Introduction

Any smooth manifold can be equipped with a smooth Riemannian metric, that is, a smoothly varying inner product on each tangent space. A Riemannian metric allows us to do geometry on the manifold, and in particular, it endows the manifold with ‘shape’. Any smooth manifold will admit a huge family of Riemannian metrics, and these metrics can display diverse geometric properties. Since each Riemannian manifold (that is, a smooth manifold equipped with a choice of Riemannian metric) has a notion of shape, it makes sense to try and describe the curvature of such an object.

There are three main ways in which the curvature is usually measured. First, there is the sectional curvature, which is a natural generalization to higher dimensions of the notion of Gaussian curvature for surfaces. The Ricci curvature is an average of sectional curvatures, and the scalar curvature can be viewed as an average of Ricci curvatures. Thus the scalar curvature is a weaker notion of curvature than the Ricci curvature, and in turn, the Ricci curvature is a weaker notion than the sectional curvature. These higher dimensional measures of curvature are more difficult to interpret than the Gaussian curvature. For example the sectional curvature controls distance in some sense, whereas the Ricci curvature is closely related to volume. The scalar curvature controls volume only locally.

A fundamental problem in Riemannian geometry is to understand which manifolds admit metrics displaying certain types of curvature characteristics. Of particular importance amongst the possible curvature characteristics are the various sign-based conditions, for example negative sectional curvature, positive Ricci curvature and so on. Existence issues for positive scalar curvature metrics are reasonably well understood. The situation for positive Ricci and positive (or non-negative) sectional curvature metrics is somewhat less clear. The theory of manifolds with negative sectional curvature is well-developed, however the existence question is far from resolved.

For the most part this existence question has been a primary focus of research. However, there is an equally intruiging secondary question. If a manifold admits a given type of metric, how are such metrics distributed among all possible Riemannian metrics on this object? For example are they rare or common?

A particularly natural question concerns connectedness: given two metrics of a given type on a particular manifold, is it possible to continuously deform one metric into the other *through metrics of the same type,* assuming the space of metrics has been equipped with a suitable topology. If the answer to this question is always yes, this shows that the set of metrics having the given type forms a connected subset of the space of all metrics. Thus it makes sense to ask about the connectedness of the subset of ‘nice’ metrics in whatever context one is working. If this set is not connected, how many connected components are there? Of course, one could also seek to uncover more subtle aspects of the topology.

Instead of working with spaces of metrics, we might instead choose to work with *moduli spaces* of metrics. For any smooth manifold M, we can consider the group of all self-diffeomorphisms of M, $\mathrm{Diff}(M)$. Let $\mathcal{R}(M)$ denote the space of all Riemannian metrics on M equipped with a suitable topology. Then $\mathrm{Diff}(M)$ acts on $\mathcal{R}(M)$ by pulling back metrics. The quotient of $\mathcal{R}(M)$ by this action is a moduli space of metrics. One can similarly form moduli spaces of metrics satisfying the various curvature conditions, as these conditions are invariant under the action of $\mathrm{Diff}(M)$. Just as for spaces of metrics, one can pose questions about the topology of these restricted curvature moduli spaces.

Our aim in writing this book is to bring together the key ideas from both recent and classical results on the topology of (moduli) spaces of Riemanian metrics. To the best of our knowledge, this is the first volume to attempt to do so. We feel that this is a timely exercise, as the field has been developing rapidly in recent years, and this looks set to continue into the future.

The book is laid out as follows. In §1 we begin with a brief introduction to the possible topologies on spaces of Riemannian metrics. We will also see that the space of Riemannian metrics can be given the structure of an infinite dimensional manifold. An object of central importance in this book is the s-invariant of Kreck and Stolz. We will meet this first in §5, however a substantial amount of background material is required before being able to present it. This background includes topics such as spin geometry, the Dirac operator and index theory, and we will develop this in §2 and §3. At the end of §3 we will consider index theory obstructions to positive scalar curvature metrics. This allows us in §4 to look at some classical results about spaces of positive scalar curvature metrics due to Hitchin and Carr, (as well as some very recent results in a similar vein). In §6 we look at applications of the Kreck-Stolz s-invariant to moduli spaces of metrics with positive scalar curvature, positive Ricci curvature and both non-negative and positive sectional curvature. In §7 we look at the 'observer moduli space', which is an alternative kind of moduli space for Riemannian metrics which offers certain advantages over the traditional notion. We then provide in §8 a survey of other results on (moduli) spaces of metrics featuring, for the most part, some form of positive curvature. In §9 we turn our attention to (moduli) spaces of metrics on compact manifolds with negative sectional curvature. In §10 we look at non-compact manifolds with complete metrics of non-negative sectional curvature. Here, results about the moduli space of such metrics are established by an analysis of their 'souls', which is a technique quite different in nature to those described for compact manifolds elsewhere in the book. Finally, in §11 we take a brief look at the Klingenberg-Sakai Conjecture, and its implications for spaces of positively pinched metrics. There are two appendices: Appendix A on K-theory, and Appendix B on the Atiyah-Patodi-Singer index theorem.

Note that we only consider (moduli) spaces of Riemannian metrics under sign-based curvature conditions: we do not explore other types of moduli space, such as moduli spaces of Einstein metrics.

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