Basic notation

Most of the notation used in this book is standard. Otherwise, it will be introduced when necessary. However, for the reader's convenience, in this section we record some basic definitions and useful notation that we will use.

Given $A \subset \mathbb{R}^d$, the characteristic function of A is defined by $\chi_A(x) = 1$ if $x \in A$ and 0 otherwise. The complementary of A is denoted either by $\mathbb{R}^d \setminus A$ or A^c. The symmetric difference of two sets A, B equals $A\Delta B = (A \setminus B) \cup (B \setminus A)$. The closure of A is denoted by $\overline{A}$, its interior by $\mathring{A}$, and its boundary by ∂A. The distance from $x \in \mathbb{R}^d$ to A is denoted by $\operatorname{dist}(x, A)$, and the diameter of A by $\operatorname{diam}(A)$. Also, for a given $\delta > 0$, $\mathcal{U}_\delta(A)$ stands for the δ-neighborhood of A. That is, $\mathcal{U}_\delta(A) = \{x \in \mathbb{R}^d : \operatorname{dist}(x, A) < \delta\}$.

An open ball centered at x with radius r is denoted by $B(x, r)$, and a closed ball by $\bar{B}(x, r)$. The radius of a ball B sometimes will be written as $r(B)$. By a cube $Q \subset \mathbb{R}^d$ (or a square, in the planar case) we mean a cube (or square) with sides parallel to the axes. We denote by $\ell(Q)$ its side length. The notation $Q(x, r)$ stands for an open cube with center x and side length $2r$. The analogous notation $\bar{Q}(x, r)$ is used for closed cubes. In general, cubes are assumed to be closed unless they are dyadic, or written in the form $Q(x, r)$, or stated otherwise.

We recall now the definition of dyadic cubes in $\mathbb{R}^d$. For $m \in \mathbb{Z}$, $\mathcal{D}_m$ is the family of cubes of the form

$$\big\{x \in \mathbb{R}^d : \, k_i\, 2^{-m} \leq x_i < (k_i + 1)\, 2^{-m} \text{ for } 1 \leq i \leq d\big\},$$

where k_i are arbitrary integers and $x = (x_1, \ldots, x_d)$. Cubes of this form are called dyadic. The family of all dyadic cubes (the so-called dyadic lattice) is written as $\mathcal{D} = \bigcup_{m \in \mathbb{Z}} \mathcal{D}_m$. Observe that, for each $m \in \mathbb{Z}$, the cubes from $\mathcal{D}_m$ form a partition of $\mathbb{R}^d$. Given $Q \in \mathcal{D}_m$, there are 2^d cubes from $\mathcal{D}_{m+1}$ which are contained in Q. They are the so-called children or sons of Q. Also, given $j \geq 0$, $\mathcal{D}_j(Q)$ denotes the family of the dyadic cubes contained in Q with side length $2^{-j}\ell(Q)$.

The line that passes through two points $x, y \in \mathbb{R}^d$ is denoted by $L_{x,y}$. Given two lines L_1, L_2, the notation $\measuredangle L_1, L_2$ stands for the angle they form (the smallest one, say). Also, for $x, y, z \in \mathbb{R}^d$, $\widehat{x, y, z}$ is the angle with vertex y and sides y, x and y, z.

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As usual in the field of harmonic analysis, the letters c and C stand for positive constants (quite often absolute constants) which may change their values at different occurrences. On the other hand, constants with subscripts, such as c_1, retain their value at different occurrences in a same chapter (and they change among different chapters). The notation $A \lesssim B$ means that there is a positive constant c such that $A \leq cB$. Also, $A \approx B$ is equivalent to $A \lesssim B \lesssim A$.

For a function $f : \mathbb{C} \to \mathbb{C}$, ∂f and $\bar{\partial} f$ stand for the complex derivatives

$$\partial f(z) = \frac{1}{2}\Big(\partial_x f(z) - i\,\partial_y f(z)\Big),$$

$$\bar{\partial} f(z) = \frac{1}{2}\Big(\partial_x f(z) + i\,\partial_y f(z)\Big).$$

All measures in this book are assumed to be Borel measures. Recall that a Radon measure μ in a metric space X is a Borel measure such that

(i) $\mu(K) < \infty$ for all compact sets $K \subset X$,

(ii) $\mu(V) = \sup\{\mu(K) : K \subset V \text{ is compact}\}$ for all open sets $V \subset X$,

(iii) $\mu(A) = \inf\{\mu(V) : A \subset V,\ V \text{ is open}\}$ for all $A \subset X$.

If A is μ-measurable and μ is Radon, then for all $\varepsilon > 0$ there exists a closed set $C \subset A$ such that $\mu(A \setminus C) < \varepsilon$.

It turns out that, in $\mathbb{R}^d$, any Borel measure which is locally finite is a Radon measure.

The closed support (or just, support) of a measure μ is denoted by $\operatorname{supp}(\mu)$. The restriction of μ to a set $A \subset \mathbb{R}^d$ is written as $\mu\lfloor A$. That is, for $B \subset \mathbb{R}^d$, one sets $\mu\lfloor A(B) = \mu(A \cap B)$. The image measure (or push forward) of μ on X under a mapping $f : X \to Y$ is a measure on Y defined by $f\#\mu(A) = \mu(f^{-1}(A))$.

Real and complex measures are also assumed to be Borel in the book. The variation of a real or complex measure ν is denoted by $|\nu|$. Its total variation is $\|\nu\| = |\nu|(\mathbb{R}^d)$. The vector space of all Borel real (or complex if we are in the complex plane) finite measures is denoted by $M(\mathbb{R}^d)$. This is a Banach space with the total variation norm. On the other hand, $M_+(\mathbb{R}^d)$ stands for the subset of positive measures from $M(\mathbb{R}^d)$.

The Lebesgue measure in $\mathbb{R}^d$ is written as $\mathcal{L}^d$, or just by dx (or dy or $dt\dots$) inside some integral if the meaning is clear from the context. On the other hand, for a rectifiable curve $\Gamma \subset \mathbb{C}$ with a parameterization $\gamma : [a, b] \to \Gamma$, dz (or dz_Γ, or $dw\dots$) stands for the usual complex measure on Γ given by the image measure of $\gamma'(t)\,dt$, with the usual orientation when Γ is closed.

For $1 \leq p \leq \infty$, we denote by $L^p(\mu)$ the Banach spaces of the μ-measurable functions $f : \mathbb{R}^d \to \mathbb{R}$ (or $\mathbb{C}$) such that the following norm is finite:

$$\|f\|_{L^p(\mu)} = \left(\int |f|^p \, d\mu\right)^{1/p}.$$

For $p = \infty$, $L^\infty(\mu)$ is also a Banach space with the norm $\|\cdot\|_{L^\infty(\mu)}$ equal to the μ-essential supremum. As usual, p' will denote the conjugate index of p, i.e. $p' = p/(p-1)$, so that $L^{p'}(\mu)$ is the dual of $L^{p'}(\mu)$ for $1 \le p < \infty$.

We also consider the weak Lebesgue spaces $L^{p,\infty}(\mu)$, for $1 \le p < \infty$. Recall that $f \in L^{p,\infty}(\mu)$ if

$$\|f\|_{L^{p,\infty}(\mu)} = \sup_{\lambda>0} \lambda\, \mu\left(\{x \in \mathbb{R}^d : |f(x)| > \lambda\}\right)^{1/p}$$

is finite. When μ coincides with the Lebesgue measure on $\mathbb{R}^d$, for short we write $\|\cdot\|_p = \|\cdot\|_{L^p(\mu)}$ and $\|\cdot\|_{p,\infty} = \|\cdot\|_{L^{p,\infty}(\mu)}$

The identity operator in a vector space such as $L^p(\mu)$ is denoted by Id.

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