17th century that algebraic and differential methods were considered seriously, even though they had always been present, in a disguised form, since antiquity.

After rapidly reviewing some results that had been known empirically by the Egyptians and the Babylonians, we show how Greek geometers of antiquity, slowly, sometimes encountering great difficulties, arrived at a coherent and powerful deductive theory allowing the rigorous proof of all of these empirical results, and many others. Famous problems—such as "squaring the circle"—induced the development of sophisticated methods. In particular, during the fourth century BC, *Eudoxus* overcame the puzzling difficulty of "incommensurable quantities" by a method which is essentially that of Dedekind cuts for handling real numbers. Eudoxus also proved the validity of a "limit process" (the *Exhaustion theorem*) which allowed him to answer questions concerning, among other things, the lengths, areas or volumes related to various curves or surfaces.

We first summarize the knowledge of the Greek geometers of the time by presenting the main aspects of *Euclid's Elements*. We then switch to further work by *Archimedes* (the circle, the sphere, the spiral, ...), *Apollonius* (the conics), *Menelaus* and *Ptolemy* (the birth of trigonometry), *Pappus* (ancestor of projective geometry), and so on.

We also review some relevant results of classical *Euclidean geometry* which were only studied several centuries after Euclid, such as additional properties of triangles and conics, Ceva's theorem, the trisectors of a triangle, stereographic projection, and so on. However, the most important new aspect in this spirit is probably the theory of inversions (a special case of a conformal mapping) developed by Poncelet during the 19th century.

We proceed with the study of projective methods in geometry. These appeared in the 17th century and had their origins in the efforts of some painters to understand the rules of perspective. In a good perspective representation, parallel lines seem to meet "at the horizon". From this comes the idea of adding "points at infinity" to the Euclidean plane, points where parallel line eventually meet. For a while, projective methods were considered simply as a convenient way to handle some Euclidean problems. The recognition of projective geometry as a respectable geometric theory in itself—a geometry where two lines in the plane always intersect—only came later. After having discussed the fundamental ideas which led to projective geometry—we focus on the amazing *Hilbert theorems*. These theorems show that the very simple classical axiomatic presentation of the projective plane forces the existence of an underlying field of coordinates. The interested reader will find in [5], Vol. II of this *Trilogy*, a systematic study of the projective spaces over a field, in arbitrary dimension, fully using the contemporary techniques of linear algebra.

Another strikingly different approach to geometry imposed itself during the 19th century: non-Euclidean geometry. Euclid's axiomatization of the plane refers—first—to four highly natural postulates that nobody thought to contest. But it also contains the famous—but more involved—"fifth postulate", forcing the uniqueness of the parallel to a given line through a given point. Over the centuries many mathematicians made considerable efforts to prove Euclid's parallel postulate from the other axioms. One way of trying to obtain such a proof was by a *reductio ad absurdum*: assume that there are several parallels to a given line through a given point, then infer as many consequences as possible from this assumption, up to the moment when you reach a contradiction. But very unexpectedly, rather than leading to a contradiction, these efforts instead led step by step to an amazing new geometric theory. When actual models of this theory were constructed, no doubt was left: mathematically, this "non-Euclidean geometry" was as coherent as Euclidean geometry. We recall first some attempts at "proving" Euclid's fifth postulate, and then develop the main characteristics of the non-Euclidean plane: the limit parallels and some properties of triangles. Next we describe in full detail two famous models of non-Euclidean geometry: the *Beltrami–Klein disc* and the *Poincaré disc*. Another model—the famous *Poincaré half plane*—will be given full attention in [6], Vol. III of this *Trilogy*, using the techniques of Riemannian geometry.

We conclude this overview of synthetic geometry with Hilbert's famous axiomatization of the plane. Hilbert has first filled in the small gaps existing in Euclid's axiomatization: essentially, the questions related to the relative positions of points and lines (on the left, on the right, between, ...), aspects that Greek geometers considered as "being intuitive" or "evident from the picture". A consequence of Hilbert's axiomatization of the Euclidean plane is the isomorphism between that plane and the Euclidean plane $\mathbb{R}^2$: this forms the link with [5], Vol. II of this *Trilogy*. But above all, Hilbert observes that just replacing the axiom on the uniqueness of the parallel by the requirement that there exist several parallels to a given line through a same point, one obtains an axiomatization of the "non-Euclidean plane", as studied in the preceding chapter.

To conclude, we recall that there are various well-known problems, introduced early in antiquity by the Greek geometers, and which they could not solve. The most famous examples are: squaring a circle, trisecting an angle, duplicating a cube, constructing a regular polygon with n sides. It was only during the 19th century, with new developments in algebra, that these ruler and compass constructions were proved to be impossible. We give explicit proofs of these impossibility results, via field theory and the theory of polynomials. In particular we prove the transcendence of π and also the Gauss–Wantzel theorem, characterizing those regular polygons which are constructible with ruler and compass. Since the methods involved are completely outside the "synthetic" approach to geometry, to which this book is dedicated, we present these various algebraic proofs in several appendices.

Each chapter ends with a section of "problems" and another section of "exercises". Problems generally cover statements which are not treated in the book, but which nevertheless are of theoretical interest, while the exercises are designed for the reader to practice the techniques and further study the notions contained in the main text.

A selective bibliography for the topics discussed in this book is provided. Certain items, not otherwise mentioned in the book, have been included for further reading.

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