Linear-Quadratic Stochastic Differential Games with General Noise Processes

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Abstract In this paper a noncooperative, two person, zero sum, stochastic differential game is formulated and solved that is described by a linear stochastic system and a quadratic cost functional for the two players. The optimal strategies for the two players are given explicitly using a relatively simple direct method. The noise process for the two player linear system can be an arbitrary square integrable stochastic process with continuous sample paths. The special case of a fractional Brownian motion noise is explicitly noted.

1 Introduction

Two person, zero sum stochastic differential gamesdeveloped as a natural generalization of (one player) stochastic control problems and minimax control problems. Two person, zero-sum stochastic differential games are often used to describe some competitive economic situations (e.g. Leong and Huang 2010). Noncooperative two person zero sum stochastic differential games where the players have actions or strategies as addditive terms in a linear stochastic differential equation with a Brownian motion use the solution of a Riccati equation to characterize the optimal feedback strategies of the two players. This Riccati equation is the same as the Riccati equation that is used for an optimal control for a controlled linear system with a Brownian motion and a cost functional that is the exponential of a quadratic functional in the state and the control (Duncan 2013, Jacobson 1973).

Research supported by NSF grants DMS 0808138 and DMS 1108884, AFOSR grants FA9550-09-12-1-0384 and FA9550-12-1-0384, and ARO grant W911NF-10-1-0248.

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F. El Ouardighi and K. Kogan (eds.), *Models and Methods in Economics and Management Science*, International Series in Operations Research & Management Science 198,
DOI: 10.1007/978-3-319-00669-7_2,

Two major methods for solving stochastic differential games are the Hamilton-Jacobi-Isaacs (HJI) equations and the backward stochastic differential equations (e.g. Buckdahn and Li 2008; Fleming and Hernandez-Hernandez 2011). Both of these approaches can present significant difficulties in their solutions. The HJI equation or more simply the Isaacs equation is a pair of nonlinear second order partial differential equations so the existence and the uniqueness of solutions is usually difficult to verify. A backward stochastic differential equation to solve a differential game problem usually presents significant difficulties to verify the existence and the uniqueness of a solution because the stochastic equation is solved backward in time but a solution is required to have a forward in time measurability.

While a basic stochastic differential game formulation occurs where the stochastic system is a linear stochastic differential equation with a Brownian motion and the cost functional is quadratic in the state and the control, empirical evidence for many physical phenomena demonstrates that Brownian motion is not a reasonable choice for the system noise and some other Gaussian process from the family of fractional Brownian motions or even other processes is more appropriate. Thus it is natural to consider these two person zero sum differential games with a linear stochastic system with the Brownian motion replaced by an arbitrary fractional Brownian motion or more generally by a square integrable process with continuous sample paths. These problems cannot be easily addressed by a partial differential equation approach (e.g. Isaacs equation) or a backward stochastic differential equation. The approach used in this paper which generalizes a completion of squares method has been used to solve a linear-quadratic control problem with fractional Brownian motion (Duncan et al. 2012) and with a general noise (Duncan and Pasik-Duncan 2010, 2011). A stochastic differential game with a scalar system having a fractional Brownian motion with the Hurst parameter in $(\frac{1}{2}, 1)$ is solved in Bayraktar and Poor (2005) by determining the Nash equilibrium for the game.

It is a pleasure for the author to dedicate this paper to Charles Tapiero on the occasion of his sixtieth birthday.

2 Game Formulation and Solution

The two person stochastic differential game is described by the following linear stochastic differential equation

$$dX(t) = AX(t)dt + BU(t)dt + CV(t)dt + FdW(t) \tag{1}$$

$$X(0) = X_0 \tag{2}$$

where $X_0 \in \mathbb{R}^n$ is not random, $X(t) \in \mathbb{R}^n$, $A \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $B \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$, $U(t) \in \mathbb{R}^m$, $U \in \mathcal{U}$, $C \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^n)$, $V(t) \in \mathbb{R}^p$, $V \in \mathcal{V}$, and $F \in \mathcal{L}(\mathbb{R}^q, \mathbb{R}^n)$. The terms U and V denote the control actions of the two players. The positive integers (m, n, p, q) are arbitrary. The process $(W(t), t \geq 0)$ is a square integrable

stochastic process with continuous sample paths that is defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and $(\mathcal{F}(t), t \in [0, T])$ is the filtration for W. The family of admissible strategies for U is $\mathcal{U}$ and for V is $\mathcal{V}$ and they are defined as follows
$\mathcal{U} = \{U : U$ is an $\mathbb{R}^m$-valued process that is progressively measurable with respect to $(\mathcal{F}(t), t \in [0, T])$ such that $U \in L^2([0, T])$ a.s.$\}$
and
$\mathcal{V} = \{V : V$ is an $\mathbb{R}^p$-valued process that is progressively measurable with respect to $(\mathcal{F}(t), t \in [0, T])$ such that $V \in L^2([0, T])$ a.s.$\}$

The cost functional J is a quadratic functional of X, U, and V that is given by

$$J^0(U, V) = \frac{1}{2}\left[\int_0^T (\langle QX(s), X(s)\rangle + \langle RU(s), U(s)\rangle - \langle SV(s), V(s)\rangle)ds + \langle MX(T), X(T)\rangle\right] \tag{3}$$

$$J(U, V) = \mathbb{E}[J^0(U, V)] \tag{4}$$

where $Q \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, $R \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^m)$, $S \in \mathcal{L}(\mathbb{R}^p, \mathbb{R}^p)$, $M \in \mathcal{L}(\mathbb{R}^n, \mathbb{R}^n)$, and $Q > 0$, $R > 0$, $S > 0$, and $M \geq 0$ are symmetric linear transformations.

Let V_+ and V_- be the upper and lower values of the stochastic differential game, that is,

$$V_+ = \sup_{\mathcal{U}} \inf_{\mathcal{V}} J(U, V) \tag{5}$$

$$V_- = \inf_{\mathcal{V}} \sup_{\mathcal{U}} J(U, V) \tag{6}$$

Restricting the control problem to the interval $[t, T]$ and letting x be the value of the state at time t, an Isaacs equation can be associated with both $V_+(t, x)$ and $V_-(t, x)$. With the Isaacs minimax condition (Isaacs 1965) satisfied then both V_+ and V_- satisfy the same equation.

The following theorem provides an explicit solution to this noncooperative two person linear quadratic game with a general noise process, W. It seems that there are no other results available for these games when W is an arbitrary stochastic process with continuous sample paths.

Theorem 2.1 *The two person zero sum stochastic differential game given by (1) and (4) has optimal admissible strategies for the two players, denoted U^* and V^*, given by*

$$U^*(t) = -R^{-1}(B^T P(t)X(t) + B^T \hat{\phi}(t)) \tag{7}$$

$$V^*(t) = S^{-1}(C^T P(t)X(t) + C^T \hat{\phi}(t)) \tag{8}$$

where $(P(t), t \in [0, T])$ is the unique positive solution of the following equation

$$-\frac{dP}{dt} = Q + PA + A^T P - P(BR^{-1}B^T - CS^{-1}C^T)P \tag{9}$$

$$P(T) = M \tag{10}$$

and it is assumed that $BR^{-1}B^T - CS^{-1}C^T > 0$ *and* $(\phi(t), t \in [0, T])$ *is the solution of the following linear stochastic equation*

$$d\phi(t) = -[(A^T - P(t)BR^{-1}B^T + P(t)CS^{-1}C^T)\phi dt + P(t)FdW(t)] \tag{11}$$

$$\phi(T) = 0 \tag{12}$$

and

$$\hat{\phi}(t) = \mathbb{E}[\phi(t)|\mathcal{F}(t)] \tag{13}$$

Proof Since $BR^{-1}B^T - CS^{-1}C^T > 0$, the Riccati Equation (9) has a unique positive solution. Let $(W(t), t \in [0, T])$ be the process W in (1). For each $n \in \mathbb{N}$, let $T_n = \{t_j^{(n)}, j \in \{0, ..., n\}\}$ be a partition of $[0, T]$ such that $0 = t_0^{(n)} < t_1^{(n)}, ... < t_n^{(n)} = T$. Assume that $T_{n+1} \supset T_n$ for each $n \in \mathbb{N}$ and that the sequence $(T_n, n \in \mathbb{N})$ becomes dense in $[0, T]$. For example the sequence $(T_n, n \in \mathbb{N})$ can be chosen as the dyadic partitions of $[0, T]$. For each $n \in \mathbb{N}$, let $(W_n(t), t \in [0, T])$ be the piecewise linear process obtained from $(W(t), t \in [0, T])$ and the partition T_n by linear interpolation, that is,

$$W_n(t) = [W(t_j^{(n)}) + \frac{W(t_{j+1}^{(n)}) - W(t_j^{(n)})}{t_{j+1}^{(n)} - t_j^{(n)}}(t - t_j^{(n)})]\mathbb{1}_{[t_j^{(n)}, t_{j+1}^{(n)})}(t) \tag{14}$$

The differential game problem is solved by constructing a sequence of differential games by using this sequence of piecewise linear approximations to the process W in (1) and using a completion of squares method from deterministic linear control to obtain a sequence of optimal strategies for the two players with the sequence of linear systems where the process W in (1) is replaced by the sequence of piecewise linear approximations, $(W_n, n \in \mathbb{N})$. Then it is shown that this sequence of a pair of optimal strategies has a limit that are optimal strategies for the two players with the system (1).

Initially a "nonadapted" game problem for the linear stochastic system (1) is formed with $(W(t), t \in [0, T])$ replaced by $(W_n(t), t \in [0, T])$ for each $n \in \mathbb{N}$ given by (14) and the strategies are not required to be adapted to $(\mathcal{F}(t), t \in [0, T])$. For fixed n this new system is solved for almost all sample paths of W_n and the solution of (1) with W_n replacing W is considered as a translation of the deterministic linear system without W_n.

Let $(W_n(t), t \in [0, T], n \in \mathbb{N})$ be this sequence of processes obtained by (14) that converges uniformly almost surely to $(W(t), t \in [0, T])$. Fix $n \in \mathbb{N}$ and consider a sample path of W_n. For this sample path the dependence on $\omega \in \Omega$ is suppressed for notational convenience. Let $(X_n(t), t \in [0, T])$ be the solution of (1) with W replaced by W_n, that is, X_n is the solution of

$$dX_n(t) = AX_n(t)dt + BU(t)dt + CV(t)dt + dW_n(t) \tag{15}$$

$$X_n(0) = X_0 \tag{16}$$

The dependence of X_n on the strategies U and V is suppressed for notational simplicity. The following approach uses completion of squares for the corresponding deterministic control problem (e.g. Yong and Zhou 1999).

Let $(P(t), t \in [), T])$ be the positive, symmetric solution to the following Riccati equation

$$-\frac{dP}{dt} = Q + PA + A^T P - P(BR^{-1}B^T - CS^{-1}C^T)P \tag{17}$$

$$P(T) = M \tag{18}$$

Recall that it is assumed that $BR^{-1}B^T - CS^{-1}C^T > 0$.

For each $n \in \mathbb{N}$ let $(\phi_n(t), t \in [0, T])$ be the solution of the linear (nonhomogeneous) differential equation

$$\begin{aligned}\frac{d\phi_n}{dt} = &-[(A^T - P(t)BR^{-1}B^T + P(t)CS^{-1}C^T]\phi_n \\ &+ P(t)F\frac{dW_n}{dt}]\end{aligned} \tag{19}$$

$$\phi_n(T) = 0 \tag{20}$$

$$\text{so that} \tag{21}$$

$$\phi_n(t) = \int_t^T \Phi_P(s, t)P(s)FdW_n(t) \tag{22}$$

$$\text{where} \tag{23}$$

$$\frac{d\Phi_P}{dt}(s, t) = -(A^T - P(t)BR^{-1}B^T + P(t)CS^{-1}C^T)\Phi_P(s, t) \tag{24}$$

$$\Phi_P(s, s) = I \tag{25}$$

By taking the differential of the process $(\langle P(t)X_n(t), X_n(t)\rangle, t \in [0, T])$ and integrating this differential expression using the Riccati equation (9) it follows that

$$\langle P(T), X_n(T), X_n(T)\rangle - \langle P(0)X_0, X_0\rangle \tag{26}$$
$$= \int_0^T ([\langle P(t)(BR^{-1}B^T - CS^{-1}C^T)P(t)X_n(t), X_n(t)\rangle$$
$$- \langle QX_n(t), X_n(t)\rangle + 2\langle B^T P(t)X_n(t), U(t)\rangle$$
$$+ 2\langle C^T P(t)X_n(t), V(t)\rangle]dt + 2\langle P(t)FdW_n(t), X_n(t)\rangle)$$

Furthermore compute the differential of $(\langle \phi_n(t), X_n(t)\rangle, t \in [0, T])$ and integrate it to obtain

$$-\langle \phi_n(0), X_0\rangle = \int_0^T (\langle (PBR^{-1}B^T - PCS^{-1}C^T)\phi_n, X_n\rangle$$
$$+ \langle \phi_n, BU\rangle + \langle \phi_n, CV\rangle)dt$$
$$- \int_0^T \langle PFdW_n, X_n\rangle + \int_0^T \langle \phi_n, FdW_n\rangle \tag{27}$$

Let J_n^0 be the corresponding expression for J^0 with X replaced by X_n. It follows directly by adding the equalities (26) and (27) and using the definition of J_n^0 that

$$J_n^0(U, V) - \frac{1}{2}\langle P(0)X_0, X_0\rangle - \langle \phi_n(0), X_0\rangle \tag{28}$$
$$= \frac{1}{2}[\int_0^T (\langle RU, U\rangle - \langle SV, V\rangle$$
$$+ 2\langle P(BR^{-1}B^T - CS^{-1}C^T)PX_n, X_n\rangle$$
$$+ 2\langle B^T PX_n, U\rangle + 2\langle PCV, X_n\rangle$$
$$+ 2\langle PBR^{-1}B^T\phi_n, X_n\rangle - 2\langle PCS^{-1}C^T\phi_n, X_n\rangle$$
$$+ 2\langle \phi_n, BU\rangle + 2\langle \phi_n, CV\rangle)dt + \langle \phi_n, dW_n\rangle)]$$
$$= \frac{1}{2}\int_0^T (|R^{-\frac{1}{2}}[RU + B^T PX_n + B^T\phi_n]|^2 dt$$
$$- |S^{\frac{1}{2}}(SV - C^T PX_n - C^T\phi_n)|^2 dt$$
$$- |R^{\frac{1}{2}}B^T\phi_n|^2 + |S^{\frac{1}{2}}C^T\phi_n|^2 dt$$
$$+ 2 < \phi_n, dW_n >)$$

Since the arbitrary strategies U and V only occur in distinct quadratic terms in (28), optimal nonadapted strategies $(\bar{U}_n^*, \bar{V}_n^*)$ for the system X_n are the following

$$\bar{U}_n^*(t) = -R^{-1}(B^T P(t)X_n(t) + B^T\phi_n(t)) \tag{29}$$
$$\bar{V}_n^*(t) = S^{-1}(C^T P(t)X_n(t) + C^T\phi_n(t)) \tag{30}$$

Since the sequence of processes $(W_n(t), t \in [0, T], n \in \mathbb{N})$ converges uniformly almost surely to the process $(W(t), t \in [0, T])$, it follows that for a fixed control U, the sequence of solutions of (15), $(X_n(t), t \in [0, T], n \in \mathbb{N})$, converges uniformly almost surely to the solution of (1), $(X(t), t \in [0, T])$. This uniform convergence almost surely follows directly by representing $(X_n(t), t \in [0, T])$ by the variation of parameters formula for the linear equation and performing an integration by parts in the system equation as follows

$$\begin{aligned} X_n(t) =& e^{tA}X_0 + \int_0^t e^{A(t-s)}BU(s)ds + \int_0^t e^{A(t-s)}CV(s)ds \\ &+ \int_0^t e^{A(t-s)}FdW_n(s) \\ =& e^{tA}X_0 + \int_0^t e^{A(t-s)}BU(s)ds + \int_0^t e^{A(t-s)}CV(s)ds \\ &+ FW_n(t) + e^{tA}\int_0^t Ae^{-As}FW_n(s)ds \end{aligned} \tag{31}$$

From the equalities (29), (30) and (28) it follows that the optimal nonadapted strategies $\bar{U}$ and $\bar{V}$ are the following

$$\bar{U}(t) = -R^{-1}(B^T P(t)X(t) + B^T\phi(t)) \tag{32}$$

$$\bar{V}(t) = S^{-1}(C^T P(t)X(t) + C^T\phi(t)) \tag{33}$$

Now consider the game problem with the original family of adapted admissible controls for U and V. Let $\phi = \hat{\phi} + \widetilde{\phi}$ where

$$\hat{\phi}(t) = \mathbb{E}[\phi(t)|\mathcal{F}(t)] \tag{34}$$

From the following equality

$$\begin{aligned} &\mathbb{E}\int_0^T (|R^{-\frac{1}{2}}[RU + B^T PX + B^T\hat{\phi} + B^T\tilde{\phi}|^2 \\ &\quad - |S^{\frac{1}{2}}[SV - C^T PX - C^T\hat{\phi} - C^T\tilde{\phi}]|^2]dt \\ =&\mathbb{E}\int_0^T [|R^{-\frac{1}{2}}[RU + B^T PX + B^T\hat{\phi}]|^2 + |R^{-\frac{1}{2}}B^T\tilde{\phi}|^2 \\ &\quad - |S^{-\frac{1}{2}}[SV - C^T PX - C^T\hat{\phi}]|^2 - |S^{-\frac{1}{2}}[C^T\tilde{\phi}|^2]dt \end{aligned} \tag{35}$$

It follows that the optimal adapted strategies U^* and V^* are

$$U^*(t) = -R^{-1}(B^T P(t)X(t) + B^T\hat{\phi}(t)) \tag{36}$$

$$V^*(t) = S^{-1}(C^T P(t)X(t) + C^T\hat{\phi}(t)) \tag{37}$$

From the proof of the theorem it follows that the noise can be any square integrable process with continuous sample paths. □

If W is a fractional Brownian motion then the conditional expectation, $\hat{\phi}$, can be explicitly expressed as a Wiener-type stochastic integral of W (cf. Duncan 2006) as follows

$$\mathbb{E}[\phi(t)|\mathcal{F}(t)] = \int_0^t u_{1/2-H} I_{t-}^{1/2-H} (I_{T-}^{H-1/2} u_{H-1/2} 1_{[t,T]} \Phi_P(.,t)P) dW \tag{38}$$

where Φ_P is the fundamental solution of the Eq. (24), $H \in (0, 1)$ is the Hurst parameter of the fractional Brownian motion and I_b^a is a fractional integral for $a > 0$ and a fractional derivative for $a < 0$ (Samko et al. 1993). The optimal cost can also be determined explicitly in terms of fractional integrals and fractional derivatives. The family of fractional Brownian motions was introduced by Kolgomorov (1940) and a statistic of these processes was introduced by Hurst (1951).

It can be verified that the relation between the Riccati equations for the linear-quadratic game with Brownian motion and the linear-exponential-quadratic control problem with Brownian motion does not extend to the corresponding problems with an arbitrary fractional Brownian motion.

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http://www.springer.com/978-3-319-00668-0

Models and Methods in Economics and Management Science
Essays in Honor of Charles S. Tapiero
El Ouardighi, F.; Kogan, K. (Eds.)
2014, XIII, 254 p. 23 illus., 7 illus. in color., Hardcover
ISBN: 978-3-319-00668-0