Chapter 2
Non-uniqueness and Uniqueness in the Cauchy Problem of Elliptic and Backward-Parabolic Equations

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Abstract In this paper we consider the non-uniqueness and the uniqueness property for the solutions to the Cauchy problem for the operators

$$\mathcal{E}u = \partial_t^2 u + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}u\big) + \beta(t,x)\partial_t u + \sum_{m=1}^{n} b_m(t,x)\partial_{x_m}u + c(t,x)u$$

and

$$\mathcal{P}u = \partial_t u + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}u\big) + \sum_{m=1}^{n} b_m(t,x)\partial_{x_m}u + c(t,x)u,$$

where $\sum_{k,l=1}^{n} a_{kl}(t,x)\xi_k\xi_l|\xi|^{-2} \geq a_0 > 0$. We study non-uniqueness and uniqueness in dependence of global and local regularity properties of the coefficients of the principal part. The global regularity will be ruled by the modulus of continuity of a_{kl} on $[0,T]$ while the local regularity will concern a bound on $|\partial_t a_{kl}(t,x)|$ on every interval $[\varepsilon, T] \subseteq (0,T]$. By suitable counterexamples we show that our conditions seem to be sharp in many cases and we compare our statements with known results in the theory of hyperbolic Cauchy problems. We make also some remarks on continuous dependence for $\mathcal{P}$.

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2.1 Introduction

In this paper we collect some results on non-uniqueness and uniqueness for the solutions to the Cauchy problem for elliptic and backward-parabolic operators.

The subject of uniqueness and non-uniqueness in the Cauchy problem has a fairly long history and, from the pioneering works of Carleman [4] up to today, a huge number of results have been obtained and many different aspects of this topic have been developed (see e.g. [45] for a not so recent bibliography).

Here we are interested in two restricted classes of differential operators, precisely second order elliptic operators of the type

$$\mathcal{E}u = \partial_t^2 u + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l} u\big) + \beta(t,x)\partial_t u + \sum_{m=1}^{n} b_m(t,x)\partial_{x_m} u + c(t,x)u$$

and backward-parabolic operators of the type

$$\mathcal{P}u = \partial_t u + \sum_{k,l=1}^{n} \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l} u\big) + \sum_{m=1}^{n} b_m(t,x)\partial_{x_m} u + c(t,x)u.$$

In both cases we will suppose that $\sum_{k,l=1}^{n} a_{kl}(t,x)\xi_k\xi_l \geq a_0|\xi|^2$, with $a_0 > 0$.

We say that $\mathcal{E}$ or $\mathcal{P}$ have the *uniqueness* property (in the space X, with respect to the oriented surface $\{t \geq 0\}$, in the point 0) if for every function $u \in X$, from the fact that u is a solution to $\mathcal{E}u = 0$ or $\mathcal{P}u = 0$ with $\mathrm{supp}(u) \subseteq \{t \geq 0\}$, it follows that $u = 0$ in a neighborhood of 0.

In turn, by *non-uniqueness* for $\mathcal{E}$ or $\mathcal{P}$ we mean that we are able to find a non-zero solution $u \in X$ to $\mathcal{E}u = 0$ or $\mathcal{P}u = 0$ such that $0 \in \mathrm{supp}(u) \subseteq \{t \geq 0\}$.

Our aim is to study the connections between the properties of non-uniqueness and uniqueness with the regularity of the coefficients of the principal part of the operators under consideration. The results of Hörmander [26, 27] and J.-L. Lions and Malgrange [34] guarantee that Lipschitz regularity for a_{kl} is sufficient for uniqueness for $\mathcal{E}$ and $\mathcal{P}$ respectively, while the counterexample of Pliš [39] and some easy modifications of it (see for instance [19]) show that non-uniqueness can occur for some particular $\mathcal{E}$ and $\mathcal{P}$ having $a_{kl} \in \bigcap_{0<\alpha<1} C^\alpha$.

The investigation we want to develop will be in the narrow interval between these two bounds and the regularity of the coefficients will be measured from two points of view: using the notion modulus of continuity (we will call it *global regularity* property) and controlling the oscillation of the coefficients (this will be called *local regularity* property).

The idea to control the oscillations of the coefficients of the principal part originates from the technique of construction of all of the known counterexamples to uniqueness. In all these constructions the uniqueness property is destroyed by sufficiently fast oscillations of the principal part coefficients around a single point. Away from this point the coefficients are smooth. Hence it is natural to think that a bound on the oscillations of the coefficients might restore the uniqueness property.

In the case of the hyperbolic Cauchy problem, the interaction of global and local regularity conditions have been extensively studied with respect to the well-posedness of the problem. The Cauchy problem for elliptic and backward-parabolic operators is severally ill-posed but, anyhow, besides this difference, we would like to compare briefly the conditions for uniqueness and non-uniqueness for elliptic and backward-parabolic to those for well-posedness in the hyperbolic theory.

Considering the hyperbolic Cauchy problem

$$\begin{cases} \mathcal{L}u = \partial_t^2 u - a(t)\partial_x^2 u = 0 \\ u(0,x) = u_0(x), u_t(0,x) = u_1(x), \end{cases} \tag{1}$$

where $a(t) \geq a_0 > 0$, it is well known (see e.g. Chap. IX in [28]) that (1) is C^∞-well-posed if one supposes $a \in \mathrm{Lip}[0,T]$. In [10] the Cauchy problem (1) was studied under the condition

$$\sup_{\substack{0<|t-s|<1 \\ t,s\in[0,T]}} \frac{|a(t)-a(s)|}{|t-s||\log(|t-s|)|} \leq C < +\infty \tag{2}$$

and (2) was proved to be sufficient for C^∞-well-posedness with the so called *loss of derivatives* (this means that there is a shifting between the Sobolev norms in the energy estimates for $\mathcal{L}$, see [32] for a detailed explanation of this phenomenon). Condition (2) means that a is *globally regular* with respect to the Log-Lipschitz modulus of continuity, for short $a \in \mathrm{LogLip}[0,T]$. A counterexample in [9] shows that one cannot weaken this condition without further assumptions (see also [42] for a recent interesting improvement of [10]).

A second possibility to weaken the Lipschitz property of a in (1) goes back to [12] where the notion of *local regularity* was first explicitly introduced (see also [43]). Precisely in [12] the coefficient a was in $C^0[0,T] \cap C^1(0,T]$ with

$$\sup_{t\in[0,T]} \left|ta'(t)\right| \leq C < +\infty. \tag{3}$$

Under this hypothesis there is again the C^∞-well-posedness with loss of derivatives and some counterexamples similar to those of [9] show that this assumption can be considered optimal.

Supposing more regularity for a away from $t = 0$, some other interesting results have been obtained. It has been proved (see e.g. [25, 43]) that one gets C^∞-well-posedness without loss of derivatives under the condition $a \in C^0[0,T] \cap C^1(0,T] \cap C^2(0,T]$ with

$$\sup_{t\in[0,T]} \left|ta'(t)\right| + \left|t^2 a''(t)\right| \leq C < +\infty,$$

while the hypothesis (see [13]) for the same result with loss of derivatives is

$$\sup_{t\in[0,T]} \left|\big(t\log(t)\big)a'(t)\right| + \left|\big(t\log(t)\big)^2 a''(t)\right| \leq C < +\infty.$$

A list of counterexamples shows that these results are sharp and, at the present, even if it should be reasonable that supposing more regularity on a (e.g. a is in $C^m(0,T]$ for $m \geq 3$) some different and weaker conditions on derivatives of a should ensure C^∞-well-posedness, only the C^2-theory has been developed.

The effect that one can weaken the assumption on a' by a logarithm by assuming a condition on a'' is called the *Log-effect*. This is connected to the classification of oscillations introduced by Reissig and Yagdjian in [40] and also studied for other types of operators, as p-evolution operators in [5]. The reader may also consult [44] for more on the matter and related questions.

In [12, 22, 32] the authors have studied the possible couplings between the global regularity and the local regularity. They have, for example, proved that a coefficient with a modulus of continuity $f(s) = s\frac{\mu(s)}{\eta(s)}$, worse than Log-Lipschitz, needs a control of oscillations precisely by $-C\frac{d}{dt}\mu(\eta^{-1}(t))$ to guarantee well-posedness in some scales of Sobolev spaces. For further information we refer to the cited papers and the references therein.

Finally we refer to [30] for a more exhaustive comparison of the hyperbolic theory to the elliptic and backward parabolic theory with respect to the question of uniqueness in the Cauchy problem.

The paper is organized as follows; in Sect. 2.2 we state several non-uniqueness theorems for $\mathcal{E}$ and $\mathcal{P}$ modeled on the well-known Pliš example in [39]. We will state theorems with global and local assumptions on the principal part coefficients and we will give an example of non-uniqueness for coefficients with non-Osgood global regularity and a certain control of the oscillations, similar to the one in [32] for hyperbolic operators.

The non-uniqueness theorems under local conditions on the principal part coefficients will show that the Log-effect does not appear in the case of elliptic and backward-parabolic operators, i.e. it is not possible to weaken the condition $\sup_{t\in[0,T]} |ta'(t)| \leq C$ by adding a condition on the second derivative. Hence, under local conditions only C^1-theory is interesting for the uniqueness of the Cauchy problem for our operators.

At the end of the section we give an outline of the construction of those counterexamples with the various changes according to the different types of conditions.

Section 2.3 contains the uniqueness counterpart to Sect. 2.2. In the first sub-part of this section we recall some important results about the uniqueness in the Cauchy problem for $\mathcal{E}$ and $\mathcal{P}$ under global regularity conditions.

The next part contains the statement and the proof of a uniqueness result for $\mathcal{E}$ and $\mathcal{P}$ under a local condition like (3) with a smallness condition on the constant. We also note how one can slightly weaken this smallness condition if one restricts the uniqueness results to solutions in certain Gevrey classes, where the Gevrey-index depends on the size of the constant.

In the third sub-section of this section we state some uniqueness theorems for $\mathcal{P}$ under the assumption $\sum_{k,l=1}^n a_{kl}(t,x)\xi_k\xi_l|\xi|^{-2} \geq 0$. They complement in some sense the results under local conditions in the case of degenerate backward-parabolicity. The corresponding theorems for degenerate elliptic operators are

Table 2.1 Comparison between hyperbolic and elliptic/backward-parabolic operators with respect to uniqueness in the Cauchy problem

Hyperbolic theory	Elliptic/backward-parabolic theory
$\mu(s) = s^{1-\alpha}$, $\alpha \in (0, 1)$	
$\lvert a'(t)\rvert \le Ct^{-(1/2)(1+1/\alpha)}$	$\lvert a'(t)\rvert \le Ct^{-1}$
$\lvert a''(t)\rvert \le Ct^{-(1+1/\alpha)}$	$\lvert a^{(k)}(t)\rvert \le Ct^{-((k-1)(1/\alpha)+k)}$, for all $k \ge 1$
$\Rightarrow$ In the elliptic/backward-parabolic theory only C^1-theory is interesting	
Log-effect	
$\lvert a'(t)\rvert \le C\frac{1}{t}\log(\frac{1}{t})$	No Log-effect
$\lvert a''(t)\rvert \le C\frac{1}{t^2}(\log(\frac{1}{t}))^2$	
Constant in the estimate of a'	
$C \in \mathbb{R}_{>0}$	$C_{ell} \in (0, 2a_0)$ and $C_{bp} \in (0, a_0)$

proved in [6, 7, 11, 36]. The last part of this section is devoted to some open problems and an out-view to possible further developments.

Table 2.1 summarizes very shortly the main differences between the hyperbolic and the elliptic/backward-parabolic operators concerning uniqueness in the Cauchy problem which are proved in Sects. 2.2 and 2.3.

The first part of the table shows that for elliptic and backward-parabolic operators the additional assumption of Hölder regularity for the principal part coefficients brings nothing with respect to the allowed oscillations of the coefficient in contrast to the hyperbolic theory. For this see Sect. 2.2.2. Besides the Log-effect, for which the reader finds a more detailed discussion in Sect. 2.2.2, it illustrates that only in the elliptic and backward-parabolic regime appears a restriction on the size of the constant in the control of the oscillations, see also Sects. 2.2.2 and 2.3.1.

In the last section of the present work we summarize some results about continuous dependence of solutions to $\mathcal{P}$ and $\mathcal{E}$ on the Cauchy data in the sense of John (see [31]). These results are only concerned with global regularity.

2.1.1 Modulus of Continuity and Related Oscillation Conditions

In this section we state some definitions which we need in the subsequent sections.

Definition 1.1 (Modulus of continuity, C^μ) We call a continuous, concave, increasing function $\mu : [0, s_0] \to [0, 1]$, $s_0 > 0$, a modulus of continuity. A function $f \in C^0(Q)$, $Q \subseteq \mathbb{R}^n$ belongs to $C^\mu(Q)$ iff

$$\exists C > 0 : \sup_{\substack{0<|x-y|\le s_0 \\ x,y\in Q}} \frac{|f(x) - f(y)|}{\mu(|x - y|)} \le C < +\infty.$$

The next definition introduces the notion of the Osgood condition. The Osgood condition first appeared in [38] where Osgood studied the uniqueness of solutions of ordinary differential equations without the Lipschitz condition.

Definition 1.2 Osgood condition A modulus of continuity is said to satisfy the Osgood condition if there exists an $s_0 > 0$ such that

$$\int_0^{s_0} \frac{ds}{\mu(s)} = +\infty. \tag{4}$$

If there exist an $s_0 > 0$ such that condition (4) fails to hold we will call μ a non-Osgood modulus of continuity.

For the sake of brevity we introduce some symbols for moduli of continuity which we are going to use in the subsequent part:

$$\begin{aligned}
&\mathrm{Log}^{-1} : \mu(s) = \left(\log\left(\frac{1}{s}\right)\right)^{-1},\\
&C^\alpha : \mu(s) = s^\alpha, \quad \alpha \in (0,1),\\
&\mathrm{Lip} : \mu(s) = s,\\
&\mathrm{Log}^{1+\varepsilon}\,\mathrm{Lip} : \mu(s) = s\left(\log\left(\frac{1}{s}\right)\right)^{1+\varepsilon},\\
&\mathrm{Log}^{[m,1+\varepsilon]}\,\mathrm{Lip} : \mu(s) = s\left(\prod_{i=1}^{m-1} \log^{[i]}\left(\frac{1}{s}\right)\right)\left(\log^{[m]}\left(\frac{1}{s}\right)\right)^{1+\varepsilon}.
\end{aligned}$$

We define $\log^{[i]}(s) := \log(\log^{[i-1]}(s))$ with $\log^{[1]}(s) = \log(s)$. The last three lines of the list above are, with $\varepsilon = 0$, examples for Osgood moduli of continuity. If one takes $\varepsilon > 0$ they are examples for non-Osgood moduli of continuity.

Definition 1.3 (Osgood distance function) For a non-Osgood modulus of continuity μ we associate a function

$$\eta(t) := \int_0^t \frac{ds}{\mu(s)}. \tag{5}$$

Remark 1.1 The function η measures essentially how far the modulus of continuity is from an Osgood modulus of continuity. The *velocity* of the function $\eta(t)$ going to 0 for $t \to 0+$ carries this information. The slower this function converges to 0 the closer is μ to an Osgood modulus of continuity.

Remark 1.2 Another way to illustrate how the function η, defined by (5) for a non-Osgood modulus of continuity, measures the difference between μ and an Osgood modulus of continuity is to say that $\sigma(s) := \mu(s)\eta(s)$ is an Osgood modulus of continuity. This can be seen as follows:

$$\begin{aligned}\int_0^{s_0} \frac{ds}{\sigma(s)} &= \lim_{\epsilon\to 0+} \int_\varepsilon^{s_0} \frac{ds}{\mu(s)\eta(s)} = \lim_{\epsilon\to 0+} \int_\varepsilon^{s_0} \frac{\eta'(s)}{\eta(s)} ds \\ &= \lim_{\epsilon\to 0+} \left(\log\big(\eta(s_0)\big) + \log\left(\frac{1}{\eta(\varepsilon)}\right)\right) = +\infty,\end{aligned}$$

where we have used the fact that $\eta'(t) = \frac{1}{\mu(t)}$ for $t > 0$.

Throughout the paper we denote all $C^\infty(Q)$ functions bounded with all their derivatives by $B^\infty(Q)$. A function defined on the n-dimensional torus $\mathbb{T}^n$ will as usual be considered as a periodic function on $\mathbb{R}^n$.

2.2 Non-uniqueness

In this section we state some counterexamples to uniqueness in the Cauchy problem for elliptic and backward-parabolic operators. Actually we will state them just for elliptic operators but they are literally also true if one replaces the elliptic by a backward-parabolic operator. In the last part of this section we will show the general scheme how to construct such counterexamples of Pliš-type.

The counterexamples will also show that the so-called Log-effect, known from the hyperbolic theory or the theory of p-evolution operators, does not occur in the Cauchy problem for elliptic and backward-parabolic operators.

2.2.1 Non-uniqueness Under Global Conditions

The first counterexample to uniqueness in the Cauchy problem for elliptic operators was presented by Pliš in [39] and it came along as quite a surprise. It shows that certain amount of global regularity is necessary for the uniqueness in the Cauchy problem for elliptic operators (and apparently also for backward-parabolic operators), at least if one regards the problem in more than two dimensions. In two dimensions the situation is different. For more information on this matter the reader may consult [3, 45] and the references therein.

The original result of Pliš is

Theorem 2.1 (Theorem 1 in [39]) *There exist five real-valued functions u, a, f, g, h such that the PDE*

$$\begin{aligned}\mathcal{E}u = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + a(t)\frac{\partial^2 u}{\partial y^2} + f(t,x,y)\frac{\partial u}{\partial x} + g(t,x,y)\frac{\partial u}{\partial y} + h(t,x,y)u \\ = 0 \qquad (6)\end{aligned}$$

is satisfied on $\mathbb{R}^3$. *Furthermore, the solution* u *vanishes identically for* $t \geq 0$ *but does not vanish identically in any neighborhood of* $t = 0$. *The coefficient* a *satisfies* $\frac{1}{2} \leq a(t) \leq \frac{3}{2}$ *belongs to* $C^\infty(\mathbb{R} \setminus \{0\}) \cap \bigcap_{0<\alpha<1} C^\alpha(\mathbb{R})$ *and the functions* u, f, g, *and* h *belong to* $B^\infty(\mathbb{R}^3)$.

Remark 2.1 In [39] Pliš proved in fact a little more than he claimed. The coefficient a which he constructed in his proof is not just in $\bigcap_{0<\alpha<1} C^\alpha(\mathbb{R})$ but in $\mathrm{Log}^2\mathrm{Lip}(\mathbb{R})$. Additionally the first derivative of a satisfies the bound $|t^2 a'(t)| \leq C$ for $t < 0$ and some $C > 0$.

Later Tarama proved in [41] local uniqueness for elliptic operators whose principal coefficients are not Lipschitz-continuous (see Sect. 2.3.1). In fact they have a modulus of continuity μ satisfying the Osgood condition (see Definition 1.2). A counterexample in [15] shows that this condition cannot be weakened from the point of view of global regularity. Precisely it states:

Theorem 2.2 (Theorem 2 in [15]) *Let* μ *be a non-Osgood modulus of continuity. Then there exist five real-valued functions* u, a, f, g, h *such that the PDE*

$$\mathcal{E}u = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + a(t)\frac{\partial^2 u}{\partial y^2} + f(t,x,y)\frac{\partial u}{\partial x} + g(t,x,y)\frac{\partial u}{\partial y} + h(t,x,y)u = 0$$

is satisfied on $\mathbb{R}^3$. *Furthermore, the solution* u *vanishes for* $t \geq 0$ *but does not vanish identically in any neighborhood of* $t = 0$. *The coefficient* a *satisfies* $1 \leq a(t) \leq 2$, *belongs to* $C^\mu(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$ *and the functions* u, f, g *and* h *belong to* $B^\infty(\mathbb{R})$.

2.2.2 Non-uniqueness Under Local Conditions

In this section we state a non-uniqueness example under a local condition on the derivatives of the principal part coefficients. This is given by

Theorem 2.3 *There exist five real-valued functions* u, a, f, g, h *such that the PDE*

$$\mathcal{E}u = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + a(t)\frac{\partial^2 u}{\partial y^2} + f(t,x,y)\frac{\partial u}{\partial x} + g(t,x,y)\frac{\partial u}{\partial y} + h(t,x,y)u = 0$$

is satisfied on $\mathbb{R}^3$. *Furthermore, the solution* u *vanishes for* $t \geq 0$ *but does not vanish identically in any neighborhood of* $t = 0$. *The coefficient* a *satisfies* $1 \leq a(t) \leq 2$, *belongs to* $C^0(\mathbb{R}) \cap C^\infty(\mathbb{R} \setminus \{0\})$ *and satisfies*

$$\forall k \geq 1\ \exists C_k > 0 : \left| \frac{d^k a}{dt^k}(t) \right| \leq C_k |t|^{-k} \quad \forall t < 0.$$

The functions u, f, g, *and* h *belong to* $B^\infty(\mathbb{R}^3)$.

Remark 2.2 From Theorem 2.3 we detect some interesting differences between the elliptic and backward-parabolic case on one side and the hyperbolic case on the other side.

Firstly in the hyperbolic case a control of the type Ct^{-1} on the first time-derivatives of the coefficients second order terms gives the well-posedness (see [12]). Here if the constant in front of t^{-1} is sufficiently large we can construct a counterexample to uniqueness (see the Theorems 3.5 and 3.6 for an estimate of this constant).

Secondly in the hyperbolic case a control of the type $C_1 t^{-1}\log(t^{-1})$ on the first time-derivatives of the second order terms and of $Ct^{-2}(\log(t^{-1}))^2$ on the second time-derivatives ensure the well-posedness (it is the so called log-*effect*, see [13, 25]). Here we can construct a counterexample to uniqueness with a control of $C_1 t^{-1}$ and $C_2 t^{-2}$ respectively. So we cannot hope for a Log-*effect*, and it is not only a matter of the choice of the constants.

2.2.3 Non-uniqueness Under a Mixed Condition

The result we are going to state in this section mixes the two kinds of conditions we have focused on in the last two sections. The interesting point is how the regularity and the oscillations interact. See also Sect. 2.3.4 for further explanations and the discussion of a uniqueness counterpart for this theorem.

Theorem 2.4 *Let μ be a non-Osgood modulus of continuity. Then there exist five real-valued functions u, a, f, g, h such that the PDE*

$$\mathcal{E}u = \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} + a(t)\frac{\partial^2 u}{\partial y^2} = f(t,x,y)\frac{\partial u}{\partial x} + g(t,x,y)\frac{\partial u}{\partial y} + h(t,x,y)u = 0$$

is satisfied on $\mathbb{R}^3$. Furthermore, the solution u vanishes identically for $t \geq 0$ but does not vanish identically in any neighborhood of $t = 0$. The coefficient a satisfies $\frac{1}{2} \leq a(t) \leq \frac{3}{2}$, belongs to $C^\mu(\mathbb{R}) \cap C^\infty(\mathbb{R}\setminus\{0\})$ and satisfies

$$\forall k \geq 1\ \exists C_k > 0 : \left|\frac{d^k a}{dt^k}(t)\right| \leq C_k \frac{(\mu(\eta^{-1}(|t|)))^k}{(\eta^{-1}(|t|))^{2k-1}} \quad \forall t < 0, \tag{7}$$

where η is the Osgood distance function (5). The functions u, f, g and h belong to $B^\infty(\mathbb{R}^3)$.

Remark 2.3 Formula (7) gives for all moduli of continuity defining spaces beneath $\bigcap_{0<\alpha<1} C^\alpha$ just the control t^{-1} for the first derivative. Hence, to think about a possible positive result with a weaker control on a' we need more regularity. The Theorems 3.5 and 3.6 show, as a counterpart to Theorem 2.3, that a control of t^{-1} ensures uniqueness without any additional regularity if the constant in the estimate is sufficiently small.

Table 2.2 Some examples of oscillation conditions related to moduli of continuity

$a \in$	$\|a'(t)\| \lesssim$	$\|a^{(k)}(t)\| \lesssim$
C^0	t^{-1}	t^{-k}
Log^{-1}	t^{-1}	$t^{-(2k-1)}$
C^α	t^{-1}	$t^{-((k(2-\alpha)-1)/(1-\alpha))}$
$Log^{1+\varepsilon}Lip$	$t^{-(1+1/\varepsilon)}$	$t^{-k(1+1/\varepsilon)}(\exp(t^{-1/\varepsilon}))^{k-1}$
$Log^{[m,1+\varepsilon]}Lip$	$t^{-k(1+1/\varepsilon)}\prod_{i=1}^{m-1}\exp^{[i]}(t^{-1/\varepsilon})$	$t^{-(1+1/\varepsilon)}(\prod_{i=1}^{m-1}\exp^{[i]}(t^{-1/\varepsilon})^k)$ $\times(\exp^{[m]}(t^{-1/\varepsilon}))^{k-1}$

Table 2.2 shows some moduli of continuity and the associated control of oscillations: $\alpha \in (0,1)$, $\varepsilon > 0$

2.2.4 Scheme of the Construction of the Counterexamples

In this section we present the scheme how to construct an operator

$$\begin{aligned}\mathcal{E}u &= \frac{\partial^2}{\partial t^2}u + \frac{\partial^2}{\partial x^2}u + a(t)\frac{\partial^2}{\partial y^2}u + f(t,x,y)\frac{\partial}{\partial x}u + g(t,x,y)\frac{\partial}{\partial y}u + c(t,x,y)u \\ &= 0 \end{aligned} \tag{8}$$

with the properties stated in the theorems of the last three sections. We will not present every detail and the reader may consult the original paper of Pliš [39] or [30].

Step 1: Auxiliary Functions and Sequences Let $A(x)$, $B(x)$, $C(x)$, and $J(x)$ be elements of $B^\infty(\mathbb{R})$ with the following properties:

$$A(x) = 1 \quad \text{for } x \le \frac{1}{5}, \qquad A(x) = 0 \quad \text{for } x \ge \frac{1}{4},$$

$$B(x) = 0 \quad \text{for } x \le 0 \text{ or } x \ge 1, \qquad B(x) = 1 \quad \text{for } \frac{1}{6} \le x \le \frac{1}{2},$$

$$C(x) = 0 \quad \text{for } x \le \frac{1}{4}, \qquad C(x) = 1 \quad \text{for } x \ge \frac{1}{3},$$

$$J(x) = -2 \quad \text{for } x \le \frac{1}{6} \text{ or } x \ge \frac{1}{2}, \qquad J(x) = 2 \quad \frac{1}{5} \le x \le \frac{1}{3}.$$

In order to control the behavior of the solution and the coefficients of (8) we need two sequences $(a_n)_n$ and $(z_n)_n$ with the properties

$$-1 < a_n < a_{n+1} \quad \text{for all } n \ge 1, \qquad \lim_{n\to+\infty} a_n = 0,$$

$$1 < z_n < z_{n+1} \quad \text{for all } n \ge 1, \qquad \lim_{n\to+\infty} z_n = +\infty.$$

Furthermore, we define $r_n := a_{n+1} - a_n$, $q_1 := 0$, $q_n := \sum_{k=2}^{n} z_k r_{k-1}$ for all $n \geq 2$, and $p_n := (z_{n+1} - z_n)r_n$, where we suppose $p_n > 1$ for all $n \geq 1$. We transport the behavior of our auxiliary functions to the intervals $[a_n, a_{n+1}]$ where we shall construct our solution and the coefficients. We introduce

$$A_n(t) = A\left(\frac{t - a_{n+1}}{r_n}\right), \qquad B_n(t) = B\left(\frac{t - a_{n+1}}{r_n}\right), \qquad C_n(t) = C\left(\frac{t - a_{n+1}}{r_n}\right),$$
$$J_n(t) = J\left(\frac{t - a_{n+1}}{r_n}\right), \quad n \geq 1.$$

Step 2: Construction of a Solution u on $[a_n, a_{n+1}]$ We define the auxiliary functions

$$\begin{aligned} v_n(t,x) &= \exp\big(-q_n - z_n(t - a_{n+1})\big)\cos(z_n x), \\ w_n(t,y) &= \exp\big(-q_n - z_n(t - a_{n+1}) + J_n(t)p_n\big)\cos(z_n y) \end{aligned} \tag{9}$$

which are solutions of $u_{tt} + u_{xx} + a(t)u_{yy} = 0$ with a suitable coefficient a, which will be determined in Step 3 of the proof. To construct a solution of (8) we define

$$u(t,x,y) = \begin{cases} v_1(t,x) : t < a_1, \\ A_n(t)v_n(t,x) + B_n(t)w_n(t,y) + C_n(t)v_{n+1}(t,x) : t \in [a_n, a_{n+1}], \\ 0 : t \geq 0. \end{cases}$$

This function is obviously in $B^\infty(\mathbb{R}^3 \setminus \{0\})$. For u to be in $C^\infty(\mathbb{R}^3)$ the condition

$$\forall \alpha, \beta, \gamma \in \mathbb{N} : \left|\partial_t^\alpha \partial_x^\beta \partial_y^\gamma u(t,x,y)\right| \xrightarrow{t \to 0+} 0$$

is necessary and sufficient. This will be implied by the condition

$$\lim_{n \to \infty} \exp(-q_n + 2p_n) z_{n+1}^\alpha p_n^\beta r_n^{-\gamma} = 0 \quad \forall \alpha, \beta, \gamma \in \mathbb{N}. \tag{10}$$

Step 3: Construction of a Suitable $a(t)$ We denote $\tilde{\mathcal{E}}u = \partial_t^2 u + \partial_x^2 u + a(t)\partial_y u$ and we get from (9) that $\tilde{\mathcal{E}}v_n = 0$ for all $n \geq 1$. In order to get $\tilde{\mathcal{E}}w_n = 0$ on $[a_n, a_{n+1}]$ for $n \geq 1$ we have to set

$$a(t) = \begin{cases} 1 : t < a_1 \text{ or } t \geq 0, \\ 1 - 2J_n'(t)p_n z_n^{-1} + [J_n'(t)]^2 p_n^2 z_n^{-2} \\ \qquad + J_n''(t)p_n z_n^{-2} : t \in [a_n, a_{n+1}] \end{cases} \tag{11}$$

The condition

$$\sup_{n \in \mathbb{N}} \big(p_n r_n^{-1} z_n^{-1} + p_n^2 r_n^{-2} z_n^{-2}\big) \leq \frac{1}{2(\|J'\|_{L^\infty} + \|J''\|_{L^\infty})} \tag{12}$$

ensures that $\mathcal{E}$ is elliptic and that $1 \leq a(t) \leq 2$.

Step 4: The Regularity of $a(t)$ *Global regularity*: In this step of the construction we show how one ensures that a has the modulus of continuity μ. By the mean value theorem we have to control $|a'(t)|$ globally. From (11) we get on $[a_n, a_{n+1}]$:

$$a'(t) = -2J_n''(t)p_n z_n^{-1} + 2J_n'(t)J_n''(t)p_n^2 z_n^{-2} + J_n'''(t)p_n z_n^{-2}$$

which we can estimate as

$$\big|a'(t)\big| \lesssim r_n^{-2} p_n z_n^{-1} + r_n^{-3} p_n^2 z_n^{-2} + r_n^{-3} p_n z_n^{-2} \lesssim r_n^{-2} p_n z_n^{-1} + r_n^{-3} p_n^2 z_n^{-2}.$$

In order to get the μ-continuity for a we need $|a'(t)| \lesssim \frac{\mu(r_n)}{r_n}$ on $[a_n, a_{n+1}]$ because in this case we will be able to establish the inequality

$$\big|a(s) - a(t)\big| \lesssim \frac{\mu(r_n)}{r_n}|t-s| \lesssim \mu\big(|t-s|\big) \quad \forall s, t \in [a_n, a_{n+1}],$$

where we use $|t-s| \le r_n$ for $s, t \in [a_n, a_{n+1}]$ and the fact that $\sigma \mapsto \frac{\mu(\sigma)}{\sigma}$ is decreasing. This will be implied by the condition

$$\sup_{n\in\mathbb{N}} \frac{r_n^{-1} p_n z_n^{-1}}{\mu(r_n)} \le C < +\infty. \tag{13}$$

Local regularity: Here we want to control the oscillations of a near $t = 0$ like in Theorem 2.3. First we derive from (11) an estimate for the behavior of the k-th derivative of a:

$$\big|a^{(k)}(t)\big| \lesssim \frac{p_n}{r_n^{k+1} z_n} + \frac{p_n}{r_n^{k+2} z_n^2} + k\frac{p_n^2}{r_n^{k+2} z_n^2} \lesssim \frac{p_n}{r_n^{k+1} z_n} + k\frac{p_n^2}{r_n^{k+2} z_n^2}.$$

It is enough to analyze the term $p_n r_n^{-(k+1)} z_n^{-1}$. The other term has a better behavior and can be handled in the same way. Our goal is now to ensure that a relation like

$$\big|F(t)a'(t)\big| = \mathcal{O}(1) \quad (t \to 0-) \tag{14}$$

holds for a certain function F. In order to do that, we have to express t in terms of our sequences. By the definition of our intervals, we can conclude that

$$t \sim -\sum_{k=n}^{+\infty} r_k.$$

Now condition (14) reads as follows

$$\sup_{n\in N}\left(F\left(-\sum_{k=n}^{+\infty} r_k\right)\frac{p_n}{r_n^{k+1} z_n}\right) \le C < +\infty, \quad \forall k \ge 1. \tag{15}$$

Remark 2.4 For Theorem 2.3 we have to take $F(t) = |t|^k$ and we have to find sequences such that

$$\sup_{n\in\mathbb{N}} \left(\sum_{j=n}^{+\infty} r_j\right)^k \frac{p_n}{r_n^{k+1} z_n} \le C < +\infty, \quad \forall k \ge 1.$$

Remark 2.5 For Theorem 2.4 we have to take $F(t) = \frac{(\eta^{-1}(t))^{2k-1}}{(\mu(\eta^{-1}(|t|)))^k}$ with $\eta(t) := \int_0^t \frac{ds}{\mu(s)}$ and we have to find sequences such that

$$\sup_{n\in\mathbb{N}} \frac{(\eta^{-1}(t_n))^{2k-1}}{(\mu(\eta^{-1}(t_n)))^k} \frac{p_n}{r_n^{k+1} z_n} \le C < +\infty, \quad \forall k \ge 1, \tag{16}$$

where $t_n := \sum_{j=n}^{+\infty} r_j$.

Step 5: Definition of Lower-Order Coefficients As lower order coefficients we define

$$f(t,x,y) := -\frac{\tilde{\mathcal{E}} u}{u^2 + (\partial_x u)^2 + (\partial_y u)^2} \frac{\partial}{\partial x} u,$$

$$g(t,x,y) := -\frac{\tilde{\mathcal{E}} u}{u^2 + (\partial_x u)^2 + (\partial_y u)^2} \frac{\partial}{\partial y} u,$$

$$h(t,x,y) := -\frac{\tilde{\mathcal{E}} u}{u^2 + (\partial_x u)^2 + (\partial_y u)^2} u.$$

This coefficients will belong to $C^\infty(\mathbb{R}^3)$ if

$$\forall \alpha, \beta, \gamma \in \mathbb{N} : \lim_{n\to+\infty} \exp(-p_n) z_{n+1}^\alpha p_n^\beta r_n^{-\gamma} = 0. \tag{17}$$

To finish the construction we give examples of sequences which fulfill the conditions (10), (12), (17) and (13) and/or (15) for a sufficiently large j:

- Pliš example: $a_n := (\log(n+j))^{-1}$, $z_n := (n+j)^3$,
- Non-Osgood regularity: $a_n := \sum_{l=n}^{+\infty} ((l+j)^2 \mu(\frac{1}{(l+j)}))^{-1}$, $z_n := (n+j)^3$,
- Oscillation control: $r_n := \rho^{-(n+j)}$, $z_n := \rho^{(n+j)}(n+j)\log(n+j)$,
- Mixing situation: $a_n := \sum_{l=n}^{+\infty} ((l+j)^2 \mu(\frac{1}{(l+j)}))^{-1}$, $z_n := (n+j)^3$.

With the choice of these sequences the construction of the counterexample is finished. As already mentioned the same construction (with small changes) also work for backward-parabolic operators, see [19, 30].

Remark 2.6 To prove (16) in the mixing situation it is essential to use the relation $t_n := \sum_{j=n}^{+\infty} r_j \sim \eta(\frac{1}{n})$. This reflects precisely the non-Osgood condition.

2.3 Uniqueness

In this section we present some complementary theorems to the theorems of Sect. 2.2. The history of uniqueness in the Cauchy problem for elliptic equations is fairly long and we attempt by no means to give a survey about this development. Here we focus on theorems which are more or less direct complements of our non-uniqueness theorems.

2.3.1 Uniqueness Under Global Conditions

First, we state results which are counterparts to the results in Sect. 2.2.1. The theorems, as proved in the original papers, hold mostly for more general solutions than stated here. But for our purpose the formulations presented here are sufficient.

In [26, 27] Hörmander has deeply investigated the question of uniqueness for the Cauchy problem for partial differential operators. One of the results is the uniqueness for solutions to elliptic operators with Lipschitz continuous coefficients in the principal part. It can be stated as

Theorem 3.1 *Suppose that* $a_{kl} \in \mathrm{Lip}([0,T]\times\mathbb{R}^n)$ *and* $\beta, b_m, c \in L^\infty([0,T]\times\mathbb{R}^n)$. *Then* $\mathcal{E}$ *has the* C^∞*-uniqueness property.*

In [41] Tarama proved that uniqueness in the Cauchy problem for elliptic operators still holds true if one weakens the Lipschitz-condition on the principal part coefficients and replaces it with the Osgood-condition. As the counterexamples of Sect. 2.2.1 show, this cannot be weakened without further assumptions. We remark again that the result of Pliš is in fact a result about the sharpness of the Osgood condition (see Remark 2.1). The result of Tarama can be stated as

Theorem 3.2 (Theorem 1.2 in [41]) *Suppose that* $a_{kl} \in C^\mu([0,T]\times\mathbb{R}^n)$ *with an Osgood modulus of continuity* μ; *suppose* $\beta, b_m, c \in L^\infty([0,T]\times\mathbb{R}^n)$. *Then* $\mathcal{E}$ *has the* C^∞*-uniqueness property.*

Remark 3.1 In comparison to the hyperbolic theory it is unknown weather the Osgood condition is sufficient for the uniqueness of the Cauchy problem or not. However, a counterexample to uniqueness in [8] shows at least that one cannot consider coefficients with regularity beneath the Osgood condition.

Similar results to Theorems 3.1 and 3.2 have been proved for the backward parabolic operator $\mathcal{P}$. The first paper in this direction was perhaps [34] where J.-L. Lions and Malgrange proved uniqueness for the solutions of the Cauchy problem for $\mathcal{P}$ under the condition that the principal part coefficients Lipschitz-continuous with respect to t and L^∞ with respect to x. Similar results can be found in [1, 2, 23, 35]. The result of J.-L. Lions and Malgrange can be stated as

Theorem 3.3 *Suppose, for $\mathcal{P}$, that $a_{kl} \in \mathrm{Lip}([0,T], L^\infty(\mathbb{R}^n))$; suppose $\beta, b_m, c \in L^\infty([0,T]\times\mathbb{R}^n)$. Then, $\mathcal{P}$ has the $\mathcal{H}$-uniqueness property, where*

$$\mathcal{H} := H^1\big([0,T], L^2(\mathbb{R}^n)\big) \cap L^2\big([0,T], H^2(\mathbb{R}^n)\big). \tag{18}$$

In [16, 19, 21] Del Santo and Prizzi proved that one can weaken the Lipschitz-regularity at least in the time variable to an Osgood modulus of continuity.

Theorem 3.4 (Theorem 1 in [19]) *Let μ be a Osgood modulus of continuity, suppose that the coefficients $a_{kl} \in C^\mu([0,T], \mathrm{Lip}(\mathbb{R}^n))$ and $b_m, c \in L^\infty([0,T]\times\mathbb{R}^n)$. Then $\mathcal{P}$ has the $\mathcal{H}$-uniqueness property, where $\mathcal{H}$ is defined by* (18).

Recently in [17] the uniqueness has been proved under a regularity in x which is below Lipschitz and the modulus of continuity in x is connected with the modulus of continuity in time.

2.3.2 Uniqueness Under Local Conditions

In this section we will prove uniqueness results for backward-parabolic and elliptic operators under a local condition as a counterpart to Sect. 2.2.2. We will not give all the details of the proofs. We perform some of the calculations for the backward-parabolic operators and the proofs for the elliptic case follow exactly the same lines. Furthermore, to be as close to the counterexamples as possible, we will state the theorems mainly for solutions which are periodic in x. We consider the operators

$$\begin{aligned}\mathcal{E}u = \partial_t^2 u + \sum_{k,l=1}^n \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}u\big) \\ + \beta(t,x)\partial_t u + \sum_{m=1}^n b_m(t,x)\partial_{x_m}u + c(t,x)u\end{aligned} \tag{19}$$

and

$$\mathcal{P}u = \partial_t u + \sum_{k,l=1}^n \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}u\big) + \sum_{m=1}^n b_m(t,x)\partial_{x_m}u + c(t,x)u \tag{20}$$

under the following assumptions:

(A1) For all $k, l = 1, \dots, n$ one has $a_{kl}(t, x) = a_{lk}(t, x)$.
(A2) There exist a constant a_0 such that

$$\sum_{k,l=1}^{n} a_{kl}(t, x)\frac{\xi_k \xi_l}{|\xi|^2} \geq a_0 > 0 \quad \forall (t, x, \xi) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}.$$

(A3) Let $a_{kl} = a_{kl}(t, x) \in C^0([0, T], L^\infty(\mathbb{R}^n)) \cap C^1((0, T], L^\infty(\mathbb{R}^n))$ and

$$\exists C \in (0, a_0) : \left| \sum_{k,l=1}^{n} \frac{\partial}{\partial t} a_{kl}(t, x)\frac{\xi_k \xi_l}{|\xi|^2} \right| \leq \frac{C}{t}$$
$$\forall (t, x, \xi) \in (0, T] \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}.$$

(A4) Let b_k, β and c belong to $L^\infty([0, T] \times \mathbb{R}^n, \mathbb{C})$ for all $k = 1, \dots, n$.

Remark 3.2 For the Cauchy problem for the elliptic operator $\mathcal{E}$ the constant C in (A3) can be chosen from the interval $(0, 2a_0)$. The conclusions of Theorem 3.5 and Theorem 3.6 are the same.

Remark 3.3 The restriction on the size of the constant in (A3) is unavoidable as long as uniqueness in C^∞ classes is concerned. A simple computation in the counterexample in Sect. 2.2.2 shows that the size is sharp.

Even if the uniqueness results will be stated for C^∞ solutions, the important property to be satisfied by the solutions under consideration is that for all $N \in \mathbb{N}$ it holds that $\lim_{t\to 0+} t^{-N}|u(t, x)| = 0$ for all $x \in \mathbb{T}^n$ (or $\mathbb{R}^n$) or similar conditions expressed in an integral form.

We set

$$\mathcal{H}_{per} := \left\{ u \in C^\infty\big([0, T], C^\infty(\mathbb{T}^n)\big) : \forall N \in \mathbb{N} : \lim_{t\to 0+} t^{-N}\big|u(t, x)\big| = 0 \;\forall x \in \mathbb{T}^n \right\},$$

where $\mathbb{T}^n$ denotes the n dimensional torus $[0, 2\pi]^n$. Furthermore, we define

$$\mathcal{H} := \left\{ u \in C^\infty\big([0, T], C^\infty(\mathbb{R}^n)\big) : \forall N \in \mathbb{N} : \lim_{t\to 0+} t^{-N}\big|u(t, x)\big| = 0 \;\forall x \in \mathbb{R}^n \right\}.$$

With this preparations we state

Theorem 3.5 (Periodic case) *Let $\mathcal{P}$ be the operator defined by* (20), *assume* (A1)–(A4) *and, moreover, assume that the coefficients a_{kl} are periodic in x. Then $\mathcal{P}$ has the $\mathcal{H}_{per}$-uniqueness property.*

Theorem 3.6 (Non-periodic case) *Let $\mathcal{P}$ be the operator defined by* (20) *and assume* (A1)–(A4). *Then $\mathcal{P}$ has the $\mathcal{H}$-compact uniqueness property, i.e. if $u \in \mathcal{H}$,*

$\operatorname{supp}(u) \subseteq \{t \geq 0\}$, $\operatorname{supp}(u) \cap (\{0\} \times \mathbb{R}^n) = \{(0,0)\}$ *and* $\mathcal{P}u = 0$ *on* $[0,T] \times \mathbb{R}^n$, *then* $u \equiv 0$ *on* $[0,T] \times \mathbb{R}^n$.

Both theorems follow from an appropriate Carleman estimate. The arguments are quite standard and we refer the reader to [45] and [30] for more details. We state the Carleman estimate only for the periodic case. The changes for the general case are easy.

Theorem 3.7 *Suppose the assumptions* (A1)–(A3) *and that the coefficients* a_{kl} *are periodic in* x. *Then there exist positive constants* $C, \gamma_0 > 0$ *and* $\sigma \in (0, \frac{1}{2})$ *such that*

$$\int_0^{T/2} t^{2(\sigma-\gamma)} \left\| \partial_t u + \sum_{k,l=1}^n \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l} u\big) \right\|^2_{L^2(\mathbb{T}^n)} dt$$
$$\geq C\left(\gamma \int_0^{T/2} t^{2(\sigma-\gamma-1)} \|u\|^2_{L^2(\mathbb{T}^n)} dt + \sum_{i=1}^n \int_0^{T/2} t^{2(\sigma-\gamma-1/2)} \|\partial_{x_i} u\|^2_{L^2(\mathbb{T}^n)} dt\right) \tag{21}$$

holds for all $u \in \mathcal{H}_{per}$ *with* $\operatorname{supp}(u) \subseteq [0, T/2] \times \mathbb{R}^n$ *and for all* $\gamma \geq \gamma_0$.

Proof To prove the Carleman estimate we put $u(t,x) = t^\gamma v(t,x)$ and we obtain

$$u_t(t,x) = \gamma t^{\gamma-1} v(t,x) + t^\gamma v_t(t,x).$$

This leads to the new equation

$$\mathcal{P}_\gamma v = v_t + \sum_{k,l=1}^n \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l} v\big) + \gamma t^{-1} v. \tag{22}$$

To be able to control some sign during our calculations we need to introduce an auxiliary weight. We multiply the operator $\mathcal{P}_\gamma$ by t^σ, where $\sigma > 0$ will be specified later, and take L^2-norms. We get

$$\int_0^{T/2} t^{2\sigma} \|\mathcal{P}_\gamma v\|^2_{L^2(\mathbb{T}^n)} dt = \int_0^{T/2} t^{2\sigma} \left\| \sum_{k,l=1}^n \partial_{x_i}\big(a_{kl}(t,x)\partial_{x_l} v\big) + \gamma t^{-1} v \right\|^2_{L^2(\mathbb{T}^n)} dt$$
$$+ 2\operatorname{Re}\int_0^{T/2} \left\langle v_t \middle| t^{2\sigma} \sum_{k,l=1}^n \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l} v\big) \right\rangle_{L^2(\mathbb{T}^n)} dt$$
$$+ 2\operatorname{Re}\int_0^{T/2} \big\langle v_t \big| t^{2\sigma-1} v \big\rangle_{L^2(\mathbb{T}^n)}.$$

By integration by parts we obtain

$$2\,\mathrm{Re}\int_0^{T/2}\left\langle v_t | t^{2\sigma-1} v\right\rangle_{L^2(\mathbb{T}^n)} = \gamma(1-2\sigma)\int_0^{T/2} t^{2(\gamma-\sigma)}\|v\|^2_{L^2(\mathbb{T}^n)}dt.$$

Here, in order to ensure the positivity of this term, we put, for an $\varepsilon > 0$, $\sigma := \frac{1}{2} - \varepsilon > 0$. Again by integration by parts we obtain

$$\begin{aligned} 2\,\mathrm{Re}\int_0^{T/2}&\left\langle v_t | t^{2\sigma}\sum_{k,l=1}^n \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}v\big)\right\rangle_{L^2(\mathbb{T}^n)} dt \\ &= \sum_{k,l=1}^n\int_0^{T/2} t^{2\sigma-1}\big\langle \partial_{x_k}v | \big(2\sigma a_{kl}(t,x) + t\partial_t a_{kl}(t,x)\big)\partial_{x_l}v\big\rangle_{L^2(\mathbb{T}^n)}. \end{aligned} \tag{23}$$

To get this integral we approximate the left hand side of (23) by

$$2\,\mathrm{Re}\int_\delta^{T/2}\left\langle v_t | t^{2\sigma}\sum_{k,l=1}^n \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l}v\big)\right\rangle_{L^2(\mathbb{T}^n)} dt, \tag{24}$$

integrate by parts and take the limit $\delta \to 0+$. The boundary terms vanish since $v \in \mathcal{H}_{per}$ and $\mathrm{supp}(v) \subseteq [0,T]\times\mathbb{R}^n$. Using (A2), (A3) and choosing ε small enough there exist a $C > 0$ such that

$$\begin{aligned} \sum_{k,l=1}^n\int_0^{T/2} &t^{2\sigma-1}\big\langle \partial_{x_k}v | \big(2\sigma a_{kl}(t,x) + t\partial_t a_{kl}(t,x)\big)\partial_{x_l}v\big\rangle_{L^2(\mathbb{T}^n)} \\ &\geq C\sum_{k,l=1}^n\int_0^{T/2} t^{2\sigma-1}\langle \partial_{x_k}v | \partial_{x_l}v\rangle_{L^2(\mathbb{T}^n)}dt. \end{aligned}$$

From that we get

$$\begin{aligned} \int_0^{T/2} t^{2\sigma}\|\mathcal{P}_\gamma v\|^2_{L^2(\mathbb{T}^n)}dt \geq \gamma(1-2\sigma)&\int_0^{T/2} t^{2(\gamma-\sigma)}\|v\|^2_{L^2(\mathbb{T}^n)}dt \\ &+ C\sum_{k,l=1}^n\int_0^{T/2} t^{2\sigma-1}\langle \partial_{x_k}v | \partial_{x_l}v\rangle_{L^2(\mathbb{T}^n)}dt \end{aligned}$$

from which we, going back to u, reach estimate (21). □

For elliptic operators $\mathcal{E}$ the Carleman estimate is essentially the same. One gets just one term more to control also lower order derivatives in t. The precise statement is

Theorem 3.8 *Suppose assumptions* (A1)–(A3). *Then there exist positive constants* $C, \gamma_0 > 0$ *and* $\sigma \in (0, 1)$ *such that*

$$\int_0^{T/2} t^{2(\sigma-\gamma)} \left\| \partial_t^2 u + \sum_{k,l=1}^n \partial_{x_k} \big(a_{kl}(t,x) \partial_{x_l} u \big) \right\|_{L^2(\mathbb{T}^n)}$$
$$\geq C\gamma \Bigg(\sum_{i=1}^n \int_0^{T/2} t^{2(\sigma-\gamma-1)} \|\partial_{x_i} u\|^2_{L^2(\mathbb{T}^n)} dt + \int_0^{T/2} \gamma t^{2(\sigma-\gamma-1)} \|\partial_t u\|^2_{L^2(\mathbb{T}^n)} dt$$
$$+ \int_0^{T/2} \gamma^2 t^{2(\sigma-\gamma-1)} \|u\|^2_{L^2(\mathbb{T}^n)} dt \Bigg)$$

holds for all $u \in \mathcal{H}_{per}$ *with* $\operatorname{supp}(u) \subseteq [0, T/2] \times \mathbb{R}^n$ *and all* $\gamma \geq \gamma_0$.

As already remarked one cannot weaken the assumption on the constant in oscillation control condition (A3). To use a bigger constant there we have to shrink the space (with respect to t) in which we are proving uniqueness. It turns out that Gevrey spaces provide us the possibility to weaken condition (A3) with respect to the constant C. First, we introduce the Gevrey spaces under consideration.

For all $s \geq 1$ we define $\gamma_t^{(s)}$ to be the space of all $C^\infty((-\infty, T], C^\infty(\mathbb{R}^n))$-functions u with $\operatorname{supp}(u) \subseteq [0, T] \times \mathbb{R}^n$ which are in the Gevrey class of index s with respect to t, uniformly in x. This means $u \in \gamma_t^{(s)}$ if and only if for all compact subsets $K \subseteq (-\infty, T] \times \mathbb{R}^n$ and for all multi-indices $\alpha \in \mathbb{N}_0^n$, there exist positive constants $C = C(u, \alpha, K)$ and $M = M(u, \alpha, K)$ such that, for all $k \in \mathbb{N}_0$

$$\sup_{(t,x)\in K} \left| \partial_x^\alpha \partial_t^k u(t,x) \right| \leq C M^k (k!)^s$$

holds true.

With this we define the spaces

$$\mathcal{H}^{(s)}_{per} := \gamma_t^{(s)} \cap C^0\big((-\infty, T], C^\infty\big(\mathbb{T}^n\big)\big)$$

and

$$\mathcal{H}^{(s)} := \gamma_t^{(s)}$$

for which we are going to state the uniqueness theorems.

With this preparations we can state our new local condition

(A3$_\alpha$) Let $a_{kl} = a_{kl}(t,x) \in C^0([0,T], L^\infty(\mathbb{R}^n)) \cap C^1((0,T], L^\infty(\mathbb{R}^n))$, $\alpha > 0$ and

$$\exists C \in \big(0, (1+2\alpha)a_0\big) : \left| \sum_{k,l=1}^n \frac{\partial}{\partial t} a_{kl}(t,x) \frac{\xi_k \xi_l}{|\xi|^2} \right| \leq \frac{C}{t}.$$

for all $(t, x, \xi) \in (0, T] \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$.

Theorem 3.9 (Periodic case) *Let $\mathcal{P}$ be the operator defined by* (20), *assume* (A1), (A2), (A3$_\alpha$), (A4) *with an $\alpha > 0$ and, moreover, that the coefficients a_{kl} are periodic in x. Then $\mathcal{P}$ has the $\mathcal{H}^{(s)}_{per}$-uniqueness property for $s < 1 + \frac{1}{\alpha}$.*

It also holds

Theorem 3.10 (Non-periodic case) *Let $\mathcal{P}$ be the operator defined by* (20), *assume* (A1), (A2), (A3$_\alpha$), (A4) *with an $\alpha > 0$. Then $\mathcal{P}$ has the $\mathcal{H}^{(s)}$-compact uniqueness property for $s < 1 + \frac{1}{\alpha}$, i.e. if $u \in \mathcal{H}^{(s)}$, $\mathrm{supp}(u) \subseteq [0, T] \times \mathbb{R}^n$, $\mathrm{supp}(u) \cap (\{0\} \times \mathbb{R}^n) = \{(0, 0)\}$ and $\mathcal{P}u = 0$ on $[0, T] \times \mathbb{R}^n$, then $u \equiv 0$ on $[0, T] \times \mathbb{R}^n$.*

Both theorems follow again from a suitable Carleman estimate and as in the former case we state the estimate just for the periodic case. To obtain a suitable Carleman estimate for our uniqueness result in the Gevrey frame we need to use a weight function which is connected with the way of going to zero for such functions. From Lemma 2 of [33] we know, that we can write every $u \in \mathcal{H}^{(s)}_{per}$ as a product of a function $v \in C_0^\infty((-\infty, T], C^\infty(\mathbb{T}^n))$ with $\mathrm{supp}(u) \subseteq [0, T] \times \mathbb{R}^n$ and the function $\exp(-\gamma t^{-\alpha})$, where s and α satisfy the relation $s < 1 + \frac{1}{\alpha}$.

With this we can state

Theorem 3.11 *Suppose assumptions* (A1), (A2), (A3$_\alpha$) *with an $\alpha > 0$ and assume, moreover, that the coefficients a_{kl} are periodic in x. Then there exist constants C, γ_0 and $\sigma \in (0, \frac{1}{2}(\alpha + 1))$ such that*

$$\begin{aligned}
&\int_0^{T/2} t^{2\sigma} e^{2\gamma t^{-\alpha}} \left\| \partial_t u + \sum_{k,l=1}^n \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l} u\big) \right\|^2_{L^2(\mathbb{T}^n)} dt \\
&\quad \geq C\left(\gamma \int_0^{T/2} t^{2\sigma-\alpha-2} e^{2\gamma t^{-\alpha}} \|w\|^2_{L^2(\mathbb{T}^n)} dt \right. \\
&\qquad \left. + \sum_{m=1}^n \int_0^{T/2} t^{2\sigma} e^{2\gamma t^{-\alpha}} \|\partial_{x_m} u\|^2_{L^2(\mathbb{T}^n)} dt \right)
\end{aligned}$$

holds for all $\gamma \geq \gamma_0$ and $u \in \mathcal{H}^{(s)}_{per}$ with $\mathrm{supp}(u) \subseteq [0, T/2] \times \mathbb{R}^n$ and $s < 1 + \frac{1}{\alpha}$.

In the elliptic case the same result as stated in Theorems 3.9 and 3.10 hold true. The constant C in the oscillation condition (A3$_\alpha$) can be chosen from the interval $(0, 2(1 + \alpha)a_0)$. The Carleman estimate has again one term more and we state it for the sake of completeness.

Theorem 3.12 *Suppose assumptions* (A1), (A2), (A3$_\alpha$) *with an $\alpha > 0$ and that the principal part coefficients a_{kl} are periodic in x. Then there exist constants C, $\gamma_0 > 0$*

and $\sigma \in (0, \frac{3}{2}(\alpha+1))$ *such that*

$$\int_0^{T/2} t^{2\sigma} e^{2\gamma t^{-\alpha}} \left\| \partial_t^2 u + \sum_{k,l=1}^n \partial_{x_k}\big(a_{kl}(t,x)\partial_{x_l} u\big) \right\|^2_{L^2(\mathbb{T}^n)} dt$$
$$\geq C\Bigg(\gamma^3 \int_0^{T/2} t^{2\sigma-3\alpha-4} e^{2\gamma t^{-\alpha}} \|u\|^2_{L^2(\mathbb{T}^n)} dt$$
$$+ \gamma^2 \int_0^{T/2} t^{2\sigma-2\alpha-2} e^{2\gamma t^{-\alpha}} \|\partial_t u\|^2_{L^2(\mathbb{T}^n)} dt$$
$$+ \gamma \sum_{i=1}^n \int_0^{T/2} t^{2\sigma-\alpha-1} e^{2\gamma t^{-\alpha}} \|\partial_{x_i} u\|^2_{L^2(\mathbb{T}^n)} dt \Bigg)$$

holds for all $\gamma \geq \gamma_0$ *and* $u \in \mathcal{H}^{(s)}$ *with* $\operatorname{supp}(u) \subseteq [0, T/2] \times \mathbb{R}^n$ *and* $s < 1 + \frac{1}{\alpha}$.

2.3.3 Degenerate Operators and Local Conditions

In this section we would like to complement the results of the last two sections with some results about degenerate elliptic and backward-parabolic operators. Since, global regularity does in general not help to ensure uniqueness, which may fail even for C^∞ coefficients (see [11, 14]) we ask about the situation for local conditions.

This section gives an overview about results which seem to be new in the literature, as far as backward-parabolic operators are concerned. The proofs follow very closely the lines of the corresponding results for elliptic operators in [7, 11, 36] and therefore we omit them here. We refer also to [30]. We suppose the assumptions (A1) and (A4) and we replace condition (A2) by

(A2′) For the principal part coefficients of $\mathcal{E}$ and $\mathcal{P}$ holds

$$\sum_{k,l=1}^n a_{kl}(t,x)\xi_k\xi_l \geq 0 \quad \forall \xi \in \mathbb{R}^n.$$

In [36] Nirenberg proved compact uniqueness for C^2-solutions of degenerate elliptic operators whose coefficients satisfy the Oleinik condition from [37]. Similar to his approach one can prove compact uniqueness for $C^{1,2}_{t,x}$-solutions of degenerate backward-parabolic operators:

Theorem 3.13 *Suppose assumptions* (A1), (A2′) *and that there exist* $C', C > 0$ *such that*

$$\sum_{k,l=1}^n \big(C' a_{kl}(t,x) + \partial_t a_{kl}(t,x)\big)\xi_k\xi_l \geq C \left| \sum_{m=1}^n b_m(t,x)\xi_m \right|^2$$

holds. Then the operator $\mathcal{P}$ has the $\mathcal{H}$-compact uniqueness property.

Inspired by the work of Nirenberg, Colombini and Del Santo have investigated this kind of condition further and have proved several (compact) uniqueness theorems for C^∞ and Gevrey solutions of the Cauchy problem for degenerate elliptic operators. As already mentioned we want to state similar results for backward-parabolic operators.

Theorem 3.14 *Suppose there exist an $\varepsilon > 0$ and a $C > 0$ such that*

$$\sum_{k,l=1}^{n} \big((1-\varepsilon)a_{kl}(t,x) + t\partial_t a_{kl}(t,x)\big)\xi_k\xi_l \geq Ct^2 \left|\sum_{m=1}^{n} b_m(t,x)\xi_m\right|^2$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^n$. Then the backward-parabolic operator $\mathcal{P}$ has the $C^{1,2}_{t,x}$-compact uniqueness property.

Using Gevrey solutions the last condition can we weakened, analogue to Theorem 3.10.

Theorem 3.15 *Let $s > 1$. Suppose there exist an $\varepsilon > 0$ and $C > 0$ such that*

$$\sum_{k,l=1}^{n} \left(\left(\frac{s}{1-s} - \varepsilon\right)a_{kl}(t,x) + t\partial_t a_{kl}(t,x)\right)\xi_k\xi_l \geq Ct^{2+s/(s-1)} \left|\sum_{m=1}^{n} b_m(t,x)\xi_m\right|^2$$

for all $(t,x) \in [0,T] \times \mathbb{R}^n$ and for all $\xi \in \mathbb{R}^n$. Then the backward-parabolic operator $\mathcal{P}$ has the $\mathcal{H}^{(s)}$-compact uniqueness property.

2.3.4 Open Problems and Further Developments

Here we would like to sketch briefly some open questions and possible further developments. Unfortunately we have no counterpart to Theorem 2.4. We expect that one can prove uniqueness in this situation if one chooses a constant in the oscillation control condition which is sufficiently small. Such a result would be analogous (of course only concerned about uniqueness) to those in [32] and [22].

Conjecture 2.3.1 *Suppose that the principal part coefficients of $\mathcal{E}$ or $\mathcal{P}$ are in $C^\mu([0,T],\mathbb{R}) \cap C^1((0,T],\mathbb{R})$ with a non-Osgood modulus of continuity μ. Furthermore, we suppose*

$$\left|\sum_{k,l=1}^{n} \frac{\partial}{\partial t}a_{kl}(t)\frac{\xi_k\xi_l}{|\xi|^2}\right| \leq C\frac{\mu(\eta^{-1}(t))}{\eta^{-1}(t)} \quad \forall (t,x,\xi) \in (0,T] \times \mathbb{R}^n \times \mathbb{R}^n \setminus \{0\}$$

holds for a sufficiently small $C > 0$, *where* $\eta(t) := \int_0^t \frac{ds}{\mu(s)}$. *Then the operators* $\mathcal{P}$ *and* $\mathcal{E}$ *have the* $\mathcal{H}$*-compact uniqueness property.*

If this can be proved one can expect a similar result for degenerate operators under a condition like

$$\sum_{k,l=1}^{n} \left(C a_{kl}(t) + \frac{\eta^{-1}(t)}{\mu(\eta^{-1}(t))} \partial_t a_{kl}(t) \right) \xi_k \xi_l \geq K f_\mu(t) \left| \sum_{m=1}^{n} b_m(t,x) \xi_m \right|^2$$

for a sufficiently small constant $C > 0$. In both cases one might also consider x-dependent coefficients.

It is clear that then similar improvements as described before can be expected if one considers Gevrey classes.

2.4 Continuous Dependence for Backward-Parabolic Operators

Since the Cauchy problem for elliptic and backward-parabolic operators is severely ill-posed one cannot expect the usual stability properties. However, for applications it is important to have some quantitative information about the nature of the dependency of solution on the Cauchy data. In his celebrated paper *Continuous dependence on data for solutions of partial differential equations with a prescribed bound* [31] John attempted this problem and introduced the notion of a well-behaved problem. In the notion of John a problem is well-behaved if *only a fixed percentage of the significant digits need be lost in determining the solution from the data*. To be a little bit more precise, that means that the solution in a space $\mathcal{H}$ depends Hölder continuously on the data in some space $\mathcal{K}$, provided they satisfy a *prescribed bound*.

In [1] Agmon and Nirenberg proved well-behavedness of the Cauchy problem for $\mathcal{P}$ in the space

$$\mathcal{H} := C^0\big([0,T], L^2(\mathbb{R}^n)\big) \cap C^0\big([0,T), H^1(\mathbb{R}^n)\big) \cap C^1\big([0,T), L^2(\mathbb{R}^n)\big)$$

with data in $L^2(\mathbb{R}^n)$. They assumed the coefficients to be Lip with respect to t and $L^\infty(\mathbb{R}^n)$ with respect to x. At about the same time Glagoleva obtained in [24] almost the same result by a different technique and time-independent coefficients. In [29] Hurd developed the technique of Glagoleva further to cover also the case where the coefficients also depend Lipschitz continuously on time. This result has been partially improved by Del Santo and Prizzi in [20]. They considered the operator $\mathcal{P}$ with coefficients depending Log-Lipschitz continuously on time. But, due to some technical difficulties arising from a commutator estimate, they had to require C^2-regularity in x. The result they got can be summarized as follows: *For every* $T' \in (0,T)$ *and* $D > 0$ *there exist* $M, N, \rho > 0$ *and* $\delta \in (0,1)$ *such that if* $u \in \mathcal{H}$ *is a solution of* $\mathcal{P}u = 0$ *on* $[0,T] \times \mathbb{R}^n$ *with* $\|u(0,\cdot)\|_{L^2(\mathbb{R}^n)} \leq \rho$ *and* $\|u(t,\cdot)\|_{L^2(\mathbb{R}^n)} \leq D$

on $[0, T]$*, then*

$$\sup_{t\in[0,T']} \left\|u(t,\cdot)\right\|^2_{L^2(\mathbb{R}^n)} \leq M \exp\left(-N\left|\log\left(\left\|u(0,\cdot)\right\|_{L^2(\mathbb{R}^n)}\right)\right|^\delta\right).$$

As one sees this dependence is weaker than Hölder continuous dependence. A counterexample in [20] shows that this result is sharp in the sense that one can in general not expect Hölder continuous dependence if the coefficients depend only Log-Lipschitz continuously on time.

The C^2 regularity with respect to x has recently been removed by use of Bony's paraproduct and could be replaced by Lipschitz continuity which is more natural in this context (see [18]).

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