

## Chapter 6

# Projective Geometry

Today, we view projective geometry as a mathematical theory in its own right a part of geometry which in any case is “highly non-Euclidean”, without any notion of distance and with very specific topological properties. But nevertheless, the origin of projective geometry is to be found *inside* Euclidean geometry. Indeed, for several centuries, projective “methods” were considered just as an efficient way to handle problems in Euclidean geometry. Let us explain this.

Everything begins with the efforts of some painters of the 15th century—in particular *Alberti* (1404–1472) and *Dürer* (1471–1528)—to determine the precise rules of perspective. Their crucial idea was that of *vanishing points*: lines which are parallel in the real world—such as the two rails of a train track—seem to converge to a single point on the horizon, and thus should be represented as such on a painting.

Two centuries later, mathematicians such as *Desargues* (1591–1661) and *Pascal* (1623–1662) turned these rules of perspective representation into mathematical techniques of proof. For example, the perspective representation of a circle is an ellipse: thus if you know the mathematical properties of the perspective representation, you should be able to infer some properties of the ellipse from corresponding properties of the circle. This is what we have already done in Sect. 4.9, proposing an efficient proof of Proposition 4.9.1. Pascal has followed this type of argument when proving his famous theorem on the hexagon inscribed in a conic. As far as the theory of projective conics is concerned, we refer the reader to Chap. 6 of [5], *Trilogy II*, where the question is treated with more adequate contemporary tools.

Mathematicians of the 17th century also observed that by cleverly using the language of “vanishing points”, they could unify many of their statements. Indeed it occurs often in Euclidean geometry that a theorem is proved for “points and lines in general position”. But this same theorem generally needs an alternative proof if, for example, two of the given lines—instead of being “in general position”—are chosen to be parallel. We exhibit in Sect. 6.2 the case of Pappus’ theorem (see Theorem 4.14.5), where nine different Euclidean configurations must be treated separately, but reduce to a single statement when using the language of vanishing points. But for most mathematicians of the 17th century (and later), “vanishing points” did not have a true mathematical existence: they were essentially a convenient way of

speaking. The proofs of the theorems remained highly Euclidean. This is somewhat analogous to the way one might consider a limit  $\lim_{x \rightarrow a} f(x)$  in analysis; one gives a meaning to this expression, including when  $a = \pm\infty$ , but this does not mean that one is using “infinite real numbers”: the limit can be handled using standard real numbers—one need not enter into the real of non-standard analysis.

However, the 17th century, with *Fermat* (1601–1655) and *Descartes* (1596–1650), had also seen the birth of analytic geometry (see [5], *Trilogy II*). During the 19th century, a sometimes violent reaction arose against the analytic methods and their heavy calculations, very far from the elegance of Greek geometry. It was also the time of many political revolutions. Various students of *Monge* (1746–1818), in particular *Poncelet* (1788–1867), *Carnot* (1753–1823) and *Brianchon* (1785–1864) were among the prominent figures of a “geometric revolution”. They developed the great principles of *projective geometry* and studied the projective transformations. They stressed in particular the importance of the famous principle of duality: *In the projective plane, for every valid statement about points and lines, there is a corresponding valid statement interchanging the roles of points and lines*. Eventually, they gave evidence that many geometric results could be inferred *synthetically*, without any need of a system of coordinates and sets of equations. In particular, *Chasles* (1793–1880) underlined the central role played by anharmonic ratios: a notion already studied by Pappus (see Sect. 4.14 and in particular, Theorem 4.14.3). A little later, *von Staudt* (1798–1867) and *Peri* (1860–1913) exhibited a theory of the real projective plane, totally independent of any Euclidean considerations.

As already mentioned, we shall study projective geometry in Chap. 6 of [5], *Trilogy II*, with adequate algebraic tools. The present chapter is intended only to exhibit how the fundamental ideas of projective geometry arose and how the axiomatic approach to projective geometry is closely connected to the algebraic approach. To this end, we first discuss the historical definition of the real projective plane in terms of the Euclidean plane augmented with points at infinity. We extend the notion of anharmonic ratio to the real projective plane and use it to prove the Desargues and Pappus theorems. These two theorems in the real case then become the key ingredients of an axiomatic system for an abstract projective plane. The main concern of this chapter is to prove the famous Hilbert’s theorems (*David Hilbert*, 1862–1943) showing that this very simple axiomatic system hides in the background the presence of an underlying “field of coordinates”. This yields the link with our algebraic treatment of projective geometry in [5], *Trilogy II*. To avoid unnecessary repetitions, we rely on this algebraic treatment in *Trilogy II* for the study of projective conics and projective transformations.

## 6.1 Perspective Representation

During the *Renaissance*, painters were calling for a precise technique which would enable them to faithfully reproduce ocular vision in their paintings.

In his treatise *Della Pictura*, *Leon Battista Alberti* (1404–1472) proposes a method to represent on a vertical canvas, a horizontal pavement comprised of square

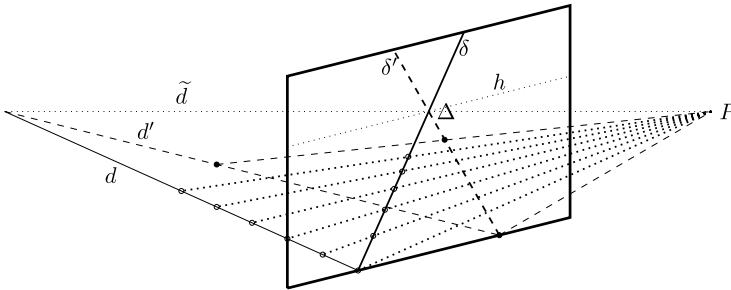


Fig. 6.1

tiles. This can be regarded as the first document on projective methods in geometry. He shows that every straight line of the pavement must be represented as a straight line on the canvas, that parallel lines of the pavement must in general converge to a “vanishing point” on the canvas, and he provides a precise geometric construction allowing the artist to determine the position of each line on the canvas. From Alberti’s work, let us pick up the fundamental ideas of projective geometry.

Imagine that your eye is situated at some point  $P$  and that you are looking at the pavement through the (presently empty) frame intended to support the canvas (see Fig. 6.1). Your line of sight can be identified with a straight line which follows the lines between the tiles of the pavement. What you will eventually draw on the canvas are all the intersection points of your line of sight with the plane of the canvas, as your line of sight follows all the lines of the pavement.

Call  $\Pi$  the plane of the pavement and  $\Gamma$  the plane of the canvas. When your line of sight follows a straight line  $d$  in the plane  $\Pi$  of the pavement, all the successive positions of your line of sight lie in the plane  $\sigma$  determined by the point  $P$  (your eye) and the line  $d$ . The representation of the line  $d$  on the canvas is thus (part of) the straight line  $\delta$ , the intersection of the two planes  $\sigma$  and  $\Gamma$ .

Repeating the same construction with another line  $d'$  of the pavement, parallel to  $d$ ; you obtain a corresponding plane  $\sigma'$  and a corresponding line  $\delta'$  on the canvas. The two planes  $\sigma$  and  $\sigma'$  intersect, since both contain the point  $P$  (your eye); their intersection is thus a straight line  $\tilde{d}$ . That line  $\tilde{d}$  is the parallel to  $d$  and  $d'$  passing through the point  $P$ . Indeed since  $P$  and  $d$  lie in  $\sigma$ , the parallel  $\tilde{d}$  still lies in  $\sigma$ ; analogously, it lies in  $\sigma'$ . Notice that this line  $\tilde{d}$  is horizontal, since it is parallel to the horizontal lines  $d$  and  $d'$ . Write  $\Delta$  for the intersection of this line  $\tilde{d}$  with the plane  $\Gamma$  of the canvas. Since  $\tilde{d}$  lies in both planes  $\sigma$  and  $\sigma'$ , the point  $\Delta$  lies on both lines  $\delta$  and  $\delta'$ . Thus on the canvas, the two lines  $\delta, \delta'$  containing the representations of the parallel lines  $d, d'$  of the pavement intersect at the point  $\Delta$ . This point  $\Delta$  does not correspond a any point of the pavement, since it corresponds to a horizontal line of sight  $\tilde{d}$ . But the representations on the canvas of all the lines of the pavement parallel to  $d$  intersect at this “vanishing point”  $\Delta$ .

You can repeat the same process with another family of parallel lines on the pavement, in another direction, and you will end up with another horizontal line  $\tilde{d}_1$  through the point  $P$ , and with a corresponding “vanishing point”  $\Delta_1$ . All the

“vanishing points” obtained from a family of parallel lines on the pavement are thus situated on the horizontal line  $h$  of the canvas, at the height of your eye. Let us temporarily call this line  $h$  “the line of horizon”: the line comprising all our points “at infinity”.

We now present a (very small) mathematical abstraction of this situation: we consider the whole plane  $\Pi$  of the pavement and the whole plane  $\Gamma$  of the canvas. Mathematically, the perspective representation of the pavement on the canvas is the central projection, with center the point  $P$ , of the plane  $\Pi$  of the pavement on the plane  $\Gamma$  of the canvas. Notice in particular that the points of the plane  $\Pi$  lying behind you are now represented by points of the canvas situated higher than your eye.

And here comes the crucial point. When your line of sight follows a line  $d$  of the pavement, further and further away in front of you, the corresponding point on the canvas travels upwards on the portion of the line  $\delta$  situated under the line of horizon, and tends to the corresponding “vanishing point”  $\Delta$  on  $\delta$ . But if you follow the same line  $d$  behind you, further and further away, the corresponding point of the canvas still travels on the same line  $\delta$ , downwards this time and above the line of horizon, and again eventually tends to the vanishing point  $\Delta$  on  $\delta$ . Taking this representation  $\delta$  of the line  $d$  seriously, you can say that traveling on  $d$  in front of you, further and further away, is equivalent to following the line  $\delta$  upwards on the canvas. This eventually leads you to some “point at infinity” corresponding to the vanishing point on  $\delta$ . Continuing further than this point at infinity (that is, if you continue to follow the line  $\delta$  upwards) you come back along the same line  $d$ , behind you. In other words, the perspective representation  $\delta$  of the line  $d$  adds to this line a single new point “at infinity”, where “both ends” of the line  $d$  join. Let us stress that there is *one single point at infinity*, not a distinct point at each “end” of the line  $d$ . This uniqueness of the “point at infinity” was already very clear in the mind of the mathematicians of the 17th century.

We need to highlight an important feature concerning the real projective plane. The presence and the uniqueness of the point at infinity modify in a striking way the “topology” of the plane. Indeed, “moving far away from each other”—towards infinity—along the two directions on a straight line now becomes the same as “moving towards each other”—at infinity—along this same line. Points which were “very far from each other” suddenly become “very close to each other”. These topological aspects of the real projective plane will be further investigated in Sect. 6.22 of [5], *Trilogy II*.

Notice nevertheless that the description above seems to contain a “gap”. We have already noticed that the points on the “line of horizon” do not represent any point of the pavement, because they correspond to lines of sight which are parallel to the plane of the pavement. On the other hand all the lines of sight parallel to the plane  $\Gamma$  of the canvas never meet this canvas. That is, drawing through the point  $P$  the vertical plane  $\Gamma_P$  parallel to  $\Gamma$ , the points on the line  $\ell$ , the intersection of  $\Gamma_P$  and  $\Pi$ , do not admit any representation in the plane  $\Gamma$ . So the perspective representation of  $\Pi$  on  $\Gamma$  has indeed added a new “line of points at infinity”, but has omitted the line  $\ell$  of  $\Pi$ . Of course, in a sensible “projective” extension of the

plane  $\Pi$ , you should keep this line  $\ell$  as well, and add further a “point at infinity” in the direction of  $\ell$ .

**Definition 6.1.1** Let  $\Pi$  be a Euclidean plane. The corresponding *projective plane*  $\mathbb{P}(\Pi)$  has for points:

- all the points of the Euclidean plane  $\Pi$ ;
- one additional point (called the “point at infinity”) for each direction in the plane  $\Pi$ .

The lines of the projective plane are:

- the lines of the Euclidean plane  $\Pi$ , each of them being augmented by the “point at infinity” corresponding to its direction;
- a line called the line “at infinity”, comprising all the “points at infinity”.

The main advantage of the projective plane is:

**Proposition 6.1.2** *In the projective plane:*

- *through two distinct points passes exactly one line;*
- *two distinct lines intersect in exactly one point.*

*Proof* Through two Euclidean points passes the corresponding Euclidean line (see Postulate 3.1.2.1). Through a Euclidean point  $P$  and a point  $Q$  at infinity passes the line through  $P$  in the direction determined by  $Q$ . Through two points at infinity passes the line at infinity.

If two Euclidean lines are not parallel, they intersect at some Euclidean point. If they are parallel, they intersect at their common “point at infinity”. Finally the “line at infinity” intersects a Euclidean line at the “point at infinity” on this Euclidean line. □

The absence of the distinction between parallel and intersecting lines will make projective geometry rather efficient and elegant. But unfortunately, Definition 6.1.1 still distinguishes between “Euclidean points” and “points at infinity”. It was only centuries later that definitions of the projective plane were given which avoid this distinction. The key to such a definition is this:

**Proposition 6.1.3** *The projective plane can equivalently be defined as follows, given an arbitrary point  $P$  in solid space.*

- *A projective point is a Euclidean line through  $P$ .*
- *The projective lines are the sets of Euclidean lines through  $P$  lying in the same Euclidean plane.*

*Proof* Let  $\Pi$  be the Euclidean plane; choose a point  $P \notin \Pi$ . Each point  $A \in \Pi$  corresponds to a unique line  $d_{AP}$  through  $P$ . This exhausts all the lines through  $P$ , with the exception of the lines through  $P$  parallel to  $\Pi$ , and there is exactly one

such line in each direction of  $\Pi$ . This exhibits the bijection with  $\mathbb{P}(\Pi)$  in the case of points.

Given a Euclidean line  $d$  in  $\Pi$ , the points of the projective line induced by  $d$  correspond via the bijection to all the lines  $d_{AP}$  through  $P$  and a point  $A \in d$ , plus the line through  $P$  parallel to  $d$  (the “point at infinity” in the direction of  $d$ ). This means precisely: all the lines through  $P$  in the plane  $\sigma$  are determined by  $d$  and  $P$  and the “line at infinity” corresponds to all the lines of the plane passing through  $P$  and parallel to  $\Pi$ .  $\square$

Proposition 6.1.3 is, essentially, the contemporary definition of the real projective plane (see Definition 6.1.1 in [5], *Trilogy II*). This definition is trivially independent (up to an isomorphism) of the choice of the point  $P$ . However, it took centuries for this elegant definition to materialize, simply because during the 17th century, a point was still a point and a line was still a line, as they were for Greek geometers. Thus a line (a Euclidean line) could by no means be a point (a projective point)! In terms of Corollary 6.1.3, the statement of Proposition 6.1.2 simply recalls that two distinct lines through  $P$  determine a unique plane, while two distinct planes through  $P$  intersect as a line. When this is more convenient, we shall freely use the alternative formulation provided by Proposition 6.1.3 instead of the more “historical” Definition 6.1.1.

## 6.2 Projective Versus Euclidean

The projective plane has thus been defined as the Euclidean plane, to which intersection points of parallel lines are added “at infinity” (see Definition 6.1.1). In this spirit, it sounds *a priori* natural to handle a projective problem via Euclidean methods, as long as it concerns Euclidean points, and via Euclidean parallel lines, as soon as points at infinity must be considered. In this way, every projective problem can be fully treated via Euclidean methods. This is what was done at the beginning of projective geometry and essentially, up to the nineteenth century. Let us comment on this method for the example of Pappus’ theorem 4.14.5.

*In the projective plane, consider:*

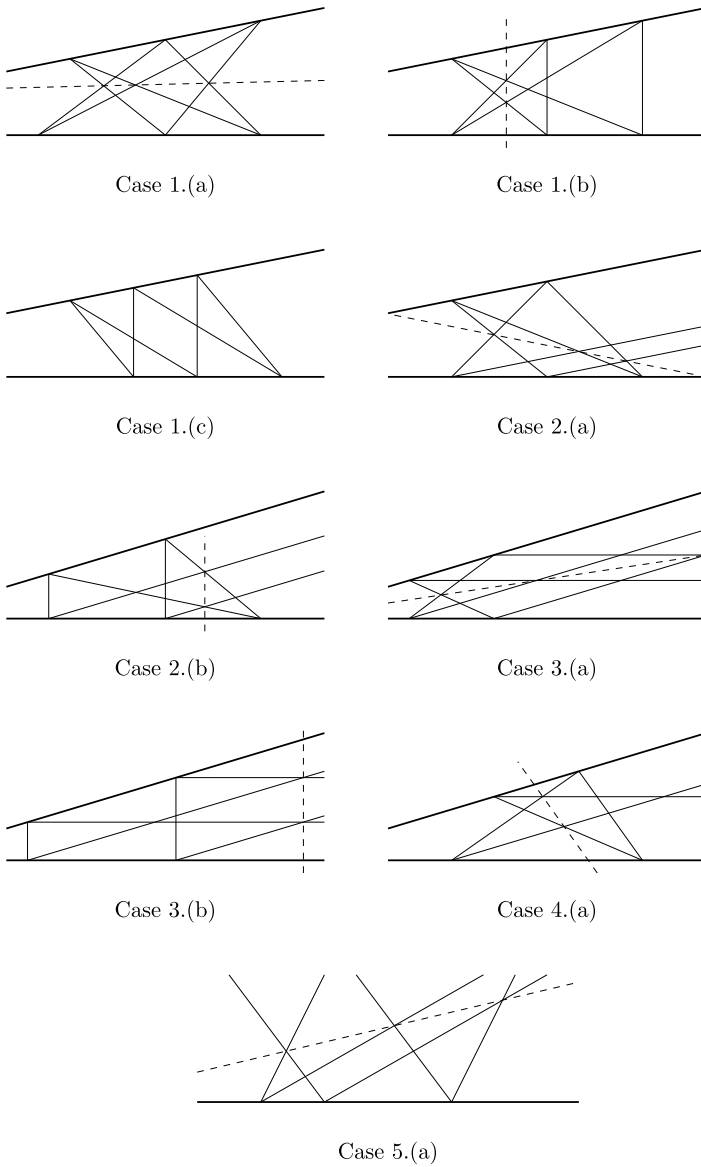
- two distinct lines  $d$  and  $d'$ ;
- three distinct points  $A, B, C$  on  $d$  but not on  $d'$ ;
- three distinct points  $A', B', C'$  on  $d'$  but not on  $d$ .

*In that case, the three points*

$$X = d_{BC'} \cap d_{B'C}, \quad Y = d_{CA'} \cap d_{C'A}, \quad Z = d_{AB'} \cap d_{A'B}$$

*are on the same line.*

This statement makes perfect sense in the projective plane: indeed all the intersection points involved exist by Proposition 6.1.2, because the assumptions force in each case the two corresponding lines to be distinct. However, to prove this projective theorem via Euclidean methods, one must consider the possibility that some of



**Fig. 6.2**

the nine points involved lie “at infinity”. Figure 6.2 reviews all nine cases, which we briefly consider below.

1. *All six points  $A, B, C, A', B', C'$  are Euclidean.*

- (a) The three points  $X, Y, Z$  are Euclidean:  
these points must be on a Euclidean line.

- (b)  $X$  is at infinity,  $Y$  and  $Z$  are not.

Then  $d_{BC'}$  and  $d_{B'C}$  are parallel. The thesis “ $X$  on  $d_{YZ}$ ” means that  $d_{YZ}$  must also be parallel to  $d_{BC'}$  and  $d_{B'C}$ .

- (c)  $X$  and  $Y$  are at infinity.

Then  $d_{BC'}$ ,  $d_{B'C}$  are parallel, as are  $d_{AC'}$  and  $d_{A'C}$ . The line  $d_{XY}$  is the line at infinity. The thesis “ $Z$  on  $d_{XY}$ ” means that  $Z$  is at infinity, that is,  $d_{AB'}$  is parallel to  $d_{B'A}$ .

2.  $A$  is at infinity,  $B, C, A', B', C'$  are not.

In this case,  $d_{AB'}$  and  $d_{AC'}$  are parallel to  $d$ .

- (a) The three points  $X, Y, Z$  are Euclidean:  
these points must be on a Euclidean line.

- (b)  $X$  is at infinity,  $Y$  and  $Z$  are not.

Then  $d_{BC'}$  and  $d_{B'C}$  are parallel. The thesis “ $X$  on  $d_{YZ}$ ” means that  $d_{YZ}$  must be parallel to  $d_{BC'}$  and  $d_{B'C}$ .

- (c)  $Y$  and  $Z$  cannot be at infinity.

In the Euclidean plane,  $d_{A'B}$  cuts  $d$  at  $B$ , thus it also cuts  $d_{AB'}$  which is parallel to  $d$ . So  $Y$  is Euclidean. Analogously for  $Z$ .

3. The points  $A$  and  $A'$  are at infinity;  $B, C, B', C'$  are not.

In this case,  $d_{AB'}$  and  $d_{AC'}$  are parallel to  $d$  while  $d_{A'B}$  and  $d_{A'C}$  are parallel to  $d'$ . Since  $A \neq A'$ ,  $d$  and  $d'$  are not parallel.

- (a) The three points  $X, Y, Z$  are Euclidean:  
these points must be on a Euclidean line.

- (b)  $X$  is at infinity,  $Y$  and  $Z$  are not.

Then  $d_{BC'}$  and  $d_{B'C}$  are parallel. The thesis “ $X$  on  $d_{YZ}$ ” means that  $d_{YZ}$  must be parallel to  $d_{BC'}$  and  $d_{B'C}$ .

- (c)  $Y$  and  $Z$  cannot be at infinity.

In the Euclidean plane,  $d_{A'B}$  is parallel to  $d'$  and  $d_{AB'}$  is parallel to  $d$ . Since  $d$  and  $d'$  are not parallel,  $d_{A'B}$  and  $d_{AB'}$  are not parallel and so intersect at the Euclidean point  $Z$ . Analogously for  $Y$ .

4. The points  $A$  and  $B'$  are at infinity;  $B, C, A', C'$  are not.

In this case, the line  $d_{AB'}$  is the line at infinity thus  $Z$  is at infinity in the direction of  $d_{A'B}$ . The line  $d_{AC'}$  is parallel to  $d$  while  $d_{B'C}$  is parallel to  $d'$ .

- (a)  $X$  and  $Y$  are Euclidean.

The thesis “ $Z$  on  $d_{XY}$ ” means that  $d_{XY}$  is parallel to  $d_{A'B}$ .

- (b)  $X$  and  $Y$  cannot be at infinity.

$X$  at infinity would mean that  $d_{B'C}$ , which contains the two points  $B'$  and  $X$  at infinity, is itself the line at infinity; this is not the case because  $C$  is Euclidean. Analogously for  $Y$ .

5.  $A$  and  $B$  are at infinity.

Then  $d$  is the line at infinity, thus  $C$  is at infinity as well. On the other hand  $A', B', C'$  are not at infinity, since they are not on  $d'$ . The lines  $d_{AB'}$ ,  $d_{AC'}$  are parallel, as are the lines  $d_{BA'}$ ,  $d_{BC'}$  and also the lines  $d_{CA'}$ ,  $d_{CB'}$ .



- (a) The three points  $X, Y, Z$  are Euclidean:  
these points must be on a Euclidean line.
- (b) None of the points  $X, Y, Z$  can be at infinity.  
 $X$  at infinity would mean  $X \in d$ , thus  $d_{BC'} = d$  since  $B$  and  $X$  would be on both lines; but  $C' \notin d$ . Analogously for  $Y$  and  $Z$ .

This completes the list of cases one must consider. Thus, to produce a Euclidean proof of Pappus' theorem, it 'suffices' to prove the nine Euclidean results listed above. As soon as any of these proofs uses the intersection point  $P$  of the two lines  $d$  and  $d'$ , the proof must be further split into two cases, one where  $P$  is Euclidean and the other when  $P$  is at infinity (i.e. when  $d$  is parallel to  $d'$ ). One then ends up with seventeen cases to consider!

### 6.3 Anharmonic Ratio

The considerations concerning Pappus' theorem, in Sect. 6.2, provide evidence that if we want to take full advantage of the projective context, where two lines always intersect, we should be able to develop proofs immediately in this context, without having to constantly switch back to a Euclidean setting. Unfortunately, in the projective plane, we cannot possibly transpose the Euclidean techniques based on equality or similarity of triangles, because the presence of "points at infinity" prevents the comparison of segments via their lengths. The key to overcoming the difficulty was largely popularized during the nineteenth century by *Michel Chasles* (1793–1880): it is the notion of an anharmonic ratio, already considered by Pappus (see Sect. 4.14) and preserved by central projections (see Theorem 4.14.3).

In the time of Pappus (and for many more centuries), a number was "by nature" positive, never negative. Of course during the nineteenth century, negative numbers became part of the mathematical world. Thus in the nineteenth century, an anharmonic ratio was given a sign.

**Definition 6.3.1** Consider a quadruple of two by two distinct points on the same Euclidean line. Their anharmonic ratio is the number

$$(A, B; C, D) = \frac{\frac{AC}{CB}}{\frac{AD}{DB}}$$

where the sign is determined as follows. Choose an arbitrary orientation on the line and attach a sign to the length of each segment  $AC, CB, AD, DB$ : the sign  $+$  if the segment has the direct orientation, the sign  $-$  if it has the reverse orientation.

Of course the sign of the anharmonic ratio does not depend on the chosen orientation of the line: choosing the opposite orientation changes all four signs, and hence does not affect the sign of the anharmonic ratio. The main advantage of having provided the anharmonic ratio with a sign is:



Fig. 6.3

**Proposition 6.3.2** *On a given Euclidean line, the equality of two anharmonic ratios*

$$(A, B; C, D) = (A, B; C, D')$$

*implies the equality of the points  $D$  and  $D'$ .*

*Proof* From

$$\frac{\frac{AC}{CB}}{\frac{AD}{DB}} = (A, B; C, D) = (A, B; C, D') = \frac{\frac{AC}{CB}}{\frac{AD'}{D'B}}$$

we obtain

$$\frac{AD}{DB} = \frac{AD'}{D'B}.$$

If the sign of these fractions is positive,  $D$  and  $D'$  are between  $A$  and  $B$  and divide the segment  $AB$  in the same proportion; thus  $D = D'$ . When the sign is negative,  $D$  and  $D'$  are outside the segment  $AB$ ; let us observe that they are on the same side of the segment. Indeed if the absolute value of the fraction is less than 1,  $D$  and  $D'$  are both on the side of  $A$ , otherwise they are both on the side of  $B$ . Again the equality of the fractions forces  $D = D'$ .  $\square$

Notice that Proposition 6.3.2 is no longer valid when signs are omitted. For example (see Fig. 6.3) let  $C$  be the middle point of the segment  $AB$ , so that

$$(A, B; C, D) = \frac{DB}{AD}.$$

Choose  $D$  inside  $AB$ , with  $AD = \frac{1}{3}AB$  and  $D'$  outside  $AB$ , with  $BD' = 2AD'$ . Then

$$(A, B; C, D) = 2, \quad (A, B; C, D') = -2.$$

**Proposition 6.3.3** *In the Euclidean plane, central projections preserve the anharmonic ratio as in Definition 6.3.1.*

*Proof* We know by Theorem 4.14.3 that central projections preserve the (unsigned) anharmonic ratios. The preservation of the sign is obvious.  $\square$

Thus it makes perfect sense to define

**Definition 6.3.4** Given four lines in the Euclidean plane, pairwise distinct and passing through the same point  $P$ , their anharmonic ratio  $(a, b; c, d)$  is the anharmonic ratio  $(A, B; C, D)$  of the corresponding four intersection points of these lines with an arbitrary common secant  $\ell$  not containing  $P$  (see Fig. 6.4).

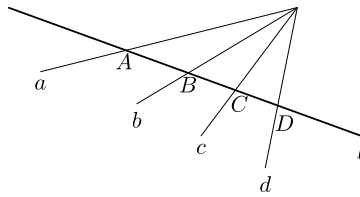


Fig. 6.4

This immediately leads to the definition of the anharmonic ratio in the projective plane:

**Definition 6.3.5** View the projective plane over a Euclidean plane  $\Pi$  as the set of lines through a point  $P \notin \Pi$  in solid space (see Proposition 6.1.3). The anharmonic ratio  $(A, B; C, D)$  of four projective points on the same projective line is their anharmonic ratio as lines in solid space.

Next observe that central projections make even better sense in the projective plane than in the Euclidean plane:

**Definition 6.3.6** Let  $S$  be a point of the projective plane and  $d, d'$  two projective lines not containing  $S$ . The central projection of  $d$  onto  $d'$ , with center  $S$ , is the mapping

$$d \longrightarrow d', \quad X \mapsto X' = d' \cap d_{SX}.$$

By Proposition 6.1.2, the central projection of  $d$  onto  $d'$  is thus defined everywhere.

**Theorem 6.3.7** *In the projective plane, central projections preserve the anharmonic ratio.*

*Proof* With the notation of Definitions 6.3.5 and 6.3.6, view  $d$  and  $d'$  as two planes through a point  $P \notin \Pi$  and  $S, A, B, C, D, A', B', C', D'$  as lines through  $P$ . Cutting the whole figure by a plane  $\sigma$  not passing through  $P$  and intersecting all the nine lines involved in the problem, the result follows at once from Proposition 6.3.3 in the plane  $\sigma$ . □

Since projective anharmonic ratios are defined via Euclidean ones, Proposition 6.3.2 extends at once:

**Proposition 6.3.8** *On a given projective line, the equality of two anharmonic ratios*

$$(A, B; C, D) = (A, B; C, D')$$

*implies the equality of the points  $D$  and  $D'$ .*

Let us also notice that, in terms of “signed” anharmonic ratios, Definition 4.9.2 must be rephrased as:

**Definition 6.3.9** A quadruple of pairwise distinct points  $(A, B; C, D)$  on a projective line is *harmonic* when its anharmonic ratio is equal to  $-1$ .

The considerations of this section show that, however, we have no notion of distance in the projective plane, thus no notion of “proportionality of segments”, it is nevertheless possible to define the notion of an “anharmonic ratio”. As the next section will show, the anharmonic ratio allows the development of elegant proofs in the projective plane, without any further reference to the Euclidean plane. But let us stress the fact that the *definition* of the anharmonic ratio in the projective plane—as in Definition 6.3.5—remains (for the time being) a very *Euclidean* one. A fully intrinsic approach to the same notion will be discussed in Sect. 6.6 of [5], *Trilogy II*.

## 6.4 The Desargues and the Pappus Theorems

*Girard Desargues* (1591–1661) was among the first mathematicians to seriously consider the possible “existence” of “points at infinity”. The theorem named after him played an important role in the development of axiomatic projective geometry (see Sect. 6.6). Of course as usual, a “triangle” consists of three points, pairwise distinct and not on the same line.

**Theorem 6.4.1** (Desargues) *In the projective plane, consider seven points  $P, A, B, C, A', B', C'$ , pairwise distinct. Assume that  $ABC$  and  $A'B'C'$  are two triangles “in perspective” from the point  $P$ , meaning that the three lines  $d_{AA'}$ ,  $d_{BB'}$ ,  $d_{CC'}$  intersect at the point  $P$  (see Fig. 6.5). In that case, the intersection points of the pairs of corresponding sides*

$$X = d_{BC} \cap d_{B'C'}, \quad Y = d_{AC} \cap d_{A'C'}, \quad Z = d_{AB} \cap d_{A'B'}$$

*are on the same projective line.*

*Proof* All intersection points considered in Fig. 6.5 exist, because we are in the projective plane (see Proposition 6.1.2). The dashed line represents the straight line through  $X$  and  $Y$ , intentionally drawn curved in order to distinguish between its (a priori arbitrary) intersection points  $Z'$  with  $d_{AB}$  and  $Z''$  with  $d_{A'B'}$ . If we prove that  $Z' = Z''$ , this point will be on the three lines  $d_{AB}$ ,  $d_{A'B'}$  and  $d_{XY}$ ; it will thus be the point  $Z$  and  $X, Y, Z$  will be on the same line as expected.

A series of central projections yields (see Theorem 6.3.7)

$$\begin{aligned} (X, Y; D'', Z') &= (C, Y; D, A) \quad (\text{center } B) \\ &= (C', Y; D', A') \quad (\text{center } P) \\ &= (X, Y; D'', Z'') \quad (\text{center } B'). \end{aligned}$$

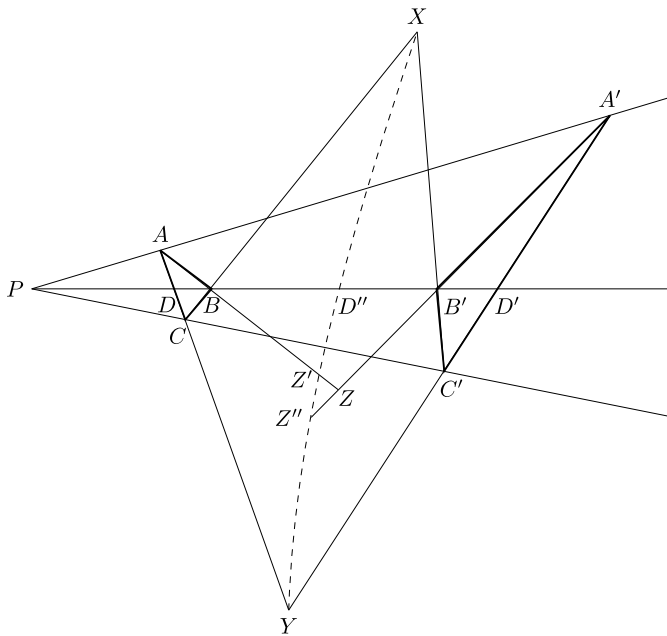


Fig. 6.5

By Proposition 6.3.8,  $Z' = Z''$ . □

Pappus’ theorem 4.14.5 also extends to the projective context and will play an important role in the axiomatization of the projective plane (see Sect. 6.6). Notice the efficiency of the proof below, compared with the multiple cases required when handling the question via Euclidean methods, as detailed in Sect. 6.2.

**Theorem 6.4.2** (Pappus) *In the projective plane, consider three points  $A, B, C$  on a line  $d$  and three points  $A', B', C'$  on another line  $d'$ . Suppose that the six points are pairwise distinct and distinct also from the intersection point of the two lines. Under these conditions the three points*

$$X = d_{BC'} \cap d_{B'C}, \quad Y = d_{AC'} \cap d_{A'C}, \quad Z = d_{AB'} \cap d_{A'B}$$

*are on the same line (see Fig. 6.6).*

*Proof* All intersection points considered in Fig. 6.5 exist, because we are in the projective plane (see Proposition 6.1.2). The dashed line represents the straight line through  $X$  and  $Y$ , intentionally drawn curved in order to distinguish its (a priori arbitrary) intersection points  $Z'$  with  $d_{AB'}$  and  $Z''$  with  $d_{A'B}$ . If we prove that  $Z' = Z''$ , this point will be on the three lines  $d_{AB'}$ ,  $d_{A'B}$  and  $d_{XY}$ ; it will thus be the point  $Z$  and  $X, Y, Z$  will be on the same line as expected.

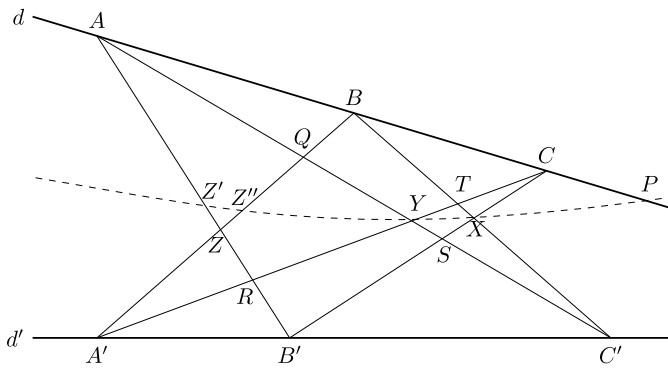


Fig. 6.6

Let us write  $P$  for the intersection of  $d$  and  $d_{XY}$ . A series of central projections yields (see Theorem 6.3.7)

$$\begin{aligned} (X, Y; P, Z') &= (X, S; C, B') \quad (\text{center } A) \\ &= (T, Y; C, A') \quad (\text{center } C') \\ &= (X, Y; P, Z'') \quad (\text{center } B). \end{aligned}$$

By Proposition 6.3.8,  $Z' = Z''$ . □

### 6.5 Axiomatic Projective Geometry

Just as Euclid had proposed (see Postulates 3.1.2) an axiomatization of plane geometry, corresponding axiomatizations have been investigated for the projective plane.

Up to now, we have established the bases of projective geometry via an essential use of Euclidean techniques. For example—following Chasles—the theory of anharmonic ratios as developed in Sect. 6.3 rests entirely on the notion of distance in the Euclidean plane. *Karl von Staudt* (1798–1867), in his treatise *Geometrie der Lage* (1847), was probably the first mathematician to develop a rigorous intrinsic approach to real projective geometry, getting rid of any Euclidean reference. *Mario Peri* (1860–1913) translated the work of von Staudt in Italian and, influenced by *Giuseppe Peano* (1858–1932), gave a system of nineteen axioms not only characterizing precisely the real projective plane, but also determining on which axioms each specific theorem depends. It was *David Hilbert* (1862–1943) who proved perhaps the most striking result concerning these axiomatizations: a very simple system of only five axioms (including the Desargues or the Pappus axiom) is sufficient to characterize the projective plane over a field, not necessarily the field of real numbers. It is to this last topic that the rest of this chapter is devoted.

In view of Postulates 3.1.2 and Proposition 6.1.2, the following axioms should certainly be part of every axiomatization of the projective plane.

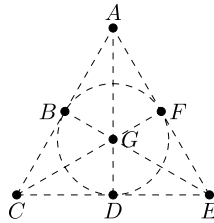


Fig. 6.7

**Definition 6.5.1** An abstract projective plane consists of:

- a set  $\mathcal{P}$ , whose elements are called “points”;
- a set  $\mathcal{L}$  of subsets of  $\mathcal{P}$ , whose elements are called “lines”.

These data must satisfy the following axioms.

**AX1** There exist three points not lying on the same line.

**AX2** Each line has at least three points.

**AX3** Through two distinct points passes exactly one line.

**AX4** Every two lines have a common point.

*Example 6.5.2* The projective plane  $\mathbb{P}(II)$ , constructed from the Euclidean plane  $II$ , is an example of an abstract projective plane.

*Proof* This follows by Postulates 3.1.2 and Proposition 6.1.2. □

However, Definition 6.5.1 is very far from an axiomatization of the projective plane constructed from a Euclidean one, as in Definition 6.1.1. Indeed, consider the following:

*Example 6.5.3* (The Fano projective plane) The seven point set

$$\mathcal{P} = \{A, B, C, D, E, F, G\}$$

with as choice of lines

$$\mathcal{D} = \{\{A, B, C\}, \{A, F, E\}, \{C, D, E\}, \\ \{A, G, D\}, \{B, G, E\}, \{C, G, F\}, \{B, D, F\}\}$$

is an abstract projective plane.

*Proof* This is straightforward to check; the Fano plane is pictured in Fig. 6.7. □

Let us now introduce a convenient point of terminology.

**Definition 6.5.4** In an abstract projective plane, we shall say that a point and a line are *incident* when the point lies on the line.

**Proposition 6.5.5** *The axioms for an abstract projective plane can equivalently (and redundantly) be stated in the form:*

- P1** *There exist three points not incident to the same line.*
- P1\*** *There exist three lines not incident to the same point.*
- P2** *Each line is incident to at least three points.*
- P2\*** *Each point is incident to at least three lines.*
- P3** *Two distinct points are incident to exactly one line.*
- P3\*** *Two distinct lines are incident to exactly one point.*

*Proof* Axiom P1 is just axiom AX1. Let  $A, B, C$  be three points not on the same line. By axiom AX3 consider the three lines  $d_{AB}, d_{BC}, d_{AC}$  through two of these points. These three lines are distinct, because the equality of two of them would imply that  $A, B, C$  are on the same line. Moreover  $d_{AB}$  and  $d_{AC}$  do not have any other intersection points than  $A$ , otherwise by axiom AX3 they would be equal and again  $A, B, C$  would have to be on the same line. Since  $A$  is not on  $d_{BC}$ , the three lines do not have any common points. This proves P1\*.

Axiom P2 is axiom AX2. Given an arbitrary point  $A$ , by axiom AX1 again, there must exist points  $B$  and  $C$  such that  $A, B, C$  are not on a line. As above, the three lines  $d_{AB}, d_{BC}, d_{AC}$  are distinct. By axiom AX2 choose a third point  $D$  on  $d_{BC}$ . The line  $d_{AD}$  is distinct from  $d_{AB}$ , otherwise  $A$  would be on  $d_{BD} = d_{BC}$ . Analogously,  $d_{AD}$  is distinct from  $d_{AC}$ . This yields three distinct lines  $d_{AB}, d_{AC}$  and  $d_{AD}$  through  $A$ . This proves P\*2.

P3 is AX3, while AX4 indicates already that two distinct lines  $d, d'$  intersect at some point  $A$ . To prove P3\*, it remains to show the uniqueness of  $A$ . If there were a second intersection point  $A'$  of  $d$  and  $d'$ , then  $d$  and  $d'$  would be equal by axiom AX3. □

Proposition 6.5.5 is the key to the so-called *duality principle* in projective geometry.

**Theorem 6.5.6** (Duality principle) *In an abstract projective plane consider an arbitrary statement, expressed in terms of points, lines and incidence, and which has been proved from the axioms. Then the dual statement, obtained when interchanging the words “point” and “line”, is a theorem as well.*

*Proof* In the formulation of Proposition 6.5.5, each axiom  $P_n^*$  is the dual of the axiom  $P_n$ . Thus all the arguments in the proof remain valid when interchanging the words “point” and “line”. □

A duality principle, based on the consideration of a conic in the real projective plane, was discovered during the 19th century, via the work of *Poncelet* on poles and polar lines with respect to a conic (see our Chap. 6 in [5], *Trilogy II*). For quite a time, it was thought that the duality principle in the real projective plane depended on the choice of conic. However, Theorem 6.5.6 shows that a duality principle exists in every abstract projective plane.



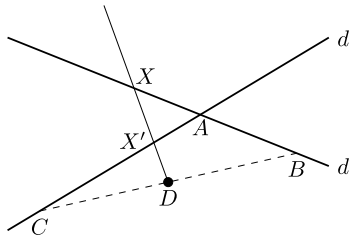


Fig. 6.8

Already with the very limited axioms of Definition 6.5.1 we have:

**Proposition 6.5.7** *In an abstract projective plane, all lines are in bijection. More precisely, given two lines  $d, d'$  and a point  $D$  not on any of them, the central projection with center  $D$  maps bijectively the line  $d$  onto the line  $d'$ .*

*Proof* Proposition 6.5.5 justifies the following arguments (see Fig. 6.8). Consider two distinct lines  $d$  and  $d'$  and their unique intersection point  $A$ . Choose  $B$  on  $d$  and  $C$  on  $d'$ , both being distinct from  $A$ . Necessarily  $B \neq C$ , otherwise  $d$  and  $d'$  would have two common points and would be equal. Choose  $D$  on  $d_{BC}$ , distinct from  $B$  and  $C$ . Once more  $D$  cannot be on  $d$ , otherwise  $d$  and  $d_{BC} = d_{BD}$  would have two common points and would be equal; this would force  $C \in d$  and thus  $C = A$ . Analogously,  $D$  is not on  $d'$ .

The central projection of  $d$  on  $d'$  with center  $D$  is a bijection from  $d$  to  $d'$ . Indeed, given  $X \in d$ ,  $d_{DX}$  is distinct from  $d'$  since  $D \notin d'$ . Thus  $d_{DX}$  intersects  $d'$  at a unique point  $X'$ . Analogously, given  $X' \in d'$ , the line  $d_{DX'}$  is distinct from  $d$  and intersects  $d$  at a unique point  $X$ . These two mappings are trivially the inverse of each other. □

**Corollary 6.5.8** *In an abstract projective plane, given two points  $A$  and  $B$ , there is a bijection between the set of lines passing through  $A$  and the set of lines passing through  $B$ . Moreover, these sets are further in bijection with the sets of points of every line.*

*Proof* The first assertion is just the dual statement (see Theorem 6.5.6) of Proposition 6.5.7. By Proposition 6.5.5, there exist three points not on the same line, thus there exists a point  $P$  not on a line  $d$ . Each point  $X$  of  $d$  then determines a line  $x = d_{PX}$  through  $P$ . Conversely every line  $x$  through  $P$  intersects  $d$  at some point  $X$ . These are inverse bijections. □

This gives us some information on the size of the finite abstract projective planes:

**Proposition 6.5.9** *If the lines of an abstract projective plane have  $n + 1$  points, then the projective plane has  $n^2 + n + 1$  points.*

*Proof* We again use Proposition 6.5.5 freely. Fixing a point  $P$ , every point  $X \neq P$  lies on a unique line through  $P$ , namely,  $d_{PX}$ . By Corollary 6.5.8 there are  $n + 1$  such lines and each of them contains  $n$  points distinct from  $P$ . Thus there are  $n(n + 1)$  points distinct from  $P$ . Together with  $P$ , this gives  $n^2 + n + 1$  points.  $\square$

It should be made clear that Proposition 6.5.9 does not claim that for every integer  $n$  (necessarily greater than 2, by axiom AX2), there exists an abstract projective plane in which the lines have  $n + 1$  points. For example, there exists a projective plane where the lines have 3 points (see Example 6.5.3), but it has been proved that there does not exist a projective plane where the lines have 7 or 11 points, while the problem of whether there exists an abstract projective plane where the lines have 13 points is still open.

In Sect. 6.7, we shall prove that for every finite field  $K$  with  $n$  elements, there exists a projective plane where each line has  $n + 1$  elements. Readers familiar with finite field theory will know that there exists a field with  $p^n$  elements, for every prime number  $p$  and every integer  $n > 0$ . The rest of this chapter will now study those abstract projective planes which necessarily arise from some field.

## 6.6 Arguesian and Pappian Planes

The axiomatics of an abstract projective plane is rather poor and as such, does not allow us to infer striking consequences. We thus have to add new axioms if we want to develop the theory further. In the very basic example of the projective plane constructed from the Euclidean plane (see Definition 6.1.1), the Desargues and the Pappus theorems are valid (see Sect. 6.4). This suggests that we should take these statements as additional axioms in an abstract projective plane.

**Definition 6.6.1** An *Arguesian plane* is an abstract projective plane satisfying the Desargues axiom:

**P4** Two triangles in perspective from a point are in perspective from a line.

Let us remark upon the above concise formulation of the Desargues axiom. “Being in perspective from a point  $P$ ” for two triangles with vertices  $ABC, A'B'C'$  has been defined in the statement of Theorem 6.4.1. “Being in perspective from a line  $p$ ”, for two triangles with sides  $abc, a'b'c'$  is just the dual notion. Looking at Fig. 6.9, one notices at once that the Desargues statement can indeed be reformulated as in Definition 6.6.1, and with this formulation, the dual of the Desargues statement is simply its converse statement.

**Proposition 6.6.2** *In an Arguesian plane:*

**P4\*** *Two triangles in perspective from a line are in perspective from a point.*

*As a consequence, the duality principle extends to Arguesian planes.*

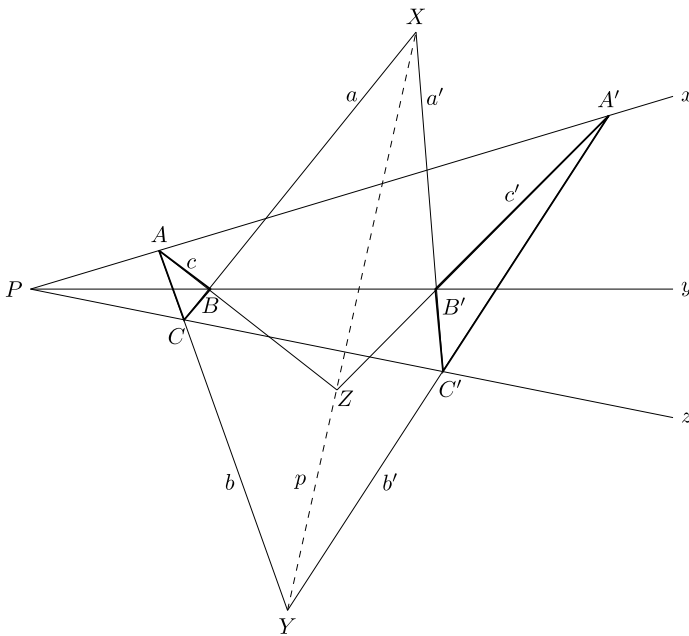


Fig. 6.9

*Proof* In Fig. 6.9, we thus assume that  $X, Y, Z$  are on a line  $P$  and we must prove that  $x, y, z$  intersect at the same point  $P$ . The assumption implies that the two triangles  $AZA'$  and  $CXC'$  are in perspective from  $Y$ . By the Desargues axiom P4 we get that

$$B = d_{AZ} \cap d_{CX}, \quad B' = d_{A'Z} \cap d_{C'X}, \quad d_{AA'} \cap d_{CC'}$$

are on the same line. In other words,  $d_{AA'} \cap d_{CC'}$  is on  $d_{BB'}$  and thus the three lines  $d_{AA'}, d_{BB'}, d_{CC'}$  intersect at the same point.  $\square$

Analogously we can consider the ‘‘Pappus axiom’’.

**Definition 6.6.3** A *Pappian plane* is an abstract projective plane satisfying the Pappus axiom:

**P5** the projective plane, consider three points  $A, B, C$  on a line  $d$  and three points  $A', B', C'$  on another line  $d'$ . Suppose that these six points are pairwise distinct and distinct also from the intersection point  $P$  of the two lines  $d$  and  $d'$ . Under these conditions the three points

$$X = d_{BC'} \cap d_{B'C}, \quad Y = d_{AC'} \cap d_{A'C}, \quad Z = d_{AB'} \cap d_{A'B}$$

are on the same line (see Fig. 6.6).

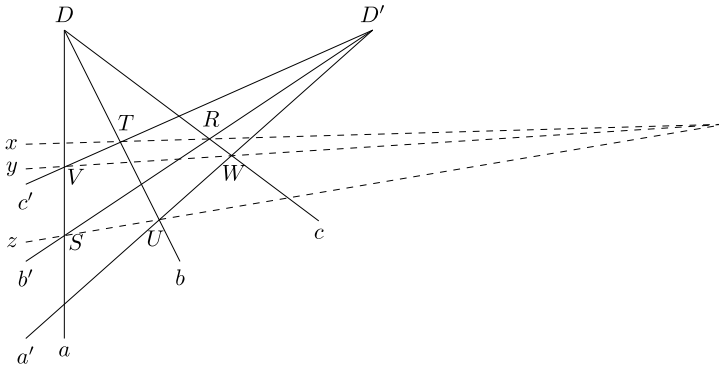


Fig. 6.10

**Proposition 6.6.4** *In a Pappian plane, the dual statement P5\* of the axiom P5 is a theorem. As a consequence, the duality principle extends to Pappian planes.*

*Proof* Consider Fig. 6.10, which defines the various intersection points to consider. We have three lines \$a, b, c\$ incident to a point \$D\$ and three other lines \$a', b', c'\$ incident to another point \$D'\$, all six lines being distinct and also distinct from the line \$d\_{DD'}\$. We consider further

$$T = b \cap c', \quad R = b' \cap c, \quad x = d_{TR}$$

and analogously for \$y\$ and \$z\$. We must prove that the three lines \$x, y, z\$ intersect at the same point. Axiom P5 applied to

- the points \$D, T, U\$ on the line \$b\$; and
- the points \$D', S, R\$ on the line \$b'\$,

implies that the points

$$V, \quad W, \quad d_{TR} \cap d_{SU}$$

are on the same line. But \$d\_{TR} \cap d\_{SU} = x \cap z\$ while \$d\_{VW} = y\$. Thus \$x \cap z\$ is on \$y\$, proving that \$x, y, z\$ intersect at the same point.  $\square$

Let us also prove that the Pappian axiom is stronger than the Arguesian axiom

**Theorem 6.6.5** (Hessenberg) *Every Pappian plane is Arguesian.*

*Proof* Consider Fig. 6.11, where the triangles \$ABC\$ and \$A'B'C'\$ are in perspective from the point \$P\$. We define as usual

$$X = d_{CB} \cap d_{C'B'}, \quad Y = d_{AC} \cap d_{A'C'}, \quad Z = d_{AB} \cap d_{A'B'};$$

we must prove that these points are on the same line.

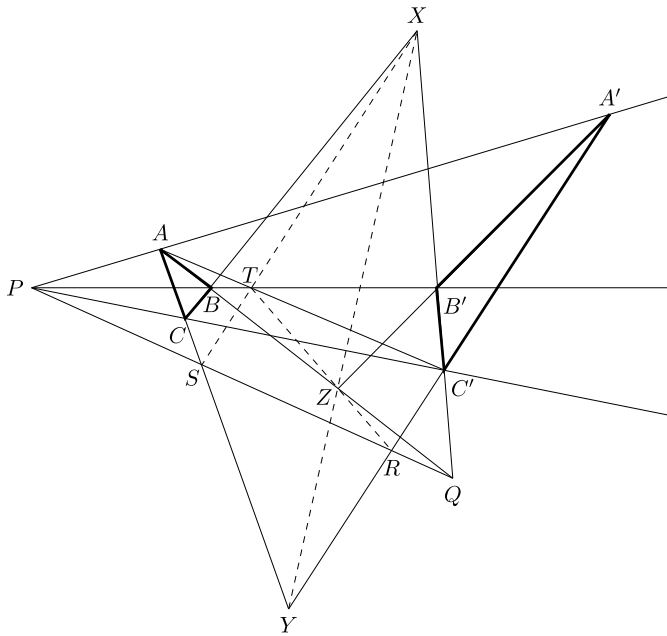


Fig. 6.11

Let us first suppose that  $A \notin d_{B'C'}$ . We define further

$$Q = d_{AB} \cap d_{B'C'}, \quad R = d_{PQ} \cap d_{A'C'},$$

$$S = d_{PQ} \cap d_{AC}, \quad T = d_{AC'} \cap d_{BB'}.$$

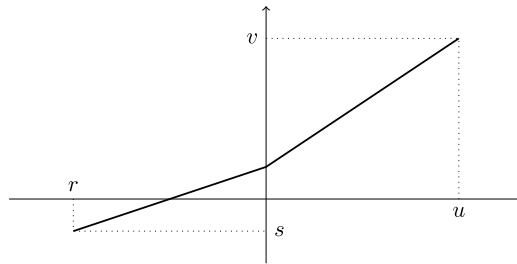
The Pappus axiom applied to the points  $(P, C', C)$  and  $(A, B, Q)$  implies that  $(S, X, T)$  are on the same line. The Pappus axiom applied to the points  $(P, A', A)$  and  $(C', Q, B')$  implies that  $(T, Z, R)$  are on the same line. If  $R \neq S$ , the Pappus axiom applied to the points  $(C', A, T)$  and  $(S, R, Q)$  implies that  $(X, Y, Z)$  are on the same line. Furthermore, if  $R = S$ , then  $R = S = Y$ , from which  $T \in d_{YZ}$  thus  $X \in d_{YZ} = d_{ST}$ .

An analogous proof holds when  $B \notin d_{A'C'}$ , or  $C \notin d_{A'B'}$ , or  $A' \notin d_{BC}$ , or  $B' \notin d_{AC}$ , or  $C' \notin d_{AB}$ . Of course one of the six possibilities must occur, otherwise  $A, B, C, A', B', C'$  would be on the same line.  $\square$

We conclude this section with a counterexample:

*Counterexample 6.6.6* An abstract projective plane which is not Arguesian.

*Proof* Let us start with the ordinary Euclidean plane  $\mathbb{R}^2$  and its usual Euclidean lines; we thus have what we shall call for now the “ordinary” projective plane as in Definition 6.1.1.



**Fig. 6.12**

Let us now introduce a new system of so-called “broken-lines” in  $\mathbb{R}^2$ . These are:

1. all the Euclidean parallels to the  $Y$ -axis:

$$X = a, \quad a \in \mathbb{R};$$

2. all the Euclidean lines with negative slope:

$$Y = aX + b, \quad a \leq 0, b \in \mathbb{R};$$

3. the “refracted lines” with positive slope, whose slope for  $X > 0$  is the double of their slope for  $X < 0$ :

$$Y = \begin{cases} aX + b & X \leq 0, \\ 2aX + b & X \geq 0, \end{cases} \quad a > 0, b \in \mathbb{R}.$$

The shape of a broken-line with positive slope is depicted in Fig. 6.12.

Notice first that two distinct points of  $\mathbb{R}^2$  can be joined by a unique broken-line. The only non-trivial case is that of two points in the following positions

$$(r, s), \quad (u, v), \quad r < 0, u > 0, s < v.$$

Two such points can only be joined by a broken-line. The “slope”  $a$  of such a line in the definition above is such that

$$a(0 - r) + 2a(u - 0) = v - s \quad \text{thus } a = \frac{v - s}{2u - r}$$

(see Fig. 6.12 again).

Two disjoint broken-lines are said to be “broken-parallel”. Just as we did in Definition 6.1.1, add a point at infinity in each “broken-direction”. It follows at once that we thus obtain an abstract projective plane in the sense of Definition 6.5.1: let us call it the “broken projective plane”.

To prove that this broken projective plane is not Arguesian, consider the Desargues configuration in Fig. 6.13. The nine points  $P, A, B, C, A', B', C', X, Y$  are all on the left hand side of the  $Y$ -axis: thus the intersection points  $X, Y$  are the same,

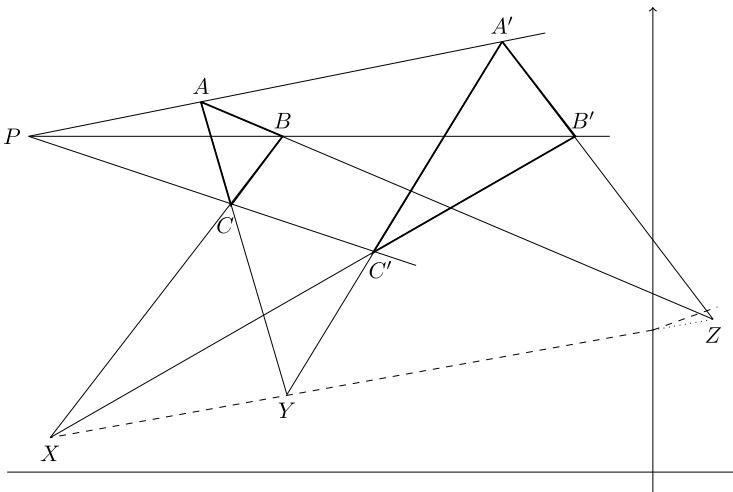


Fig. 6.13

whether working in the ordinary projective plane or in the broken plane. Moreover, the lines  $d_{AB}$  and  $d_{A'B'}$  have a negative slope: thus the point  $Z$  is also the same in the ordinary projective plane and in the broken plane. By Desargues' theorem in the ordinary projective plane (see Theorem 6.4.1), the three points  $X, Y, Z$  are on an ordinary Euclidean line. But since  $Z$  is on the right hand side of the  $Y$ -axis and  $d_{XY}$  has a strictly positive slope, the broken line through  $X$  and  $Y$  differs from the Euclidean line on the right hand side of the  $Y$ -axis; therefore, the broken line  $d_{XY}$  does not pass through  $Z$ .  $\square$

We shall provide in Counterexample 6.8.4 an example of an Arguesian plane which is not Pappian.

### 6.7 The Projective Plane over a Skew Field

This section is entirely devoted to describing a very fundamental example of a projective plane.

*Example 6.7.1* The projective plane over a skew field.

*Proof* We fix a skew field  $K$  (of course,  $K$  may be commutative, but we do not require it). Using linear algebra (with which we assume some basic familiarity), we shall construct the  $K$ -projective plane in the spirit of what has been done in Sect. 6.1 and in particular, Proposition 6.1.3. Let us recall that the usual projective plane was obtained there as the set of those lines of solid (i.e. three dimensional) space passing through a fixed point  $P$ , while the projective lines were determined by the planes passing through  $P$ .

The vector space  $K^3$  will play the role of the three-dimensional “solid space”; as point  $P \in K^3$ , we choose the origin of  $K^3$ . We choose as lines  $d$  through  $P$  the 1-dimensional vector subspace of  $K^3$ . However, as the multiplication of  $K$  is not necessarily commutative, we have to be more precise:  $d$  will be a *right* vector subspace, that is, an additive subgroup such that

$$d \subseteq K^3, \quad \forall v \in d \forall k \in K \quad vk \in d.$$

Such a line  $d$  is entirely determined by any non-zero vector on it:

$$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \neq \begin{pmatrix} a \\ b \\ c \end{pmatrix} \in d \implies d = \left\{ \begin{pmatrix} ak \\ bk \\ ck \end{pmatrix} \mid k \in K \right\}.$$

Trivially, two non-zero vectors define the same line  $d$  precisely when they are right proportional. Therefore the set  $\mathbb{P}(K)$  of lines through the origin can equivalently be defined as

$$\mathbb{P}(K) = \frac{\left\{ \begin{pmatrix} a \\ b \\ c \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \mid a, b, c \in K \right\}}{\approx}$$

where  $\approx$  is the equivalence relation defined by

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \approx \begin{pmatrix} a' \\ b' \\ c' \end{pmatrix} \quad \text{iff} \quad \exists k \in K \quad \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a'k \\ b'k \\ c'k \end{pmatrix}.$$

We shall write

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \text{equivalence class of } \begin{pmatrix} a \\ b \\ c \end{pmatrix}.$$

The set  $\mathbb{P}(K)$  is our candidate for the  $K$ -projective plane.

The projective lines are now chosen to be the sets of vector lines through the origin, lying in the same vectorial plane. A vectorial plane  $\sigma \subseteq K^3$  is determined by an equation of degree 1 with coefficients in  $K$ :

$$\sigma = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mid ux + vy + wz = 0 \right\}, \quad (0 \ 0 \ 0) \neq (u \ v \ w), \quad u, v, w \in K.$$

Two triples of coefficients determine the same vectorial plane precisely when they are proportional on the left:

$$(u \ v \ w) \approx (u' \ v' \ w') \quad \text{when } \exists k \in K \quad (u \ v \ w) = (ku' \ kv' \ kw').$$

We can thus identify the set of lines with the quotient

$$\frac{\{(u \ v \ w) \neq (0 \ 0 \ 0) \mid u, v, w \in K\}}{\approx}.$$



Again we write

$$[u \ v \ w] = \text{equivalence class of } (u \ v \ w).$$

We then have

$$[u \ v \ w] \quad \text{and} \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \text{are incident} \quad \text{when} \quad ua + vb + wc = 0.$$

This incidence relation is trivially compatible with the two equivalence relations  $\approx$  above.

Let us prove that we have so defined an abstract projective plane. We refer to the notation of Definition 6.5.1.

**AX1** The vector space  $K^3$  has dimension 3, thus there exist three linearly independent vectors and therefore three vector lines not in the same vector plane.

**AX2** A vector plane  $\sigma$  contains two linearly independent vectors  $v, w$ . But  $v + w \in \sigma$  cannot be a multiple of  $v$  or  $w$ , since  $v + w = vk$  for  $k \in K$  would imply  $w = v(k - 1)$ , contradicting the linear independence of  $v$  and  $w$ . Thus  $\sigma$  contains three pairwise linearly independent vectors  $v, w, v + w$  and these determine three distinct vector lines.

**AX3** Given two distinct vector lines  $d, d'$  and two vectors  $0 \neq v \in d, 0 \neq w \in d'$ , the unique vector plane containing  $d$  and  $d'$  is the set of linear combinations  $vk + v'k'$  of  $v$  and  $w$ .

**AX4** In terms of dimensions, two vector planes  $\sigma$  and  $\sigma'$  yield

$$\begin{aligned} \dim \sigma = 2 = \dim \sigma', \quad \dim(\sigma + \sigma') &\leq \dim K^3 = 3, \\ 4 = \dim \sigma + \dim \sigma' = \dim(\sigma + \sigma') + \dim(\sigma \cap \sigma') &\leq 3 + \dim(\sigma \cap \sigma'). \end{aligned}$$

Thus  $\dim(\sigma \cap \sigma') \geq 1$  and  $\sigma \cap \sigma'$  contains at least one vector line.

Of course, an analogous proof could have been developed using left vector subspaces. □

**Proposition 6.7.2** *The projective plane over a skew field is Arguesian.*

*Proof* We refer to Fig. 6.9 and use the notation of Example 6.7.1. Let us put

$$P = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix}, \quad A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}, \quad C = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

Since  $A'$  is on the projective line through  $P$  and  $A$ , a vector  $w \in K^3$  determining  $A'$  is linear combination of two vectors  $u, v \in K^3$  determining respectively  $P$  and  $A$ ,

and analogously for  $B'$  and  $C'$ . Thus

$$A' = \begin{bmatrix} p_1k + a_1l \\ p_2k + a_2l \\ p_3k + a_3l \end{bmatrix}, \quad B' = \begin{bmatrix} p_1k' + b_1l' \\ p_2k' + b_2l' \\ p_3k' + b_3l' \end{bmatrix}, \quad C' = \begin{bmatrix} p_1k'' + c_1l'' \\ p_2k'' + c_2l'' \\ p_3k'' + c_3l'' \end{bmatrix}$$

with  $k, l, k', l', k'', l'' \in K$ .

Let us now observe that trivially

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} lk^{-1} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} l'(k')^{-1} = \begin{pmatrix} p_1k + a_1l \\ p_2k + a_2l \\ p_3k + a_3l \end{pmatrix} k^{-1} - \begin{pmatrix} p_1k' + b_1l' \\ p_2k' + b_2l' \\ p_3k' + b_3l' \end{pmatrix} (k')^{-1}.$$

The left hand side indicates that the corresponding point is on the projective line  $d_{AB}$ , while the right hand side indicates that it is on the projective line  $d_{A'B'}$ . This is thus the point  $Z$ . Doing the same thing for  $X$  and  $Y$  we conclude that

$$\begin{aligned} X &= \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} l'(k')^{-1} - \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} l''(k'')^{-1}, \\ Y &= \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} l''(k'')^{-1} - \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} lk^{-1}, \\ Z &= \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} lk^{-1} - \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} l'(k')^{-1}. \end{aligned}$$

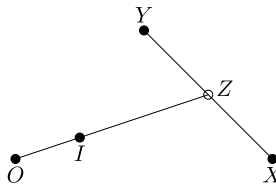
Observe that the sum of the corresponding first two vectors yields the opposite of the third one. Thus the three corresponding vectors are linearly dependent, that is, lie in the same vector plane. In other words,  $X, Y, Z$  are on the same projective line.  $\square$

## 6.8 The Hilbert Theorems

This section shows that every Arguesian plane is the projective plane over a skew field; the Arguesian plane is Pappian precisely when the field is commutative. These results are due to *David Hilbert* (1862–1943). Throughout this section, we continue to use the notation  $d_{AB}$  to indicate the line through two points  $A$  and  $B$ .

**Lemma 6.8.1** *In an abstract projective plane, there exist four points  $O, X, Y, I$  such that three of them are not on the same line.*

*Proof* We refer to the axiomatics 6.5.5 and Fig. 6.14. By Axiom P1, there exist three points  $O, X, Y$  not on the same line. By Axiom P2, there is a point  $Z$



**Fig. 6.14**

on  $d_{XY}$ , distinct from  $X$  and  $Y$ . Still by Axiom P2, there is a point  $I$  on  $d_{OZ}$  distinct from  $O$  and  $Z$ . Observe that  $d_{OZ} \neq d_{XY}$  because  $O \notin d_{XY}$ . By Axiom P3\*,  $d_{OZ} \cap d_{XY} = \{Z\}$ , thus  $I \notin d_{XY}$ . By Axiom P3,  $I \in d_{OX}$  would force  $d_{OX} = d_{OI}$  thus  $X \in d_{OI} \cap d_{XY} = \{Z\}$ , which is not the case; thus  $I \notin d_{OX}$ . Analogously,  $I \notin d_{OY}$ . Thus  $I$  is on none of the three lines  $d_{OX}, d_{OY}, d_{XY}$ . But then  $O \notin d_{XI}$  since this would force  $I \in d_{OX}$ , and analogously for the other possibilities.  $\square$

**Theorem 6.8.2** (Hilbert) *An abstract projective plane is Arguesian if and only if it is the projective plane over a skew field.*

*(Of course the skew field in this statement may be commutative.)*

*Proof* The implication in one direction is the content of Proposition 6.7.2. Conversely, let us consider an Arguesian plane  $(\mathcal{P}, \mathcal{D})$ . In this proof, we fix once and for all a quadruple  $(O, X, Y, I)$  of points as in Lemma 6.8.1. We define further the points

$$I_X = d_{IY} \cap d_{OX}, \quad I_Y = d_{IX} \cap d_{OY}, \quad Z = d_{OI} \cap d_{XY}.$$

To clarify the scheme of this long proof, we split it into several steps.  $\square$

**Step 1. Borrowing our Intuition from the Euclidean Case** The proof will be better understood by borrowing one’s intuition from the case of the projective plane constructed from the Euclidean plane (see Definition 3.1.44 and Theorem 6.4.1). In that case, view

- $O$  as the origin of the coordinate axes in the Euclidean plane;
- $X$  and  $Y$  as the points at infinity in the directions of the two coordinate axes;
- $d_{OI}$  as the “first diagonal”.

In this very special case, we thus think of

- $d_{XY}$  as being the line at infinity;
- two projective lines intersecting at a point on  $d_{XY}$  as parallel Euclidean lines;
- $Z$  as the point at infinity in the direction of the first diagonal  $d_{OI}$ .

**Step 2. The System of Coordinates** In the projective plane constructed from the Euclidean plane, the “field of coordinates” (i.e. the field of real numbers) is in bijection with all the Euclidean points of an arbitrary line, that is, all the points with

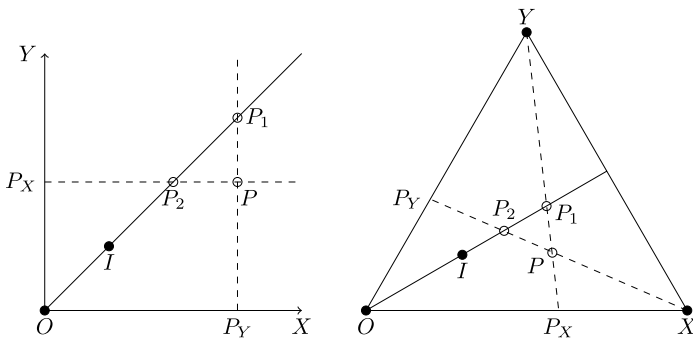


Fig. 6.15

the exception of the point at infinity. Of course any line can do the job: but since symmetry often facilitates computation, we shall work with the first diagonal.

In the case of an abstract Arguesian plane, we thus define

$$K = d_{OI} \setminus \{Z\}.$$

Let us first prove that  $K$  provides a “system of Cartesian coordinates for the points not at infinity”, that is, there is a bijection

$$\mathcal{P} \setminus d_{XY} \cong K^2.$$

In the Euclidean case, the two coordinates  $(P_X, P_Y)$  of a point  $P$  are obtained by drawing through  $P$  the parallels to the axes (see the left hand diagram of Fig. 6.15). In the projective case we thus define, given  $P \in \mathcal{P} \setminus d_{XY}$  (see the right hand diagram of Fig. 6.15)

$$P_1 = d_{OI} \cap d_{YP}, \quad P_2 = d_{OI} \cap d_{XP}.$$

Let us prove that this defines a bijection

$$\Gamma: \mathcal{P} \setminus d_{XY} \rightarrow K^2, \quad P \mapsto (P_1, P_2).$$

First  $P_1 \neq Z$ , otherwise one would have  $d_{XY} = d_{PY}$  and thus  $P \in d_{XY}$ ; thus  $P_1 \in K$  and analogously for  $P_2$ . The mapping  $\Gamma$  is therefore well defined.

The mapping  $\Gamma$  is injective. Indeed if  $\Gamma(P) = \Gamma(Q)$ , from  $P_1 = Q_1$  we deduce that  $d_{PY} = d_{QY}$  thus  $Q \in d_{PY}$ . Analogously  $Q \in d_{PX}$ . Thus by Proposition 6.5.5

$$Q = d_{PX} \cap d_{PY} = P.$$

The mapping  $\Gamma$  is also surjective. Given  $(P_1, P_2) \in K^2$ , the two lines  $d_{YP_1}$  and  $d_{XP_2}$  are distinct, otherwise they would be the line  $d_{XY}$ , which would contradict  $P_i \in K$ . Thus these two lines intersect at a unique point  $P$  (see Proposition 6.5.5) and then by definition of  $\Gamma$ ,  $\Gamma(P) = (P_1, P_2)$ .

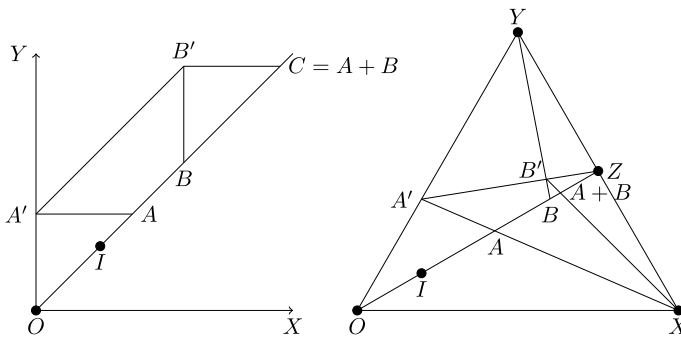


Fig. 6.16

Of course we know by Proposition 6.5.7 that the central projection

$$p_Y : d_{OI} \rightarrow d_{OX}$$

with center  $Y$  is a bijection; it fixes  $O$  and interchanges  $Z$  and  $X$ . Thus  $p_Y$  yields an isomorphism

$$p_X : K = (d_{OI} \setminus \{Z\}, O, I) \xrightarrow{\cong} (d_{OX} \setminus \{X\}, O, I_X)$$

mapping  $P_I$  onto  $P_{I_X}$ , as in Fig. 6.15 again. An analogous argument holds for the “second axis”  $d_{OY}$  via the central projection  $p_X$  with center  $X$ . This shows the equivalence of our mapping  $\Gamma$  with the more usual  $(X, Y)$ -system of coordinates of a point.

*From now on, in this proof, we leave to the reader the straightforward arguments showing, via Proposition 6.5.5, that the various points considered are correctly defined as intersections of two distinct lines.*

**Step 3. Addition on  $K$**  Our next concern is to provide  $K$  with an addition. Consider first the Euclidean case (see the left hand diagram of Fig. 6.16) and choose two points  $A, B$  on the first diagonal. Draw  $AA'$  parallel to the  $X$ -axis,  $BB'$  parallel to the  $Y$ -axis,  $A'B'$  parallel to the first diagonal and finally  $B'C$  parallel to the  $X$ -axis again. This yields two parallelograms  $OA'B'B$  and  $AA'B'C$ . By Proposition 3.1.38

$$OC = OA + AC = OA + A'B' = OA + OB.$$

This provides an addition defined by  $A + B = C$ .

In the projective case (right hand diagram of Fig. 6.16) we thus define

$$A' = d_{XA} \cap d_{OY}, \quad B' = d_{A'Z} \cap d_{YB}, \quad A + B = d_{XB'} \cap d_{OZ}.$$

**Step 4. The Zero Element** In the definition of addition, if  $A = O$ , then  $A' = O$ ,  $B' = B$  and  $A + B = B$ . Furthermore, if  $B = O$ ,  $B' = A'$  and  $A + B = A$ . This proves that  $O$  is a zero element for the addition defined in Step 3.

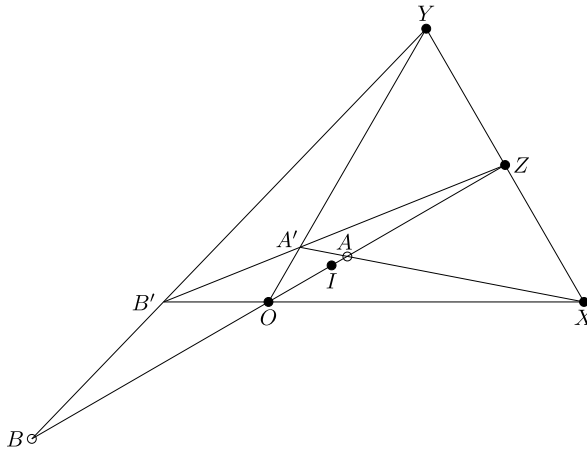


Fig. 6.17

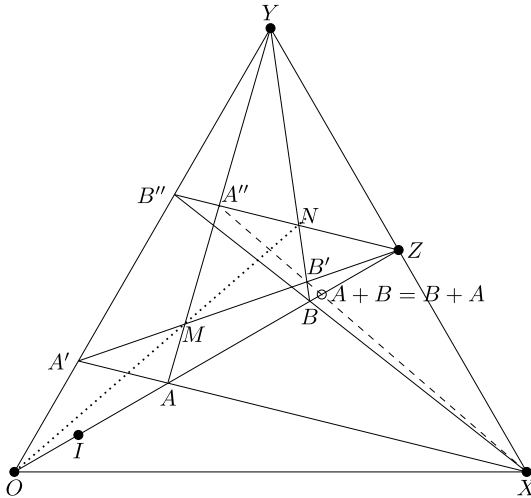


Fig. 6.18

**Step 5. The Opposites** Given  $A \in K$ , define

$$A' = d_{XA} \cap d_{OY}, \quad B' = d_{A'Z} \cap d_{OX}, \quad B = d_{YB'} \cap d_{OZ}.$$

The notation is compatible with that of Step 3 and Fig. 6.17 proves at once that  $A + B = 0$ . The equality  $B + A = 0$  will follow from the commutativity of addition.

**Step 6. Commutativity of Addition** By Step 4, it suffices to prove the commutativity of addition for two non-zero elements  $A, B \in K$  and, of course, for two distinct elements! We refer to Fig. 6.18: it reproduces first the right hand diagram of

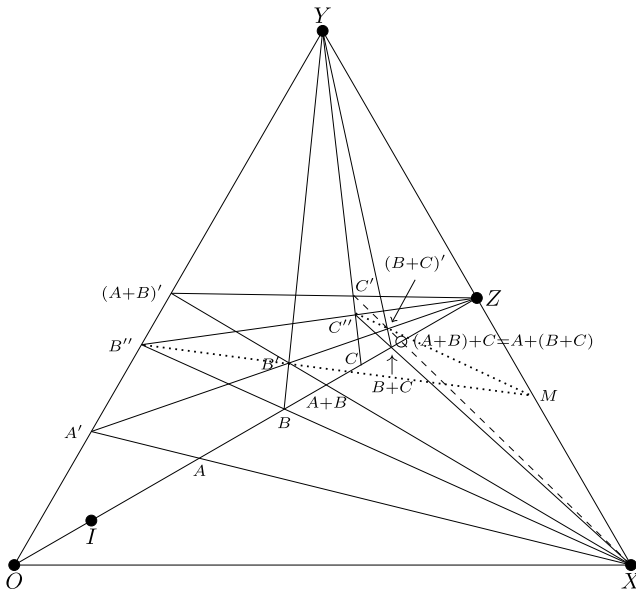


Fig. 6.19

Fig. 6.16 defining  $A + B$ , then considers further the corresponding points needed to define  $B + A$

$$B'' = d_{XB} \cap d_{OY}, \quad A'' = d_{B''Z} \cap d_{YA}, \quad B + A = d_{XA''} \cap d_{OZ}.$$

Proving the equality  $A + B = B + A$  thus reduces to proving that the three points  $B', A'', X$  are on the same line.

The two triangles  $A'B''Z$  and  $ABY$  are in perspective from the point  $X$ ; by the Desargues axiom P4, the three points  $O, M, N$  are on the same line. Therefore the two triangles  $BB''N$  and  $ZYM$  are in perspective from the point  $O$  and by the Desargues axiom P4, the three points  $B', A'', X$  are on the same line.

**Step 7. Associativity of Addition** We refer to Fig. 6.19 where, following the definition in Step 3, one constructs successively

- $A', B', A + B$ ;
- $(A + B)', C', (A + B) + C$ ;
- $B'', C'', B + C$ ;
- $A', (B + C)', A + (B + C)$ .

The equality  $(A + B) + C = A + (B + C)$  then reduces to the fact that the three points  $C', (B + C)'$  and  $X$  are on the same line.

The two triangles  $BXY$  and  $ZC''(B + C)'$  are in perspective from the point  $B + C$ ; by the Desargues axiom P4 the three points

$$B'', \quad B', \quad M = d_{XY} \cap d_{C''(B+C)'}$$

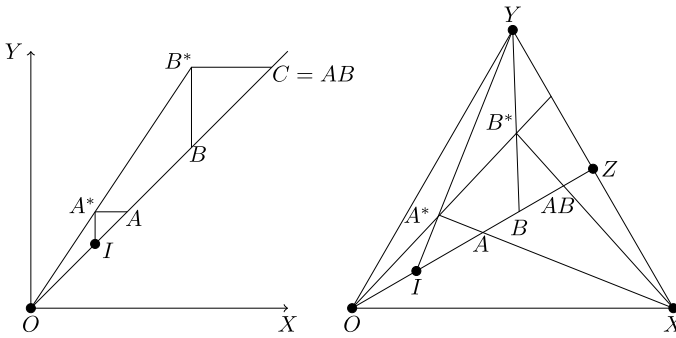


Fig. 6.20

are on the same line. Therefore the triangles  $(A + B)'ZB'$  and  $YC''M$  are in perspective from the point  $B''$  and the Desargues axiom P4 implies as expected that the three points  $C', X, (B + C)'$  are on the same line.

**Step 8. Multiplication on  $K$**  We have already provided  $K$  with the structure of an abelian group  $(K, +, O)$ . Let us next define a multiplication on  $K$ . The left hand diagram of Fig. 6.20 presents the situation in the Euclidean case, given two points  $A, B$  in  $K$ .

- $A^*$  is the intersection of the parallel to the  $X$  axis through  $A$  and the parallel to the  $Y$ -axis through  $I$ .
- $B^*$  is the intersection of  $d_{OA^*}$  and the parallel to the  $Y$ -axis through  $B$ .
- $C$  is the intersection of the first diagonal with the parallel to the  $X$ -axis through  $B^*$ .

By Propositions 3.1.31 and 3.6.4, we have three pairs of similar triangles:  $B^*OC$  and  $A^*OA$ ,  $B^*BC$  and  $A^*IA$ ,  $B^*BO$  and  $A^*IO$ . This implies (see Definition 3.6.3)

$$\frac{OC}{OA} = \frac{B^*C}{A^*A} = \frac{B^*B}{A^*I} = \frac{OB}{OI}.$$

If we now think of the segment  $OI$  as being the unit length, we obtain  $OC = OA \cdot OB$ . Therefore it is sensible to define  $C = AB$ , the product of the two points  $A$  and  $B$  in  $K$ .

Let us now switch to the projective case, pictured on the right hand side of Fig. 6.20. Given  $A, B \in K$ , we define

$$A^* = d_{AX} \cap d_{IY}, \quad B^* = d_{OA^*} \cap d_{BY}, \quad AB = d_{B^*X} \cap d_{OI}.$$

**Step 9. The Zero and the Unit for the Multiplication** If  $A = 0$ , then  $A^* \in d_{OX}$ , thus  $B^* \in d_{OX}$  and therefore  $AB = O$ . If  $B = O$  then  $B^* = O$  and thus  $AB = O$ . So we have proved that  $OB = O = AO$ .

If  $A = I$ , then  $A^* = I$ ,  $B^* = B$  and  $AB = B$ . If  $B = I$ , then  $A^* = B^*$  and  $AB = A$ . This proves that  $IB = B$  and  $AI = A$ , thus  $I$  is a unit for the multiplication.



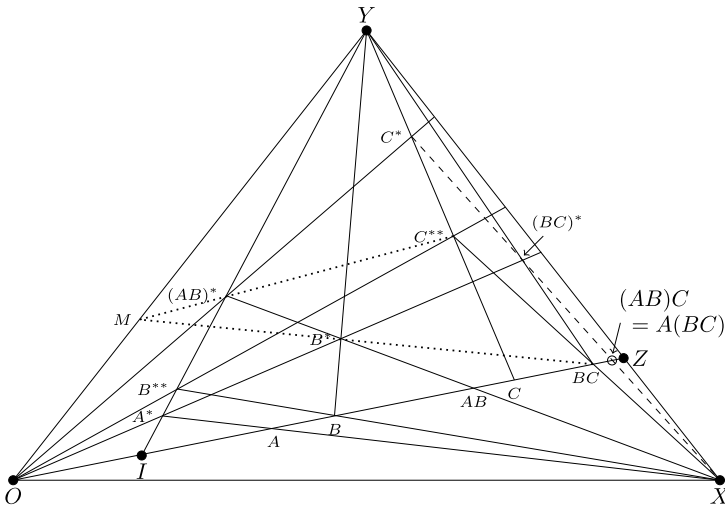


Fig. 6.21

**Step 10. Associativity of Multiplication** We refer to Fig. 6.21 where, following the definition in Step 9, one constructs successively

- $A^*, B^*, AB$ ;
- $(AB)^*, C^*, (AB)C$ ;
- $B^{**}, C^{**}, BC$ ;
- $A^*, (BC)^*, A(BC)$ .

The thesis then reduces to proving that the three points  $C^*, (BC)^*, X$  are on the same line.

The triangles  $BB^*(BC)$  and  $B^{**}(AB)^*C^{**}$  are in perspective from the point  $X$ ; by the Desargues axiom P4, the points

$$Y, \quad O, \quad M = d_{B^*(BC)} \cap d_{(AB)^*C^{**}}$$

are on the same line. Therefore the triangles  $O(AB)^*B^*$  and  $YC^{**}(BC)$  are in perspective from the point  $M$  and by Desargues' axiom P4,  $C^*, (BC)^*$  and  $X$  are on the same line.

**Step 11. The Inverses** We refer to Fig. 6.22. Given  $A \in K$ , let us first successively construct

$$B = d_{AX} \cap d_{IY}, \quad C = d_{BO} \cap d_{IX}, \quad D = d_{CY} \cap d_{OI}.$$

With the notation of Step 8, the construction of  $AD$  is obtained as follows:

$$A^* = B, \quad D^* = C, \quad AD = I.$$

Thus  $D$  is right inverse to  $A$ .

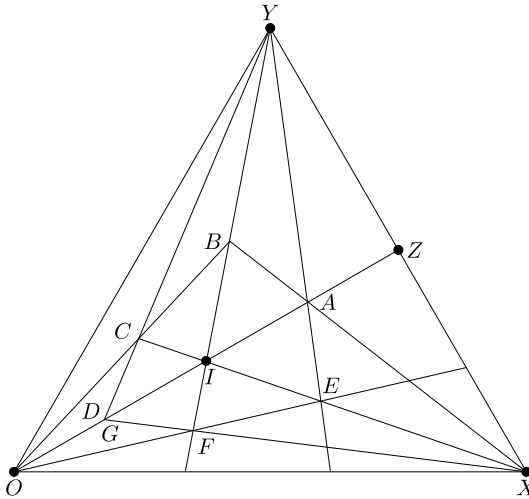


Fig. 6.22

Let us now construct instead

$$E = d_{AY} \cap d_{XI}, \quad F = d_{EO} \cap d_{IY}, \quad G = d_{FX} \cap d_{OI}.$$

Still using the notation of Step 8, the construction of  $GA$  is obtained as follows:

$$G^* = F, \quad A^* = E, \quad GA = I.$$

Thus  $G$  is left inverse to  $A$ .

It is well-known that, in view of Steps 9 and 10, the left and the right inverse of  $A$  are equal:

$$D = ID = (GA)D = G(AD) = GI = G.$$

**Step 12. The Distributivity Laws** By Step 9, the first distributivity law

$$(A + B)C = AC + BC$$

requires a proof only when  $A \neq O, B \neq O, C \neq O, C \neq I$ . We refer to Fig. 6.23 where, following the constructions of Steps 3 and 8, we have defined successively:

- $A', B', A + B$ ;
- $(A + B)^*, C^*, (A + B)C$ ;
- $A^*, C^{**}, AC$ ;
- $B^*, C^{***}, BC$ ;
- $(AC)', (BC)', AC + BC$ .

The thesis thus reduces to proving that the three points  $C^*, (BC)'$  and  $X$  are on the same line.

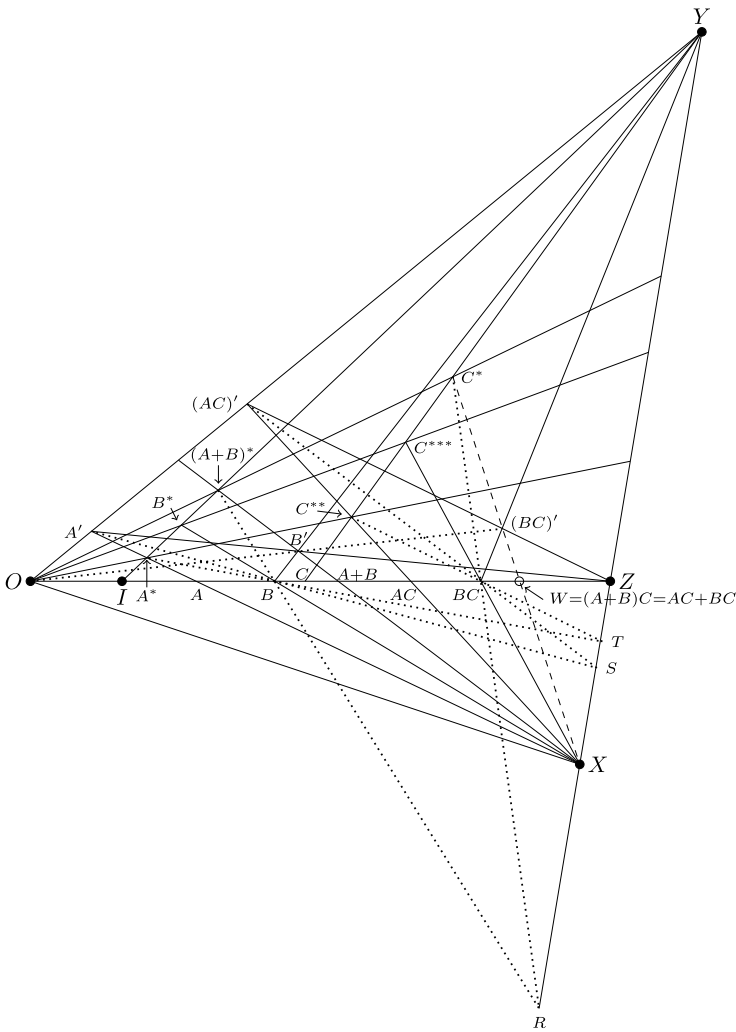


Fig. 6.23

The triangles  $B^*A^*B$  and  $C^{***}C^{**}(BC)$  are in perspective from the point  $O$ ; by Desargues' axiom P4, the three points

$$Y, \quad X, \quad T = d_{A^*B} \cap d_{C^{**}(BC)}$$

are on the same line. The triangles  $A'A^*B$  and  $(AC)'C^{**}(BC)$  are in perspective from the point  $O$  as well; by Desargues' axiom P4, the three points

$$X, \quad S = d_{A'B} \cap d_{(AC)'(BC)}, \quad T$$

are on the same line, namely, the line  $d_{XY}$  since we know already that  $d_{XT} = d_{XY}$ . All this implies that the two triangles  $A'B'B$  and  $(AC)'(BC)'(BC)$  are in perspective from the line  $d_{XY}$ ; by the dual P4\* of Desargues' axiom (see Proposition 6.6.2) the three lines

$$d_{A'(AC)'}, \quad d_{B'(BC)'}, \quad d_{B(BC)}$$

intersect at the same point; since  $d_{A'(AC)'} \cap d_{B(BC)} = O$ , this means that the three points  $O$ ,  $B'$ ,  $(BC)'$  are on the same line. Then the two triangles  $(A+B)^*B^*B$  and  $C^*C^{***}(BC)$  are also in perspective from the point  $O$ ; once more Desargues' axiom P4 implies that the three points

$$Y, \quad R = d_{(A+B)^*B} \cap d_{C^*(BC)}, \quad X$$

are on the same line. It remains to observe that the two triangles  $(A+B)^*B'B$  and  $C^*(BC)'(BC)$  are in perspective from the point  $O$ , from which by Desargues' axiom P4, we deduce the collinearity of the three points

$$W = d_{(A+B)^*B'} \cap d_{C^*(BC)'}, \quad R, \quad Y.$$

Since  $d_{RY} = d_{XY}$ , this proves that

$$W \in d_{XY} \cap d_{(A+B)^*B'} = X.$$

Thus  $X = W \in d_{C^*(BC)'}$  and as expected, the three points  $C^*$ ,  $(BC)'$  and  $X$  are on the same line.

To complete the proof that we have provided  $K$  with the structure of a field, it remains to check the second distributivity law

$$A(B+C) = AB + AC.$$

Again by Step 9 it suffices to prove this when  $A \neq O$ ,  $B \neq O$ ,  $C \neq O$ ,  $C \neq I$ . We refer to Fig. 6.24 where, following the constructions of Steps 3 and 8, we have defined successively:

- $B', C', B+C$ ;
- $A^*, (B+C)^*, A(B+C)$ ;
- $A^*, B^*, AB$ ;
- $A^*, C^*, AC$ ;
- $(AB)', (AC)', AB+AC$ .

The thesis thus reduces to proving that the three points  $(B+C)^*$ ,  $(AC)'$  and  $X$  are on the same line.

Let us write  $R = d_{OA^*} \cap d_{XY}$ . The triangles  $YZC^*$  and  $B'BR$  are in perspective from the point  $O$ ; by the Desargues axiom P4, the points

$$X, \quad U = d_{YC^*} \cap d_{B'R}, \quad V = d_{ZC^*} \cap d_{BR}$$

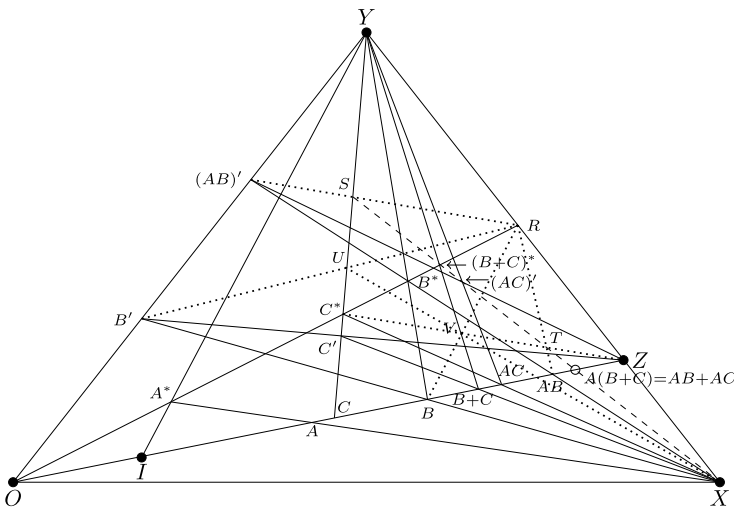


Fig. 6.24

are on the same line. The triangles  $UXC'$  and  $RBZ$  are in perspective from the point  $B'$ ; by the Desargues axiom P4 the three points

$$V' = d_{UX} \cap d_{RB}, \quad Y, \quad B + C$$

are on the same line. But since  $V \in d_{UX} \cap d_{RB}$ , we necessarily have  $V' = V$ . The triangles  $YZC^*$  and  $(AB)'(AB)R$  are in perspective from the point  $O$ ; by the Desargues axiom P4 the three points

$$X, \quad S = d_{YC^*} \cap d_{(AB)'R}, \quad T = d_{ZC^*} \cap d_{(AB)R}$$

are on the same line. Thus  $d_{SX} = d_{TX}$ . The triangles  $Y(AC)C^*$  and  $(AB)'ZR$  are in perspective from the point  $O$ ; by the Desargues axiom P4 the three points

$$(AC)', \quad S, \quad X$$

are on the same line. Thus  $(AC)' \in d_{SX}$ . Finally the triangles  $B^*R(AB)$  and  $YVZ$  are in perspective from the point  $B$ ; by the Desargues axiom P4, the three points

$$(B + C)^*, \quad X, \quad T$$

are on the same line. Thus  $(B + C)^* \in d_{TX}$ . Therefore  $(AC)', (B + C)^*$  and  $X$  are on the same line  $d_{SX} = d_{TX}$ .

**Step 13. The Equation of a Line** Going back to Step 2 and using its notation, it follows at once that given a point  $P \in \mathcal{P} \setminus d_{XY}$

- $P_1 = O$  if and only if  $P \in d_{OY}$ ;
- $P_2 = O$  if and only if  $P \in d_{OX}$ .

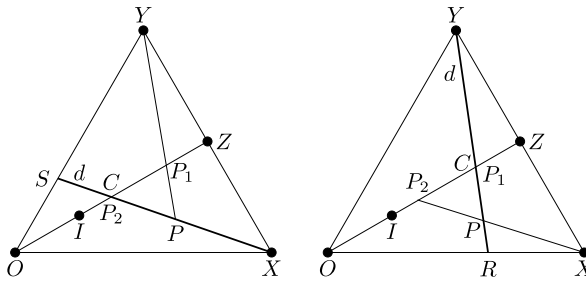


Fig. 6.25

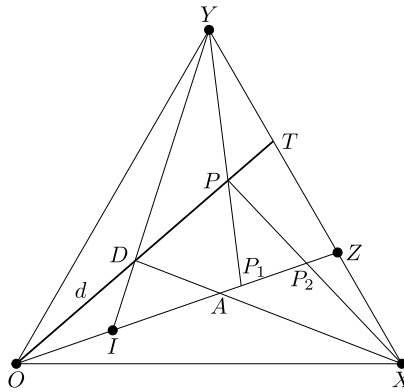


Fig. 6.26

Thus the trace on  $\mathcal{P} \setminus d_{XY}$  of

- $d_{OY}$  admits the equation  $P_1 = O$ ;
- $d_{OX}$  admits the equation  $P_2 = O$ .

Next consider a line  $d \neq d_{OX}$  passing through  $X$ , thus intersecting  $d_{OY}$  at some point  $S$  distinct from  $O$  and  $Y$  (see the left hand diagram of Fig. 6.25). Putting  $C = d \cap d_{OI} \in K$ , we have  $P_2 = C$ . Conversely if a point  $P$  is such that  $P_2 = C$ , then  $P \in d_{CX} = d$ . Thus  $P_2 = C$  is the equation of the trace on  $\mathcal{P} \setminus d_{XY}$  of the line  $d_{CX}$ . Analogously,  $P_1 = C$  is the equation of the trace of the line  $d_{YC}$ .

Consider next a line  $d$  passing through  $O$  but distinct from  $d_{OX}$  and  $d_{OY}$ . Write

$$D = d \cap d_{IY}, \quad A = d_{DX} \cap d_{OI}.$$

Given a point  $P \in d$ ,  $P \notin d_{XY}$ , as depicted in Fig. 6.26, with the notation of Step 8 the definition of  $AP_1$  yields

$$A^* = D, \quad P_1^* = P, \quad P_2 = AP_1.$$

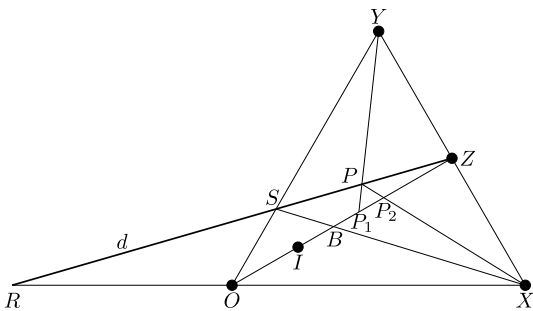


Fig. 6.27

Conversely an arbitrary point  $P$  is the intersection of  $d_{YP_1}$  and  $d_{XP_2}$  (see Step 2). If  $P_2 = AP_1$ , then by Step 8

$$A^* = D, \quad P_1^* = d_{OD} \cap d_{P_1Y}, \quad P_2 = d_{P_1^*X} \cap d_{OI}.$$

Thus  $P$  and  $P_1^*$  are both on  $d_{P_1Y}$  and  $d_{P_2X}$ , proving the equality  $P_1^* = P$ . So  $P \in d_{OD} = d$ . This proves that  $P \in d$  if and only if  $P_2 = AP_1$ . Thus  $d$  admits the equation  $P_2 = AP_1$ . Conversely our argument shows that  $P_2 = AP_1$  is the equation of the trace of the line  $d = d_{OD}$ , where  $D = d_{AX} \cap d_{IY}$ .

Next we consider the case of a line  $d$  passing through  $Z$  but not through  $O$ ; the line  $d$  cuts  $d_{OX}$  at a point  $R$  and  $d_{OY}$  at a point  $S$  (see Proposition 6.5.5). Consider  $B = d_{OI} \cap d_{SX}$  (see Fig. 6.27). With the notation of Step 3 we have

$$S = B', \quad P = P', \quad P_2 = B + P_1.$$

Conversely an arbitrary point  $P$  is the intersection of  $d_{YP_1}$  and  $d_{XP_2}$  (see Step 2). If  $P$  is such that  $P_2 = B + P_1$ , then by Step 2

$$B' = S, \quad P' = d_{SZ} \cap d_{P_1Y}, \quad P_2 = d_{P'_X} \cap d_{OI}.$$

Then  $P$  and  $P'_X$  are both on  $d_{P_1Y}$  and  $d_{P_1YX}$ , thus  $P'_1 = P$ . So  $P \in d_{SZ} = d$ . This proves that  $P \in d$  if and only if  $P_2 = B + P_1$ ; thus  $d$  admits the equation  $P_2 = B + P_1$ . Conversely, our argument shows that  $P_2 = B + P_1$  is the equation of the line  $d_{SZ}$ , where  $S = d_{BX} \cap d_{OY}$ .

There still remains the “general” case of a line  $d$  not containing any of the points  $O, X, Y, Z$ . Let us write  $R, S, T$  for its intersections with  $d_{OX}, d_{OY}, d_{XY}$ . We further define

$$D = d_{OT} \cap d_{IY}, \quad A = d_{A^*X} \cap d_{OI}, \quad B = d_{SX} \cap d_{OI}.$$

Using the notation of Steps 3 and 8 we then have

- $A^* = d_{AX} \cap d_{OI} = D, P_1^* = d_{P_1Y} \cap d_{OD}, AP_1 = d_{P_1^*X} \cap d_{OI};$
- $B' = d_{BX} \cap d_{OY} = S, (AP_1)' = d_{SZ} \cap d_{Y(AP_1)}, B + AP_1 = d_{(AP_1)'X} \cap d_{OI}.$

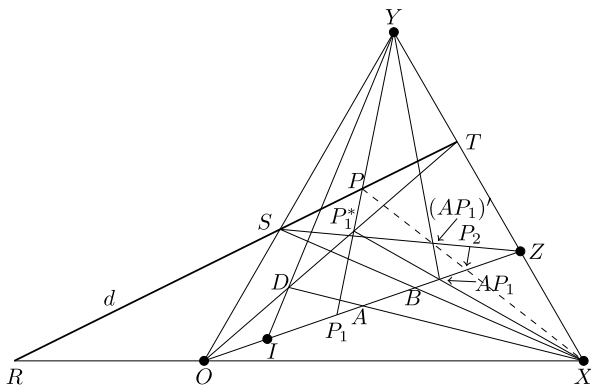


Fig. 6.28

Let us prove first that  $P_2 = B + AP_1$ , that is, that the points  $P, (AP_1)', X$  are on the same line. The two triangles  $PS(AP_1)'$  and  $P_1^*O(AP_1)$  are in perspective from the point  $Y$ ; by the Desargues axiom P4, the three points

$$T, \quad X' = d_{P(AP_1)'} \cap d_{P_1^*(AP_1)}, \quad Z$$

are on the same line. But then  $X$  and  $X'$  are both on  $d_{TZ}$  and  $d_{P_1^*(AP_1)}$ , thus  $X = X'$ . Therefore indeed,  $(AP_1)'$  and  $P_2$  are both on  $d_{PX}$ .

Conversely assume that a point  $P$  is such that  $P_2 = B + AP_2$ . We must prove that  $P \in d$ , that is, that the three points  $S, P, T$  are on the same line. Still referring to Fig. 6.28, by Definition of  $AP_1$ , the two triangles  $PS(AP_1)'$  and  $P_1^*O(AP_1)$  are in perspective from the point  $Y$ ; by the Desargues axiom P4 and the assumption  $P_2 = B + AP_1$ , the three points

$$T' = d_{PS} \cap d_{P_1^*O}, \quad X, \quad Z$$

are on the same line. But then  $T$  and  $T'$  are both on  $d_{XZ}$  and  $d_{OA^*} = d_{OP_1^*}$ , thus  $T = T'$ . This proves that  $T = T', P, S$  are on the same line and  $P \in d$ .

Thus in this last “general” case,  $P \in d$  if and only if  $P_2 = B + AP_1$ , so that the trace on  $\mathcal{P} \setminus d_{XY}$  of the line  $d$  admits the equation  $P_2 = B + AP_1$ . Conversely our argument shows that  $P_2 = B + AP_1$  is the equation of the trace of the line  $d_{ST}$  where

$$S = d_{BX} \cap d_{OY}, \quad A^* = d_{AX} \cap d_{IY}, \quad T = d_{A^*O} \cap d_{XY}.$$

Putting together all the above results, we conclude that every line  $d \neq d_{XY}$  has a trace on  $\mathcal{P} \setminus d_{XY}$  admitting an equation of one of the forms

$$P_1 = k, \quad P_2 = k, \quad P_2 = kP_1, \quad P_2 = k + k'P_1, \quad k, k' \in K$$

and each such equation is the equation of the trace of a projective line. Of course multiplying or dividing all the coefficients of an equation by the same non-zero



element of  $K$  does not change the solutions of the equation. Therefore the above results can be summarized by saying that every line  $d \neq d_{XY}$  has a trace on  $\mathcal{P} \setminus d_{XY}$  admitting an equation of the form

$$k_1 P_1 + k_2 P_2 = k_0, \quad k_0, k_1, k_2 \in K, \quad (k_1, k_2) \neq (0, 0)$$

and each such equation is that of the trace of some projective line. Trivially, the various cases considered in this step of the proof exhaust all the possibilities of such a linear equation.

**Step 14. The Homogeneous Coordinates** We are close to the conclusion of the proof. By Step 2, we have a bijection

$$\mathcal{P} \setminus d_{XY} \xrightarrow{\cong} K^2, \quad P \mapsto \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}.$$

On the other hand, using the notation of Example 6.7.1, we have an obvious injection

$$K^2 \hookrightarrow \mathbb{P}(K), \quad \begin{pmatrix} a \\ b \end{pmatrix} \mapsto \begin{bmatrix} a \\ b \\ 1 \end{bmatrix}.$$

Since a projective point is an equivalence class “up to a multiple”, this injection identifies  $K^2$  with those projective points whose third component is non-zero. Putting these two mappings together, we get an inclusion

$$\mathcal{P} \setminus d_{XY} \hookrightarrow \mathbb{P}(K), \quad P \mapsto \begin{bmatrix} P_1 \\ P_2 \\ 1 \end{bmatrix}$$

identifying the points of  $\mathcal{P} \setminus d_{XY}$  with those projective points whose third coordinate is non-zero.

With this notation, we can rephrase the result of step 13 by saying that the traces of the projective lines  $d \neq d_{XY}$  on  $\mathcal{P} \setminus d_{XY}$  are precisely the subsets admitting an equation

$$k_1 P_1 + k_2 P_2 + k_3 1 = 0, \quad (0, 0) \neq (k_1, k_2) \in K^2,$$

or equivalently

$$k_1 P_1 + k_2 P_2 + k_3 P_3 = 0, \quad \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix} \in \mathbb{P}(K), \quad P_3 \neq 0, \quad (0, 0) \neq (k_1, k_2).$$

Let us recall further that the equations

$$k_1 P_1 + k_2 P_2 + k_3 P_3 = 0, \quad (k_1, k_2, k_3) \neq (0, 0, 0)$$

describe exactly all the projective lines in  $\mathbb{P}(K)$  (see Example 6.7.1).

Let us now prove that we can extend the inclusion

$$\mathcal{P} \setminus d_{XY} \hookrightarrow \mathbb{P}(K)$$

to a bijection

$$\mathcal{P} \xrightarrow{\cong} \mathbb{P}(K)$$

compatible with the notion of a projective line. We have no choice: the points of  $d_{XY}$  must now be identified with those projective points whose third component is zero.

Consider first  $T \in d_{XY}$ , with  $T \neq X$ ,  $T \neq Y$ . Given a projective line  $d \neq d_{XY}$  containing  $T$ , let us adopt the notation of Fig. 6.28 and consider the points

$$S = d \cap d_{OY}, \quad D = d_{OT} \cap d_{IY}, \quad A = d_{OI} \cap d_{DX}, \quad B = d_{OI} \cap d_{SX}.$$

In the various cases of Step 13 with a point  $T \in d \cap d_{XY}$ , the trace of  $d$  admits each time the equation

$$P_2 = B + AP_1, \quad A \neq O.$$

Indeed:

- in the situation of Fig. 6.26,  $S = O$ ,  $B = O$ ;
- in the situation of Fig. 6.27,  $T = Z$ ,  $D = I$ ,  $A = I$ ;
- the situation of Fig. 6.28 is precisely our guideline for the present argument.

With the notation above, the point  $T$  is thus on each projective line  $d$  with an equation

$$AP_1 - P_2 + B = 0.$$

If we want this point to correspond to a point of  $\mathbb{P}(K)$  with a zero third component and satisfying the corresponding equations in  $\mathbb{P}(K)$

$$T = \begin{bmatrix} T_1 \\ T_2 \\ 0 \end{bmatrix}, \quad AT_1 - T_2 + BO = 0$$

we have to identify  $T$  with the point

$$T = \begin{bmatrix} 1 \\ A \\ 0 \end{bmatrix} \in \mathbb{P}(K), \quad A \neq O.$$

It remains to consider the two trivial cases  $T = X$  and  $T = Y$ , which must be identified with

$$X \mapsto \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad Y \mapsto \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

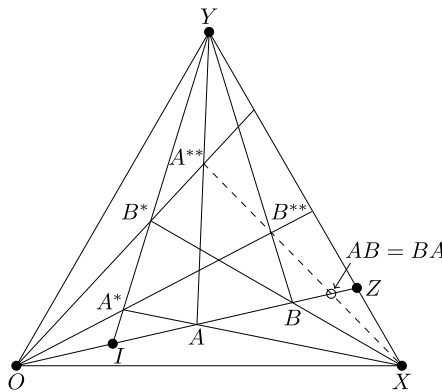


Fig. 6.29

in order to be, respectively, on each line of equation  $P_2 = C$  through  $X$  or  $P_1 = C$  through  $Y$ .

**Theorem 6.8.3** (Hilbert) *An abstract projective plane is Pappian if and only if it is the projective plane over a commutative field.*

*Proof* A Pappian plane is Arguesian by Theorem 6.6.5, thus it is the projective plane on a skew field by Theorem 6.8.2. We must prove that the multiplication, as defined in Step 8 of the proof of Theorem 6.8.2, is commutative.

Of course  $AB = BA$  as soon as  $A = B$  or if at least one of the two points is equal to  $0 = O$  or  $1 = I$ . In the other cases, and with the notation of the proof of Theorem 6.8.2, we consider Fig. 6.29, and construct successively

- $A^*, B^{**}, AB$ ;
- $B^*, A^{**}, BA$ .

The equality  $AB = BA$  reduces to the collinearity of the three points  $A^{**}, B^{**}$  and  $X$ . This is the case by Pappus' axiom P5 (see Definition 6.6.3) applied to the points  $(O, A, B)$  and  $(Y, B^*, A^*)$ .

Conversely, we must prove that Pappus' axiom P5 holds in the projective plane over a commutative field. This is Theorem 6.9.1 in [5], *Trilogy II*: indeed the whole of Chap. 6 of [5], *Trilogy II* is devoted to the study of projective spaces over a commutative field. □

*Counterexample 6.8.4* An Arguesian plane which is not Pappian.

*Proof* By Theorems 6.8.2 and 6.8.3, it suffices to consider the projective plane  $\mathbb{P}(K)$  over a skew field  $K$  whose multiplication is not commutative. For example, the field  $K = \mathbb{H}$  of quaternions. □

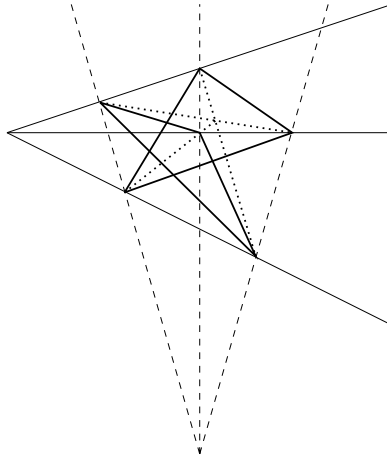


Fig. 6.30

## 6.9 Problems

**6.9.1** Show that the real projective plane can be defined as the quotient of the sphere which identifies two diametrically opposite points; the projective lines are the subsets of the quotient corresponding to the great circles.

**6.9.2** In the projective plane constructed from the Euclidean plane, consider six distinct points  $P, Q, A, B, C, D$  on a conic. Prove that the two anharmonic ratios of lines

$$(d_{PA}, d_{PB}; d_{PC}, d_{PD}) = (d_{QA}, d_{QB}; d_{QC}, d_{QD})$$

are equal. (Hint: first prove the result for the circle via Proposition 3.3.5 and infer it for an arbitrary conical section via a central projection.)

**6.9.3** Show, via counterexamples, that the four axioms for a projective plane in Definition 6.5.1 are independent (i.e. none of these axioms is a consequence of the other three).

**6.9.4** In a projective plane, given three distinct points  $A, B, C$  on an arbitrary line  $d$  and three distinct points  $A', B', C'$  on an arbitrary line  $d'$ , prove that there exists a composite of central projections mapping  $d$  onto  $d'$ ,  $A$  to  $A'$ ,  $B$  to  $B'$  and  $C$  to  $C'$ .

**6.9.5** Prove that in an abstract projective plane, Pappus' theorem is equivalent to the following statement: *If two triangles are in perspective from two distinct points, they are in perspective from a third one.* (See Fig. 6.30.)

**6.9.6** Prove that the field associated with an Arguesian plane is unique up to isomorphism (see Theorem 6.8.2).

## 6.10 Exercises

**6.10.1** Given the anharmonic ratio  $\rho = (A, B; C, D)$  of four points on a Euclidean line, calculate in terms of  $\rho$  the anharmonic ratio of every permutation of the four points.

**6.10.2** In the Euclidean plane, show that two intersecting lines and their two bisectors constitute a harmonic quadruple of lines.

**6.10.3** In the real projective plane, prove that a mapping between two projective lines which preserves the anharmonic ratio is necessarily a composite of central projections. (Hint: use Problem 6.9.4.)

**6.10.4** Construct the Arguesian plane with 13 points.

**6.10.5** In Desargues' theorem, and using the notation of Fig. 6.9, find a configuration in which the four points  $P, X, Y, Z$  are on the same line.

**6.10.6** Using the Pappus or the Desargues theorem show how, in the Euclidean plane, one can draw a straight line through two very distant points using a very short ruler.



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