Chapter 1

The Riemannian adiabatic limit

The purpose of this chapter is to study the adiabatic limit of the Levi-Civita connection on a fibred manifold. This study was initiated in [B86a], and continued in Bismut-Cheeger [BC89], Berline-Getzler-Vergne [BeGeV92], Berthomieu-Bismut [BerB94] and Bismut [B97].

This chapter is organized as follows. In Section 1.1, we introduce a smooth proper submersion $p : M \rightarrow S$.

In Section 1.2, we construct a family of Riemannian metrics $g_{\epsilon}^{TM}$, and we study the limit as $\epsilon \rightarrow 0$ of the corresponding Levi-Civita connection and of related tensors.

Finally, in Section 1.3, we construct a trilinear form $\rho_0$ on the tangent bundle $TM$.

1.1 A smooth submersion

Let $M, S$ be smooth manifolds. Let $p : M \rightarrow S$ be a smooth submersion with compact fibre $X$. Let $TX = TM/S$ denote the relative tangent bundle. We have the exact sequence of smooth vector bundles on $M$,

$$0 \rightarrow TX \rightarrow TM \xrightarrow{p_*} p^* TS \rightarrow 0. \quad (1.1.1)$$

Let $g^{TM}$ be a smooth Riemannian metric on $TM$, let $g^{TX}$ be its restriction to $TX$. Let $T^HM$ be the orthogonal bundle to $TX$ in $TM$ with respect to $g^{TM}$, so that $TM$ splits orthogonally as

$$TM = TX \oplus T^HM. \quad (1.1.2)$$

Clearly $p_*$ induces the isomorphism

$$T^HM = p^* TS. \quad (1.1.3)$$
By (1.1.2), (1.1.3), we get
\[ TM = TX \oplus p^*TS. \] (1.1.4)

If \( A \in TS \), let \( A^H \in T^HM \) correspond to \( A \) via (1.1.3).

Let \( g^{TM} \) be the metric induced by \( g^{TM} \) on \( T^HM \). Then \( g^{TM} \) can be viewed as a metric on \( p^*TS \).

Let \( P^TX, P^{TM} \) be the projections from \( TM \) on \( TX, T^HM \) with respect to the splitting (1.1.2).

Let \( \nabla^{TM,LC} \) be the Levi-Civita connection on \((TM, g^{TM})\).

By the results of [B86a, section 1], \((T^HM, g^{TX})\) uniquely determines a metric preserving connection \( \nabla^{TX,LC} \) on \( TX \). The connection \( \nabla^{TX,LC} \) is the projection of \( \nabla^{TM,LC} \) on \( TX \) with respect to the splitting (1.1.2), the crucial point being that it only depends on \((T^HM, g^{TX})\). The restriction of \( \nabla^{TX,LC} \) to one given fibre \( X \) is the Levi-Civita connection on the tangent bundle of the fibre.

If \( A \) is a smooth section of \( TS \), let \( L_A^H \) be the Lie derivative operator associated with the vector field \( A^H \). Then \( L_A^H \) acts on the tensor algebra of \( TX \), and this action is a tensor also in \( A \in TS \). By [B97, Theorem 1.1], if \( A, E \) are smooth sections of \( TS, TX \), then
\[ \nabla^{TX,LC}_{A^H} E = [A^H, E] + \frac{1}{2} (g^{TX})^{-1} (L_A^H g^{TX}) E. \] (1.1.5)

Let \( g^{TS} \) be a smooth Riemannian metric on \( TS \). Let \( \nabla^{TS,LC} \) be the Levi-Civita connection on \((TS, g^{TS})\). We can write \( g^{TM} \) in the form
\[ g^{TM} = g^{TS}k, \] (1.1.6)
where \( k \) is a smooth section of \( p^*\text{End}(TS) \) over \( M \) which is self-adjoint and positive with respect to \( g^{TS} \).

1.2 The limit of the Levi-Civita connection as \( \epsilon \to 0 \)

For \( \epsilon > 0 \), let \( g^{TM}_\epsilon \) be the metric on \( TM \) given by
\[ g^{TM}_\epsilon = g^{TM} + \frac{1}{\epsilon} p^* g^{TS}. \] (1.2.1)

Using (1.1.4), (1.1.6), we can rewrite (1.2.1) in the form
\[ g^{TM}_\epsilon = g^{TX} \oplus g^{TS} \left( \frac{1}{\epsilon} + k \right). \] (1.2.2)

Then \( g^{TM}_\epsilon \) still induces the metric \( g^{TX} \) on \( TX \), and \( T^HM \) is the orthogonal bundle to \( TX \) with respect to \( g^{TM}_\epsilon \). Let \( \nabla^{TM,LC}_\epsilon \) be the Levi-Civita connection on \( TM \) with respect to the metric \( g^{TM}_\epsilon \). By the above, \( \nabla^{TM,LC}_\epsilon \) projects on \( TX \) as the fixed connection \( \nabla^{TX,LC}_\epsilon \).
1.2. The limit of the Levi-Civita connection as $\epsilon \to 0$

With respect to the splitting (1.1.4) of $TM$, $\nabla_{\epsilon}^{TM,LC}$ can be written in the form

$$\nabla_{\epsilon}^{TM,LC} = \begin{bmatrix} \nabla_{TX}^{LC} & S_{TX,LC}^{TM,LC} \\ -S_{TX,LC}^{TM,LC} & \nabla_{\epsilon}^{TS} \end{bmatrix}. \quad (1.2.3)$$

Let $\nabla_{s,\epsilon}^{TS}$ be the connection on $p^{*}TS$,

$$\nabla_{s,\epsilon}^{TS} = \nabla^{TS,LC} + \frac{\epsilon}{2} (1 + \epsilon k)^{-1} \nabla_{\epsilon}^{TS,LC} k. \quad (1.2.4)$$

Set

$$\nabla_{s,\epsilon}^{TM,LC} = \begin{bmatrix} \nabla_{TX}^{LC} & 0 \\ 0 & \nabla_{s,\epsilon}^{TS} \end{bmatrix}. \quad (1.2.5)$$

Then $\nabla_{s,\epsilon}^{TM,LC}$ is an Euclidean connection on $(TM, g^{TM})$.

Set

$$\nabla_{s}^{TM,LC} = \begin{bmatrix} \nabla_{TX}^{LC} & 0 \\ 0 & \nabla_{TS,LC} \end{bmatrix}. \quad (1.2.6)$$

Then $\nabla_{s}^{TM,LC}$ is an Euclidean connection on $TX \oplus p^{*}TS$ equipped with the metric $g^{TX} \oplus p^{*}g^{TS}$.

By (1.2.4), (1.2.5), as $\epsilon \to 0$,

$$\nabla_{s,\epsilon}^{TM,LC} = \nabla_{s}^{TM,LC} + O(\epsilon). \quad (1.2.7)$$

In (1.2.7), $O(\epsilon)$ indicates that if $K$ is a compact subset of $M$, for any $k \in \mathbb{N}$, the coefficients of the considered operator and its derivatives of order $\leq k$ can be dominated by $C_{K,k} \epsilon$. In the whole book, a similar notation will be used for other tensors.

Let $T_{s,\epsilon}, T_{s}$ be the torsions of $\nabla_{s,\epsilon}^{TM,LC}, \nabla_{s}^{TM,LC}$. Since $\nabla_{TX}^{LC}$ is fibrewise torsion free, $T_{s,\epsilon}, T$ both vanish on $TX \times TX$. By (1.2.7), we get

$$T_{s,\epsilon} = T_{s} + O(\epsilon). \quad (1.2.8)$$

By (1.2.4), for $A, B \in TM$,

$$\frac{\partial}{\partial \epsilon} |_{\epsilon=0} (A, B) = \frac{1}{2} \left( \nabla_{A}^{TS} k_{p^{*}B} - \nabla_{B}^{TS} k_{p^{*}A} \right). \quad (1.2.9)$$

By (1.2.6), $T_{s}$ takes its values in $TX$. Since $\nabla_{TS}$ is torsion free, if $A, B \in TS$,

$$T_{s} \left( A^{H}, B^{H} \right) = -P_{TX} \left[ A^{H}, B^{H} \right]. \quad (1.2.10)$$

By [B97, Theorem 1.1] or by (1.1.5), if $A \in TS, E \in TX$,

$$T_{s} \left( A^{H}, E \right) = \frac{1}{2} \left( g^{TX} \right)^{-1} \left( L_{A^{H}} g^{TX} \right) E. \quad (1.2.11)$$

In particular if $A \in TS$, and if $E, F, \in TX$, then

$$\langle T_{s} \left( A^{H}, E \right), F \rangle_{g^{TX}} = \langle E, T_{s} \left( A^{H}, F \right) \rangle_{g^{TX}}. \quad (1.2.12)$$
By the above, we recover the known fact that the tensor $T_s$ depends only on $(g^{TX}, T^HM)$.

Set
$$S^{TM}_{s,\epsilon} = \nabla^{TM,LC}_\epsilon - \nabla^{TM,LC}_{s,\epsilon}. \tag{1.2.13}$$

By (1.2.3), (1.2.5), $S^{TM}_{s,\epsilon}$ is of the form
$$S^{TM}_{s,\epsilon} = \begin{bmatrix} 0 & S^{TX,LC}_\epsilon \\ -S^{TX,LC}_\epsilon & S^{TS}_\epsilon \end{bmatrix}. \tag{1.2.14}$$

Since $\nabla^{TM,LC}_\epsilon$ is torsion free, if $A, B \in TM$, then
$$T_{s,\epsilon} (A, B) = -S^{TM}_{s,\epsilon} (A) B + S^{TM}_{s,\epsilon} (B) A. \tag{1.2.15}$$

Moreover, if $A, B, C \in TM$, we have the classical identity
$$2 \langle S^{TM}_{s,\epsilon} (A) B, C \rangle_{g^{TM}_\epsilon} + \langle T_{s,\epsilon} (A, B), C \rangle_{g^{TM}_\epsilon}$$
$$+ \langle T_{s,\epsilon} (C, A), B \rangle_{g^{TM}_\epsilon} - \langle T_{s,\epsilon} (B, C), A \rangle_{g^{TM}_\epsilon} = 0. \tag{1.2.16}$$

Using (1.2.8), (1.2.9), the fact that $T_s$ takes its values in $TX$, and (1.2.16), we find that there is a smooth section $S^{TX,LC}$ of $T^*M \otimes \text{Hom}(p^*TS, TX)$ such that as $\epsilon \to 0$,
$$S^{TX,LC}_\epsilon = S^{TX,LC} + \mathcal{O}(\epsilon), \quad S^{TX,LC*}_\epsilon = \mathcal{O}(\epsilon), \quad S^{TS}_\epsilon = \mathcal{O}(\epsilon). \tag{1.2.17}$$

In the sequel, we identify $S^{TX,LC}$ with the corresponding element of $T^*M \otimes \text{End}(TM)$ that vanishes on $TX$. By (1.2.14), (1.2.17), as $\epsilon \to 0$,
$$S^{TM}_{s,\epsilon} = S^{TX,LC} + \mathcal{O}(\epsilon). \tag{1.2.18}$$

By (1.2.8), (1.2.15), and (1.2.17), if $A, B \in TM$,
$$T_s (A, B) = -S^{TX,LC}_\epsilon (A) P^{TM} B + S^{TX,LC}_\epsilon (B) P^{TM} A. \tag{1.2.19}$$

By (1.2.16), (1.2.18), if $A \in TM, B \in TS, C \in TX$, then
$$2 \langle S^{TX,LC}_\epsilon (A) B^H, C \rangle_{g^{TX}_\epsilon} + \langle T_s (A, B^H), C \rangle_{g^{TX}_\epsilon} + \left( \frac{\partial}{\partial \epsilon} T_{s,\epsilon|\epsilon=0} (C, A), B^H \right)_{g^{TS}_\epsilon}$$
$$- \langle T_s (B^H, C), P^{TX} A \rangle_{g^{TX}_\epsilon} - \left( \frac{\partial}{\partial \epsilon} T_{s,\epsilon|\epsilon=0} (B^H, C), P^{TM} A \right)_{g^{TS}_\epsilon} = 0. \tag{1.2.20}$$

By (1.2.9), we can rewrite (1.2.20) in the form
$$2 \langle S^{TX,LC}_\epsilon (A) B^H, C \rangle_{g^{TX}_\epsilon} + \langle T_s (A, B^H), C \rangle_{g^{TX}_\epsilon}$$
$$- \langle T_s (B^H, C), P^{TX} A \rangle_{g^{TX}_\epsilon} + \langle \nabla^{TS}_C kB, p_* A \rangle_{g^{TS}_\epsilon} = 0. \tag{1.2.21}$$
1.3 The trilinear form \( \rho_0 \)

Set

\[
\nabla^{TM,LC}_0 = \nabla^{TM,LC}_s + S^{TX,LC}.
\]  

(1.2.22)

Equivalently,

\[
\nabla^{TM,LC}_0 = \begin{bmatrix} \nabla^{TX,LC} & S^{TX,LC} \\ 0 & \nabla^{TS,LC} \end{bmatrix}.
\]  

(1.2.23)

By (1.2.7), (1.2.13), and (1.2.18), as \( \epsilon \to 0 \),

\[
\nabla^{TM,LC}_\epsilon = \nabla^{TM,LC}_0 + O(\epsilon).
\]  

(1.2.24)

Since \( \nabla^{TM,LC}_\epsilon \) is torsion free, \( \nabla^{TM,LC}_0 \) is also torsion free. Equation (1.2.19) is a reformulation of this fact.

1.3 The trilinear form \( \rho_0 \)

**Definition 1.3.1.** For \( A, B, C \in TM \), set

\[
\rho_\epsilon (A, B, C) = \langle S^{TM}_{s,\epsilon} (A) B, C \rangle_{g^{TM}_\epsilon}.
\]  

(1.3.1)

If \( A, B, C \in TM \), let \( \rho_0 (A, B, C) \in R \) be defined by

\[
2\rho_0 (A, B, C) + \langle T_s (A, B), P^{TX} C \rangle_{g^{TX}} + \langle T_s (C, A), P^{TX} B \rangle_{g^{TX}} - \langle T_s (B, C), P^{TX} A \rangle_{g^{TX}} = 0.
\]  

(1.3.2)

**Proposition 1.3.2.** As \( \epsilon \to 0 \),

\[
\rho_\epsilon = \rho_0 + O(\epsilon).
\]  

(1.3.3)

Moreover, if \( A \in TX, B, C \in TM \), \( \rho_0 (A, B, C) \) does not depend on \( g^{TS} \), and is given by

\[
2\rho_0 (A, B, C) + \langle T_s (A, B), P^{TX} C \rangle_{g^{TX}} + \langle T_s (C, A), P^{TX} B \rangle_{g^{TX}} - \langle T_s (B, C), P^{TX} A \rangle_{g^{TX}} = 0.
\]  

(1.3.4)

**Proof.** Equation (1.3.3) follows from (1.2.9), (1.2.16), and (1.3.1). When \( A \in TX \), \( p_* A = 0 \), and the last two terms in the left-hand side of (1.3.2) do vanish, so that we get (1.3.4). This shows that \( \rho_0 (A, B, C) \) does not depend on \( g^{TS} \). The proof of our proposition is completed. \( \square \)

**Definition 1.3.3.** Let \( S^{TM}_0 \in T^* M \otimes \text{End}(TM) \) be such that if \( A, B, C \in TM \), then

\[
\langle S^{TM}_0 (A) B, C \rangle_{g^{TX} \oplus g^{TS}} = \rho_0 (A, B, C).
\]  

(1.3.5)
By (1.2.21), (1.3.2), $S^{TM}_0$ can be written in the form

$$S^{TM}_0 = \begin{bmatrix} 0 & S^{TX,LC} \\ -S^{TX,LC*} & S^{TS} \end{bmatrix}.$$  \hspace{1cm} (1.3.6)

As the notation indicates, $S^{TX,LC*}$ is the adjoint of $S^{TX,LC}$.

Remark 1.3.4. The trilinear form $\rho_0$ was already obtained in [B86a, section 1 c)] when the metric $g^{TM}$ defines a Riemannian submersion.
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