

Chapter 2

Pseudo-hoops

A generalization of pseudo-BL algebras was given in [148], where the *pseudo-hoops* were defined and studied. Pseudo-hoops were originally introduced by Bosbach in [15] and [16] under the name of *complementary semigroups*. It was proved that a pseudo-hoop has the pseudo-divisibility condition and it is a meet-semilattice, so a bounded $R\ell$ -monoid can be viewed as a bounded pseudo-hoop together with the join-semilattice property. In other words, a bounded pseudo-hoop is a meet-semilattice ordered residuated, integral and divisible monoid.

In this chapter we present the main notions and results regarding pseudo-hoops from [148], we prove new properties of these structures, we introduce the notions of a join-center and cancellative-center of a pseudo-hoop and we define and study algebras on subintervals of pseudo-hoops.

2.1 Definitions and Properties

Definition 2.1 A *pseudo-hoop* is an algebra $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ of the type $(2, 2, 2, 0)$ such that, for all $x, y, z \in A$:

- $(psHOOP_1)$ $x \odot 1 = 1 \odot x = x$;
- $(psHOOP_2)$ $x \rightarrow x = x \rightsquigarrow x = 1$;
- $(psHOOP_3)$ $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$;
- $(psHOOP_4)$ $(x \odot y) \rightsquigarrow z = y \rightsquigarrow (x \rightsquigarrow z)$;
- $(psHOOP_5)$ $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

In the sequel, we will agree that \odot has higher priority than the operations $\rightarrow, \rightsquigarrow$, and those higher than \wedge and \vee .

If the operation \odot is commutative, or equivalently $\rightarrow = \rightsquigarrow$, then the pseudo-hoop is said to be *hoop*. Properties of hoops were studied in [1, 10, 15, 16].

On the pseudo-hoop A we define $x \leq y$ iff $x \rightarrow y = 1$ (equivalent to $x \rightsquigarrow y = 1$) and \leq is a partial order on A .

In the sequel we will refer to the pseudo-hoop $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ by its universe A .

Proposition 2.1 *In every pseudo-hoop $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ the following hold:*

(*pshoop-c₁*) $x \odot y \leq z$ iff $x \leq y \rightarrow z$ iff $y \leq x \rightsquigarrow z$ (*pseudo-residuation*);

(*pshoop-c₂*) $(A, \odot, 1)$ is a monoid and $x \leq y$ implies $x \odot z \leq y \odot z, z \odot x \leq z \odot y$;

(*pshoop-c₃*) (A, \leq) is a meet-semilattice with

$$\begin{aligned} x \wedge y &= (x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \rightsquigarrow y) \\ &= y \odot (y \rightsquigarrow x) \quad (\text{pseudo-divisibility}); \end{aligned}$$

(*pshoop-c₄*) the element 1 is the greatest element of A and

$$1 \rightarrow x = 1 \rightsquigarrow x = x, \quad x \rightarrow 1 = x \rightsquigarrow 1 = 1;$$

(*pshoop-c₅*) $x \leq (x \rightarrow y) \rightsquigarrow y, x \leq (x \rightsquigarrow y) \rightarrow y$;

(*pshoop-c₆*) $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z), x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$.

Proof

(*pshoop-c₁*) We have $x \odot y \leq z$ iff $x \odot y \rightarrow z = 1$ iff $x \rightarrow (y \rightarrow z) = 1$ iff $x \leq y \rightarrow z$. Similarly, $x \odot y \leq z$ iff $y \leq x \rightsquigarrow z$.

(*pshoop-c₂*) By (*psHOOP₁*), 1 is the neutral element of A .

Consider $u \in A$. We have:

$$\begin{aligned} (x \odot y) \odot z \leq u &\quad \text{iff} \quad x \odot y \leq z \rightarrow u \quad \text{iff} \\ x \leq y \rightarrow (z \rightarrow u) &= y \odot z \rightarrow u \quad \text{iff} \quad x \odot (y \odot z) \leq u. \end{aligned}$$

It follows that $(x \odot y) \odot z = x \odot (y \odot z)$, so \odot is associative.

Thus $(A, \odot, 1)$ is a monoid.

From $y \odot z \leq y \odot z$ we have $y \leq z \rightarrow y \odot z$. Since $x \leq y$, we get $x \leq z \rightarrow y \odot z$, hence $x \odot z \leq y \odot z$. Similarly, $z \odot x \leq z \odot y$.

(*pshoop-c₃*) From $x \rightarrow y \leq x \rightarrow y$ and $y \rightarrow x \leq y \rightarrow x$ we get $(x \rightarrow y) \odot x \leq y$ and $(y \rightarrow x) \odot y \leq x$, respectively. Hence, if we denote $x \wedge y = (x \rightarrow y) \odot x = (y \rightarrow x) \odot y$, it follows that $x \wedge y$ is a lower bound of $\{x, y\}$.

Let u be another lower bound of $\{x, y\}$, that is, $u \leq x$ and $u \leq y$.

From $u \rightarrow x = 1 = u \rightarrow u$ we get $(u \rightarrow x) \odot u \leq u \leq y$.

But $(u \rightarrow x) \odot u = (x \rightarrow u) \odot x$, hence $(x \rightarrow u) \odot x \leq y$, so $x \rightarrow u \leq x \rightarrow y$.

We have $(x \rightarrow u) \odot x = (u \rightarrow x) \odot u = 1 \odot u = u$.

It follows that $u = (x \rightarrow u) \odot x \leq (x \rightarrow y) \odot x = x \wedge y$. Thus $x \wedge y$ is the greatest lower bound of $\{x, y\}$ with respect to the order \leq .

We conclude that every pseudo-hoop is a meet-semilattice with respect to the order \leq , where the meet operation is given by

$$x \wedge y = (x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x).$$

(*pshoop-c₄*) Taking $y = 1$ in the equality $x \wedge y = (x \rightarrow y) \odot x = (y \rightarrow x) \odot y$, we get $x \wedge 1 = (x \rightarrow 1) \odot x = 1 \rightarrow x$. It follows that $1 \rightarrow x = x \wedge 1 \leq x$.

On the other hand, from $x \odot 1 = x$ we have $x \leq 1 \rightarrow x$.

Hence $1 \rightarrow x = x$ and similarly $1 \rightsquigarrow x = x$.

It follows that $x = 1 \rightarrow x = (x \rightarrow 1) \odot x = x \wedge 1 \leq 1$ for all $x \in A$.

We conclude that 1 is the greatest element of A .

From $1 \odot x = x \leq 1$ we get $1 \leq x \rightarrow 1$. But $x \rightarrow 1 \leq 1$, hence $x \rightarrow 1 = 1$.

Similarly, $x \rightsquigarrow 1 = 1$.

(*pshoop-c5*) From $(x \rightarrow y) \odot x = x \wedge y \leq y$, we get $x \leq (x \rightarrow y) \rightsquigarrow y$.

Similarly, from $x \odot (x \rightsquigarrow y) = x \wedge y \leq y$, we have $x \leq (x \rightsquigarrow y) \rightarrow y$.

(*pshoop-c6*) Applying (*pshoop-c3*), we get:

$$(y \rightarrow z) \odot (x \rightarrow y) \odot x = (y \rightarrow z) \odot (x \wedge y) \leq (y \rightarrow z) \odot y = y \wedge z \leq z.$$

It follows that $(y \rightarrow z) \odot (x \rightarrow y) \leq x \rightarrow z$, so $x \rightarrow y \leq (y \rightarrow z) \rightsquigarrow (x \rightarrow z)$.

Similarly, $x \rightsquigarrow y \leq (y \rightsquigarrow z) \rightarrow (x \rightsquigarrow z)$. \square

Proposition 2.2 *Every pseudo-hoop is a pseudo-BCK(pP) algebra which is a meet-semilattice satisfying the pseudo-divisibility property.*

Proof Suppose that $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-hoop.

We will prove that $(A, \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCK(pP) algebra.

(*psBCK*₁) follows from (*pshoop-c6*);

(*psBCK*₂) follows from (*pshoop-c5*);

(*psBCK*₃) follows from (*psHOOP*₂);

(*psBCK*₄) follows from (*pshoop-c4*);

(*psBCK*₅) and (*psBCK*₆) follow by the definition of \leq and from the fact that \leq is a partial order on A .

The (pP) condition is a consequence of (*pshoop-c1*) and Theorem 1.4.

Thus $(A, \leq, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-BCK(pP) algebra. \square

It follows that all properties proved for a pseudo-BCK(pP) algebra are also valid in the case of pseudo-hoops.

Definition 2.2 A *bounded pseudo-hoop* is an algebra $(A, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ such that $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-hoop and $0 \leq x$ for all $x \in A$.

Proposition 2.3 ([148]) *In any pseudo-hoop A , the following property holds for all $x, y \in A$:*

(*pshoop-c7*) $[((y \rightarrow x) \rightsquigarrow x) \rightarrow y] \rightarrow (y \rightarrow x) = y \rightarrow x$ and $[((y \rightsquigarrow x) \rightarrow x) \rightsquigarrow y] \rightsquigarrow (y \rightsquigarrow x) = y \rightsquigarrow x$.

Proof Let $z = (y \rightarrow x) \rightsquigarrow x$. By (*pshoop-c5*) and (*psbck-c22*) we have $y \leq z$ and $z \rightarrow x = y \rightarrow x$. It follows that

$$\begin{aligned} (z \rightarrow y) \rightarrow (y \rightarrow x) &= (z \rightarrow y) \rightarrow (z \rightarrow x) = (z \rightarrow y) \odot z \rightarrow x = (z \wedge y) \rightarrow x \\ &= y \rightarrow x. \end{aligned}$$

Similarly, $[(y \rightsquigarrow x) \rightarrow x] \rightsquigarrow y = y \rightsquigarrow x$. \square

Let $(A, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ be a bounded pseudo-hoop.

We define two negations $-$ and \sim : for all $x \in A$, $x^- := x \rightarrow 0$, $x^\sim := x \rightsquigarrow 0$.

A bounded pseudo-hoop A is called *good* if $x^{-\sim} = x^{\sim-}$ for all $x \in A$.

If $x^{-\sim} = x^{\sim-} = x$ for all $x \in A$, then the bounded pseudo-hoop A is said to have the *pseudo-double negation* condition, (pDN) for short.

Obviously, any bounded pseudo-hoop with (pDN) is good.

Proposition 2.4 *If A is a good pseudo-hoop, then for all $x, y \in A$:*

- (ps hoop-c₈) $(x^{-\sim} \rightarrow x)^\sim = (x^{-\sim} \rightsquigarrow x)^- = 0$;
- (ps hoop-c₉) $(x \rightarrow y)^{-\sim} = x^{-\sim} \rightarrow y^{-\sim}$ and $(x \rightsquigarrow y)^{-\sim} = x^{-\sim} \rightsquigarrow y^{-\sim}$;
- (ps hoop-c₁₀) $(x \wedge y)^{-\sim} = x^{-\sim} \wedge y^{-\sim}$;
- (ps hoop-c₁₁) $x \rightarrow y^- = x^{-\sim} \rightarrow y^-$ and $x \rightsquigarrow y^\sim = x^{-\sim} \rightsquigarrow y^\sim$.

Proof

(ps hoop-c₈) From $0 \leq x$ applying (psbck-c₁₀) we get $x^{-\sim} \rightarrow 0 \leq x^{-\sim} \rightarrow x$, that is, $x^- \leq x^{-\sim} \rightarrow x$. Hence $(x^{-\sim} \rightarrow x)^\sim \leq x^{-\sim}$.

It follows that:

$$\begin{aligned} (x^{-\sim} \rightarrow x)^\sim &= (x^{-\sim} \rightarrow x)^\sim \wedge x^{-\sim} = x^{-\sim} \odot [(x^{-\sim} \rightsquigarrow (x^{-\sim} \rightarrow x)^\sim)] \\ &= x^{-\sim} \odot [(x^{-\sim} \rightarrow x) \odot x^{-\sim}]^\sim = x^{-\sim} \odot (x^{-\sim} \wedge x)^\sim \\ &= x^{-\sim} \odot x^\sim = 0 \end{aligned}$$

(we applied the axiom $v \odot (v \rightsquigarrow u) = u \wedge v$ for $v = x^{-\sim}$ and $u = (x^{-\sim} \rightarrow x)^\sim$). Similarly, $x^{-\sim} \rightsquigarrow 0 \leq x^{-\sim} \rightsquigarrow x$, so $x^\sim \leq x^{-\sim} \rightsquigarrow x$. Hence $(x^{-\sim} \rightsquigarrow x)^- \leq x^{-\sim}$.

Therefore

$$\begin{aligned} (x^{-\sim} \rightsquigarrow x)^- &= (x^{-\sim} \rightsquigarrow x)^- \wedge x^{-\sim} = [x^{-\sim} \rightarrow (x^{-\sim} \rightsquigarrow x)^-] \odot x^{-\sim} \\ &= [x^{-\sim} \odot (x^{-\sim} \rightsquigarrow x)]^- \odot x^{-\sim} = (x^{-\sim} \wedge x)^- \odot x^{-\sim} \\ &= x^- \odot x^{-\sim} = 0. \end{aligned}$$

Thus $(x^{-\sim} \rightarrow x)^\sim = (x^{-\sim} \rightsquigarrow x)^- = 0$.

(ps hoop-c₉) Applying the axioms of a pseudo-hoop we have:

$$(x \wedge y) \odot (y \rightsquigarrow 0) \leq y \odot (y \rightsquigarrow 0) = y \wedge 0 = 0.$$

But $x \wedge y = (x \rightarrow y) \odot x$, hence $(x \rightarrow y) \odot x \odot (y \rightsquigarrow 0) \leq 0$.

It follows that $x \odot (y \rightsquigarrow 0) \leq (x \rightarrow y) \rightsquigarrow 0$. Applying (*psbck-c1*) we get

$$[(x \rightarrow y) \rightsquigarrow 0] \rightarrow 0 \leq x \odot (y \rightsquigarrow 0) \rightarrow 0 = x \rightarrow [(y \rightsquigarrow 0) \rightarrow 0].$$

Thus $(x \rightarrow y)^{\sim} \leq x \rightarrow y^{\sim}$. On the other hand we have:

$$(x \rightarrow y^{\sim})^{\sim} = (x \odot y^{\sim})^{\sim} = (x \odot y^{\sim})^{-} = x \rightarrow y^{\sim}.$$

Replacing x with $x \rightarrow y^{\sim}$ and y with $x \rightarrow y$ in the above identity we have:

$$(x \rightarrow y^{\sim}) \rightarrow (x \rightarrow y)^{\sim} = [(x \rightarrow y^{\sim}) \rightarrow (x \rightarrow y)^{\sim}]^{\sim}.$$

Since $x \rightarrow y \leq (x \rightarrow y)^{\sim}$, applying (*psbck-c10*) we get:

$$(x \rightarrow y^{\sim}) \rightarrow (x \rightarrow y)^{\sim} \geq (x \rightarrow y^{\sim}) \rightarrow (x \rightarrow y), \quad \text{so} \\ [(x \rightarrow y^{\sim}) \rightarrow (x \rightarrow y)^{\sim}]^{\sim} \geq [(x \rightarrow y^{\sim}) \rightarrow (x \rightarrow y)]^{\sim}.$$

Taking into consideration the above identity and applying (*psHOOP3*) we get:

$$(x \rightarrow y^{\sim}) \rightarrow (x \rightarrow y)^{\sim} \geq [(x \rightarrow y^{\sim}) \rightarrow (x \rightarrow y)]^{\sim} \\ = [(x \rightarrow y^{\sim}) \odot x \rightarrow y]^{\sim} = (x \wedge y^{\sim} \rightarrow y)^{\sim}.$$

But $x \wedge y \leq y \leq y^{\sim}$, hence by (*psbck-c1*) we have $y^{\sim} \rightarrow y \leq x \wedge y^{\sim} \rightarrow y$. Applying (*pshoop-c8*) we get:

$$(x \rightarrow y^{\sim}) \rightarrow (x \rightarrow y)^{\sim} \geq (y^{\sim} \rightarrow y)^{\sim} = 0^{-} = 1.$$

It follows that $(x \rightarrow y^{\sim}) \rightarrow (x \rightarrow y)^{\sim} = 1$, that is, $x \rightarrow y^{\sim} \leq (x \rightarrow y)^{\sim}$.

We conclude that $(x \rightarrow y)^{\sim} = x \rightarrow y^{\sim}$.

From (*psbck-c19*) we have $x \rightarrow y^{\sim} = x^{\sim} \rightarrow y^{\sim}$, hence $(x \rightarrow y)^{\sim} = x^{\sim} \rightarrow y^{\sim}$. Similarly, $(x \rightsquigarrow y)^{\sim} = x^{\sim} \rightsquigarrow y^{\sim}$.

(*pshoop-c10*) From $x \wedge y \leq x, y$ we get $(x \wedge y)^{\sim} \leq x^{\sim}, y^{\sim}$.

Hence $(x \wedge y)^{\sim} \leq x^{\sim} \wedge y^{\sim}$.

On the other hand, applying Proposition 1.23 and (*pshoop-c9*) we have:

$$(x \wedge y)^{\sim} = ((x \rightarrow y) \odot x)^{\sim} \geq (x \rightarrow y)^{\sim} \odot x^{\sim} \\ = (x^{\sim} \rightarrow y^{\sim}) \odot x^{\sim} = x^{\sim} \wedge y^{\sim}.$$

We conclude that $(x \wedge y)^{\sim} = x^{\sim} \wedge y^{\sim}$.

(*pshoop-c11*) Applying (*psbck-c19*), (*pshoop-c9*) and (*psbck-c37*) we have:

$$x^{\sim} \rightarrow y^{-} = x^{\sim} \rightarrow y^{\sim} = (x \rightarrow y)^{\sim} \\ = ((x \odot y)^{-})^{\sim} = (x \odot y)^{-} = x \rightarrow y^{-}.$$

Similarly, $x \rightsquigarrow y^{\sim} = x^{\sim} \rightsquigarrow y^{\sim}$. □

Proposition 2.5 ([148]) *Let A be a pseudo-hoop and I an arbitrary set. Then:*

- (1) $x \odot (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \odot y_i)$;
- (2) $(\bigvee_{i \in I} y_i) \odot x = \bigvee_{i \in I} (y_i \odot x)$;
- (3) $x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \wedge y_i)$,

whenever the arbitrary unions exist.

Proof

- (1) Since $y_i \leq \bigvee_{i \in I} y_i$ for all $i \in I$, it follows that $x \odot y_i \leq x \odot (\bigvee_{i \in I} y_i)$ for all $i \in I$. Thus $\bigvee_{i \in I} (x \odot y_i) \leq x \odot (\bigvee_{i \in I} y_i)$.

Let $z \in A$ such that $\bigvee_{i \in I} (x \odot y_i) \leq z$. It follows that $x \odot y_i \leq z$ for all $i \in I$, so $y_i \leq x \rightsquigarrow z$ for all $i \in I$. Thus $\bigvee_{i \in I} y_i \leq x \rightsquigarrow z$, so $x \odot (\bigvee_{i \in I} y_i) \leq z$.

So, for any $z \in A$, we have proved that $\bigvee_{i \in I} (x \odot y_i) \leq z$ implies $x \odot (\bigvee_{i \in I} y_i) \leq z$.

Taking $z = \bigvee_{i \in I} (x \odot y_i)$, we get $x \odot (\bigvee_{i \in I} y_i) \leq \bigvee_{i \in I} (x \odot y_i)$.

We conclude that $x \odot (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \odot y_i)$.

- (2) Similar to (1).

- (3) From $y_i \leq \bigvee_{i \in I} y_i$, we get $x \wedge y_i \leq x \wedge \bigvee_{i \in I} y_i$ for all $i \in I$.

Hence $\bigvee_{i \in I} (x \wedge y_i) \leq x \wedge (\bigvee_{i \in I} y_i)$.

Conversely, we have:

$$\begin{aligned} x \wedge \left(\bigvee_{i \in I} y_i \right) &= \left(\bigvee_{i \in I} y_i \right) \wedge x = \left(\bigvee_{i \in I} y_i \right) \odot \left(\bigvee_{i \in I} y_i \rightsquigarrow x \right) \\ &= \bigvee_{i \in I} \left(y_i \odot \left(\bigvee_{i \in I} y_i \rightsquigarrow x \right) \right). \end{aligned}$$

From (psbck-c₁) we have $\bigvee_{i \in I} y_i \rightsquigarrow x \leq y_i \rightsquigarrow x$ for all $i \in A$, so $y_i \odot (\bigvee_{i \in I} y_i \rightsquigarrow x) \leq y_i \odot (y_i \rightsquigarrow x) = y_i \wedge x = x \wedge y_i$ for all $i \in I$.

It follows that

$$\bigvee_{i \in I} \left(y_i \odot \left(\bigvee_{i \in I} y_i \rightsquigarrow x \right) \right) \leq \bigvee_{i \in I} (x \wedge y_i), \quad \text{so } x \wedge \left(\bigvee_{i \in I} y_i \right) \leq \bigvee_{i \in I} (x \wedge y_i).$$

Thus $x \wedge (\bigvee_{i \in I} y_i) = \bigvee_{i \in I} (x \wedge y_i)$. □

Proposition 2.6 ([148]) *Let A be a pseudo-hoop and $H \subseteq A$. The following are equivalent:*

- (a) H is a compatible deductive system;
- (b) H is a normal filter (i.e. $x \odot H = H \odot x$ for all $x \in A$).

Proof

- (a) \Rightarrow (b) Consider $y \in x \odot H$, $y = x \odot h$ with $x \in A$ and $h \in H$.

It follows that $x \odot h = y = x \wedge y = (x \rightarrow y) \odot x$.

From $h \leq x \rightsquigarrow x \odot h = x \rightsquigarrow y$ and $h \in H$ we have $x \rightsquigarrow y \in H$.

Since H is a compatible deductive system of A , we get $x \rightarrow y \in H$.

Consequently, if we let $h' = x \rightarrow y$, it follows that $y = h' \odot x \in H \odot x$.

Thus $x \odot H \subseteq H \odot x$.

Similarly, $H \odot x \subseteq x \odot H$, so $x \odot H = H \odot x$ for all $x \in A$.

- (b) \Rightarrow (a) Assume $x \rightsquigarrow y \in H$, so $x \wedge y = x \odot (x \rightsquigarrow y) \in x \odot H = H \odot x$, that is, $x \wedge y = h \odot x$ for some $h \in H$.

It follows that $x \rightarrow y = x \rightarrow x \wedge y = x \rightarrow h \odot x$. Since $h \leq x \rightarrow h \odot x$ and $h \in H$, we get $x \rightarrow y \in H$. Similarly, from $x \rightarrow y \in H$ we get $x \rightsquigarrow y \in H$.

Thus H is a compatible deductive system of A . \square

If $x, y \in A$, then we define the *pseudo-joins* of x and y by:

$$x \cup_1 y = ((x \rightarrow y) \rightsquigarrow y) \wedge ((y \rightarrow x) \rightsquigarrow x),$$

$$x \cup_2 y = ((x \rightsquigarrow y) \rightarrow y) \wedge ((y \rightsquigarrow x) \rightarrow x).$$

We will also use the notation:

$$x \vee_1 y = (x \rightarrow y) \rightsquigarrow y \quad \text{and} \quad x \vee_2 y = (x \rightsquigarrow y) \rightarrow y.$$

Obviously, $x \cup_1 y = (x \vee_1 y) \wedge (y \vee_1 x)$ and $x \cup_2 y = (x \vee_2 y) \wedge (y \vee_2 x)$.

Proposition 2.7 ([148]) *In any bounded pseudo-hoop A the following hold:*

- (1) $x \cup_1 0 = x \cup_2 0 = 0 \cup_1 x = 0 \cup_2 x = x$;
- (2) $x \cup_1 1 = x \cup_2 1 = 1 \cup_1 x = 1 \cup_2 x = 1$;
- (3) $x \cup_1 x = x \cup_2 x = x$.

Proof

- (1) $x \cup_1 0 = ((x \rightarrow 0) \rightsquigarrow 0) \wedge ((0 \rightarrow x) \rightsquigarrow x) = x^{-\sim} \wedge x = x$ (by (psbck-c₁₄)).

Similarly, $x \cup_2 0 = 0 \cup_1 x = 0 \cup_2 x = x$.

- (2) $x \cup_1 1 = ((x \rightarrow 1) \rightsquigarrow 1) \wedge ((1 \rightarrow x) \rightsquigarrow x) = 1 \wedge 1 = 1$.

Similarly, $x \cup_2 1 = 1 \cup_1 x = 1 \cup_2 x = 1$.

- (3) This is obvious. \square

Proposition 2.8 ([148]) *In any bounded pseudo-hoop A the following hold:*

- (1) $x \cup_1 y = y \cup_1 x$ and $x \cup_2 y = y \cup_2 x$;
- (2) $x, y \leq x \cup_1 y$ and $x, y \leq x \cup_2 y$;
- (3) $x \leq y$ iff $x \cup_1 y = y$;
- (4) $x \leq y$ iff $x \cup_2 y = y$.

Proof

- (1) This is obvious.

- (2) By (pshoop-c₅) and (psbck-c₆) we have $x, y \leq (x \rightarrow y) \rightsquigarrow y$ and $x, y \leq (y \rightarrow x) \rightarrow x$. Hence $x, y \leq x \cup_1 y$. Similarly, $x, y \leq x \cup_2 y$.

- (3) If $x \leq y$, then $(x \rightarrow y) \rightsquigarrow y = 1 \rightsquigarrow y = y$.
Hence $x \cup_1 y = y \wedge [(y \rightarrow x) \rightsquigarrow x] = y$, since by (*pshoop-c5*), $y \leq (y \rightarrow x) \rightsquigarrow x$.
Conversely, suppose that $x \cup_1 y = y$.
It follows that $x \wedge y = x \wedge (x \cup_1 y) = x$, by (2). Thus $x \leq y$.
- (4) Similar to (3). □

Proposition 2.9 ([148]) *Let A be a pseudo-hoop. The following are equivalent:*

- (a) \cup_1 is associative;
(b) for all $x, y, z \in A$, $x \leq y$ implies $x \cup_1 z \leq y \cup_1 z$;
(c) for all $x, y, z \in A$, $x \cup_1 (y \wedge z) \leq (x \cup_1 y) \wedge (x \cup_1 z)$;
(d) \cup_1 is the join operation on A .

Proof

- (a) \Rightarrow (d) We have $x, y \leq x \cup_1 y$, so $x \cup_1 y$ is an upper bound of $\{x, y\}$.

Let $z \in A$ such that $x, y \leq z$. By (a), $(x \cup_1 y) \cup_1 z = x \cup_1 (y \cup_1 z)$.

From $y \leq z$ and (*pshoop-c5*) we have:

$$y \cup_1 z = ((y \rightarrow z) \rightsquigarrow z) \wedge ((z \rightarrow y) \rightsquigarrow y) = (1 \rightsquigarrow z) \wedge ((z \rightarrow y) \rightsquigarrow y) = z.$$

It follows that $x \cup_1 (y \cup_1 z) = x \cup_1 z = ((x \rightarrow z) \rightsquigarrow z) \wedge ((z \rightarrow x) \rightsquigarrow x) = z$.

Thus $x \cup_1 y \leq (x \cup_1 y) \cup_1 z = z$. Hence $x \cup_1 y$ is the l.u.b. of $\{x, y\}$, so $x \vee y$ exists and $x \vee y = x \cup_1 y$.

- (d) \Rightarrow (a) Applying (d) we have $(x \cup_1 y) \cup_1 z = (x \vee y) \vee z = x \vee (y \vee z) = x \cup_1 (y \cup_1 z)$.

Thus \cup_1 is associative.

- (b) \Rightarrow (d) We have $x, y \leq x \cup_1 y$, so $x \cup_1 y$ is an upper bound of $\{x, y\}$.

Let $z \in A$ such that $x, y \leq z$. From $x \leq z$, applying (b) we obtain:

$$x \cup_1 y \leq z \cup_1 y = ((z \rightarrow y) \rightsquigarrow y) \wedge ((y \rightarrow z) \rightsquigarrow z) = z.$$

We conclude that $x \cup_1 y$ is the l.u.b. of $\{x, y\}$, so $x \vee y$ exists and $x \vee y = x \cup_1 y$.

- (d) \Rightarrow (b) Let $x, y, z \in A$ such that $x \leq y$.

It follows that $x \cup_1 z = x \vee z \leq y \vee z = y \cup_1 z$.

- (c) \Rightarrow (d) We have $x, y \leq x \cup_1 y$, so $x \cup_1 y$ is an upper bound of $\{x, y\}$.

Let $z \in A$ such that $x, y \leq z$. We have $x \cup_1 (y \wedge z) \leq (x \cup_1 y) \wedge (x \cup_1 z)$.

Since $x \cup_1 z = ((x \rightarrow z) \rightsquigarrow z) \wedge ((z \rightarrow x) \rightsquigarrow x) = z$, we get $x \cup_1 y \leq (x \cup_1 y) \wedge z \leq z$. Thus $x \cup_1 y$ is the l.u.b. of $\{x, y\}$, so $x \vee y$ exists and $x \vee y = x \cup_1 y$.

- (d) \Rightarrow (c) For all $x, y, z \in A$, $x \cup_1 (y \wedge z) = x \vee (y \wedge z)$.

Obviously, $y \wedge z \leq y$ implies $x \vee (y \wedge z) \leq x \vee y$.

From $x \leq x \vee z$ and $y \wedge z \leq z \leq x \vee z$ we get $x \vee (y \wedge z) \leq x \vee z$.

It follows that $x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z)$.

Hence $x \cup_1 (y \wedge z) = x \vee (y \wedge z) \leq (x \vee y) \wedge (x \vee z) = (x \cup_1 y) \wedge (x \cup_1 z)$. □

Proposition 2.10 ([148]) *Let A be a pseudo-hoop. The following are equivalent:*

- (a) \cup_2 is associative;
- (b) for all $x, y, z \in A$, $x \leq y$ implies $x \cup_2 z \leq y \cup_2 z$;
- (c) for all $x, y, z \in A$, $x \cup_2 (y \wedge z) \leq (x \cup_2 y) \wedge (x \cup_2 z)$;
- (d) \cup_2 is the join operation on A .

Proof The proof is similar to that of Proposition 2.9. □

Remark 2.1 Suppose that \cup_1 is associative. By Proposition 2.9 it follows that \cup_1 is the join operation on A , that is, \vee exists and $\vee = \cup_1$.

Applying Proposition 2.5(3) we get that (A, \wedge, \vee) is a distributive lattice.

The same result is obtained using \cup_2 and Proposition 2.10.

Theorem 2.1 *Every bounded locally finite pseudo-hoop has property (pDN).*

Proof Let A be a bounded locally finite pseudo-hoop and $x \in A$. If $x = 0$, then $0^{\sim\sim} = 0^{\sim-} = 0$. Suppose $x \neq 0$. We prove that $x^{\sim\sim} = x$. By (psbck-c14) we have $x \leq x^{\sim\sim}$. Suppose that $x^{\sim\sim} \not\leq x$, hence $x^{\sim\sim} \rightarrow x \neq 1$. Since A is locally finite, there is an $n \in \mathbb{N}$, $n \geq 1$, such that $(x^{\sim\sim} \rightarrow x)^n = 0$. We have:

$$\begin{aligned}
 (x^{\sim\sim} \rightarrow x) \rightarrow x^- &= (x^{\sim\sim} \rightarrow x) \rightarrow x^{\sim\sim-} = (x^{\sim\sim} \rightarrow x) \rightarrow (x^{\sim\sim} \rightarrow 0) \\
 &= (x^{\sim\sim} \rightarrow x) \odot x^{\sim\sim} \rightarrow 0 = (x \wedge x^{\sim\sim}) \rightarrow 0 \\
 &= x \rightarrow 0 = x^-, \\
 (x^{\sim\sim} \rightarrow x)^2 \rightarrow x^- &= (x^{\sim\sim} \rightarrow x) \rightarrow ((x^{\sim\sim} \rightarrow x) \rightarrow x^-) \\
 &= (x^{\sim\sim} \rightarrow x) \rightarrow x^- = x^-.
 \end{aligned}$$

By induction we get $(x^{\sim\sim} \rightarrow x)^n \rightarrow x^- = x^-$. Thus $0 \rightarrow x^- = x^-$, so $x^- = 1$. Hence $x \leq x^{\sim\sim} = 0$, that is, $x = 0$, a contradiction. Therefore $x^{\sim\sim} \leq x$, so $x^{\sim\sim} = x$. Similarly, $x^{\sim-} = x$. Thus A satisfies (pDN). □

We recall the notion of an ordinal sum of pseudo-hoops.

Let A_1 and A_2 be pseudo-hoops such that $A_1 \cap A_2 = \{1\}$. We set $A = A_1 \cup A_2$ and we define the operations \odot , \rightarrow , \rightsquigarrow on A as follows:

$$x \odot y := \begin{cases} x \odot_i y & \text{if } x, y \in A_i, i = 1, 2 \\ x & \text{if } x \in A_1 \setminus \{1\}, y \in A_2 \\ y & \text{if } x \in A_2, y \in A_1 \setminus \{1\} \end{cases}$$

$$x \rightarrow y := \begin{cases} x \rightarrow_i y & \text{if } x, y \in A_i, i = 1, 2 \\ y & \text{if } x \in A_2, y \in A_1 \setminus \{1\} \\ 1 & \text{if } x \in A_1 \setminus \{1\}, y \in A_2 \end{cases}$$

$$x \rightsquigarrow y := \begin{cases} x \rightsquigarrow_i y & \text{if } x, y \in A_i, i = 1, 2 \\ y & \text{if } x \in A_2, y \in A_1 \setminus \{1\} \\ 1 & \text{if } x \in A_1 \setminus \{1\}, y \in A_2. \end{cases}$$

Then $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ is a pseudo-hoop called the *ordinal sum* of A_1 and A_2 and we denote it by $A = A_1 \oplus A_2$. The construction can of course be extended to arbitrarily many systems of pseudo-hoops.

Definition 2.3 A pseudo-hoop A is called:

- (1) *simple* if $\{1\}$ is the unique proper normal filter of A ;
- (2) *strongly simple* if $\{1\}$ is the unique proper filter of A .

Obviously, any strongly simple pseudo-hoop is simple.

When A is a hoop, since filters and normal filters coincide, the notions of simple and strongly simple hoop coincide.

Proposition 2.11 ([148]) *For any pseudo-hoop A the following are equivalent:*

- (a) A is strongly simple;
- (b) for all $x \in A$, if $x \neq 1$ then $[x] = A$;
- (c) for all $x, y \in A$, if $x \neq 1$ then there exists an $n \in \mathbb{N}$, $n > 0$, such that $y \geq x^n$;
- (d) for all $x, y \in A$, if $x \neq 1$ then there exists an $n \in \mathbb{N}$, $n > 0$, such that $x \rightarrow^n y = 1$ for some $n \in \mathbb{N}$, $n \geq 1$;
- (e) for all $x, y \in A$, if $x \neq 1$ then there exists an $n \in \mathbb{N}$, $n \geq 1$, such that $x \rightsquigarrow^n y = 1$ for some $n \in \mathbb{N}$, $n > 0$.

Proof (a) \Leftrightarrow (b) is obvious.

By Lemma 1.9 and Proposition 1.33 any one of the conditions (c), (d) and (e) is equivalent to condition (b). \square

Lemma 2.1 ([148]) *In any strongly simple pseudo-hoop A the following hold for all $x, y \in A$:*

- (1) $y \rightarrow x = x$ implies $x = 1$ or $y = 1$;
- (2) $y \rightsquigarrow x = x$ implies $x = 1$ or $y = 1$.

Proof

- (1) Consider $x, y \in A$ such that $y \rightarrow x = x$. Applying $(psHOOP_3)$, it follows by induction that $y^n \rightarrow x = x$ for all $n \in \mathbb{N}$, $n > 0$. If $y \neq 1$, then according to Proposition 2.11(c), there exists an $n_0 \in \mathbb{N}$, $n_0 > 0$, such that $y^{n_0} \leq x$, that is, $y^{n_0} \rightarrow x = 1$. Hence $x = 1$.
- (2) Similar to (1). \square

Definition 2.4 A pseudo-hoop $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ is said to be *cancellative* if the monoid $(A, \odot, 1)$ is cancellative, that is, $x \odot a = y \odot a$ implies $x = y$ and $a \odot x = a \odot y$ implies $x = y$ for all $x, y, a \in A$.

Proposition 2.12 ([148]) *A pseudo-hoop A is cancellative iff the following identities hold:*

$$(C_1) \quad y \rightarrow x \odot y = x;$$

$$(C_2) \quad y \rightsquigarrow y \odot x = x,$$

for all $x, y \in A$.

Proof Suppose that A is cancellative.

It follows that $x \odot y = y \wedge (x \odot y) = (y \rightarrow x \odot y) \odot y$, hence $x = y \rightarrow x \odot y$.

Similarly, $y \odot x = y \wedge (y \odot x) = y \odot (y \rightsquigarrow y \odot x)$, hence $x = y \rightsquigarrow y \odot x$.

Conversely, suppose that A satisfies (C_1) and (C_2) .

If $x \odot z = y \odot z$, then applying (C_1) twice we get $x = z \rightarrow x \odot z = z \rightarrow y \odot z = y$.

Similarly, from $z \odot x = z \odot y$ and (C_2) it follows that $x = z \rightsquigarrow z \odot x = z \rightsquigarrow z \odot y = y$. \square

Example 2.1 ([148]) Let $\mathbf{G} = (G, \vee, \wedge, +, -, 0)$ be an arbitrary ℓ -group and \mathbf{G}^- be the negative cone of \mathbf{G} , that is, $\mathbf{G}^- = \{x \in G \mid x \leq 0\}$.

On \mathbf{G}^- we define the following operations:

$$x \odot y := x + y,$$

$$x \rightarrow y := (y - x) \wedge 0,$$

$$x \rightsquigarrow y := (-x + y) \wedge 0.$$

Then $\mathbf{G}^- = (G^-, \odot, \rightarrow, \rightsquigarrow, 0)$ is a cancellative pseudo-hoop.

We shall verify the conditions $(psHOOP_1)$ – $(psHOOP_5)$.

Consider $x, y, z \in G^-$.

$$(psHOOP_1) \quad x \odot 0 = x + 0 = x = 0 + x = 0 \odot x.$$

$$(psHOOP_2) \quad x \rightarrow x = x \rightsquigarrow x = 0 \wedge 0 = 0.$$

$$(psHOOP_3) \quad x \odot y \rightarrow z = [z - (x + y)] \wedge 0 = (z - y - x) \wedge 0 \text{ and}$$

$$\begin{aligned} x \rightarrow (y \rightarrow z) &= [(z - y) \wedge 0 - x] \wedge 0 = (z - y - x) \wedge (-x) \wedge 0 \\ &= (z - y - x) \wedge 0 \end{aligned}$$

(since $-x \geq 0$, we have $(-x) \wedge 0 = 0$). Thus $x \odot y \rightarrow z = x \rightarrow (y \rightarrow z)$.

$$(psHOOP_4) \quad \text{Similar to } (psHOOP_3).$$

$$(psHOOP_5) \quad (x \rightarrow y) \odot x = (y - x) \wedge 0 + x = y \wedge x.$$

Similarly, $(y \rightarrow x) \odot y = x \wedge y$, $x \odot (x \rightsquigarrow y) = y \wedge x$, $y \odot (y \rightsquigarrow x) = x \wedge y$.

Thus $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

It follows that \mathbf{G}^- is a pseudo-hoop.

We will verify conditions (C_1) and (C_2) .

If $x, y \in G^-$, then $y \rightarrow x \odot y = (x + y - y) \wedge 0 = x \wedge 0 = x$ and $y \rightsquigarrow y \odot x = (-y + y + x) \wedge 0 = x \wedge 0 = x$.

Thus \mathbf{G}^- is a cancellative pseudo-hoop.

Proposition 2.13 ([148]) *Let A be a cancellative pseudo-hoop. Then for all $x, y, z \in A$ the following hold:*

- (1) $x \rightarrow y = x \odot z \rightarrow y \odot z$ and $x \rightsquigarrow y = z \odot x \rightsquigarrow z \odot y$;
 (2) $x \odot z \leq y \odot z$ iff $x \leq y$ and $z \odot x \leq z \odot y$ iff $x \leq y$.

Proof

- (1) By (C_1) and $(psHOOP_3)$ we get

$$x \rightarrow y = x \rightarrow (z \rightarrow y \odot z) = x \odot z \rightarrow y \odot z.$$

By (C_2) and $(psHOOP_4)$ we get

$$x \rightsquigarrow y = x \rightsquigarrow (z \rightsquigarrow z \odot y) = z \odot x \rightsquigarrow z \odot y.$$

- (2) Applying (1), $x \odot z \leq y \odot z$ iff $x \odot z \rightarrow y \odot z = 1$ iff $x \rightarrow y = 1$ iff $x \leq y$.
 The second inequality can be proved in the same way. \square

Definition 2.5 A pseudo-hoop $(A, \odot, \rightarrow, \rightsquigarrow, 1)$ is said to be *Wajsberg* if it satisfies the following conditions:

- (Wa_1) $(x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x$;
 (Wa_2) $(x \rightsquigarrow y) \rightarrow y = (y \rightsquigarrow x) \rightarrow x$,

i.e. $x \vee_1 y = y \vee_1 x$ and $x \vee_2 y = y \vee_2 x$.

Remark 2.2 Taking $y = 0$ in (Wa_1) and (Wa_2) , it follows that a bounded Wajsberg pseudo-hoop satisfies (pDN).

Example 2.2 ([148]) Let $\mathbf{G} = (G, \vee, \wedge, +, -, 0)$ be an arbitrary ℓ -group. For an arbitrary element $u \in G$, $u \geq 0$ define on the set $G[u] = [0, u]$ the operations:

$$x \odot y := (x - u + y) \vee 0,$$

$$x \rightarrow y := (y - x + u) \wedge u,$$

$$x \rightsquigarrow y := (u - x + y) \wedge u.$$

Then $\mathbf{G}[u] = (G[u], \odot, \rightarrow, \rightsquigarrow, u)$ is a bounded Wajsberg pseudo-hoop.

Indeed, we check the conditions $(psHOOP_1)$ – $(psHOOP_5)$.

$(psHOOP_1)$ $x \odot u = (x - u + u) \vee 0 = x \vee 0 = x$ and $u \odot x = (u - u + x) \vee 0 = x \vee 0 = x$, since $x \geq 0$.

$(psHOOP_2)$ $x \rightarrow x = x \rightsquigarrow x = u \wedge u = u$.

$(psHOOP_3)$ Applying the properties of an ℓ -group we have:

$$\begin{aligned} x \odot y \rightarrow z &= [(x - u + y) \vee 0] \rightarrow z \\ &= [z - (x - u + y) \vee 0 + u] \wedge u \end{aligned}$$

$$\begin{aligned}
&= (z - y + u - x + u) \wedge (z + u) \wedge u \\
&= (z - y + u - x + u) \wedge u \quad \text{and} \\
x \rightarrow (y \rightarrow z) &= [(y \rightarrow z) - x + u] \wedge u = [(z - y + u) \wedge u - x + u] \wedge u \\
&= (z - y + u - x + u) \wedge (u - x + u) \wedge u \\
&= (z - y + u - x + u) \wedge u.
\end{aligned}$$

Thus $(x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z)$.

$(psHOOP_4)$ Can be proved in a similar way as $(psHOOP_3)$.

$(psHOOP_5)$ We have:

$$\begin{aligned}
(x \rightarrow y) \odot x &= [(x \rightarrow y) - u + x] \vee 0 = [(y - x + u) \wedge u - u + x] \vee 0 \\
&= [(y - x + u - u + x) \wedge (u - u + x)] \vee 0 = (y \wedge x) \vee 0 \\
&= y \wedge x.
\end{aligned}$$

Similarly, $(y \rightarrow x) \odot y = x \wedge y$, $x \odot (x \rightsquigarrow y) = y \wedge x$, $y \odot (y \rightsquigarrow x) = x \wedge y$.

Thus $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$.

It follows that $\mathbf{G}[u] = (G[u], \odot, \rightarrow, \rightsquigarrow, u)$ is a pseudo-hoop.

Obviously, it is bounded.

We will prove that $\mathbf{G}[u]$ satisfies conditions (Wa_1) and (Wa_2) .

Let $x, y \in G[u]$. We have:

$$\begin{aligned}
(x \rightarrow y) \rightsquigarrow y &= [u - (x \rightarrow y) + y] \wedge u = [u - (y - x + u) \wedge u + y] \wedge u \\
&= [(u - u + x - y + y) \vee (u - u + y)] \wedge u \\
&= (x \vee y) \wedge u = x \vee y \quad \text{and} \\
(y \rightarrow x) \rightsquigarrow x &= [u - (y \rightarrow x) + x] \wedge u = [u - (x - y + u) \wedge u + x] \wedge u \\
&= [(u - u + y - x + x) \vee (u - u + x)] \wedge u \\
&= (y \vee x) \wedge u = y \vee x = x \vee y.
\end{aligned}$$

Thus $(x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x$, hence $\mathbf{G}[u]$ satisfies (Wa_1) .

We can similarly prove that condition (Wa_2) is also satisfied.

Therefore $\mathbf{G}[u] = (G[u], \odot, \rightarrow, \rightsquigarrow, u)$ is a bounded Wajsberg pseudo-hoop.

Proposition 2.14 ([148]) *Let A be a Wajsberg pseudo-hoop. Then for all $x, y \in A$ the following hold:*

- (cw₁) $x \cup_1 y = (x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x$;
- (cw₂) $x \cup_2 y = (x \rightsquigarrow y) \rightarrow y = (y \rightsquigarrow x) \rightarrow x$;
- (cw₃) \cup_1 and \cup_2 are associative;
- (cw₄) $x \vee y = x \cup_1 y = x \cup_2 y$.

Proof

(cw₁) This follows from the definition of \cup_1 and (Wa₁).

(cw₂) This follows from the definition of \cup_2 and (Wa₂).

(cw₃) If $x \leq y$ and $z \in A$, then applying (psbck-c₁) twice we get $y \rightarrow z \leq x \rightarrow z$ and $(x \rightarrow z) \rightsquigarrow z \leq (y \rightarrow z) \rightsquigarrow z$, that is, $x \cup_1 z \leq y \cup_1 z$.

By Proposition 2.9, \cup_1 is associative. Similarly, \cup_2 is associative.

(cw₄) This follows by Remark 2.1. □

Corollary 2.1 *If A is a Wajsberg pseudo-hoop, then*

$$x \vee y \rightarrow y = x \rightarrow y \quad \text{and} \quad x \vee y \rightsquigarrow y = x \rightsquigarrow y$$

for all $x, y \in A$.

Definition 2.6 A pseudo-hoop A is called *basic* if it satisfies the following conditions:

(Ba₁) $(x \rightarrow y) \rightarrow z \leq ((y \rightarrow x) \rightarrow z) \rightarrow z$;

(Ba₂) $(x \rightsquigarrow y) \rightsquigarrow z \leq ((y \rightsquigarrow x) \rightsquigarrow z) \rightsquigarrow z$.

We say that a pseudo-hoop is *representable* if it can be represented as a subdirect product of linearly ordered pseudo-hoops (see Chap. 3).

It is straightforward to verify that any linearly ordered pseudo-hoop and hence any representable pseudo-hoop is basic.

Proposition 2.15 ([148]) *Let A be a basic pseudo-hoop. For any $x, y, z \in A$, the following hold:*

(1) $(x \rightarrow y) \cup_1 (y \rightarrow x) = 1$ and $(x \rightsquigarrow y) \cup_2 (y \rightsquigarrow x) = 1$;

(2) $x \rightarrow y = (x \cup_1 y) \rightarrow y$ and $x \rightsquigarrow y = (x \cup_2 y) \rightsquigarrow y$;

(3) $(x \cup_1 y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$ and $(x \cup_2 y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$.

Proof

(1) Let $u = (x \rightarrow y) \cup_1 (y \rightarrow x)$. According to (Ba₁) we have $(x \rightarrow y) \rightarrow u \leq ((y \rightarrow x) \rightarrow u) \rightarrow u$. Applying Proposition 2.8(2) we have $x \rightarrow y, y \rightarrow x \leq u$, hence $(x \rightarrow y) \rightarrow u = (y \rightarrow x) \rightarrow u = 1$. It follows that $1 \leq 1 \rightarrow u = u$, so $u = 1$, that is, $(x \rightarrow y) \cup_1 (y \rightarrow x) = 1$. Similarly, $(x \rightsquigarrow y) \cup_2 (y \rightsquigarrow x) = 1$.

(2) Since $x \leq x \cup_1 y$, applying (psbck-c₁) we get $(x \cup_1 y) \rightarrow y \leq x \rightarrow y$.

From (ps hoop-c₅) and (psbck-c₁) it follows that

$$x \rightarrow y \leq ((x \rightarrow y) \rightsquigarrow y) \rightarrow y \leq (x \cup_1 y) \rightarrow y,$$

since $x \cup_1 y = ((x \rightarrow y) \rightsquigarrow y) \wedge ((y \rightarrow x) \rightsquigarrow x) \leq (x \rightarrow y) \rightsquigarrow y$.

Hence $x \rightarrow y = (x \cup_1 y) \rightarrow y$.

We can prove in the same manner that $x \rightsquigarrow y = (x \cup_2 y) \rightsquigarrow y$.

(3) Since $x, y \leq x \cup_1 y$, applying (*psbck-c*₁) we have $(x \cup_1 y) \rightarrow z \leq x \rightarrow z$ and $(x \cup_1 y) \rightarrow z \leq y \rightarrow z$. Hence $(x \cup_1 y) \rightarrow z \leq (x \rightarrow z) \wedge (y \rightarrow z)$.

Let $u = [(x \rightarrow z) \wedge (y \rightarrow z)] \rightsquigarrow [(x \cup_1 z) \rightarrow z]$. We will prove that $u = 1$.

We have:

$$\begin{aligned} & [(x \rightarrow z) \wedge (y \rightarrow z)] \odot [(x \cup_1 y) \rightarrow y] \odot (x \cup_1 y) \\ &= [(x \rightarrow z) \wedge (y \rightarrow z)] \odot [(x \cup_1 y) \wedge y] \\ &= [(x \rightarrow z) \wedge (y \rightarrow z)] \odot y \leq (y \rightarrow z) \odot y = y \wedge z \leq z, \quad \text{so} \\ & [(x \rightarrow z) \wedge (y \rightarrow z)] \odot [(x \cup_1 y) \rightarrow y] \leq (x \cup_1 y) \rightarrow z. \end{aligned}$$

It follows that $(x \cup_1 y) \rightarrow y \leq [(x \rightarrow z) \wedge (y \rightarrow z)] \rightsquigarrow [(x \cup_1 y) \rightarrow z] = u$.

Applying (2) it follows that $x \rightarrow y = (x \cup_1 y) \rightarrow y \leq u$, that is, $(x \rightarrow y) \rightarrow u = 1$.

Similarly, $(y \rightarrow x) \rightarrow u = 1$.

By (*Ba*₁) we get $1 = (x \rightarrow y) \rightarrow u \leq ((y \rightarrow x) \rightarrow u) \rightarrow u = 1 \rightarrow u = u$, so $u = 1$.

Hence $(x \rightarrow z) \wedge (y \rightarrow z) \leq (x \cup_1 z) \rightarrow z$.

We conclude that $(x \cup_1 y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z)$.

Similarly, $(x \cup_2 y) \rightsquigarrow z = (x \rightsquigarrow z) \wedge (y \rightsquigarrow z)$. □

Proposition 2.16 ([148]) *Let A be a basic pseudo-hoop. Then, for any $x, y \in A$, $x \vee y$ exists and $x \vee y = x \cup_1 y = x \cup_2 y$. The lattice (A, \wedge, \vee) is distributive.*

Proof We have $x, y \leq x \cup_1 y$ and $x, y \leq x \cup_2 y$. Let $z \in A$ such that $x, y \leq z$, that is, $x \rightarrow z = y \rightarrow z = 1$.

According to Proposition 2.15(3) we have $(x \cup_1 y) \rightarrow z = (x \rightarrow z) \wedge (y \rightarrow z) = 1 \wedge 1 = 1$, so $x \cup_1 y \leq z$. Similarly, $x \cup_2 y \leq z$. Thus $x \vee y = x \cup_1 y = x \cup_2 y$.

Finally, applying Proposition 2.5(3) we conclude that (A, \wedge, \vee) is a distributive lattice. □

Proposition 2.17 ([148]) *Let A be a pseudo-hoop. The following are equivalent:*

- (a) A is a basic pseudo-hoop;
- (b) \cup_1 and \cup_2 are associative and $(x \rightarrow y) \cup_1 (y \rightarrow x) = 1$ for all $x, y \in A$;
- (c) \cup_1 and \cup_2 are associative and $(x \rightsquigarrow y) \cup_2 (y \rightsquigarrow x) = 1$ for all $x, y \in A$.

Proof

(a) \Rightarrow (b) Applying Proposition 2.16 it follows that $\vee = \cup_1 = \cup_2$ is the join operation on A . Taking into consideration Propositions 2.9 and 2.10 we get that \cup_1 and \cup_2 are associative. The second assertion follows by Proposition 2.15(1).

(b) \Rightarrow (a) By Remark 2.1 we have $\vee = \cup_1 = \cup_2$. Applying (*psbck-c*₂₄) and (*psbck-c*₁₂) we get

$$\begin{aligned} ((x \rightarrow y) \rightarrow z) \odot ((y \rightarrow x) \rightarrow z) &\leq ((x \rightarrow y) \rightarrow z) \wedge ((y \rightarrow x) \rightarrow z) \\ &= ((x \rightarrow y) \vee (y \rightarrow x)) \rightarrow z = 1 \rightarrow z = z. \end{aligned}$$

Hence $((x \rightarrow y) \rightarrow z) \leq ((y \rightarrow x) \rightarrow z) \rightarrow z$, that is, (Ba_1) .

(Ba_2) can be proved similarly.

(a) \Leftrightarrow (c) follows in the same manner as (a) \Leftrightarrow (b). \square

Proposition 2.18 ([148]) *In any basic pseudo-hoop A the following hold:*

- (1) $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$ and $(y \wedge z) \odot x = (y \odot x) \wedge (z \odot x)$;
- (2) $(x \rightarrow y) \rightarrow (y \rightarrow x) = y \rightarrow x$ and $(x \rightsquigarrow y) \rightsquigarrow (y \rightsquigarrow x) = y \rightsquigarrow x$.

Proof

- (1) According to Proposition 2.16, \vee exists, $\vee = \cup_1 = \cup_2$ and (A, \wedge, \vee) is distributive. Applying Propositions 2.15, 2.5 we get:

$$\begin{aligned} (x \odot y) \wedge (x \odot z) &= [(x \odot y) \wedge (x \odot z)] \odot 1 \\ &= [(x \odot y) \wedge (x \odot z)] \odot [(y \rightsquigarrow z) \vee (z \rightsquigarrow y)] \\ &= [((x \odot y) \wedge (x \odot z)) \odot (y \rightsquigarrow z)] \\ &\quad \vee [((x \odot y) \wedge (x \odot z)) \odot (z \rightsquigarrow y)] \\ &\leq [x \odot y \odot (y \rightsquigarrow z)] \vee [x \odot z \odot (z \rightsquigarrow y)] \\ &= [x \odot (y \wedge z)] \vee [x \odot (z \wedge y)] = x \odot (y \wedge z). \end{aligned}$$

On the other hand, from $x \odot (y \wedge z) \leq x \odot y$ and $x \odot (y \wedge z) \leq x \odot z$ we get $x \odot (y \wedge z) \leq (x \odot y) \wedge (x \odot z)$.

Thus $x \odot (y \wedge z) = (x \odot y) \wedge (x \odot z)$ and similarly $(y \wedge z) \odot x = (y \odot x) \wedge (z \odot x)$.

- (2) According to $(psbck-c_6)$ we have $y \rightarrow x \leq (x \rightarrow y) \rightarrow (y \rightarrow x)$.

Applying Proposition 2.15, we have

$$\begin{aligned} 1 &= (y \rightarrow x) \cup_1 (x \rightarrow y) \\ &= [((y \rightarrow x) \rightarrow (x \rightarrow y)) \rightsquigarrow (x \rightarrow y)] \\ &\quad \wedge [((x \rightarrow y) \rightarrow (y \rightarrow x)) \rightsquigarrow (y \rightarrow x)]. \end{aligned}$$

Hence $(x \rightarrow y) \rightarrow (y \rightarrow x) \leq y \rightarrow x$.

Thus $(x \rightarrow y) \rightarrow (y \rightarrow x) = y \rightarrow x$.

Similarly, $(x \rightsquigarrow y) \rightsquigarrow (y \rightsquigarrow x) = y \rightsquigarrow x$. \square

Proposition 2.19 ([148]) *Any Wajsberg pseudo-hoop is a basic pseudo-hoop.*

Proof Let $x, y \in A$. By $(pshoop-c_7)$, (Wa_1) and $(psbck-c_{22})$ we have:

$$\begin{aligned} y \rightarrow x &= [((y \rightarrow x) \rightsquigarrow x) \rightarrow y] \rightarrow (y \rightarrow x) \\ &= [((x \rightarrow y) \rightsquigarrow y) \rightarrow y] \rightarrow (y \rightarrow x) = (x \rightarrow y) \rightarrow (y \rightarrow x). \end{aligned}$$

Applying (cw_1) we get

$$\begin{aligned} (x \rightarrow y) \cup_1 (y \rightarrow x) &= ((x \rightarrow y) \rightarrow (y \rightarrow x)) \rightsquigarrow (y \rightarrow x) \\ &= (y \rightarrow x) \rightsquigarrow (y \rightarrow x) = 1. \end{aligned}$$

By (cw_3) it follows that \cup_1 and \cup_2 are associative and applying Proposition 2.17 we get that A is a basic pseudo-hoop. \square

Proposition 2.20 ([148]) *Let A be a basic pseudo-hoop satisfying the conditions:*

$$\begin{aligned} y \rightarrow x = x \quad &\text{implies} \quad x = 1 \text{ or } y = 1 \quad \text{and} \\ y \rightsquigarrow x = x \quad &\text{implies} \quad x = 1 \text{ or } y = 1 \end{aligned}$$

for all $x, y \in A$. Then A is a linearly ordered Wajsberg pseudo-hoop.

Proof Consider $x, y \in A$. Applying Proposition 2.18(2) we have $(x \rightarrow y) \rightarrow (y \rightarrow x) = y \rightarrow x$. Taking into consideration the hypothesis, we get $x \rightarrow y = 1$ or $y \rightarrow x = 1$, that is, $x \leq y$ or $y \leq x$. It follows that A is a linearly ordered pseudo-hoop.

We will now prove that A is a Wajsberg pseudo-hoop.

Let $x, y \in A$. If $x = y$, then (Wa_1) is obvious. Assume $x \neq y$. Since A is linear, we can suppose that $x < y$. It follows that $(x \rightarrow y) \rightsquigarrow y = 1 \rightsquigarrow y = y$.

By $(pshoop-c_7)$ we have $[((y \rightarrow x) \rightsquigarrow x) \rightarrow y] \rightarrow (y \rightarrow x) = y \rightarrow x$, so by hypothesis and the fact that $y \rightarrow x \neq 1$, we get $((y \rightarrow x) \rightsquigarrow x) \rightarrow y = 1$, hence $(y \rightarrow x) \rightsquigarrow x \leq y$. But, from $(pshoop-c_5)$ we have $y \leq (y \rightarrow x) \rightsquigarrow x$, so $(y \rightarrow x) \rightsquigarrow x = y$. Hence $(x \rightarrow y) \rightsquigarrow y = (y \rightarrow x) \rightsquigarrow x$.

Thus A satisfies (Wa_1) . Similarly, A satisfies (Wa_2) .

We conclude that A is a Wajsberg pseudo-hoop. \square

Corollary 2.2 *Every strongly simple basic pseudo-hoop is a linearly ordered Wajsberg pseudo-hoop.*

Proof This follows from Lemma 2.1 and Proposition 2.20. \square

Example 2.3 ([148]) The pseudo-hoop \mathbf{G}^- from Example 2.1 is a basic pseudo-hoop. Indeed, consider $x, y, z \in G^-$. We have

$$(x \rightarrow y) \rightarrow z = (z - (x \rightarrow y)) \wedge 0 = (z - (y - x) \wedge 0) \wedge 0 = [(z - x + y) \vee z] \wedge 0.$$

Similarly, $(y \rightarrow x) \rightarrow z = [(z - y + x) \vee z] \wedge 0$. It follows that:

$$\begin{aligned} ((y \rightarrow x) \rightarrow z) \rightarrow z &= [z - (((z - y + x) \vee z) \wedge 0)] \wedge 0 \\ &= [(z - (z - y + x) \vee z) \vee z] \wedge 0 \end{aligned}$$

$$\begin{aligned}
&= [(z - x + y - z) \wedge 0] \vee z] \wedge 0 \\
&= [(z - x + y - z) \vee z] \wedge (0 \vee z) \wedge 0 \\
&= [(z - x + y - z) \vee z] \wedge 0.
\end{aligned}$$

Since $z \leq 0$, we have $0 \leq -z$, hence $z - x + y \leq z - x + y - z$. Thus

$$\begin{aligned}
(x \rightarrow y) \rightarrow z &= [(z - x + y) \vee z] \wedge 0 \leq [(z - x + y - z) \vee z] \wedge 0 \\
&= ((y \rightarrow x) \rightarrow z) \rightarrow z.
\end{aligned}$$

It follows that \mathbf{G}^- satisfies (Ba_1) and similarly \mathbf{G}^- satisfies (Ba_2) .

We conclude that \mathbf{G}^- is a basic pseudo-hoop.

Example 2.4 The pseudo-hoop $\mathbf{G}[\mathbf{u}]$ from Example 2.2 is a basic pseudo-hoop. Indeed, we have proved that $\mathbf{G}[\mathbf{u}]$ is a Wajsberg pseudo-hoop and applying Proposition 2.19 it follows that $\mathbf{G}[\mathbf{u}]$ is a basic pseudo-hoop.

Definition 2.7 An element a of a pseudo-hoop A is said to be an *idempotent* if $a^2 = a$. The set of all idempotents of A is denoted by $\text{Id}(A)$.

A pseudo-hoop A is called an *idempotent pseudo-hoop* if $\text{Id}(A) = A$, that is, all elements of A are idempotent.

On the other hand, an idempotent pseudo-hoop A is a Gödel pseudo-hoop, that is, a pseudo-hoop with condition (Gödel) ($a \odot a = a$ for all $a \in A$).

Lemma 2.2 (Proposition 3.1 in [106]) *If $a \in \text{Id}(A)$, then for all $x \in A$ we have:*

- (1) $a \odot x = a \wedge x = x \odot a$;
- (2) $a \rightarrow x = a \rightsquigarrow x$.

Proof

- (1) We have:

$$a \odot x \leq a \wedge x = a \odot (a \rightsquigarrow x) = a \odot a \odot (a \rightsquigarrow x) = a \odot (a \wedge x) \leq a \odot x.$$

Thus $a \odot x = a \wedge x$ and similarly $x \odot a = a \wedge x$.

- (2) For an arbitrary $z \in A$ we have:

$$z \leq a \rightarrow x \quad \text{iff} \quad z \odot a \leq x \quad \text{iff} \quad a \odot z \leq x \quad \text{iff} \quad z \leq a \rightsquigarrow x,$$

that is, $a \rightarrow x = a \rightsquigarrow x$. □

Remark 2.3

- (1) Representable Brouwerian algebras are idempotent basic hoops and generalized Boolean algebras are idempotent Wajsberg hoops ([198]).
- (2) Any bounded idempotent pseudo-hoop A is good.

Indeed, applying the identity $a \rightarrow x = a \rightsquigarrow x$ for $x = 0$, we get $a^- = a^\sim$, so $a^{-\sim} = a^{--} = a^{\sim-}$ for all $a \in A$.

2.2 Join-Center and Cancellative-Center of Pseudo-hoops

We introduce the notions of join-center and cancellative-center of a pseudo-hoop and we prove some of their properties.

Definition 2.8 If A is a pseudo-hoop, then the set $JC(A) = \{a \in A \mid a \vee x \text{ exists for all } x \in A\}$ is called the *join-center* of A .

Obviously, $1 \in JC(A)$ and, if A is bounded, then $0 \in JC(A)$.

If A is a Wajsberg pseudo-hoop, then $JC(A) = A$.

Definition 2.9 If A is a pseudo-hoop, then the set $CC(A) = \{a \in A \mid x \odot a = y \odot a \text{ implies } x = y \text{ and } a \odot x = a \odot y \text{ implies } x = y \text{ for all } x, y \in A\}$ is called the *cancellative-center* of A .

Obviously, $1 \in CC(A)$ and $0 \notin CC(A)$.

If A is a cancellative pseudo-hoop, then $CC(A) = A$.

Proposition 2.21 *If A is a pseudo-hoop and $a \in CC(A)$, then the following hold for all $x \in A$:*

- (1) $a \rightarrow (x \odot a) = x$;
- (2) $a \rightsquigarrow (a \odot x) = x$.

Proof

- (1) We have $x \odot a = a \wedge (x \odot a) = (a \rightarrow (x \odot a)) \odot a$.

Taking into consideration that $a \in CC(A)$, we get $a \rightarrow (x \odot a) = x$.

- (2) Similarly, from $a \odot x = a \wedge (a \odot x) = a \odot (a \rightsquigarrow (a \odot x))$ we have $a \rightsquigarrow (a \odot x) = x$. □

Corollary 2.3 *If A is a pseudo-hoop and $a \in CC(A)$, then*

$$a^n \rightarrow a^{n+1} = a^n \rightsquigarrow a^{n+1} = a \quad \text{for all } n \in \mathbb{N}.$$

Proof Applying Proposition 2.21(1) and (*psHOOP*₃) for $x = a, a^2, a^3, \dots$ we get:

$$\begin{aligned} a &= a \rightarrow a^2 = a \rightarrow (a \rightarrow a^3) = a^2 \rightarrow a^3 = a^2 \rightarrow (a \rightarrow a^4) \\ &= a^3 \rightarrow a^4 = \dots = a^n \rightarrow a^{n+1}. \end{aligned}$$

Similarly, by Proposition 2.21(2) and (*psHOOP*₄) we get $a = a^n \rightsquigarrow a^{n+1}$. □

Proposition 2.22 *If A is a pseudo-hoop and $a \in CC(A)$, then the following hold for all $x, y \in A$:*

- (1) $x \rightarrow y = (x \odot a) \rightarrow (y \odot a)$;

- (2) $x \rightsquigarrow y = (a \odot x) \rightsquigarrow (a \odot y)$;
 (3) $x \leq y$ iff $x \odot a \leq y \odot a$ iff $a \odot x \leq a \odot y$.

Proof

- (1) Applying Proposition 2.21(1) and (*psHOOP*₃) we get:

$$x \rightarrow y = x \rightarrow (a \rightarrow (y \odot a)) = (x \odot a) \rightarrow (y \odot a).$$

- (2) Similarly, $x \rightsquigarrow y = x \rightsquigarrow (a \rightsquigarrow (a \odot y)) = (a \odot x) \rightsquigarrow (a \odot y)$.
 (3) Applying (1) we have $x \odot a \leq y \odot a$ iff $(x \odot a) \rightarrow (y \odot a) = 1$ iff $x \rightarrow y = 1$ iff $x \leq y$. Similarly, applying (2) we get $a \odot x \leq a \odot y$ iff $x \leq y$. \square

2.3 Algebras on Subintervals of Pseudo-hoops

The problem of introducing an MV-algebra structure and a pseudo-MV algebra structure on subintervals of algebras was solved in [37] and respectively in [195] and [196]. It was proved in [105] that for a bounded $R\ell$ -monoid or a pseudo-BL algebra A , for any $a, b \in A$, $a \leq b$, the subinterval $[a, b]$ can be endowed with a structure of the same kind as that on A . For the case of an FL_w -algebra A it was proved in [55] that, if a, b with $a \leq b$ belonging to the Boolean center of A , then the subinterval $[a, b]$ of A can be endowed with a structure of an FL_w -algebra. In this section we will establish some conditions on $a, b \in A$ for the subinterval $[a, b]$ of A to be endowed with a structure of a pseudo-hoop.

Theorem 2.2 *Let $(A, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ be a bounded pseudo-hoop and $a \in JC(A)$. Then the algebra $A_a^1 = ([a, 1], \odot_a^1, \rightarrow_a^1, \rightsquigarrow_a^1, a, 1)$ is a bounded pseudo-hoop, where $x \odot_a^1 y := (x \odot y) \vee a$, $x \rightarrow_a^1 y := x \rightarrow y$ and $x \rightsquigarrow_a^1 y := x \rightsquigarrow y$.*

Proof First, we observe that $a \leq y \leq x \rightarrow y$, $x \rightsquigarrow y$ implies $x \rightarrow y$, $x \rightsquigarrow y \in [a, 1]$ for all $x, y \in [a, 1]$. We will check conditions (*psHOOP*₁)–(*psHOOP*₅) from the definition of a pseudo-hoop:

(*psHOOP*₁) For all $x \in [a, 1]$ we have:

$$\begin{aligned} x \odot_a^1 1 &= (x \odot 1) \vee a = x \vee a = x \quad \text{and} \\ 1 \odot_a^1 x &= (1 \odot x) \vee a = x \vee a = x. \end{aligned}$$

(*psHOOP*₂) $x \rightarrow_a^1 x = x \rightarrow x = 1$ and $x \rightsquigarrow_a^1 x = x \rightsquigarrow x = 1$;

(*psHOOP*₃)

$$\begin{aligned} x \odot_a^1 y \rightarrow_a^1 z &= (x \odot y) \vee a \rightarrow z = (x \odot y \rightarrow z) \wedge (a \rightarrow z) \\ &= (x \odot y) \rightarrow z = x \rightarrow (y \rightarrow z) = x \rightarrow_a^1 (y \rightarrow_a^1 z) \end{aligned}$$

(since $a \leq z$, it follows that $a \rightarrow z = 1$).

(*psHOOP*₄) can be proved in a similar way as (*psHOOP*₃).

(*psHOOP*₅) Since $(x \rightarrow y) \odot x = (y \rightarrow x) \odot y = x \odot (x \rightsquigarrow y) = y \odot (y \rightsquigarrow x)$, we get

$$\begin{aligned} [(x \rightarrow y) \odot x] \vee a &= [(y \rightarrow x) \odot y] \vee a \\ &= [x \odot (x \rightsquigarrow y)] \vee a = [y \odot (y \rightsquigarrow x)] \vee a, \end{aligned}$$

that is,

$$(x \rightarrow_a^1 y) \odot_a^1 x = (y \rightarrow_a^1 x) \odot_a^1 y = x \odot_a^1 (x \rightsquigarrow_a^1 y) = y \odot_a^1 (y \rightsquigarrow_a^1 x).$$

Thus $A_a^1 = ([a, 1], \odot_a^1, \rightarrow_a^1, \rightsquigarrow_a^1, a, 1)$ is a bounded pseudo-hoop. \square

Obviously, $A = A_0^1$ and $\{1\} = A_1^1$.

Theorem 2.3 *Let $(A, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ be a bounded pseudo-hoop, $a \in CC(A)$ and $A_0^a = ([0, a], \odot_0^a, \rightarrow_0^a, \rightsquigarrow_0^a, 0, a)$, where: $x \odot_0^a y := x \odot (a \rightsquigarrow y)$, $x \rightarrow_0^a y := (x \rightarrow y) \odot a$ and $x \rightsquigarrow_0^a y := a \odot (x \rightsquigarrow y)$. Then A_0^a is a bounded pseudo-hoop.*

Proof We will verify the axioms of a pseudo-hoop:

(*psHOOP*₁) For all $x \in [0, a]$ we have:

$$\begin{aligned} x \odot_0^a a &= x \odot (a \rightsquigarrow a) = x \odot 1 = x \quad \text{and} \\ a \odot_0^a x &= a \odot (a \rightsquigarrow x) = a \wedge x = x. \end{aligned}$$

(*psHOOP*₂) For all $x \in [0, a]$ we have:

$$\begin{aligned} x \rightarrow_0^a x &= (x \rightarrow x) \odot a = 1 \odot a = a \quad \text{and} \\ x \rightsquigarrow_0^a x &= a \odot (x \rightsquigarrow x) = a \odot 1 = a. \end{aligned}$$

(*psHOOP*₃) First of all we note that $x \odot_0^a y = (a \rightarrow x) \odot y$.

Indeed, from $(a \rightarrow x) \odot a \odot (a \rightsquigarrow y) = (a \rightarrow x) \odot a \odot (a \rightsquigarrow y)$ we get $(a \wedge x) \odot (a \rightsquigarrow y) = (a \rightarrow x) \odot (a \wedge y)$. Hence $x \odot (a \rightsquigarrow y) = (a \rightarrow x) \odot y$, that is, $x \odot_0^a y = (a \rightarrow x) \odot y$.

Applying the rules of calculus in pseudo-hoops, we get:

$$\begin{aligned} x \odot_0^a y \rightarrow_0^a z &= x \rightarrow_0^a (y \rightarrow_0^a z) \quad \text{iff} \\ (a \rightarrow x) \odot y \rightarrow_0^a z &= x \rightarrow_0^a (y \rightarrow z) \odot a \quad \text{iff} \\ [(a \rightarrow x) \odot y \rightarrow z] \odot a &= [x \rightarrow (y \rightarrow z) \odot a] \odot a \quad \text{iff} \\ (a \rightarrow x) \odot y \rightarrow z &= x \rightarrow (y \rightarrow z) \odot a \quad \text{iff} \\ (a \rightarrow x) \rightarrow (y \rightarrow z) &= x \rightarrow (y \rightarrow z) \odot a. \end{aligned}$$

For any $u \in A$ we have:

$$\begin{aligned}
u \leq (a \rightarrow x) \rightarrow (y \rightarrow z) &\Rightarrow u \odot (a \rightarrow x) \leq y \rightarrow z \\
&\Rightarrow u \odot (a \rightarrow x) \odot a \leq (y \rightarrow z) \odot a \\
&\Rightarrow u \odot (a \wedge x) \leq (y \rightarrow z) \odot a \\
&\Rightarrow u \odot x \leq (y \rightarrow z) \odot a \\
&\Rightarrow u \leq x \rightarrow (y \rightarrow z) \odot a.
\end{aligned}$$

Conversely,

$$\begin{aligned}
u \leq x \rightarrow (y \rightarrow z) \odot a &\Rightarrow u \odot x \leq (y \rightarrow z) \odot a \\
&\Rightarrow u \odot (a \wedge x) \leq (y \rightarrow z) \odot a \\
&\Rightarrow u \odot (a \rightarrow x) \odot a \leq (y \rightarrow z) \odot a \\
&\Rightarrow u \odot (a \rightarrow x) \leq y \rightarrow z \\
&\Rightarrow u \leq (a \rightarrow x) \rightarrow (y \rightarrow z).
\end{aligned}$$

Since u is arbitrary, it follows that $(a \rightarrow x) \rightarrow (y \rightarrow z) = x \rightarrow (y \rightarrow z) \odot a$.

Thus $x \odot_0^a y \rightarrow_0^a z = x \rightarrow_0^a (y \rightarrow_0^a z)$.

(*psHOOP*₄) This can be proved in a similar way as (*psHOOP*₃).

(*psHOOP*₅) For all $x, y \in [0, a]$ we have:

$$\begin{aligned}
(x \rightarrow_0^a y) \odot_0^a x &= (y \rightarrow_0^a x) \odot_0^a y \quad \text{iff} \\
[(x \rightarrow y) \odot a] \odot_0^a x &= [(y \rightarrow x) \odot a] \odot_0^a y \quad \text{iff} \\
(x \rightarrow y) \odot a \odot (a \rightsquigarrow x) &= (y \rightarrow x) \odot a \odot (a \rightsquigarrow y) \quad \text{iff} \\
(x \rightarrow y) \odot (a \wedge x) &= (y \rightarrow x) \odot (a \wedge y) \quad \text{iff} \\
(x \rightarrow y) \odot x &= (y \rightarrow x) \odot y.
\end{aligned}$$

The last identity is true, since $(A, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ is a pseudo-hoop. The remaining identities in (*psHOOP*₅) can be proved in a similar manner as the above.

We conclude that A_0^a is a bounded pseudo-hoop. \square

Theorem 2.4 *Let $(A, \odot, \rightarrow, \rightsquigarrow, 0, 1)$ be a bounded pseudo-hoop, $a, b \in CC(A) \cap JC(A)$, $a \leq b$ and $A_a^b = ([a, b], \odot_a^b, \rightarrow_a^b, \rightsquigarrow_a^b, a, b)$, where: $x \odot_a^b y := (x \odot (b \rightsquigarrow y)) \vee a$, $x \rightarrow_a^b y := (x \rightarrow y) \odot b$ and $x \rightsquigarrow_a^b y := b \odot (x \rightsquigarrow y)$. Then A_a^b is a bounded pseudo-hoop.*

Proof According to Theorem 2.2, the algebra $([a, 1], \odot_a^1, \wedge, \vee, \rightarrow_a^1, \rightsquigarrow_a^1, a, 1)$ with the operations $x \odot_a^1 y = (x \odot y) \vee a$, $x \rightarrow_a^1 y = x \rightarrow y$ and $x \rightsquigarrow_a^1 y = x \rightsquigarrow y$ is a

bounded pseudo-hoop. Let $x, y \in [a, b]$. Since $x \leq b$, by *(psbck-c₁)* we have $b \rightarrow y \leq x \rightarrow y$, hence

$$(x \rightarrow y) \odot b \geq (b \rightarrow y) \odot b = b \wedge y \geq a.$$

Similarly, $b \rightsquigarrow y \leq x \rightsquigarrow y$, so:

$$b \odot (x \rightsquigarrow y) \geq b \odot (b \rightsquigarrow y) = b \wedge y \geq a.$$

By Theorem 2.3 it follows that the algebra $([a, b], \odot_a^b, \rightarrow_a^b, \rightsquigarrow_a^b, a, b)$ is a bounded pseudo-hoop with the operations:

$$x \odot_a^b y = x \odot_a^1 (b \rightsquigarrow_a^1 y) = x \odot_a^1 (b \rightsquigarrow y) = (x \odot (b \rightsquigarrow y)) \vee a,$$

$$x \rightarrow_a^b y = (x \rightarrow_a^1 y) \odot_a^1 b = (x \rightarrow y) \odot_a^1 b = ((x \rightarrow y) \odot b) \vee a = (x \rightarrow y) \odot b,$$

$$x \rightsquigarrow_a^b y = b \odot_a^1 (x \rightsquigarrow_a^1 y) = b \odot_a^1 (x \rightsquigarrow y)$$

$$= (b \odot (x \rightsquigarrow y)) \vee a = b \odot (x \rightsquigarrow y). \quad \square$$



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