

# Chapter 2

## Linear Operators on Banach Spaces

Let  $(\mathcal{X}, \|\cdot\|)$  and  $(\mathcal{Y}, \|\cdot\|_1)$  be two Banach spaces over the same field  $\mathbb{F}$ . A mapping  $A : D(A) \subset \mathcal{X} \rightarrow \mathcal{Y}$  satisfying

$$A(\alpha x + \beta y) = \alpha Ax + \beta Ay$$

for all  $x, y \in D(A)$  and  $\alpha, \beta \in \mathbb{F}$ , is called a linear operator or a linear transformation.

In this chapter we study various properties of bounded and unbounded linear operators needed in the sequel.

### 2.1 Bounded Linear Operators

**Definition 2.1.** A linear operator  $A : \mathcal{X} \mapsto \mathcal{Y}$  satisfying the following property: there exists  $K \geq 0$  such that

$$\|Ax\|_1 \leq K\|x\|$$

for all  $x \in \mathcal{X}$ , is called a bounded (or continuous) linear operator.

The collection of all bounded linear operators from  $\mathcal{X}$  into  $\mathcal{Y}$  will be denoted by  $B(\mathcal{X}, \mathcal{Y})$  with  $B(\mathcal{X}, \mathcal{X}) := B(\mathcal{X})$ .

#### 2.1.1 Examples of Bounded Linear Operators

*Example 2.2.* Let  $\mathcal{X} = \mathcal{Y} = \mathbb{F}^n$  equipped with its natural norm given by

$$\|x\| = (x_1^2 + x_2^2 + \dots + x_n^2)^{1/2}$$

for all  $x = (x_1, x_2, \dots, x_n) \in \mathbb{F}^n$ .

Consider the canonical base for  $\mathbb{F}^n$ , that is,  $e_1 = (1, 0, 0, \dots, 0)$ ,  $e_2 = (0, 1, 0, \dots, 0)$ , ...,  $e_n = (0, 0, \dots, 1)$ . Thus every  $x \in \mathbb{F}^n$  can be written in the orthonormal basis  $(e_j)_{j=1, \dots, n}$  as follows:

$$x = \sum_{j=1}^n x_j e_j \text{ for some } x_j \in \mathbb{F}, j = 1, 2, \dots, n.$$

In particular, there exists  $a_{ij} \in \mathbb{K}$  for  $i, j = 1, 2, \dots, n$  such that

$$Ae_j = \sum_{i=1}^n a_{ij} e_i.$$

We will show that every linear operator from  $\mathbb{F}^n$  into  $\mathbb{F}^n$  is necessarily bounded. Indeed, let  $A : \mathbb{F}^n \mapsto \mathbb{F}^n$  be a linear operator. Then, for all  $x, y \in \mathbb{F}^n$  with

$$x = \sum_{j=1}^n x_j e_j \text{ and } y = \sum_{j=1}^n y_j e_j,$$

we have

$$\begin{aligned} \|Ax - Ay\| &= \left\| \sum_{j=1}^n (x_j - y_j) Ae_j \right\| \\ &\leq C \max(|x_1 - y_1|, |x_2 - y_2|, \dots, |x_n - y_n|) \\ &\leq C \|x - y\|, \end{aligned}$$

where  $C = \sum_{j=1}^n \|Ae_j\| = \sum_{j=1}^n \left( \sum_{i=1}^n |a_{ij}|^2 \right)^{1/2} < \infty$ . Therefore,  $A$  is continuous.

*Example 2.3.* Fix  $p, q \geq 1$  such that  $p^{-1} + q^{-1} = 1$ . Let  $\mathcal{X} = L^p(\mathbb{R}^n)$  and  $\mathcal{Y} = L^1(\mathbb{R}^n)$  be equipped with their natural topologies. Consider the so-called *multiplication operator* defined by  $Af = Qf$  where  $Q \in L^q(\mathbb{R}^n)$ . Using Hölder's inequality (Proposition 1.81), it easily follows that  $\|Af\|_1 \leq \|Q\|_q \|f\|_p$  for all  $f \in L^p(\mathbb{R})$  and hence  $A \in B(L^p(\mathbb{R}^n), L^1(\mathbb{R}^n))$ .

*Example 2.4.* Fix  $a, b \in \mathbb{R}$  with  $a < b$ . Let  $\mathcal{X} = \mathcal{Y} = C[a, b]$ , that is, the collection of all continuous functions from  $[a, b]$  into  $\mathbb{F}$ , which we equip with the sup-norm, that is,  $\|\cdot\|_\infty$ . Consider the so-called *Volterra operator* defined by

$$(Af)(t) = \int_a^t f(s) ds$$

for all  $t \in [a, b]$  and  $f \in C[a, b]$ . It can be easily seen that  $\|Af\|_\infty \leq (b - a) \|f\|_\infty$  for all  $f \in C[a, b]$ , which yields  $A \in B(C[a, b])$ .

*Example 2.5.* Let  $\Omega \subset \mathbb{R}^N$  be a bounded closed subset. Fix  $m \in \mathbb{Z}_+$  and let  $C^{(m)}(\Omega)$  be the collection of all functions  $f : \Omega \mapsto \mathbb{F}$  such that  $D^\alpha f$  exists and belongs to  $C(\Omega)$  for  $|\alpha| \leq m$ . The space  $C^{(m)}(\Omega)$  equipped with the norm

$$\|f\|_{m,\infty} := \max_{|\alpha| \leq m} \sup_{x \in \Omega} |D^\alpha f(x)|$$

is a Banach space.

Now let  $\mathcal{X} = C(\Omega)$  be equipped with its corresponding sup-norm and let  $\mathcal{Y} = C^{(m)}(\Omega)$  be equipped with its above-mentioned norm. Consider the differential operator given by

$$Af = \sum_{|\alpha| \leq m} a_\alpha D^\alpha f(x),$$

where the coefficients  $a_\alpha \in \mathbb{F}$  are constants.

Obviously,  $A$  is a linear operator from  $C^{(m)}(\Omega)$  to  $C(\Omega)$ . Moreover, it is not hard to show that  $A$  is continuous.

### 2.1.2 Properties of Bounded Operators

The identity and zero operators of  $\mathcal{X}$  will be respectively denoted by  $I$  and  $O$  and are defined by  $I(x) = x$  and  $O(x) = 0$  for all  $x \in \mathcal{X}$ .

Now, if  $A : \mathcal{X} \mapsto \mathcal{Y}$  is a bounded linear operator, then its kernel and range are respectively defined by  $N(A) = \{x \in \mathcal{X} : Ax = 0\}$  and  $R(A) = \{Ax : x \in \mathcal{X}\}$ .

**Proposition 2.6.** *If  $A, B \in B(\mathcal{X}, \mathcal{Y})$  and  $\gamma \in \mathbb{F}$ , then the following properties hold,*

- (a)  $A + B \in B(\mathcal{X}, \mathcal{Y})$ .
- (b)  $\gamma A \in B(\mathcal{X}, \mathcal{Y})$ .
- (c) If  $A, B \in B(\mathcal{X})$ , then  $AB, BA \in B(\mathcal{X})$ .

*Proof.* (a) Using the linearity of both  $A$  and  $B$  it follows that

$$\begin{aligned} (A + B)(\lambda x + \mu y) &= A(\lambda x + \mu y) + B(\lambda x + \mu y) \\ &= \lambda Ax + \mu Ay + \lambda Bx + \mu By \\ &= \lambda(A + B)x + \mu(A + B)y, \end{aligned}$$

and hence  $A + B$  is linear.

Similarly, using the triangle inequality and the continuity of both  $A$  and  $B$ , we obtain

$$\|(A + B)x\| = \|Ax + Bx\|$$

$$\begin{aligned}
&\leq \|Ax\| + \|Bx\| \\
&\leq K\|x\| + K'\|x\| \\
&= (K + K')\|x\|,
\end{aligned}$$

which yields  $A + B$  is continuous.

- (b) The proof is obvious and hence is omitted.  
(c) We will prove it for  $AB$  as the proof for  $BA$  is quite similar. Using the linearity of both  $A$  and  $B$  and (b) it follows that

$$(AB)(\lambda x + \mu y) = A[B(\lambda x + \mu y)] = A[\lambda Bx + \mu By] = \lambda(AB)x + \mu(AB)y$$

and hence  $AB$  is linear. Similarly, using the continuity of both  $A$  and  $B$ , we obtain that

$$\|(AB)x\| = \|A(Bx)\| \leq K\|Bx\| = (KK')\|x\|,$$

which yields  $AB$  is bounded.

In view of Proposition 2.6,  $B(\mathcal{X}, \mathcal{Y})$  is a vector space. Moreover, if  $A \in B(\mathcal{X}, \mathcal{Y})$ , then we define

$$\|A\| = \sup_{0 \neq x \in \mathcal{X}} \frac{\|Ax\|_1}{\|x\|}$$

which turns out to be a norm on  $B(\mathcal{X}, \mathcal{Y})$  commonly called *operator-norm*. Further, it can be shown that

$$\begin{aligned}
\|A\| &= \sup_{0 \neq x \in \mathcal{X}} \frac{\|Ax\|_1}{\|x\|} \\
&= \sup_{x \in \mathcal{X}} \{\|Ax\|_1 : \|x\| \leq 1\} \\
&= \sup_{x \in \mathcal{X}} \{\|Ax\|_1 : \|x\| = 1\}.
\end{aligned}$$

By definition of the operator-norm,  $\|Ax\|_1 \leq \|A\| \cdot \|x\|$  for all  $x \in \mathcal{X}$ . Moreover, if  $A, B \in B(\mathcal{X}, \mathcal{Y})$ , then

$$\|AB\| \leq \|A\| \|B\|.$$

**Theorem 2.7.** *The space  $B(\mathcal{X}, \mathcal{Y})$  is a Banach space when it is equipped with its operator norm.*

*Proof.* In view of Proposition 2.6 it follows that  $B(\mathcal{X}, \mathcal{Y})$  is a normed vector space. Now let  $(A_n)_{n \in \mathbb{N}} \subset B(\mathcal{X}, \mathcal{Y})$  be a Cauchy sequence, that is,  $\|A_n - A_m\| \rightarrow 0$  as

$n, m \rightarrow \infty$ . To complete the proof we have to show that there exists  $A \in B(\mathcal{X}, \mathcal{Y})$  such that  $\|A_n - A\| \rightarrow 0$  as  $n \rightarrow \infty$ . Indeed, for all  $x \in \mathcal{X}$ , we have,

$$\|(A_n - A_m)x\|_1 \leq \|A_n - A_m\| \cdot \|x\| \rightarrow 0$$

as  $n, m \rightarrow \infty$  and so  $(A_n x)_{n \in \mathbb{N}}$  is a Cauchy sequence in  $\mathcal{Y}$ . Since  $\mathcal{Y}$  is a Banach space, it follows that there exists  $\xi \in \mathcal{Y}$  such that  $\|A_n x - \xi\|_1 \rightarrow 0$  as  $n \rightarrow \infty$ . Thus

$$Ax = \xi = \lim_{n \rightarrow \infty} A_n x$$

defines a linear operator  $A : \mathcal{X} \mapsto \mathcal{Y}$ . It remains to show that  $A$  is bounded. Using the fact that  $(A_n)_{n \in \mathbb{N}}$  is a Cauchy sequence it follows that  $\sup_{n \in \mathbb{N}} \|A_n\| < \infty$ . Consequently,

$$\|Ax\|_1 = \lim_{n \rightarrow \infty} \|A_n x\|_1 \leq K \|x\|$$

for all  $x \in \mathcal{X}$ , that is,  $A \in B(\mathcal{X}, \mathcal{Y})$ .

**Theorem 2.8.** *If  $A : \mathcal{X} \rightarrow \mathcal{Y}$  is a linear operator, then the following statements are equivalent:*

- (a)  $A$  is continuous.
- (b)  $A$  is continuous at 0.
- (c) There exists a constant  $C > 0$  such that  $\|Au\|_1 \leq C \cdot \|u\|$  for each  $u \in \mathcal{X}$ .

*Proof.* It is not hard to see that (a) yields (b). Now if (b) holds, then there exists  $\eta > 0$  such that  $\|Ax\|_1 \leq 1$  whenever  $\|x\| \leq \eta$ . Thus for each  $0 \neq x \in \mathcal{X}$ , we have

$$\left\| \frac{\eta x}{\|x\|} \right\| = \eta.$$

Now

$$1 \geq \left\| A \left( \frac{\eta x}{\|x\|} \right) \right\|_1 = \frac{\eta \|Ax\|_1}{\|x\|},$$

and hence  $\|Ax\|_1 \leq \eta^{-1} \|x\|$ , that is (c) holds.

Now if (c) holds, then

$$\|Ax - Ax_0\|_1 = \|A(x - x_0)\|_1 \leq C \|x - x_0\|.$$

Consequently, for each  $\varepsilon > 0$  one can find an  $\eta = \varepsilon C^{-1}$  such that  $\|Ax - Ax_0\|_1 < \varepsilon$  whenever  $\|x - x_0\| \leq \eta$ . Therefore,  $A$  is continuous at  $x_0$ , which yields  $A$  is continuous on  $\mathcal{X}$  as  $x_0$  is arbitrary.

In the rest of this chapter, the spaces  $(\mathcal{H}_1, \langle \cdot, \cdot \rangle_1, \|\cdot\|_1)$ ,  $(\mathcal{H}_2, \langle \cdot, \cdot \rangle_2, \|\cdot\|_2)$ , and  $(\mathcal{H}_3, \langle \cdot, \cdot \rangle_3, \|\cdot\|_3)$  stand for Hilbert spaces over the same field  $\mathbb{F}$ .

### 2.1.2.1 Adjoint for Bounded Operators

Let  $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ . For each  $y \in \mathcal{H}_2$ , the functional  $x \mapsto \xi_y(x) := \langle Ax, y \rangle_2$  is linear and bounded. Therefore, from the Riesz representation theorem (Theorem 1.108), there exists a unique  $y^* \in \mathcal{H}_1$  such that

$$\langle Ax, y \rangle_2 = \xi_y(x) = \langle x, y^* \rangle_1$$

for all  $x \in \mathcal{H}_1, y \in \mathcal{H}_2$ .

The transformation  $A^* : \mathcal{H}_2 \mapsto \mathcal{H}_1$  defined by  $y \mapsto y^* = A^*y$  is called the adjoint of the linear operator  $A$ . In view of the above,

$$\langle Ax, y \rangle_2 = \langle x, A^*y \rangle_1$$

for all  $x \in \mathcal{H}_1, y \in \mathcal{H}_2$ .

**Proposition 2.9.** *If  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is a bounded linear operator, then  $A^* \in B(\mathcal{H}_2, \mathcal{H}_1)$ . Furthermore,  $\|A\| = \|A^*\|$ .*

*Proof.* First of all, let us show that  $A^*$  is a bounded linear operator. Indeed, given  $v, w \in \mathcal{H}_2$  and  $\alpha, \beta \in \mathbb{F}$ , we have

$$\begin{aligned} \langle u, A^*(\alpha v + \beta w) \rangle_1 &= \langle Au, \alpha v + \beta w \rangle_2 \\ &= \overline{\alpha} \langle Au, v \rangle_2 + \overline{\beta} \langle Au, w \rangle_2 \\ &= \overline{\alpha} \langle u, A^*v \rangle_1 + \overline{\beta} \langle u, A^*w \rangle_1 \\ &= \langle u, \alpha A^*v \rangle_1 + \langle u, \beta A^*w \rangle_1 \\ &= \langle u, \alpha A^*v + \beta A^*w \rangle_1 \end{aligned}$$

for all  $u \in \mathcal{H}_1$ . Hence  $A^*(\alpha v + \beta w) = \alpha A^*v + \beta A^*w$ , that is,  $A^*$  is a linear operator from  $\mathcal{H}_2$  into  $\mathcal{H}_1$ .

Now

$$\begin{aligned} \|A^*u\|_1^2 &= \langle A^*u, A^*u \rangle_1 \\ &= \langle AA^*u, u \rangle_2 \\ &\leq \|AA^*u\|_2 \cdot \|u\|_2 \\ &\leq \|A\| \cdot \|A^*u\|_1 \cdot \|u\|_2 \end{aligned}$$

and hence  $\|A^*u\|_1 \leq \|A\| \cdot \|u\|_2$ , that is,  $\|A^*\| \leq \|A\|$ . Similarly,  $\|(A^*)^*\| \leq \|A^*\|$ . Now using the fact  $(A^*)^* = A$  it follows that  $\|A\| \leq \|A^*\|$ . Combining, we obtain the desired result.

**Proposition 2.10 ([18]).** *If  $A : \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is a bounded linear operator, then  $A^*A \in B(\mathcal{H}_1)$  and  $AA^* \in B(\mathcal{H}_2)$ . Moreover,  $\|AA^*\| = \|A^*A\| = \|A^*\|^2 = \|A\|^2$ .*

**Proposition 2.11 ([18]).** *If  $A, B$  are bounded linear operators on  $\mathcal{H}$  and if  $\lambda \in \mathbb{C}$ , then,*

- (a)  $I^* = I$ .
- (b)  $O^* = O$ .
- (c)  $(A + B)^* = A^* + B^*$ .
- (d)  $(\lambda A)^* = \overline{\lambda} A^*$ .
- (e)  $(AB)^* = A^* B^*$ .

**Definition 2.12.** A bounded linear operator  $A : \mathcal{H} \mapsto \mathcal{H}$  is called self-adjoint or symmetric if  $A = A^*$ .

**Proposition 2.13 ([70]).** *If  $A \in B(\mathcal{H}_1, \mathcal{H}_2)$ , then both  $AA^*$  and  $A^*A$  are self-adjoint.*

**Proposition 2.14 ([18]).** *Let  $A : \mathcal{H} \mapsto \mathcal{H}$  be a bounded self-adjoint operator. Then the following properties hold:*

- (a)  $\langle Ax, x \rangle \in \mathbb{R}$  for all  $x \in \mathcal{H}$ .
- (b)  $\|A\| = \sup_{x \neq 0} \frac{|\langle Ax, x \rangle|}{\|x\|^2}$ .
- (c) *If  $B \in B(\mathcal{H})$  is also self-adjoint and if  $AB = BA$ , then  $AB$  is also self-adjoint.*

*Example 2.15.* Let  $\mathcal{H}_1 = \mathcal{H}_2 = L^2[a, b]$ , which we equip with the  $L^2$ -topology. Consider the integral operator defined by

$$(Af)(t) = \int_a^b K(t, s)f(s)ds$$

for all  $f \in L^2[a, b]$ , where  $K \in L^2([a, b] \times [a, b])$ .

It can be easily seen that  $A$  is a bounded linear operator on  $L^2[a, b]$ . Moreover, the adjoint of  $A$  is given by

$$(A^*f)(t) = \int_a^b \overline{K(s, t)}f(s)ds$$

for all  $f \in L^2[a, b]$ . In particular,  $A$  is self-adjoint if and only if

$$K(t, s) = \overline{K(s, t)}$$

for all  $t, s \in [a, b]$ .

### 2.1.2.2 The Inverse Operator

**Definition 2.16.** An operator  $A \in B(\mathcal{X})$  is called invertible if there exists  $B \in B(\mathcal{X})$  such that

$$AB = BA = I.$$

In that event, the operator  $B$  is called the inverse operator of  $A$  and denoted by  $B = A^{-1}$ .

**Theorem 2.17 ([18]).** *If  $A \in B(\mathcal{X})$  and if  $\|A\| < 1$ , then the linear operator  $I - A$  is invertible and its inverse is given by*

$$(I - A)^{-1} = \sum_{k=0}^{\infty} A^k.$$

*Proof.* First of all, observe that  $(I - A)(I + A^2 + \cdots + A^n) = I - A^{n+1}$  and that  $\|A^{n+1}\| \leq \|A\|^{n+1} \rightarrow 0$  as  $n \rightarrow \infty$  as  $\|A\| < 1$ . Consequently,

$$\lim_{n \rightarrow \infty} (I - A)(I + A^2 + \cdots + A^n) = I \text{ in } B(\mathcal{X}).$$

Again from  $\|A\| < 1$  and the fact that  $B(\mathcal{X})$  is a Banach algebra, then the quantity

$$S := \lim_{n \rightarrow \infty} (I + A^2 + \cdots + A^n)$$

does exist and  $(I - A)S = I$ . In fact, we have

$$(I - A)S = I + (I - A)[S - (I + A^2 + \cdots + A^n)].$$

On other hand,

$$\begin{aligned} \|(I - A)[S - (I + A^2 + \cdots + A^n)]\| &\leq \|I - S\| \\ &\cdot \|S - (I + A^2 + \cdots + A^n)\| \\ &\rightarrow 0 \text{ as } n \rightarrow \infty, \end{aligned}$$

hence  $(I - A)S = I$ .

Thus  $(I - A)$  is invertible and  $(I - A)^{-1} = S$ , where

$$S = \sum_{k=0}^{\infty} A^k \text{ (} A^0 \text{ being } I\text{)}.$$

Observe that if  $A, B \in B(\mathcal{X})$  are invertible, so is  $AB$  with  $(AB)^{-1} = B^{-1}A^{-1}$ .



**Proposition 2.18 ([102]).** *If  $A \in B(\mathcal{X})$  is invertible and if  $B \in B(\mathcal{X})$  is given such that*

$$\|A - B\| < \frac{1}{\|A^{-1}\|},$$

*then  $B$  is invertible and its inverse is given by*

$$B^{-1} = \sum_{k=0}^{\infty} [A^{-1}(A - B)]^k A^{-1}$$

*and*

$$\|A^{-1} - B^{-1}\| \leq \frac{\|A^{-1}\|^2 \cdot \|A - B\|}{1 - \|A^{-1}\| \cdot \|A - B\|}.$$

*Proof.* It suffices to write  $B = A[I - A^{-1}(A - B)]$ . From  $\|A^{-1}(A - B)\| < 1$  and Theorem 2.17 it follows that  $I - A^{-1}(A - B)$  is invertible. Using the fact that  $A$  is invertible it follows that  $A[I - A^{-1}(A - B)]$  is invertible. Further, using Theorem 2.17 it follows that

$$B^{-1} = [I - A^{-1}(A - B)]^{-1} A^{-1} = \sum_{k=0}^{\infty} [A^{-1}(A - B)]^k A^{-1}.$$

Moreover,

$$\|A^{-1} - B^{-1}\| \leq \frac{\|A^{-1}\|^2 \cdot \|A - B\|}{1 - \|A^{-1}\| \cdot \|A - B\|}.$$

### 2.1.2.3 Spectrum for Bounded Linear Operators

**Definition 2.19.** If  $A : \mathcal{X} \rightarrow \mathcal{X}$  is a bounded linear operator, then its spectrum  $\sigma(A)$  is defined by

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is not invertible}\}.$$

Similarly, the resolvent  $\rho(A)$  of  $A$  is defined by  $\rho(A) = \mathbb{C} \setminus \sigma(A)$ , that is, the collection of all  $\lambda \in \mathbb{C}$  such that the operator  $A - \lambda I$  is one-to-one ( $N(A - \lambda I) = 0$ ), onto ( $R(A - \lambda I) = \mathcal{X}$ ), and bounded.

*Example 2.20 ([18]).* Let  $q : [\alpha, \beta] \rightarrow \mathbb{C}$  be a continuous function. Define the bounded linear operator  $M_q$  on  $\mathcal{X} = L^2([\alpha, \beta])$  by

$$(M_q \phi)(s) = q(s)\phi(s), \quad \forall s \in [\alpha, \beta].$$

It can be shown that  $\lambda I - M_q$  is invertible on  $L^2([\alpha, \beta])$  if and only if

$$\lambda - q(s) \neq 0, \quad \forall s \in [\alpha, \beta]. \quad (2.1)$$

The inverse  $(\lambda I - M_q)^{-1}$  of  $\lambda I - M_q$  is defined by

$$[(\lambda I - M_q)^{-1}] \psi(s) = \left[ \frac{1}{\lambda - q(s)} \right] \psi(s)$$

with the following estimate:

$$\|(\lambda I - M_q)^{-1}\| \leq \max_{s \in [\alpha, \beta]} \left[ \frac{1}{|\lambda - q(s)|} \right].$$

The spectrum  $\sigma(M_q)$  of  $M_q$  is given by

$$\sigma(M_q) = \{q(s) : s \in [\alpha, \beta]\}.$$

#### 2.1.2.4 Compact Operators

**Definition 2.21 ([102]).** An operator  $A \in B(\mathcal{H}_1, \mathcal{H}_2)$  is said to be compact if for each sequence  $(x_n)_{n \in \mathbb{N}}$  in  $\mathcal{H}_1$  with  $\|x_n\|_1 = 1$  for each  $n \in \mathbb{N}$ , the sequence  $(Ax_n)_{n \in \mathbb{N}}$  has a subsequence which converges in  $\mathcal{H}_2$ .

The collection of all compact operators from  $\mathcal{H}_1$  to  $\mathcal{H}_2$  is denoted by  $\mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ .

**Theorem 2.22 ([102]).** If  $K, L \in B(\mathcal{H}_1, \mathcal{H}_2)$  are compact linear operators, then

- (a)  $\alpha L$  is compact.
- (b)  $K + L$  is compact.
- (c) If  $A \in B(\mathcal{H}_3, \mathcal{H}_1)$  and  $B \in B(\mathcal{H}_2, \mathcal{H}_3)$ , then  $KA$  and  $BK$  are compact.

*Proof.* (a) This is straightforward.

(b) Let  $(u_n) \in \mathcal{H}_1$  with  $\|u_n\|_1 = 1$ . Since  $K$  is compact,  $(Ku_n)_{n \in \mathbb{N}}$  has a convergent subsequence  $(Ku_{n_k})_{k \in \mathbb{N}}$ . Similarly,  $(Lu_n)_{n \in \mathbb{N}}$  has a convergent subsequence  $(Lu_{n_k})_{k \in \mathbb{N}}$ . Therefore,  $((K + L)u_{n_k})_{k \in \mathbb{N}}$  converges.

(c) Let  $(v_n)_{n \in \mathbb{N}} \subset \mathcal{H}_3$  with  $\|v_n\|_3 = 1$  for each  $n \in \mathbb{N}$ . Thus  $(Av_n)_{n \in \mathbb{N}}$  is bounded. Now since  $K$  is compact, it is clear  $(KA v_n)_{n \in \mathbb{N}}$  has a convergent subsequence and hence  $KA$  is a compact operator.

Let  $(w_n)_{n \in \mathbb{N}} \subset \mathcal{H}_1$  with  $\|w_n\|_1 = 1$  for each  $n \in \mathbb{N}$ . Now since  $K$  is compact, then  $(Kw_n)_{n \in \mathbb{N}}$  has a convergent subsequence, say,  $(Kw_{n_k})_{k \in \mathbb{N}}$ . Now by the continuity of  $B$  it follows that  $(BKw_{n_k})_{k \in \mathbb{N}}$  converges and hence  $BK$  is a compact operator.

*Example 2.23.* In  $\mathcal{H} = L^2[a, b]$ , define the integral operator  $A$  by

$$(Af)(t) := \int_a^b V(t, \tau)f(\tau)d\tau \text{ for each } f \in L^2[a, b].$$

Assuming that  $V \in L^2([a, b] \times [a, b])$ , it can be shown that  $A$  is compact.

**Definition 2.24.** A bounded linear operator is of finite rank if its image is a finite-dimensional Banach space. The collection of all finite rank operators from  $\mathcal{H}_1$  into  $\mathcal{H}_2$  is denoted  $\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2)$ .

**Proposition 2.25 ([102]).** Let  $A : \mathcal{H}_1 \mapsto \mathcal{H}_2$  be an operator of finite rank  $n$ . Then there exist vectors  $x_1, x_2, \dots, x_n$  in  $\mathcal{H}_1$  and vectors  $y_1, y_2, \dots, y_n$  in  $\mathcal{H}_2$  such that

$$Ax = \sum_{k=1}^n \langle x, x_k \rangle y_k.$$

The vectors  $y_1, y_2, \dots, y_n$  can be chosen to be an orthonormal base for  $R(A)$ .

**Proposition 2.26.**  $\mathcal{F}(\mathcal{H}_1, \mathcal{H}_2) \subset \mathcal{K}(\mathcal{H}_1, \mathcal{H}_2)$ .

**Proposition 2.27 ([102]).** An operator  $A \in B(\mathcal{H}_1, \mathcal{H}_2)$  is compact if and only if its adjoint  $A^*$  is compact.

*Proof.* If  $A$  is compact, then  $AA^*$  is compact. Thus for any  $(x_n)_{n \in \mathbb{N}} \subset \mathcal{H}_2$  with  $\|x_n\|_2 = 1$  there exists a subsequence  $(x_{n_k})_{k \in \mathbb{N}}$  of  $(x_n)_{n \in \mathbb{N}}$  such that  $(AA^*x_{n_k})_{k \in \mathbb{N}}$  converges. Now using the fact

$$\|A^*x_{n_k} - A^*x_{n_l}\| \leq 2\|AA^*x_{n_k} - AA^*x_{n_l}\|$$

it follows that  $(A^*x_{n_k})_{k \in \mathbb{N}}$  is a Cauchy sequence, which converges as  $\mathcal{H}_1$  is complete. Therefore,  $A^*$  is compact. Similarly, if  $A^*$  is compact, then using the above proved result, it follows that  $A = (A^*)^*$  is compact.

## 2.2 Unbounded Linear Operators

Let  $A : D(A) \subset \mathcal{X} \mapsto \mathcal{Y}$  be a linear operator. If  $A$  is not continuous, then  $A$  is called an unbounded operator. The collection of all unbounded linear operators from  $\mathcal{X}$  into  $\mathcal{Y}$  will be denoted  $\mathcal{U}(\mathcal{X}, \mathcal{Y})$  with  $\mathcal{U}(\mathcal{X}, \mathcal{X}) = \mathcal{U}(\mathcal{X})$ . Because of the domain involved, elements of  $\mathcal{U}(\mathcal{X})$  need to be manipulated with a great care. For instance, if  $A, B \in \mathcal{U}(\mathcal{X}, \mathcal{Y})$ , then their (algebraic) sum  $A + B$  may or may be trivial depending on  $D(A) \cap D(B)$ . Note that the (algebraic) sum operator  $A + B$  is defined by

$$D(A + B) = D(A) \cap D(B) \text{ and } (A + B)x = Ax + Bx$$

for all  $x \in D(A) \cap D(B)$ .

Similarly, the product operator  $AB$  is defined by,

$$D(AB) = \{x \in D(B) : Bx \in D(A)\} \text{ and } ABx = A(Bx)$$

for all  $x \in D(AB)$ .

As for bounded linear operators, if  $A : D(A) \subset \mathcal{X} \mapsto \mathcal{Y}$  is an unbounded linear operator, then we define its kernel  $N(A)$  and range  $R(A)$  as follows:

$$N(A) := \{x \in D(A) : Ax = 0\} \text{ and } R(A) = \{Ax : x \in D(A)\}.$$

It is clear that  $N(A) \subset \mathcal{X}$  while  $R(A) \subset \mathcal{Y}$ . The operator  $A$  is said to be *one-to-one* if  $N(A) = 0$ . The operator  $A$  is said to be *onto* if  $R(A) = \mathcal{Y}$ . Of course, a linear operator  $A$  is *bijective* or *invertible* if it is one-to-one and onto. If  $A$  is invertible, then its inverse will be denoted by  $A^{-1}$ . If  $A$  is invertible, then  $D(A^{-1}) = R(A)$  and  $R(A^{-1}) = D(A)$ .

### 2.2.1 Examples of Unbounded Operators

*Example 2.28 ([136]).* Let  $\mathcal{X} = \mathcal{Y} = L^2(\mathbb{R})$  be equipped with its natural topology and consider the one-dimensional Laplace operator defined by

$$D(A) = H^2(\mathbb{R}) \text{ and } Au = -u''$$

for all  $u \in H^2(\mathbb{R})$ .

Consider the sequence of functions defined by  $\psi_n(t) = e^{-n|t|}$ ,  $n = 1, 2, \dots$ . Clearly, for each  $n = 1, 2, \dots$ ,  $\psi_n \in D(A) = H^2(\mathbb{R})$ . Furthermore,

$$\|\psi_n\|_2^2 = \int_{-\infty}^{+\infty} e^{-2n|t|} dt = \frac{1}{n}$$

and

$$\|A\psi_n\|_2^2 = \int_{-\infty}^{+\infty} n^4 e^{-2n|t|} dt = n^3.$$

Therefore,  $\frac{\|A\psi_n\|_2}{\|\psi_n\|_2} = n \rightarrow \infty$  as  $n$  goes to  $\infty$ , that is,  $A$  is an unbounded linear operator on  $L^2(\mathbb{R})$ .

*Example 2.29 ([136]).* Let  $\mathcal{X} = \mathcal{Y} = L^2(0, 1)$  be equipped with its natural topology and consider the derivative operator defined by

$$D(A) = C^1(0, 1) \text{ and } Au = u'$$

for all  $u \in C^1(0, 1)$ , where  $C^1(0, 1)$  is the collection of continuously differentiable functions over  $(0, 1)$ .

Consider the sequence of functions defined by  $\phi_n(t) = t^n$ ,  $n = 1, 2, \dots$ . Clearly, for each  $n = 1, 2, \dots$ ,  $\phi_n \in C^1(0, 1)$ . Furthermore,

$$\|\phi_n\|_2^2 = \int_0^1 t^{2n} dt = \frac{1}{2n+1},$$

and

$$\|A\phi_n\|_2^2 = \int_0^1 n^2 t^{2n-2} dt = \frac{n^2}{2n-1}.$$

Here again,

$$\frac{\|A\phi_n\|_2}{\|\phi_n\|_2} = n\sqrt{\frac{2n+1}{2n-1}} \rightarrow \infty$$

as  $n$  goes to  $\infty$ , that is,  $A$  is an unbounded linear operator on  $L^2(\mathbb{R})$ .

### 2.2.2 Closed Linear Operators

Equip  $\mathcal{X} \times \mathcal{Y}$  with its natural norm, that is,  $\|(x, y)\| = (\|x\|^2 + \|y\|^2)^{1/2}$  for all  $(x, y) \in \mathcal{X} \times \mathcal{Y}$ .

**Definition 2.30.** If  $A : D(A) \subset \mathcal{X} \rightarrow \mathcal{Y}$  is a linear operator, then its graph is defined by

$$\mathcal{G}(A) = \{(x, Ax) \in \mathcal{X} \times \mathcal{Y} : x \in D(A)\}.$$

**Definition 2.31.** If  $A, B$  belong to  $\mathcal{U}(\mathcal{X}, \mathcal{Y})$ , then  $A$  is said to be an extension of  $B$  if  $D(B) \subset D(A)$  and  $Au = Bu$  for all  $u \in D(B)$ . We then denote this by  $B \subset A$ . Moreover,  $B \subset A$  if and only if  $\mathcal{G}(B) \subset \mathcal{G}(A)$ .

**Definition 2.32.** A linear operator  $A : D(A) \subset \mathcal{X} \mapsto \mathcal{X}$  is called closed if its graph  $\mathcal{G}(A) \subset \mathcal{X} \times \mathcal{X}$  is closed. Namely, if  $u_n \in D(A)$  such that  $u_n \rightarrow u$  and  $Au_n \rightarrow v$  in  $\mathcal{X}$  as  $n \rightarrow \infty$ , then  $u \in D(A)$  and  $Ax = v$ . The collection of all closed operators on  $\mathcal{X}$  will be denoted  $cl(\mathcal{X})$ .

It is clear that an operator  $A : D(A) \subset \mathcal{X} \mapsto \mathcal{Y}$  is closed if and only if  $\mathcal{G}(A) \subset \mathcal{X} \times \mathcal{Y}$  is closed.

### 2.2.2.1 Examples of Closed Linear Operators

*Example 2.33 ([70]).* Every  $A \in B(\mathcal{X})$  is closed. Indeed, let  $(u_n)_{n \in \mathbb{N}} \in D(A)$  such that  $u_n \rightarrow u$  with  $Au_n \rightarrow v$  in  $\mathcal{X}$  as  $n \rightarrow \infty$ . Now since  $A$  is bounded, therefore  $D(A) = \mathcal{X}$ . Again, from the continuity of  $A$  it is clear that  $u \in \mathcal{X}$  and  $Au = v$ .

*Example 2.34.* If  $A \in cl(\mathcal{X})$  and  $B \in B(\mathcal{X})$ , then  $A + B \in cl(\mathcal{X})$ . Indeed, suppose  $(u_n)_{n \in \mathbb{N}} \in D(A + B) = D(A)$  such that  $u_n \rightarrow u$  and  $(A + B)u_n \rightarrow v$  in  $\mathcal{X}$  as  $n \rightarrow \infty$ . Now since  $B$  is bounded it follows that  $Au_n \rightarrow v - Bu$  in  $\mathcal{X}$  as  $n \rightarrow \infty$ . Since  $A$  is closed, then  $u \in D(A)$  and  $Au = v - Bu$ .

*Example 2.35.* This is an illustration of Example 2.34. Let  $\mathcal{X} = L^2(\mathbb{R}^n)$  and define  $A$  and  $B$  by

$$D(A) = H^2(\mathbb{R}^n), \quad Au = -\Delta u, \quad \forall u \in H^2(\mathbb{R}^n),$$

and

$$D(B) = \{u \in L^2(\mathbb{R}^n) : \gamma(x)u \in L^2(\mathbb{R}^n)\}, \quad Bu = \gamma(x)u, \quad \forall u \in D(B),$$

where  $\Delta$  is the  $n$ -dimensional Laplace operator defined by

$$\Delta = \sum_{k=1}^n \frac{\partial^2}{\partial x_k^2}$$

and that  $\gamma \in L^\infty(\mathbb{R}^n)$ . It is then clear that  $B \in B(L^2(\mathbb{R}^n))$  and hence  $-\Delta + \gamma \in cl(L^2(\mathbb{R}^n))$  with  $D(-\Delta + \gamma) = H^2(\mathbb{R}^n)$ .

*Example 2.36.* Let  $\mathcal{X} = \mathcal{Y} = C[a, b]$  where  $a, b \in \mathbb{R}$  with  $a < b$ . Define  $C_c^1(a, b)$  to be the set of all  $f \in C^1[a, b]$  with  $\text{Supp}(f) \subset (a, b)$  where

$$\text{Supp}(f) = \overline{\{x \in [a, b] : f(x) \neq 0\}}.$$

Setting  $Au = u'$  with  $D(A) = C_c^1(a, b)$  it can be easily shown that  $A \notin cl(C[a, b])$ .

## 2.2.3 Spectral Theory for Linear Operators

### 2.2.3.1 Basic Definitions

**Definition 2.37.** If  $A \in cl(\mathcal{X})$ , then  $\rho(A)$  the resolvent set of  $A$  is defined by

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A : D(A) \mapsto \mathcal{X} \text{ is bijective, and } (\lambda I - A)^{-1} \in B(\mathcal{X})\},$$

and  $\sigma(A)$  the spectrum of  $A$  is the complement of the resolvent set  $\rho(A)$  in  $\mathbb{C}$ .

Note that if  $\lambda \in \rho(A)$ , then the operator-valued function  $R(\lambda, A) := (\lambda I - A)^{-1} : \rho(A) \mapsto \mathcal{B}(\mathcal{X})$  is called the resolvent of the operator  $A$ . It should be mentioned that  $\rho(A) \neq \emptyset$  if  $A \in cl(\mathcal{X})$ .

*Example 2.38.* Let  $\mathcal{X} = \mathcal{Y} = C[a, b]$  and let  $Au = u'$  for all  $u \in D(A) = \{u \in C^1[a, b] : u(b) = 0\}$ . We want to compute  $\rho(A)$ . For that, we have to find conditions on  $\lambda \in \mathbb{C}$  such that

$$(\lambda I - A)u = f \tag{2.2}$$

where  $\lambda \in \mathbb{C}$ ,  $u \in D(A)$ , and  $f \in C[a, b]$ , can be solved.

This is equivalent to solving the differential equation  $u' = \lambda u - f$  where  $u \in C^1[a, b]$  and  $u(b) = 0$ , which is also equivalent to

$$u(t) = \int_t^b e^{(t-s)\lambda} f(s) ds$$

for all  $t \in [a, b]$ .

Thus for each  $\lambda \in \mathbb{C}$ , Eq. (2.2) can be solved and its solution is given above. This means that  $\rho(A) = \mathbb{C}$ , which yields  $\sigma(A) = \emptyset$ . Let us also mention that the operator  $(\lambda I - A)^{-1}$  is then defined by

$$(\lambda I - A)^{-1}v(t) = R(\lambda, A)v(t) = \int_t^b e^{(t-s)\lambda} v(s) ds$$

for each  $v \in C[a, b]$ .

**Proposition 2.39 ([18]).** Let  $A, B \in cl(\mathcal{X})$ .

- (a) If  $\lambda, \mu \in \rho(A)$ , then  $R(\lambda, A) - R(\mu, A) = (\mu - \lambda)R(\lambda, A)R(\mu, A)$ . Furthermore,  $R(\lambda, A)$  and  $R(\mu, A)$  commute.
- (b) If  $D(A) \subset D(B)$ , then for all  $\lambda \in \rho(A) \cap \rho(B)$  we have

$$R(\lambda, A) - R(\lambda, B) = R(\lambda, A)(A - B)R(\lambda, B).$$

- (c) If  $D(A) = D(B)$ , then for all  $\lambda \in \rho(A) \cap \rho(B)$  we have

$$R(\lambda, A) - R(\lambda, B) = R(\lambda, A)(A - B)R(\lambda, B) = R(\lambda, B)(A - B)R(\lambda, A).$$

*Proof.* (a) Clearly

$$\begin{aligned} R(\lambda, A) - R(\mu, A) &= R(\lambda, A)[(\mu I - A) - (\lambda I - A)]R(\mu, A) \\ &= (\mu - \lambda)R(\lambda, A)R(\mu, A). \end{aligned}$$

Similarly

$$\begin{aligned} R(\lambda, A) R(\mu, A) &= \frac{1}{\mu - \lambda} [R(\lambda, A) - R(\mu, A)] \\ &= \frac{1}{\lambda - \mu} [R(\mu, A) - R(\lambda, A)] \\ &= R(\mu, A) R(\lambda, A). \end{aligned}$$

(b) We have

$$\begin{aligned} R(\lambda, A) - R(\lambda, B) &= R(\lambda, A)[(\lambda I - B) - (\lambda I - A)]R(\lambda, B) \\ &= R(\lambda, A)(A - B)R(\lambda, B). \end{aligned}$$

(c) We have

$$\begin{aligned} R(\lambda, A) - R(\lambda, B) &= R(\lambda, A)[(\lambda I - B) - (\lambda I - A)]R(\lambda, B) \\ &= R(\lambda, A)(A - B)R(\lambda, B) \\ &= R(\lambda, B)(A - B)R(\lambda, A). \end{aligned}$$

### 2.2.3.2 Self-Adjoint Linear Operators

**Definition 2.40 ([57]).** If  $A : D(A) \subset \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is a densely defined linear operator, then its adjoint denoted by  $A^*$  is defined in a unique fashion by

$$D(A^*) = \{v \in \mathcal{H}_2 : u \mapsto \langle Au, v \rangle_2 \text{ is } \mathcal{H}_1\text{---continuous over } D(A)\},$$

and

$$\langle Au, v \rangle_2 = \langle u, A^*v \rangle_1, \quad \forall u \in D(A), \quad v \in D(A^*).$$

**Proposition 2.41 ([102]).** If  $A : D(A) \subset \mathcal{H}_1 \rightarrow \mathcal{H}_2$  is a densely defined  $\overline{D(A)} = \mathcal{H}_1$  unbounded linear operator, then  $A^*$  is closed.

**Proposition 2.42 ([160]).** If  $A, B \in \mathcal{U}(\mathcal{H})$  are densely defined, then

- (a)  $A^*B^* \subset (BA)^*$ .
- (b) If  $B \in \mathcal{B}(\mathcal{H})$ , then  $A^*B^* = (BA)^*$ .
- (c) If  $A + B$  is densely defined, we have  $(A + B)^* \supset A^* + B^*$ .

*Proof.* (a) Let us show that the operators  $A^*B^*$  and  $BA$  are adjoint to each other. Indeed, let  $u \in D(A^*B^*)$  and  $v \in D(BA)$ . Clearly,  $u \in D(B^*)$  such that  $B^*u \in D(A^*)$ . Similarly,  $v \in D(A)$  such that  $Av \in D(B)$ . Using the definition of the adjoint it follows

$$\langle A^*B^*u, v \rangle = \langle B^*u, Av \rangle = \langle u, BAv \rangle.$$



- (b) Using (a) it is enough to show that  $D(BA)^* \subset D(A^*B^*)$ . Indeed, let  $u \in D(BA)^*$ . Now since  $B^*$  is bounded it follows that for all  $v \in D(BA) = D(A)$ , we obtain  $\langle (BA)^*u, v \rangle = \langle u, BAv \rangle = \langle B^*u, v \rangle$ , and hence  $B^*u \in D(A^*)$ , that is,  $u \in D(A^*B^*)$ .
- (c) Let  $u \in D(A^* + B^*) = D(A^*) \cap D(B^*)$ . Clearly, for all  $v \in D(A + B) = D(A) \cap D(B)$ , we have

$$\langle (A^* + B^*)u, v \rangle = \langle A^*u, v \rangle + \langle B^*u, v \rangle = \langle u, Av \rangle + \langle u, Bv \rangle = \langle u, (A + B)v \rangle$$

and hence  $u \in D((A + B)^*)$  and  $(A + B)^*u = A^*u + B^*u$ .

**Definition 2.43.** An operator  $A : D(A) \subset \mathcal{H} \mapsto \mathcal{H}$  is called self-adjoint if  $A = A^*$ .

**Theorem 2.44 ([160]).** Let  $A : D(A) \subset \mathcal{H} \mapsto \mathcal{H}$  be a closed linear operator. Then  $AA^*$  and  $A^*A$  are self-adjoint linear operators on  $\mathcal{H}$ .

**Definition 2.45.** A linear operator  $A$  is said to have a compact resolvent if  $\rho(A) \neq \emptyset$  and  $R(\lambda, A) = (\lambda I - A)^{-1}$  is a compact operator for all  $\lambda \in \rho(A)$ . In particular,  $A$  is said to have a compact inverse whether  $A^{-1}$  is compact.

**Theorem 2.46 ([102]).** Suppose  $\mathcal{H}$  is an infinite dimensional Hilbert space and  $A : D(A) \subset \mathcal{H} \mapsto \mathcal{H}$  is self-adjoint and has a compact inverse. Then,

- (a) There exists an orthonormal basis  $(f_n)_{n \in \mathbb{N}}$  of  $\mathcal{H}$  consisting of eigenvectors of  $A$ . If the sequence  $(\mu_n)_{n \in \mathbb{N}}$  are the corresponding eigenvalues, then  $\mu_n$  is a real number and  $|\mu_n| \rightarrow \infty$  as  $n \rightarrow \infty$ . The numbers of repetitions of  $\mu_n$  in the sequence  $(\mu_n)_{n \in \mathbb{N}}$  is finite and equals the dimension of  $N(\mu_n I - A)$ .

(b)  $D(A) = \left\{ x \in \mathcal{H} : \sum_{n=0}^{\infty} |\mu_n|^2 \cdot |\langle x, f_n \rangle|^2 < \infty \right\}$ .

(c)  $Ax = \sum_{n=0}^{\infty} \mu_n \langle x, f_n \rangle f_n$  for all  $x \in D(A)$ .

*Example 2.47.* Let  $\lambda = (\lambda_n)_{n \in \mathbb{N}}$  be a real-valued sequence. Define the diagonal operator  $D_\lambda$  on  $l^2$  by  $D(D_\lambda) = \{x = (x_k)_{k \in \mathbb{N}} \in l^2 : (\lambda_k x_k)_{k \in \mathbb{N}} \in l^2\}$ ,  $D_\lambda x = (\lambda_k x_k)_{k \in \mathbb{N}}$  for all  $x \in D(D_\lambda)$ . It can be easily shown that  $D_\lambda$  is a self-adjoint linear operator. Moreover, for all  $\mu \in \rho(D_\lambda)$ , the linear operator  $(\mu I - D_\lambda)^{-1}$  is compact if and only if  $\lim_{n \rightarrow \infty} |\lambda_n| = \infty$ .

*Example 2.48.* In  $L^2(a, b)$  consider the so-called Neumann Laplace operator defined by

$$D(\Delta_N) = \{u \in H^2(a, b) : u'(a) = u'(b) = 0\} \text{ and } \Delta_N u = u''$$

for all  $u \in D(\Delta_N)$ .

It can be shown that the operator  $\Delta_N$  is self-adjoint and has compact resolvent.

## 2.2.4 Sectorial Linear Operators

### 2.2.4.1 Basic Definitions

**Definition 2.49 ([137]).** A linear operator  $A : D(A) \subset \mathcal{X} \rightarrow \mathcal{X}$  (not necessarily densely defined) is said to be sectorial if the following hold: there exist constants  $\zeta \in \mathbb{R}$ ,  $\theta \in (\frac{\pi}{2}, \pi)$ , and  $M > 0$  such that

- (a)  $\rho(A) \supset S_{\theta, \zeta} := \{\lambda \in \mathbb{C} : \lambda \neq \zeta, |\arg(\lambda - \zeta)| < \theta\}$ , and  
 (b)  $\|R(\lambda, A)\| \leq \frac{M}{|\lambda - \zeta|}$  for each  $\lambda \in S_{\theta, \zeta}$ .

Most of the unbounded linear operators encountered in the literature are sectorial. Observe that if  $A$  is a sectorial operator, then  $A$  is necessarily closed as  $\rho(A) \neq \emptyset$ . Thus the space  $(D(A), \|\cdot\|_A)$ , where  $\|x\|_A = \|x\| + \|Ax\|$  for all  $x \in D(A)$ , is a Banach space. The norm  $\|\cdot\|_A$  is called the graph norm of  $A$ .

**Proposition 2.50 ([137]).** Let  $A$  be a linear operator on  $\mathcal{X}$  such that  $\rho(A)$  contains the half-plane  $\{\lambda \in \mathbb{C} : \Re \lambda \geq \zeta\}$ , and  $\|\lambda R(\lambda, A)\| \leq M$  for  $\Re \lambda \geq \zeta$  with  $\zeta \in \mathbb{R}$  and  $M > 0$ . Then  $A$  is sectorial.

### 2.2.4.2 Examples of Sectorial Operators

*Example 2.51.* Let  $\Omega \subset \mathbb{R}^n$  be a bounded open subset with  $C^2$  boundary  $\partial\Omega$ . Let  $\mathcal{X} = L^2(\Omega)$  and define the second-order differential operator

$$Au = \Delta u \text{ for all } u \in D(A) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

It can be shown that the linear operator  $A$  is sectorial.

*Example 2.52 ([18]).* Let  $\Omega \subset \mathbb{R}^N$  be a bounded open subset whose boundary  $\partial\Omega$  is of class  $C^2$ . Let  $n(x)$  denote the outer normal to  $\Omega$  for each  $x \in \partial\Omega$ . Consider the differential operator defined by

$$A_0 u(x) = \sum_{i,j=1}^N a_{ij}(x) \frac{\partial u}{\partial x_i \partial x_j} + \sum_{i=1}^N b_i(x) \frac{\partial u}{\partial x_i} + c(x)u(x),$$

where the coefficients  $a_{ij}$  and  $b_i$  and  $c$  are real, bounded, and continuous on  $\overline{\Omega}$ . Moreover, we suppose that for each  $x \in \overline{\Omega}$ , the matrix  $(a_{ij}(x))_{i,j=1,\dots,N}$  is symmetric and strictly positive definite, that is,

$$\sum_{i,j=1}^N a_{ij}(x) \xi_i \xi_j \geq \omega |\xi|^2$$

for all  $x \in \overline{\Omega}$  and  $\xi \in \mathbb{R}^N$ .

**Theorem 2.53 ([137]).** *Let  $p > 1$ .*

- (a) *Let  $A_p : W^{2,p}(\mathbb{R}^N) \mapsto L^p(\mathbb{R}^N)$  be the linear operator defined by  $A_p u = A_0 u$ . Then the operator  $A_p$  is sectorial in  $L^p(\mathbb{R}^N)$  and the domain  $D(A_p)$  is dense in  $L^p(\mathbb{R}^N)$ .*
- (b) *Let  $A_0$  be defined as above and let  $A_p$  be the linear operator defined by*

$$D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega) \text{ and } A_p u = A_0 u$$

*for all  $u \in D(A_p)$ . Then the linear operator  $A_p$  is sectorial in  $L^p(\Omega)$ . Moreover,  $D(A_p)$  is dense in  $L^p(\Omega)$ .*

- (c) *Let  $A_0$  be defined as above and let  $A_p$  be the linear operator defined by*

$$D(A_p) = \{u \in W^{2,p}(\Omega) : Bu|_{\partial\Omega} = 0\}, \quad A_p u = A_0 u, \quad u \in D(A_p)$$

*where*

$$Bu(x) = b_0 u(x) + \sum_{i=1}^N b_i(x) \frac{\partial u}{\partial x_i}$$

*with the coefficients  $b_i$  ( $i = 1, \dots, N$ ) are in  $C^1(\overline{\Omega})$  and the condition*

$$\sum_{i=1}^N b_i(x) n_i(x) \neq 0 \quad x \in \partial\Omega$$

*holds. Then  $A_p$  is sectorial in  $L^p(\Omega)$  and  $D(A_p)$  is dense in  $L^p(\Omega)$ .*

### 2.2.5 Semigroups of Linear Operators

**Definition 2.54.** A family of bounded linear operators  $(T(t))_{t \in \mathbb{R}^+} : \mathcal{X} \rightarrow \mathcal{X}$  is said to be a semigroup if the following hold,

- (a)  $T(0) = I$ ; and  
 (b)  $T(t+s) = T(t)T(s)$  for all  $s, t \geq 0$ .

If in addition,

- (c)  $\lim_{t \searrow 0} \|T(t) - I\| = 0$ , then the semigroup  $T(t)$  is said to be uniformly continuous.

**Definition 2.55.** A semigroup of bounded linear operators  $(T(t))_{t \in \mathbb{R}^+} : \mathcal{X} \mapsto \mathcal{X}$  is said to be a  $c_0$ -semigroup (or strongly continuous semigroup of bounded linear operators) if

$$\lim_{t \searrow 0} \|T(t)x - x\| = 0$$

for each  $x \in \mathcal{X}$ .

Observe that if  $(T(t))_{t \in \mathbb{R}^+} : \mathcal{X} \rightarrow \mathcal{X}$  is a semigroup of bounded linear operators, one can associate with it an operator  $(D(A), A)$  called its infinitesimal generator defined by

$$D(A) := \left\{ u \in \mathcal{X} : \lim_{t \searrow 0} \frac{T(t)u - u}{t} \text{ exists} \right\}, \quad (2.3)$$

and

$$Au := \lim_{t \searrow 0} \frac{T(t)u - u}{t} \quad (2.4)$$

for every  $u \in D(A)$ .

*Remark 2.56.* Note that an operator  $A$  is the infinitesimal generator of a uniformly continuous semigroup of bounded linear operators  $(T(t))_{t \in \mathbb{R}^+}$  if and only if  $A$  is bounded. In that event, it can be shown that

$$T(t) = e^{tA} = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}.$$

*Example 2.57 ([18]).* Suppose that  $\mathcal{X} = (BUC(\mathbb{R}), \|\cdot\|_{\infty})$  is the Banach space of bounded uniformly continuous functions on the real number line equipped with the sup norm. Define

$$(S(t)\varphi)(\sigma) = \varphi(t + \sigma)$$

for all  $\varphi \in BUC(\mathbb{R})$ . Then,  $(S(t))_{t \in \mathbb{R}}$  is a  $c_0$ -semigroup with  $\|S(t)\| \leq 1$  for each  $t \in [0, \infty)$ . Moreover, its infinitesimal generator  $A$  is defined by

$$D(A) = H^1(\mathbb{R}) \text{ and } A\varphi = \varphi'$$

for all  $\varphi \in H^1(\mathbb{R})$ .

*Example 2.58 ([18]).* Let  $1 \leq p < \infty$  and let  $\mathcal{X} = L^p(\mathbb{R})$  equipped with its natural norm  $\|\cdot\|_p$ . Define  $(S(0))u(x) = u(x)$  for all  $x \in \mathbb{R}$ , and

$$(S(t))u(x) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} e^{-\frac{|x-y|^2}{4t}} u(y) dy, \quad t > 0, \quad x \in \mathbb{R}.$$

Then  $S(t)$  is a  $c_0$ -semigroup satisfying  $\|S(t)u\|_p \leq \|u\|_p$  and whose infinitesimal generator  $A_p$  is defined by

$$D(A_p) = W^{2,p}(\mathbb{R}) \text{ and } A_p u = u''$$

for all  $u \in D(A_p)$ .

*Example 2.59 ([18]).* This is a generalization of Example 2.58. Let  $1 \leq p < \infty$  and let  $\mathcal{X} = L^p(\mathbb{R}^N)$  (or  $BC(\mathbb{R}^N, \mathbb{C})$  equipped with the sup norm) equipped with its natural norm  $\|\cdot\|_p$ . Define  $(S(0)u)(x) = u(x)$  for all  $x \in \mathbb{R}^N$ , and

$$(S(t)u)(x) = \frac{1}{(4\pi t)^{N/2}} \int_{-\infty}^{\infty} e^{-\frac{\|x-y\|^2}{4t}} u(y) dy, \quad t > 0, \quad x \in \mathbb{R}^N.$$

Then  $S(t)$  is a  $c_0$ -semigroup satisfying  $\|S(t)u\|_p \leq \|u\|_p$  whose infinitesimal generator  $A_p$  is defined by

$$D(A_p) = W^{2,p}(\mathbb{R}^N) \quad \text{and} \quad A_p u = \Delta u$$

for all  $u \in D(A_p)$ .

### 2.2.5.1 Basic Properties of Semigroups

**Theorem 2.60 ([148]).** Let  $(T(t))_{t \in \mathbb{R}^+} : \mathcal{X} \rightarrow \mathcal{X}$  be a semigroup of bounded linear operators, then

- (a) there are constants  $C, \zeta$  such that  $\|T(t)\| \leq C e^{\zeta t}$ ,  $t \in \mathbb{R}^+$ ;
- (b) the infinitesimal generator  $A$  of the semigroup  $T(t)$  is a densely defined closed operator;
- (c) the map  $t \mapsto T(t)x$  which goes from  $\mathbb{R}^+$  into  $\mathcal{X}$  is continuous for every  $x \in \mathcal{X}$ ;
- (d) the differential equation given by

$$\frac{d}{dt} T(t)x = AT(t)x = T(t)Ax,$$

holds for every  $x \in D(A)$ ;

- (e) for every  $x \in \mathcal{X}$ , then  $T(t)x = \lim_{s \searrow 0} (\exp(tA_s))x$ , with

$$A_s x := \frac{T(s)x - x}{s},$$

where the above convergence is uniform on every compact subset of  $\mathbb{R}^+$ ; and

- (f) if  $\lambda \in \mathbb{C}$  with  $\Re \lambda > \zeta$ , then the integral

$$R(\lambda, A)x := (\lambda I - A)^{-1}x = \int_0^{\infty} e^{-\zeta t} T(t)x dt,$$

defines a bounded linear operator  $R(\lambda, A)$  on  $\mathcal{X}$  whose range is  $D(A)$  and

$$(\lambda I - A) R(\lambda, A) = R(\lambda, A) (\lambda I - A) = I.$$

If  $\zeta = 0$  in (a) above, then the corresponding semigroup is uniformly bounded. Moreover, if  $C = 1$ , then  $(T(t))_{t \in \mathbb{R}^+}$  is said to be a  $c_0$ -semigroup of contractions.

**Theorem 2.61 (Hille–Yosida [148]).** *Let  $A : D(A) \rightarrow \mathcal{X}$  be an unbounded linear operator in a Banach space  $\mathcal{X}$ . Then  $A$  is the infinitesimal generator of a  $c_0$ -semigroup of contractions  $(T(t))_{t \in \mathbb{R}^+}$  if and only if:*

- (a)  $A$  is a densely defined closed operator; and
- (b) the resolvent  $\rho(A)$  of  $A$  contains  $\mathbb{R}^+$  and

$$\|(\lambda I - A)^{-1}\| \leq \frac{1}{\lambda}, \quad \forall \lambda > 0. \quad (2.5)$$

**Definition 2.62.** Let  $\mathcal{X}$  be a Banach space. The family of bounded operators  $(T(t))_{t \in \mathbb{R}} : \mathcal{X} \rightarrow \mathcal{X}$  is said to be a  $c_0$ -group if the following statements hold:

- (a)  $T(0) = I$ .
- (b)  $T(t+s) = T(t)T(s)$  for every  $s, t \in \mathbb{R}$ .
- (c)  $\lim_{t \rightarrow 0} \|T(t)x - x\| = 0$  for  $x \in \mathcal{X}$ .

**Theorem 2.63 ([148]).** *Let  $A : D(A) \rightarrow \mathcal{X}$  be a linear operator on  $\mathcal{X}$ . Then  $A$  is the infinitesimal generator of a  $c_0$ -group of bounded linear operators  $(T(t))_{t \in \mathbb{R}}$  satisfying  $\|T(t)\| \leq C e^{\zeta|t|}$  if and only if:*

- (a)  $A$  is a densely defined closed operator; and
- (b) every  $\lambda \in \mathbb{R}$  such that  $|\lambda| \geq \zeta$  is in  $\rho(A)$  and that for such a  $\lambda$ , the following holds:

$$\|(\lambda I - A)^{-n}\| \leq \frac{C}{(|\lambda| - \zeta)^n}. \quad (2.6)$$

**Proposition 2.64 ([137]).** *Let  $A$  be a sectorial operator and let  $T(t)$  be the analytic semigroup associated with it. Then the following hold:*

- (a)  $T(t)u \in D(A^k)$  for all  $t > 0$ ,  $u \in \mathcal{X}$ ,  $n \in \mathbb{N}$ . If  $D(A^n)$ , then

$$A^n T(t)u = T(t)A^n u, \quad t \geq 0;$$

- (b) there exist constants  $M_0, M_1, \dots$  such that

$$\begin{aligned} \|T(t)\| &\leq M_0 e^{\zeta t}, \quad t > 0, \text{ and} \\ \|t^n (A - \zeta I)^n T(t)\| &\leq M_n e^{\zeta t}, \quad t > 0; \text{ and} \end{aligned}$$

- (c) the mapping  $t \rightarrow T(t)$  belongs to  $C^\infty((0, \infty), B(\mathcal{X}))$  and

$$\frac{d^n}{dt^n} T(t) = A^n T(t), \quad t > 0, \quad \forall n \in \mathbb{N}.$$

**Proposition 2.65 ([137]).** *Let  $(T(t))_{t>0}$  be a family of bounded linear operators on  $\mathcal{X}$  such that  $t \mapsto T(t)$  is differentiable, and*

- (a)  $T(t+s) = T(t)T(s)$  for all  $t, s > 0$ ;  
 (b) there exist  $\zeta \in \mathbb{R}, M_0, M_1 > 0$  such that

$$\|T(t)\| \leq M_0 e^{\zeta t}, \quad \|tT'(t)\| \leq M_1 e^{\zeta t}, \quad \forall t > 0;$$

- (c) either (a) there exists  $t > 0$  such that  $T(t)$  is one-to-one, or (b) for every  $x \in \mathcal{X}$ ,  
 $s - \lim_{t \rightarrow 0} T(t)x = x$ .

Then  $t \mapsto T(t)$  is analytic in  $(0, \infty)$  with values in  $B(\mathcal{X})$ , and there exists a unique sectorial operator  $A : D(A) \subset \mathcal{X} \rightarrow \mathcal{X}$  such that  $(T(t))_{t \geq 0}$  is the semigroup associated with  $A$ .

## 2.2.6 Intermediate Spaces

### 2.2.6.1 Fractional Powers of Sectorial Operators

Let  $A$  be a sectorial linear operator on  $\mathcal{X}$  whose associated analytic semigroup  $T(t)$  satisfies the following: for all  $t > 0$ ,

$$\|T(t)\| \leq M_0 e^{-\omega t}, \quad \|tAT(t)\| \leq M_1 e^{-\omega t},$$

where  $M_0, M_1, \omega > 0$ .

For each  $\alpha > 0$  one defines the fractional powers of  $-A$  implicitly by

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^{+\infty} t^{\alpha-1} T(t) dt, \quad (2.7)$$

where  $\Gamma$  is defined by

$$\Gamma(x) := \int_0^{+\infty} e^{-xt} t^{x-1} dt$$

for each  $x > 0$ .

**Lemma 2.66 ([18]).** *For all  $\alpha, \beta > 0$ , then the following properties hold:*

- (a)  $(-A)^{-\alpha} (-A)^{-\beta} = A^{-(\alpha+\beta)}$ .  
 (b)  $\lim_{\alpha \rightarrow 0} (-A)^{-\alpha} = I$  in the strong operator topology.

*Proof.* (a) We have,

$$\begin{aligned}
(-A)^{-\alpha}(-A)^{-\beta} &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{+\infty} \int_0^{+\infty} t^{\alpha-1}s^{\beta-1}T(t)T(s)dt ds \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{+\infty} t^{\alpha-1} \int_t^{+\infty} (r-t)^{\beta-1}T(r)dr dt \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^{+\infty} \int_0^r t^{\alpha-1}(r-t)^{\beta-1}dt T(r)dr \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 \tau^{\alpha-1}(1-\tau)^{\beta-1}d\tau \int_0^\infty r^{\alpha+\beta-1}T(r)dr \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_0^{+\infty} r^{\alpha+\beta-1}T(r)dr \\
&= (-A)^{-\alpha-\beta}.
\end{aligned}$$

(b) Since  $(-A)^{-\alpha}$  is one-to-one, if  $v \in D(A)$ , there exists  $u \in \mathcal{X}$  such that  $v = (-A)^{-\alpha}u$ . Thus  $(-A)^{-\alpha}v - v = (-A)^{-1-\alpha}u - (-A)^{-1}u \rightarrow 0$  as  $\alpha \rightarrow 0$  by the fact that  $(-A)^{-\alpha}$  is continuous with respect to uniform operator norm.

*Remark 2.67 ([18]).*

(a) Let  $\alpha \in (0, 1)$ . Using the fact that

$$(\lambda I - A)^{-1} = \int_0^\infty e^{-\lambda t} T(t) dt,$$

then Eq. (2.7) can be rewritten as

$$(-A)^{-\alpha} = \frac{\sin(\pi\alpha)}{\pi} \int_0^{+\infty} t^{-\alpha} (tI - A)^{-1} dt. \quad (2.8)$$

(b) The operator  $(-A)^{-\alpha}$  is one-to-one, and hence has an inverse, which obviously is  $(-A)^\alpha$ . The operator  $(-A)^\alpha$  is closed with domain  $D((-A)^\alpha) = R((-A)^{-\alpha})$ .

The operators  $(-A)^\alpha$  are called fractional powers of  $-A$ .

(c) If  $\alpha > \beta$ , then  $D((-A)^\alpha) \subset D((-A)^\beta)$ .

(d)  $D((-A)^\alpha)$  is endowed with the norm  $\|u\|_\alpha = \|(-A)^\alpha u\|$  for each  $u \in D((-A)^\alpha)$ .

(e)  $(-A)^\alpha$  commutes with  $T(t)$  on  $D((-A)^\alpha)$  with

$$\|T(t)\|_{B(D((-A)^\alpha))} \leq M_0 e^{-\omega t}, \quad t > 0.$$



### 2.2.6.2 The Spaces $D_A(\alpha)$ , $D_A(\alpha, \infty)$ , and $D_A(\alpha + n, p)$

Let  $A$  be a sectorial and let  $(T(t))_{t \geq 0}$  be the analytic semigroup associated with it. Clearly,  $(T(t))_{t \geq 0}$  maps  $(0, \infty)$  into  $B(\mathcal{X})$  and there exist  $M_0, M_1 > 0$  [137] with

$$\|T(t)\| \leq M_0 e^{\omega t}, \quad t > 0, \quad (2.9)$$

$$\|t(A - \omega)T(t)\| \leq M_1 e^{\omega t}, \quad t > 0. \quad (2.10)$$

**Definition 2.68.** Let  $\alpha \in (0, 1)$ . A Banach space  $(\mathcal{X}_\alpha, \|\cdot\|_\alpha)$  is said to be an intermediate space between  $D(A)$  and  $\mathcal{X}$ , or a space of class  $\mathcal{J}_\alpha$ , if  $D(A) \subset \mathcal{X}_\alpha \subset \mathcal{X}$  and there is a constant  $c > 0$  such that

$$\|x\|_\alpha \leq C \|x\|^{1-\alpha} \|x\|_A^\alpha, \quad x \in D(A), \quad (2.11)$$

where  $\|\cdot\|_A$  is the graph norm of  $A$ .

Concrete examples of the intermediate spaces  $\mathcal{X}_\alpha$  include  $D(A^\alpha)$  for  $\alpha \in (0, 1)$ , the domains of the fractional powers of  $A$ , the real interpolation spaces  $D_A(\alpha, \infty)$ ,  $\alpha \in (0, 1)$ , defined as follows:

$$\left\{ \begin{array}{l} D_A(\alpha, \infty) := \left\{ x \in X : [x]_\alpha = \sup_{0 < t \leq 1} \|t^{1-\alpha}(A - \omega)e^{-\omega t}T(t)x\| < \infty \right\} \\ \|x\|_\alpha = \|x\| + [x]_\alpha, \end{array} \right.$$

and the abstract Hölder spaces  $D_A(\alpha) := \overline{D(A)}^{\|\cdot\|_\alpha}$ .

One should point out that  $D_A(\alpha, \infty)$  is characterized by the behavior of the quantity  $t \mapsto \|t^{1-\alpha}AT(t)u\|$  near  $t = 0$ . Moreover, all the spaces  $D_A(\alpha, \infty)$  are subspaces of  $\overline{D(A)}$ . Namely, the following embedding hold with equivalent norms:

$$D(A) \subset D_A(\beta, \infty) \subset D_A(\alpha, \infty) \subset \overline{D(A)}$$

for all  $0 < \alpha < \beta < 1$ .

If  $\alpha \in (0, 1)$ , it is not very hard to see that  $D_A(\alpha, \infty)$  can be characterized as being the subspace of all  $u \in \mathcal{X}$  such that

$$[[u]]_\alpha = \sup_{t \in (0, 1]} t^{-\alpha} \|T(t)u - u\| < \infty.$$

Furthermore, the norm defined by  $u \mapsto \|u\| + [[u]]_\alpha$  is equivalent to the natural norm of  $D_A(\alpha, \infty)$ .

More generally, if  $\alpha \in (0, 1)$  and  $p \in [1, \infty]$ , we define the real interpolation space  $D_A(\alpha, p)$  as follows:

$$\begin{cases} D_A(\alpha, p) := \left\{ x \in X : t \mapsto v(t) = \|t^{1-\alpha-\frac{1}{p}} A T(t)x\| \in L^p(0, 1) \right\} \\ \|x\|_{\alpha, p} = \|x\| + \|v\|_{L^p(0, 1)}, \end{cases}$$

and for  $n \in \mathbb{N}$ ,

$$D_A(\alpha + n, p) = \{x \in D(A^n) : A^n x \in D_A(\alpha, p)\}, \quad \|x\|_{\alpha+n, p} = \|x\| + \|A^n x\|_{\alpha, p}.$$

**Proposition 2.69 ([137]).** For  $\alpha \in (0, 1)$  and  $1 \leq p \leq \infty$  and for  $(\alpha, p) = (1, \infty)$ , then

$$D_A(\alpha, p) = (\mathcal{X}, D(A))_{\alpha, p}$$

with equivalent norms. Moreover, for  $0 < \alpha < 1$ , then

$$D_A(\alpha) = (\mathcal{X}, D(A))_{\alpha}.$$

**Proposition 2.70 ([137]).** For  $\alpha \in (0, 1)$ , then

$$D_A(\alpha, 1) \subset D((-A)^\alpha) \subset (\mathcal{X}, D(A))_{\alpha, p}.$$

*Proof.* First of all, note that  $D((-A)^\alpha)$  belongs to the class  $J_\alpha$ . Notice that for each  $u \in D(A)$ ,  $(-A)^\alpha u = (-A)^{-(1-\alpha)}(-Au)$  and hence for each  $\lambda > 0$ ,

$$\begin{aligned} \|(-A)^\alpha u\| &= \frac{1}{\Gamma(1-\alpha)} \left\| \left( \int_0^\lambda + \int_\lambda^\infty \right) t^{-\alpha} A e^{tA} u dt \right\| \\ &\leq \frac{1}{\Gamma(1-\alpha)} \left( \frac{M_0}{1-\alpha} \|Au\| \lambda^{1-\alpha} + \frac{M_1}{\alpha} \|u\| \lambda^{-\alpha} \right). \end{aligned}$$

Letting  $\lambda = \frac{\|u\|}{\|Au\|}$  it follows that

$$\|(-A)^\alpha u\| \leq c \|Au\|^\alpha \|u\|^{1-\alpha}.$$

It remains to prove that  $D(-A)^\alpha$  is continuously embedded in  $D_A(\alpha, \infty)$ . For that let  $u \in D((-A)^\alpha)$  and let  $v = (-A)^\alpha u$ . So for  $0 < \xi \leq 1$ , we have

$$\begin{aligned} \|\xi^{1-\alpha} A e^{\xi A} u\| &= \|\xi^{1-\alpha} A e^{\xi A} (-A)^\alpha v\| \\ &\leq \frac{\xi^{1-\alpha}}{\Gamma(\alpha)} \left\| \int_0^\infty t^{\alpha-1} A e^{(\xi+t)A} v dt \right\| \end{aligned}$$

$$\begin{aligned}
&\leq \frac{M_1 \xi^{1-\alpha}}{\Gamma(\alpha)} \int_0^\infty \frac{\xi^{\alpha-1}}{\xi+t} dt \|v\| \\
&\leq \frac{M_1}{\Gamma(\alpha)} \int_0^\infty \frac{s^{\alpha-1}}{1+s} ds \|(-A)^\alpha u\|
\end{aligned}$$

and hence  $u \in D_A(\alpha, \infty)$ .

**Lemma 2.71 (Diagana and Nelson [75]).** *Suppose  $A$  is sectorial. If  $n < m$ , then,*

$$D_A(\alpha + m, p) \hookrightarrow D_A(\alpha + n, p).$$

*Proof.* Indeed,  $x \in D_A(\alpha + m, p)$  yields  $x \in D(A^m) \subset D(A^n)$  and  $A^m x \in D_A(\alpha, p)$ . Now using the facts that  $A$  is invertible and  $n - m < 0$  it follows that

$$\begin{aligned}
\|t^{1-\alpha-\frac{1}{p}} AT(t) A^n x\| &= \|t^{1-\alpha-\frac{1}{p}} AT(t) A^{n-m} A^m x\| \\
&\leq \|A^{n-m}\| \cdot \|t^{1-\alpha-\frac{1}{p}} AT(t) A^m x\| \in L^p(0, 1)
\end{aligned}$$

and hence  $A^n x \in D_A(\alpha, p)$ , that is,  $x \in D_A(\alpha + n, p)$ .

In this book, the bound of the injection  $D_A(\alpha + m, p) \hookrightarrow D_A(\alpha + n, p)$  will be denoted by  $C'$ .

**Lemma 2.72 (Diagana and Nelson [75]).** *Fix  $\alpha \in (0, 1)$  and  $1 \leq p \leq \infty$ . Let  $A : D(A) \subset \mathcal{X} \mapsto \mathcal{X}$  be a sectorial linear operator. Suppose that the analytic semigroup  $(T(t))_{t \geq 0}$  associated with  $A$  is exponentially stable, that is, there exists  $M, \omega > 0$  such that*

$$\|T(t)\| \leq M e^{-\delta t} \quad \text{for } t \geq 0. \quad (2.12)$$

*Then for all  $m \in \mathbb{N}$ , the function  $t \mapsto \|A^m T(t)\|_{B(D_A(\alpha+m-1, p), \mathcal{X})}$  belongs to  $L^1(0, \infty)$ , that is, there exists a measurable function  $H : (0, \infty) \mapsto (0, \infty)$  with*

$$\theta_0 := \int_0^\infty H(t) dt < \infty$$

*such that*

$$\|A^m T(t)\|_{B(D_A(\alpha+m-1, p), \mathcal{X})} \leq H(t), \quad t > 0.$$

*Proof.* Let  $x \in D_A(\alpha + m - 1, p)$  for  $\alpha \in (0, 1)$ ,  $p \in [1, \infty]$ , and  $n \in \mathbb{N}$ . From [137, Corollary 2.2.3] one obtains that  $D_A(\alpha, p) \hookrightarrow D_A(\alpha, \infty)$ . Now using the definition of  $D_A(\alpha, \infty)$  it follows that

$$\begin{aligned}
\|A^m T(t)x\| &= \|t^{\alpha-1} t^{1-\alpha} AT(t) A^{m-1} x\| \\
&\leq t^{\alpha-1} \|x\|_{\alpha+m-1, \infty} \\
&\leq K t^{\alpha-1} \|x\|_{\alpha+m-1, p}
\end{aligned}$$

for a.e.  $t \in (0, 1)$ .

Since  $\alpha \in (0, 1)$  it follows that  $t \mapsto A^m T(t) \in L^1((0, 1), B(D_A(\alpha + m - 1, p), \mathcal{X}))$ .

Now using the exponential stability of  $T(t)$ , it follows that there exists  $C_m > 0$  such that for all  $x \in D_A(\alpha + m - 1, p)$  we have

$$\|A^m T(t)x\| \leq C_m t^{-m} e^{-\delta t} \|x\| \leq K'_m t^{-m} e^{-\delta t} \|x\|_{\alpha+m-1,p}, \quad t > 0.$$

In particular,

$$\|A^m T(t)x\| \leq K'_m t^{-m} e^{-\delta t} \|x\|_{\alpha+m-1,p} \leq K'_m e^{-\delta t} \|x\|_{\alpha+m-1,p}, \quad t \geq 1$$

and hence  $t \mapsto A^m T(t) \in L^1([1, \infty), B(D_A(\alpha + m - 1, p), \mathcal{X}))$ .

### 2.2.6.3 Hyperbolic Semigroups

**Definition 2.73 ([15]).** Let  $A$  be a sectorial operator on  $\mathcal{X}$  and let  $(T(t))_{t \geq 0}$  be the analytic semigroup associated with it. The semigroup  $(T(t))_{t \geq 0}$  is said to be hyperbolic if there exist a projection  $P$  and constants  $M, \delta > 0$  such that each  $T(t)$  commutes with  $P$ ,  $N(P)$  is invariant with respect to  $T(t)$ ,  $T(t) : R(Q) \mapsto R(Q)$  is invertible, and

$$\|T(t)Px\| \leq M e^{-\delta t} \|x\| \quad \text{for } t \geq 0, \quad (2.13)$$

$$\|T(t)Qx\| \leq M e^{\delta t} \|x\| \quad \text{for } t \leq 0, \quad (2.14)$$

where  $Q := I - P$  and  $T(t) := (T(-t))^{-1}$  for  $t < 0$ .

Recall that an analytic semigroup  $(T(t))_{t \geq 0}$  is hyperbolic if and only if (see [90])

$$\sigma(A) \cap i\mathbb{R} = \emptyset. \quad (2.15)$$

For the hyperbolic analytic semigroup  $T(t)$ , we can easily check that estimations similar to Eqs. (2.13) and (2.14) hold also with norms  $\|\cdot\|_\alpha$ . In fact, as the part of  $A$  in  $R(Q)$  is bounded, it follows from the inequality Eq. (2.14) that

$$\|AT(t)Qx\| \leq c' e^{\delta t} \|x\| \quad \text{for } t \leq 0.$$

In view of the above, there exists a constant  $c(\alpha) > 0$  such that

$$\|T(t)Qx\|_\alpha \leq c(\alpha) e^{\delta t} \|x\| \quad \text{for } t \leq 0. \quad (2.16)$$

Similarly,

$$\|T(t)Px\|_\alpha \leq \|T(1)\|_{B(\mathcal{X}, \mathcal{X}_\alpha)} \|T(t-1)Px\| \quad \text{for } t \geq 1,$$

and then from Eq. (2.13), we obtain

$$\|T(t)Px\|_\alpha \leq M' e^{-\delta t} \|x\|, \quad t \geq 1,$$

where  $M'$  depends on  $\alpha$ .

Clearly,

$$\|T(t)Px\|_\alpha \leq M'' t^{-\alpha} \|x\|,$$

and hence there exist constants  $M(\alpha) > 0$  and  $\gamma > 0$  such that

$$\|T(t)Px\|_\alpha \leq M(\alpha) t^{-\alpha} e^{-\gamma t} \|x\| \quad \text{for } t > 0. \quad (2.17)$$

We need the next lemma, which will be very crucial for our computations.

**Lemma 2.74 (Diagana [57]).** *Let  $0 < \alpha, \beta < 1$ . Then*

$$\|AT(t)Qx\|_\alpha \leq c e^{\delta t} \|x\|_\beta \quad \text{for } t \leq 0, \quad (2.18)$$

$$\|AT(t)Px\|_\alpha \leq c t^{\beta-\alpha-1} e^{-\gamma t} \|x\|_\beta \quad \text{for } t > 0. \quad (2.19)$$

*Proof.* As for Eq. (2.16), the fact that the part of  $A$  in  $R(Q)$  is bounded yields

$$\|AT(t)Qx\| \leq c e^{\delta t} \|x\|_\beta, \quad \|A^2 T(t)Qx\| \leq c e^{\delta t} \|x\|_\beta \quad \text{for } t \leq 0, \quad (2.20)$$

since  $\mathcal{X}_\beta \hookrightarrow \mathcal{X}$ . Hence, from Eq. (2.11) there is a constant  $c(\alpha) > 0$  such that

$$\|AT(t)Qx\|_\alpha \leq c(\alpha) e^{\delta t} \|x\|_\beta \quad \text{for } t \leq 0. \quad (2.21)$$

Furthermore,

$$\|AT(t)Px\|_\alpha \leq \|AT(1)\|_{B(\mathcal{X}, \mathcal{X}_\alpha)} \|T(t-1)Px\| \quad (2.22)$$

$$\leq c e^{-\delta t} \|x\|_\beta \quad \text{for } t \geq 1. \quad (2.23)$$

Now for  $t \in (0, 1]$ , by Eq. (2.11), one has

$$\|AT(t)Px\|_\alpha \leq c t^{-\alpha-1} \|x\|,$$

and

$$\|AT(t)Px\|_\alpha \leq c t^{-\alpha} \|Ax\|,$$

for each  $x \in D(A)$ . Thus, by the Reiteration Theorem (see [137]), it follows that

$$\|AT(t)Px\|_\alpha \leq c t^{\beta-\alpha-1} \|x\|_\beta$$

for every  $x \in \mathcal{X}_\beta$  and  $0 < \beta < 1$ , and hence, there exist constants  $M(\alpha) > 0$  and  $\gamma > 0$  such that

$$\|T(t)Px\|_\alpha \leq M(\alpha)t^{\beta-\alpha-1}e^{-\gamma t}\|x\|_\beta \quad \text{for } t > 0.$$

## 2.3 Evolution Families

### 2.3.1 Evolution Families and Their Estimates

In this section we study time-dependent linear operators of the form  $(A(t), D(A(t)))_{t \in \mathbb{R}}$  as well as their associated evolution families. For more on evolution families and related topics, we refer to [137].

**Definition 2.75.** A family of bounded linear operators  $\{U(t, s) : t, s \in \mathbb{R}, t \geq s\}$  on  $\mathcal{X}$  is called an evolution family whether the following hold:

- (a)  $U(t, s)U(s, r) = U(t, r)$  for  $t, s, r \in \mathbb{R}$  such that  $t \geq s \geq r$ ;
- (b)  $U(t, t) = I$  for  $t \in \mathbb{R}$ ; and
- (c) for each  $x \in \mathcal{X}$ , the function  $(t, s) \mapsto U(t, s)x$  is continuous for  $t \geq s$ .

Evolution families (also called “evolution systems,” “evolution operators,” “evolution processes,” “propagators,” or “fundamental solutions”) play a crucial role especially when it comes to studying some partial differential equations.

Classical examples of evolution families include  $U$  defined by  $U(t, s) = T(t - s)$  for all  $t, s \in \mathbb{R}$  with  $t \geq s$ , where  $(T(t))_{t \geq 0}$  is a strongly continuous semigroup.

Another example of an evolution family consists of  $U(t, s) = f(t)f^{-1}(s)$  for all  $t, s \in \mathbb{R}$  with  $t \geq s$ , where  $f : \mathbb{R} \mapsto \mathbb{R}$  is a function satisfying,  $f(t) \neq 0$  for all  $t \in \mathbb{R}$ .

**Definition 2.76.** An evolution family  $\{U(t, s) : t, s \in \mathbb{R}, t \geq s\}$  on  $\mathcal{X}$  is said to be exponentially bounded if there exist  $M, \omega > 0$  such that

$$\|U(t, s)\| \leq Me^{\omega(t-s)}, \quad t, s \in \mathbb{R}, \quad t \geq s. \quad (2.24)$$

It can be easily seen that the evolution equation  $U(t, s) = T(t - s)$  for all  $t, s \in \mathbb{R}$  with  $t \geq s$ , where  $(T(t))_{t \geq 0}$  is a strongly continuous semigroup, is an example of an exponentially bounded evolution family.

**Definition 2.77.** An evolution family  $\{U(t, s) : t, s \in \mathbb{R}, t \geq s\}$  on  $\mathcal{X}$  is said to be exponentially stable if there exists  $M, \omega > 0$  such that

$$\|U(t, s)\| \leq Me^{-\omega(t-s)}, \quad t, s \in \mathbb{R}, \quad t \geq s. \quad (2.25)$$

It can be easily seen that the evolution equation  $U(t, s) = T(t - s)$  for all  $t, s \in \mathbb{R}$  with  $t \geq s$ , where  $(T(t))_{t \geq 0}$  is an exponentially stable  $C_0$ -semigroup, is an example of an exponentially stable evolution family.

### 2.3.2 Acquistapace–Terreni Conditions

**Definition 2.78.** A family of linear operators  $(A(t))_{t \in \mathbb{R}}$  is said to satisfy the Acquistapace–Terreni conditions whether there exist  $\lambda_0 \geq 0$  and the constants  $\phi \in (\frac{\pi}{2}, \pi)$ ,  $L, K \geq 0$ , and  $\mu, \nu \in (0, 1]$  with  $\mu + \nu > 1$  such that

$$\Sigma_\phi \cup \{0\} \subseteq \rho(A(t) - \lambda_0) \ni \lambda, \quad \|R(\lambda, A(t) - \lambda_0)\| \leq \frac{K}{1 + |\lambda|} \quad (2.26)$$

and

$$\|(A(t) - \lambda_0)R(\lambda_0, A(t) - \lambda_0) [R(\lambda_0, A(t)) - R(\lambda_0, A(s))]\| \leq L|t - s|^\mu |\lambda|^{-\nu} \quad (2.27)$$

for  $t, s \in \mathbb{R}$ ,  $\lambda \in \Sigma_\phi := \{\lambda \in \mathbb{C} \setminus \{0\} : |\arg \lambda| \leq \phi\}$ .

For a given family of closed linear operators  $\{A(t) : t \in \mathbb{R}\}$  on  $\mathcal{X}$ , the existence of an evolution family associated with it is not always guaranteed. However, if the family  $A(t)$  satisfies Acquistapace–Terreni conditions, then the family of linear operators  $A(t)$  has an evolution family associated with it such that  $U(t, s)\mathcal{X} \subseteq D(A(t))$  for all  $t, s \in \mathbb{R}$ . Moreover, the following hold: for  $t > s$ , the mapping  $(t, s) \mapsto U(t, s) \in B(\mathcal{X})$  is continuously differentiable in  $t$  with  $\partial_t U(t, s) = A(t)U(t, s)$ . Furthermore, there exists a constant  $C' > 0$  which depends on constants in Eqs. (2.26) and (2.27) such that

$$\|A^k(t)U(t, s)\| \leq C'(t - s)^{-k} \quad (2.28)$$

for  $0 < t - s \leq 1$  and  $k = 0, 1$ .

*Remark 2.79.* (a) In the particular case of a constant domain  $D(A(t))$ , one can replace Eq. (2.27) (see for instance [148]) with the following: there exist constants  $L$  and  $0 < \mu \leq 1$  such that

$$\|(A(t) - A(s))R(\lambda_0, A(r))\| \leq L|t - s|^\mu, \quad s, t, r \in \mathbb{R}. \quad (2.29)$$

(b) The Acquistapace–Terreni conditions were introduced in the literature by Acquistapace and Terreni in [1, 2] for  $\lambda_0 = 0$ .

**Definition 2.80.** An evolution family  $\{U(t, s) : t \geq s \text{ with } t, s \in \mathbb{R}\} \subset B(\mathcal{X})$  is said to have an *exponential dichotomy* (or is *hyperbolic*) if there are projections  $P(t)$  ( $t \in \mathbb{R}$ ) that are uniformly bounded and strongly continuous in  $t$  (and we then let  $Q(t) = I - P(t)$ ) and constants  $\delta > 0$  and  $N \geq 1$  such that

- (a) (i)  $U(t, s)P(s) = P(t)U(t, s)$  for all  $t \geq s$ ;  
 (b) the restriction  $U_Q(t, s) : Q(s)\mathcal{X} \rightarrow Q(t)\mathcal{X}$  of  $U(t, s)$  is invertible for all  $t \geq s$   
 (and we then set  $U_Q(s, t) := U_Q(t, s)^{-1}$ ); and  
 (c)  $\|U(t, s)P(s)\| \leq Ne^{-\delta(t-s)}$  and  $\|U_Q(s, t)Q(t)\| \leq Ne^{-\delta(t-s)}$  for  $t \geq s$  and  $t, s \in \mathbb{R}$ .

### 2.3.2.1 Estimates for the Evolution Family $U(t, s)$

This subsection is devoted to the study of some estimates related to  $U(t, s)$ . For that, we introduce the corresponding interpolation spaces for  $A(t)$ . We refer the reader to [90, 137] for proofs and further information on these spaces.

Let  $A$  be a sectorial operator on  $\mathcal{X}$  and let  $\alpha \in (0, 1)$ . Define the real interpolation space

$$\mathcal{X}_\alpha^A := \{x \in \mathcal{X} : \|x\|_\alpha^A := \sup_{r>0} \|r^\alpha(A - \zeta)R(r, A - \zeta)x\| < \infty\},$$

which, by the way, is a Banach space when endowed with the norm  $\|\cdot\|_\alpha^A$ . For convenience we further write

$$\mathcal{X}_0^A := \mathcal{X}, \quad \|x\|_0^A := \|x\|, \quad \mathcal{X}_1^A := D(A), \quad \text{and} \quad \|x\|_1^A := \|(\zeta - A)x\|.$$

We will need the space  $\widehat{\mathcal{X}}^A := \overline{D(A)}$ . In particular, we will frequently be using the following continuous embedding:

$$D(A) \hookrightarrow \mathcal{X}_\beta^A \hookrightarrow D((\zeta - A)^\alpha) \hookrightarrow \mathcal{X}_\alpha^A \hookrightarrow \widehat{\mathcal{X}}^A \subset \mathcal{X}, \quad (2.30)$$

for all  $0 < \alpha < \beta < 1$ , where the fractional powers are defined in the usual way.

In general,  $D(A)$  is not dense in the spaces  $\mathcal{X}_\alpha^A$  and  $\mathcal{X}$ . However, we have the following continuous injection:

$$\mathcal{X}_\beta^A \hookrightarrow \overline{D(A)}^{\|\cdot\|_\alpha^A} \quad (2.31)$$

for  $0 < \alpha < \beta < 1$ .

Given the operators  $A(t)$  for  $t \in \mathbb{R}$ , satisfying Acquistapace–Terreni conditions, we set

$$\mathcal{X}_\alpha^t := \mathcal{X}_\alpha^{A(t)}, \quad \widehat{\mathcal{X}}^t := \widehat{\mathcal{X}}^{A(t)}$$

for  $0 \leq \alpha \leq 1$  and  $t \in \mathbb{R}$ , with the corresponding norms. Then the embedding in Eq. (2.30) holds with constants independent of  $t \in \mathbb{R}$ . These interpolation spaces are of class  $\mathcal{J}_\alpha$  and hence there is a constant  $l(\alpha)$  such that

$$\|y\|_\alpha^t \leq l(\alpha) \|y\|^{1-\alpha} \|A(t)y\|^\alpha, \quad y \in D(A(t)). \quad (2.32)$$

We will need the following estimates to establish some of the results of this book.



**Proposition 2.81 (Baroun et al. [15]).** For  $x \in \mathcal{X}$ ,  $0 \leq \alpha \leq 1$ , and all  $t > s$ , the following hold:

(a) There is a constant  $c(\alpha)$ , such that

$$\|U(t,s)P(s)x\|_{\alpha}^t \leq c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \quad (2.33)$$

(b) There is a constant  $m(\alpha)$ , such that

$$\|\tilde{U}_{\mathcal{Q}}(s,t)Q(t)x\|_{\alpha}^s \leq m(\alpha)e^{-\delta(t-s)}\|x\|. \quad (2.34)$$

*Proof.* (a) Using Eq. (2.32) we obtain

$$\begin{aligned} \|U(t,s)P(s)x\|_{\alpha}^t &\leq l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,s)P(s)x\|^{\alpha} \\ &\leq l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,t-1)U(t-1,s)P(s)x\|^{\alpha} \\ &\leq l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,t-1)\|^{\alpha}\|U(t-1,s)P(s)x\|^{\alpha} \\ &\leq l(\alpha)N' e^{-\delta(t-s)(1-\alpha)}e^{-\delta(t-s-1)\alpha}\|x\| \\ &\leq c'(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}(t-s)^{\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\| \end{aligned}$$

for  $t-s \geq 1$  and  $x \in \mathcal{X}$ .

Since  $(t-s)^{\alpha}e^{-\frac{\delta}{2}(t-s)} \rightarrow 0$  as  $t \rightarrow \infty$  it easily follows that

$$\|U(t,s)P(s)x\|_{\alpha}^t \leq c_1(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\|.$$

If  $0 < t-s \leq 1$ , we have

$$\begin{aligned} \|U(t,s)P(s)x\|_{\alpha}^t &\leq l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U(t,s)P(s)x\|^{\alpha} \\ &\leq l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U\left(t,\frac{t+s}{2}\right)U\left(\frac{t+s}{2},s\right)P(s)x\|^{\alpha} \\ &\leq l(\alpha)\|U(t,s)P(s)x\|^{1-\alpha}\|A(t)U\left(t,\frac{t+s}{2}\right)\|^{\alpha}\|U\left(\frac{t+s}{2},s\right)P(s)x\|^{\alpha} \\ &\leq l(\alpha)Ne^{-\delta(t-s)(1-\alpha)}2^{\alpha}(t-s)^{-\alpha}e^{-\frac{\delta\alpha}{2}(t-s)}\|x\| \\ &\leq l(\alpha)Ne^{-\frac{\delta}{2}(t-s)(1-\alpha)}2^{\alpha}(t-s)^{-\alpha}e^{-\frac{\delta\alpha}{2}(t-s)}\|x\| \\ &\leq c_2(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|, \end{aligned}$$

and hence

$$\|U(t,s)P(s)x\|_{\alpha}^t \leq c(\alpha)(t-s)^{-\alpha}e^{-\frac{\delta}{2}(t-s)}\|x\| \quad \text{for } t > s.$$

(ii) We have,

$$\begin{aligned}
\|\tilde{U}_Q(s,t)Q(t)x\|_\alpha^s &\leq l(\alpha)\|\tilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)\tilde{U}_Q(s,t)Q(t)x\|^\alpha \\
&\leq l(\alpha)\|\tilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)Q(s)\tilde{U}_Q(s,t)Q(t)x\|^\alpha \\
&\leq l(\alpha)\|\tilde{U}_Q(s,t)Q(t)x\|^{1-\alpha}\|A(s)Q(s)\|^\alpha\|\tilde{U}_Q(s,t)Q(t)x\|^\alpha \\
&\leq l(\alpha)Ne^{-\delta(t-s)(1-\alpha)}\|A(s)Q(s)\|^\alpha e^{-\delta(t-s)\alpha}\|x\| \\
&\leq m(\alpha)e^{-\delta(t-s)}\|x\|.
\end{aligned}$$

In the last inequality we made use of the fact that  $\|A(s)Q(s)\| \leq c$  for some constant  $c \geq 0$  (see [153, Proposition 3.18]).

*Remark 2.82.* It should be mentioned that if  $U(t,s)$  is exponentially stable, then  $P(t) = I$  and  $Q(t) = I - P(t) = 0$  for all  $t \in \mathbb{R}$ . In that case, Eq. (2.33) still holds and can be rewritten as follows: for all  $x \in \mathcal{X}$ ,

$$\|U(t,s)x\|_\alpha^t \leq c(\alpha)e^{-\frac{\delta}{2}(t-s)}(t-s)^{-\alpha}\|x\|. \quad (2.35)$$

We will need the following technical lemma later on.

**Lemma 2.83 (Diagana [76]).** *Let  $\alpha \in (0, 1)$ . Suppose that the evolution family  $(U(t,s))_{t \geq s}$  not only satisfies Acquistapace–Terreni conditions but also is exponentially stable. Then, there exists a function  $H_{\alpha,\delta} : (0, \infty) \mapsto (0, \infty)$  with  $H_{\alpha,\delta} \in L^1(0, \infty)$  such that*

$$\|A(t)U(t,s)x\| \leq H_{\alpha,\delta}(t-s)\|x\|_\alpha, \quad t > s \quad (2.36)$$

for all  $x \in \mathcal{X}_\alpha$ .

*Proof.* First of all, note that there exists a constant  $C > 0$  such that

$$\|A(t)U(t,s)\|_{B(\mathcal{X}_\alpha, \mathcal{X})} \leq C(t-s)^{\alpha-1} \quad (2.37)$$

for all  $t, s \in \mathbb{R}$  such that  $0 < t-s \leq 1$  (see [152]).

Now for  $t-s \geq 1$ , using Eq. (2.28) in the case when  $k = 1$ , we obtain that

$$\begin{aligned}
\|A(t)U(t,s)x\| &= \|A(t)U(t,t-1)U(t-1,s)x\| \\
&\leq \|A(t)U(t,t-1)\|_{B(\mathcal{X})}\|U(t-1,s)x\| \\
&\leq C'\|U(t-1,s)x\| \\
&\leq C'Ne^\delta e^{-\delta(t-s)}\|x\| \\
&\leq M'C'Ne^\delta e^{-\delta(t-s)}\|x\|_\alpha
\end{aligned}$$

for all  $x \in \mathcal{X}_\alpha$ , where  $M'$  is the bound of the continuous injection  $\mathcal{X}_\alpha \hookrightarrow \mathcal{X}$ .

Choosing the function  $H_{\alpha,\delta} : (0, \infty) \mapsto (0, \infty)$  as

$$H_{\alpha,\delta}(t) = \begin{cases} Ct^{\alpha-1} & \text{if } t \in (0, 1] \\ N'e^{-\delta t} & \text{if } t \in (1, \infty) \end{cases}$$

where  $N' = M'C'Ne^{\delta}$ , one can easily check that  $H_{\alpha,\delta} \in L^1(0, \infty)$  and that

$$\|A(t)U(t,s)x\| \leq H_{\alpha,\delta}(t-s)\|x\|_{\alpha}, \quad t > s$$

for all  $x \in \mathcal{X}_{\alpha}$ .

## Bibliographical Notes

The material of Sect. 2.1 is taken from the following sources: Diagana [57], and Bezandry and Diagana [18]. Additional relevant references include the following: Conway [40], Diagana [56], Naylar and Sell [141], Pazy [148], Rudin [151], Weidmann [160], Locker [136], Lunardi [137], and Engel and Nagel [90].

The material of Sects. 2.2 and 2.3 is taken from the following sources: Baroun et al. [15], Bezandry and Diagana [18], Diagana [76], Diagana and Nelson [75], and Lunardi [137]. Additional relevant references include Engel and Nagel [90], Schnaubelt [152, 153], and Chicone and Latushkin [38].



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