2. Clifford algebras and spin

The aim of this chapter is to develop the concept of a spin group and certain related ideas such as spin structures, spinor representations and spinor bundles. These all play a crucial role in positive scalar curvature geometry, which we will explore in subsequent sections. The importance of these concepts for our purposes is due to the fact that 'spin geometry' provides the setting in which we can define and analyze a certain first order linear differential operator called the Dirac operator. It is this operator which turns out to be intimately related to the scalar curvature. As the name suggests, the Dirac operator arose from work of the physicist Paul Dirac. Leaving the physics on one side, we can motivate the mathematics behind this operator by posing the following question: can we find a first order differential operator D whose square is equal to the Laplacian? In $\mathbb{R}^3$ for example, this amounts to solving the equation

$$D^2 = (\alpha\partial/\partial x + \beta\partial/\partial y + \gamma\partial/\partial z)^2 = -(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2).$$

Thus we are required to find coefficient functions α, β and γ satisfying the relations

$$\alpha^2 = \beta^2 = \gamma^2 = -1;$$

$$\alpha\beta = -\beta\alpha; \quad \alpha\gamma = -\gamma\alpha; \quad \beta\gamma = -\gamma\beta.$$

A moment's thought shows that these relations cannot be satisfied by scalar functions. On the other hand they *can* be solved if we allow the coefficients to be *matrices,* and working within certain matrix algebras turns out to be the right setting in which to understand this problem. These matrix algebras are called 'Clifford algebras', and we will begin below by defining these. (Note that it is not obvious from the definition that Clifford algebras are matrix algebras.) The concepts of spin groups and spin geometry will then emerge naturally from Clifford algebras.

The material in this chapter (and the next) is presented in detail in the book [LM]. We therefore present only an outline of the key ideas and results, referring the interested reader to sections I and II of [LM] for the technical details.

§2.1 *Clifford algebras*

The real Clifford algebra Cl_n is formed as follows. Consider the tensor algebra

$$T_n := \mathbb{R} \oplus \mathbb{R}^n \oplus (\mathbb{R}^n \otimes \mathbb{R}^n) \oplus (\mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n) \oplus (\mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n \otimes \mathbb{R}^n) \oplus ....$$

Thus elements of T_n are sums of tensor products of elements in $\mathbb{R}^n$.

Within this algebra is an ideal I_n generated by elements of the form $v \otimes v + |v|^2.1$ for $v \in \mathbb{R}^n$. The Clifford algebra Cl_n is given by $Cl_n := T_n/I_n$. This means that the elements of Cl_n are essentially sums of products of elements in $\mathbb{R}^n$ subject to the relation that $v^2 = -|v|^2$ for all $v \in \mathbb{R}^n$, where we have suppressed the tensor product symbol.

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Applying this relation to a vector $v+w$ we see that $vw + wv = |v|^2 + |w|^2 - |v+w|^2$. In particular for orthonormal vectors α, β we have $\alpha^2 = \beta^2 = -1$, $\alpha\beta = -\beta\alpha$. Notice that these are precisely the sort of relations which arise in the 'Dirac problem' above, which justifies our claim that Clifford algebras are the right setting in which to view that problem.

Clearly, like the tensor algebra, the Clifford algebra is a *graded* algebra, graded by lengths of products of vectors. Moreover there is an obvious analogy with the exterior algebra $\Lambda^*\mathbb{R}^n$, which is defined as T_n/E_n where E_n is the ideal generated by elements of the form $v \otimes v$, i.e. so that $v^2 = 0$ for all $v \in \mathbb{R}^n$. Clearly the subspace $\Lambda^p\mathbb{R}^n$ corresponds to the products of length p in Cl_n, and we have an isomorphism $Cl_n \cong \Lambda^*\mathbb{R}^n$ of graded vector spaces - but not of algebras.

Note that we can define the complex Clifford algebra $\mathbb{C}l_n$ in the same way, just substituting complex scalars for real and using the complex inner product $\langle \sum z_i e_i, \sum w_j e_j \rangle = \sum z_i w_i$.

It is not difficult to see that Cl_n (respectively $\mathbb{C}l_n$) has the structure of a real (respectively complex) vector space of dimension 2^n. A basis is given by

$$\{1, e_1, ..., e_n\} \cup \{e_{i_i} \cdot e_{i_2} \cdot ... \cdot e_{i_k} \,|\, i_1 < i_2 < ... < i_k,\ 2 \leq k \leq n\}$$

where $\{e_i\}$ is the standard basis for $\mathbb{R}^n$ (or $\mathbb{C}^n$). This choice of basis determines a linear isomorphism with $\mathbb{R}^{2^n}$ (or $\mathbb{C}^{2^n}$), and using this we can introduce a topology onto the Clifford algebra which makes the isomorphism into a homeomorphism. It is easy to see that all operations in the Clifford algebra are continuous with respect to this topology, and thus Clifford algebras are topological algebras in a natural way.

It turns out that Clifford algebras - up to algebra isomorphism - are actually all familiar matrix algebras. Let $F(n)$ denote the algebra of $(n \times n)$-matrices over F, where F can be $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$. For small values of n we have the following identifications:

	1	2	3	4	5	6	7	8
Cl_n	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{H} \oplus \mathbb{H}$	$\mathbb{H}(2)$	$\mathbb{C}(4)$	$\mathbb{R}(8)$	$\mathbb{R}(8) \oplus \mathbb{R}(8)$	$\mathbb{R}(16)$
$\mathbb{C}l_n$	$\mathbb{C} \oplus \mathbb{C}$	$\mathbb{C}(2)$	$\mathbb{C}(2) \oplus \mathbb{C}(2)$	$\mathbb{C}(4)$	$\mathbb{C}(4) \oplus \mathbb{C}(4)$	$\mathbb{C}(8)$	$\mathbb{C}(8) \oplus \mathbb{C}(8)$	$\mathbb{C}(16)$

These isomorphisms are not obvious but give useful insights into the Clifford algebras, and in particular to *Clifford modules*. Note that the first few of these isomorphisms can be figured out by hand, and the rest inductively using certain periodicity relations: 8-periodic in the real case $Cl_{n+8} \cong Cl_n \otimes_{\mathbb{R}} Cl_2$, and 2-periodic in the complex case $\mathbb{C}l_{n+2} \cong \mathbb{C}l_n \otimes_{\mathbb{C}} \mathbb{C}l_2$. Using some basic isomorphisms of matrix groups in conjunction with the periodicity isomorphisms, it is not difficult to see that in all cases *the complex Clifford algebras are just complexifications of the real,* i.e. $\mathbb{C}l_n \cong Cl_n \otimes \mathbb{C}$. Therefore for many purposes it suffices to consider the real algebras, and just complexify where necessary.

A *Clifford module* is a vector space which admits a (left, say) linear action from some Clifford algebra: $Cl_n \times V \to V$. For an element $\sigma \in Cl_n$ and $v \in V$, forming a 'product' $\sigma \cdot v$ is called *Clifford multiplication.*

Every Clifford module is either irreducible or splits into a direct sum of irreducible submodules. Using the identification of Clifford algebras with matrix algebras above gives an easy way to describe the irreducible Clifford modules.

Theorem 2.1.1. *(1) If a Clifford algebra is isomorphic to a matrix algebra $F(n)$, then (up to equivalence) there is a unique irreducible Clifford module, namely F^n, with the canonical action of $F(n)$ on F^n given by left matrix multiplication. (2) If a Clifford algebra is isomorphic to a matrix algebra $F(n) \oplus F(n)$, then (up to equivalence) there are two irreducible Clifford modules, both F^n, but with the Clifford algebra action given by the canonical action of $F(n)$ on F^n by one of the factors in $F(n) \oplus F(n)$ with the other factor acting trivially.*

Notice in the table above that there are two irreducible Clifford modules in dimensions 3 mod 4 in the real case, and in every odd dimension in the complex case. In all other dimensions there is a unique irreducible Clifford module. *This is a general phenomenon which holds in all dimensions.* It will be of great significance later.

We next consider splittings of the Clifford algebra. There are two ways of doing this: a way which works for all Clifford algebras, and a more important phenomenon which only works in dimensions 0 mod 4 in the real case and in all even dimensions in the complex case.

Every Clifford algebra has a splitting into *even* and *odd* parts: $Cl_n = Cl_n^0 \oplus Cl_n^1$ (and similarly in the complex case). As a vector space Cl_n^0 is spanned by the even length products of vectors, and Cl_n^1 is spanned by the odd length products. It is easy to see that Cl_n^0 is a subalgebra, but Cl_n^1 is not; moreover $Cl_n^i \cdot Cl_n^j \subset Cl^{i+j \ mod \ 2}$. An alternative way of viewing this splitting is to consider the algebra isomorphism $\alpha : Cl_n \to Cl_n$ induced by the map $-\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$. It is easy to see that $\alpha^2 = \mathrm{id}$, and that Cl_n^0 and Cl_n^1 are respectively just the $+1$ and -1 eigenspaces of α.

For any n we have an algebra isomorphism $Cl_{n-1} \cong Cl_n^0$, which is induced by the linear map $\mathbb{R}^{n-1} \to Cl_n^0$ given by mapping $e_i \mapsto e_n \cdot e_i$ where $\{e_1, ..., e_n\}$ is the standard orthonormal basis for $\mathbb{R}^n$, and $\mathbb{R}^{n-1} = \mathrm{Span}\{e_1, ..., e_{n-1}\}$. In the complex case we have the analogous isomorphism $\mathbb{C}l_{n-1} \cong \mathbb{C}l_n^0$.

The second way of splitting the Clifford algebra involves the use of a volume element $\omega := e_1 \cdot ... \cdot e_n$. Note that this element is independent of the choice of (oriented) basis once an orientation for $\mathbb{R}^n$ has been fixed. For $\mathbb{C}l_n = Cl_n \otimes \mathbb{C}$ we introduce a complex volume element $\omega_C := i^{\lfloor (n+1)/2 \rfloor} e_1 \cdot ... \cdot e_n$. This means that the complex volume form is real in dimensions 0 and 3 mod 4. More specifically we have $\omega_C = -\omega$ in dimensions 3 and 4 mod 8, and $\omega_C = \omega$ in dimensions 7 and 0 mod 8.

These volume elements have the following properties:

$$\omega^2 = 1 \text{ if } n \equiv 0, 3 \text{ mod } 4;$$

$$e \cdot \omega = (-1)^{n-1} \omega \cdot e \text{ for } e \in \mathbb{R}^n;$$

$$\omega_C^2 = 1 \text{ for all } n;$$

$$e \cdot \omega_C = (-1)^{n-1}\omega_C \cdot e \text{ for all } n.$$

When the square of the volume element is 1, this shows that (left) multiplication by the volume elements splits the Clifford algebra in these dimensions into a sum of the ± 1-eigenspaces. We write $Cl_n = Cl_n^+ \oplus Cl_n^-$ in the real case, and similarly in the complex case. This is only a vector space splitting in general.

The fact that ω commutes with elements of $\mathbb{R}^n$ in dimensions 3 mod 4 in the real case shows that multiplication by ω is an algebra isomorphism (or a module isomorphism in the case of Clifford modules) in these dimensions. Thus the splitting into $Cl_n^\pm$ is a splitting into *subalgebras*, each of which are ideals in Cl_n. These subalgebras are isomorphic, as can be seen from the above table.

In dimensions 0 mod 4 notice that multiplication of $Cl_n^\pm$ by $e \in \mathbb{R}^n$ swaps the subspaces, i.e.

$$\mathbb{R}^n \times Cl_n^\pm \to Cl_n^\mp.$$

This phenomenon occurs for the splitting $Cl_n^0 \oplus Cl_n^1$ under multiplication by elements of Cl_n^1.

In the complex case we getting splittings in all dimensions. When n is odd ω_C is central, and the splitting is into isomorphic subalgebras, as illustrated in the table. When n is even we have a vector space splitting $\mathbb{C}l_n^\pm$ only, and this exhibits the same swapping behaviour under Clifford multiplication by elements of $\mathbb{C}^n$ as dimensions 0 mod 4 in the real case, namely

$$\mathbb{C}^n \times \mathbb{C}l_n^\pm \to \mathbb{C}l_n^\mp.$$

Note that if we compare the splittings $Cl_n = Cl_n^0 \oplus Cl_n^1$ and $Cl_n = Cl_n^+ \oplus Cl_n^-$ in the dimensions where the latter splittings occur, we find that the two splittings are 'diagonal' with respect to one another. To see this note that the map $\alpha : Cl_n \to Cl_n$ swaps the $Cl_n^\pm$, since n is odd and therefore $\alpha(\omega) = -\omega$. This means that for $\phi \in Cl_n^+$ we have $\omega\alpha(\phi) = -\alpha(\omega)\alpha(\phi) = -\alpha(\omega\phi) = -\alpha(\phi)$. Thus $\alpha(Cl_n^+) = Cl_n^-$, and vice versa. From this we see that $Cl_n^0 = \{\phi \oplus \alpha(\phi) \mid \phi \in Cl_n^+\} \subset Cl_n^+ \oplus Cl_n^-$. Similarly in the complex case.

The splitting of Clifford algebras has implications for Clifford modules. Firstly note that a Cl_n-module W is said to be $\mathbb{Z}_2$-graded if W admits a vector space splitting $W = W^0 \oplus W^1$ such that the Clifford multiplication satisfies $Cl^i \cdot W^j \subset W^{i+j}$, with the indices interpreted modulo 2. Notice that this makes both W^0 and W^1 modules for the even part Cl_n^0 (but not for the full algebra). Not every Clifford module will admit such a splitting: a pre-requisite is that W has a splitting into two Cl_{n-1}-modules (since $Cl_n^0 \cong Cl_{n-1}$). However there is natural equivalence between the categories of $\mathbb{Z}_2$-graded modules for Cl_n and ungraded modules for Cl_{n-1}. In one direction we just take W^0; in the other we form $Cl_n \otimes_{Cl_{n-1}} W$ for an ungraded Cl_{n-1}-module W, with the action of Cl_n being the usual left action on the first factor. This Cl_n-module splits as $(Cl_n^0 \otimes_{Cl_{n-1}} W) \oplus (Cl_n^1 \otimes_{Cl_{n-1}} W)$, giving a $\mathbb{Z}_2$-grading. These $\mathbb{Z}_2$-graded modules have a useful link with K-theory, which is outlined in Appendix A.

Now consider a module V for the real Clifford algebra Cl_n. When the Clifford algebra volume element ω satisfies $\omega^2 = 1$, i.e. when $n \equiv 0, 3 \bmod 4$, we get a splitting of V into the ± 1-eigenspaces for multiplication by ω: $V^{\pm} = (1 \pm \omega)/2 \cdot V$.

In the case $n \equiv 3 \bmod 4$ we have $V = V^+ \oplus V^-$, and any non-zero element $e \in \mathbb{R}^n$ has the effect (via Clifford multiplication) $e : V^{\pm} \to V^{\pm}$, as a consequence of the centrality of ω. Moreover both $V^{\pm}$ are still modules for Cl_n. In particular this means that if V is irreducible, then either $V = V^+$ or $V = V^-$, i.e. V must be just a single eigenspace. Thus there are precisely two inequivalent (consider the action of ω!) irreducible modules for Cl_n when $n \equiv 3 \bmod 4$, as indicated in the theorem above.

When $n \equiv 0 \bmod 4$ we get $e : V^+ \to V^-$ and $e : V^- \to V^+$. (We see this phenomenon when studying Dirac operators.) For this reason the $V^{\pm}$ are not modules for Cl_n, however since ω commutes with Cl_n^0 we see that they are in fact modules for $Cl_n^0 \cong Cl_{n-1}$. In the case that V is an irreducible Cl_n-module, the resulting Cl_{n-1} modules $V^{\pm}$ are precisely the two inequivalent irreducible modules discussed in the $n \equiv 3 \bmod 4$ case above. Notice also that for $n \equiv 0 \bmod 4$ the splitting $V^{\pm}$ gives V a $\mathbb{Z}_2$-graded structure. In fact there are two such structures, depending on whether we label V^+ as V^0 or V^1.

In the case of a module W for a complex Clifford algebra $\mathbb{C}l_n$, analogous observations apply: we obtain a splitting $W = W^+ \oplus W^-$ for all n, and when n is odd this is a splitting into submodules. For n even Clifford multiplication gives a map $\mathbb{C}^n \times W^{\pm} \to W^{\mp}$, where $W^{\pm}$ can be viewed as modules for the subalgebra $\mathbb{C}l_n^0 \cong \mathbb{C}l_{n-1}$. Similar comments about irreducibilty apply: for n odd and W irreducible, we have two possibilities for W depending on whether the complex volume element acts as $+1$ or -1. For n even we obtain a splitting of an irreducible module W into the two irreducible modules for $\mathbb{C}l_{n-1} \cong \mathbb{C}l_n$.

§2.2 *The Spin groups*

It is well-known that any element of $\mathrm{SO}(n)$ can be expressed as a product of an even number of reflections across hyperplanes. This is not difficult to see: by a change of basis any rotation matrix can be put into block diagonal form where the non-trivial diagonal blocks are all (2×2)-rotation matrices. The claim will then follow if we can show that any two-dimensional rotation can be expressed as a product of two reflections. An elementary calculation in plane geometry shows that a reflection through a line making an angle of θ with the positive x-axis followed by a reflection through a line making an angle ϕ is equivalent to an anticlockwise rotation through an angle $2(\theta - \phi)$. This also demonstrates the extent to which the representation of rotations by reflections - even in two dimensions - is non-unique.

Given a rotation in $\mathbb{R}^n$, express this as a product of reflections $r_1 .... r_m$. For each r_i we can find a unit vector $v_i \in \mathbb{R}^n$ orthogonal to the hyperplane of reflection (and therefore determining both the hyperplane and the reflection). Thus we could express the rotation by the string $v_1 .... v_m$. Notice that we could replace any of the v_i by $-v_i$ without changing the corresponding rotation.

Let us interpret the string $v_1 .... v_m$ as a product inside the Clifford algebra Cl_n. By the above observation, the element $-v_1 ... v_m$ also represents the same rotation (any minus signs can be carried to the front since we are now working in an algebra). Thus there are two expressions giving the same rotation. As noted above, the initial choice of reflections $r_1....r_m$ is not unique, however the remarkable thing is that the corresponding Clifford algebra representations will be equal (in the algebra) to either $v_1....v_m$ or $-v_1....v_m$. (This can be seen, for example, from the short exact sequence below.) *Thus the Clifford algebra is an appropriate setting in which to consider rotations.*

Consider the multiplicative subgroup of Cl_n generated by the collection of unit vectors in $\mathbb{R}^n$. (We need to keep things real here as we are investigating $\mathrm{SO}(n)$.) Within this subgroup we can consider the even length products of unit vectors. It is not difficult to see that this has the structure of a multiplicative group.

Definition 2.2.1. *For each $n \in \mathbb{N}$, the spin group Spin(n) is given by*

$$\mathit{Spin}(n) := \{v_1 \cdot \ldots \cdot v_r \mid v_i \in \mathbb{R}^n \text{ is a unit vector and } r \text{ is even}\}.$$

Since for any unit vector v we have $v \cdot v = -1$ and $(-v) \cdot v = +1$, we see that every spin group contains the scalars ± 1. (It is not difficult to see that these are the only pure scalars in $\mathrm{Spin}(n)$: the pure scalars must form a multiplicative subgroup of $\mathrm{Spin}(n)$, and so must be a multiplicative subgroup of the non-zero real numbers. Working with respect to an orthonormal basis for $\mathbb{R}^n$ we see that the scalar term of any Clifford product of unit vectors must lie in $[-1, 1]$, and the only multiplicative group within this interval is $\{\pm 1\}$.) Identifying $\{\pm 1\}$ with $\mathbb{Z}_2$, the correspondence between $\mathrm{Spin}(n)$ and $\mathrm{SO}(n)$ can be expressed via the following short exact sequence of groups

$$0 \to \mathbb{Z}_2 \to \mathrm{Spin}(n) \to \mathrm{SO}(n) \to 0.$$

Recall from §2.1 that the Clifford algebra is a topological space in a natural way. The induced topology then makes $\mathrm{Spin}(n)$ into a topological group, and with respect to this topology the map $\pi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ in the above exact sequence is a continuous homomorphism. Thus $\mathrm{Spin}(n)$ is a topological double cover of $\mathrm{SO}(n)$, and since $\pi_1(\mathrm{SO}(n)) \cong \mathbb{Z}_2$ for $n \geq 3$, we see that $\mathrm{Spin}(n)$ is *simply-connected* for $n \geq 3$. Moreover, as the universal cover of a manifold, we see that $\mathrm{Spin}(n)$ is also a manifold, and moreover a Lie group with respect to the smooth structure it inherits from $Cl_n \cong \mathbb{R}^{2^n}$ (or equivalently by pulling back the smooth structure from $\mathrm{SO}(n)$ via π).

In low dimensions note that there are certain group isomorphisms which express $\mathrm{Spin}(n)$ in terms of other Lie groups. We have:

$$\mathrm{Spin}(1) = \mathbb{Z}_2;$$

$$\mathrm{Spin}(2) = S^1;$$

$$\mathrm{Spin}(3) \cong \mathrm{SU}(2) \cong \mathrm{Sp}_1 \cong S^3;$$

$$\mathrm{Spin}(4) \cong \mathrm{Spin}(3) \times \mathrm{Spin}(3);$$

$$\mathrm{Spin}(5) \cong \mathrm{Sp}(2);$$

$$\mathrm{Spin}(6) \cong \mathrm{SU}(4).$$

Consider a fibre bundle E with fibre F and base B. If $\{U_\alpha\}$ is a covering of B by open balls, we can view E as constructed from products $U_\alpha \times F$ by gluing the fibres over overlap regions $U_\alpha \cap U_\beta$ using maps $\theta_{\alpha\beta} : U_\alpha \cap U_\beta \to G$, where $G \subset \mathrm{Diff}F$ is a fixed Lie group (the 'structure' group) and where the maps $\theta_{\alpha\beta}$ must satisfy the relation $\theta_{\alpha\beta}\theta_{\beta\gamma}\theta_{\gamma\alpha} = 1$. Thus

$$E = \Big(\coprod_\alpha U_\alpha \times F\Big)/\sim$$

where $(u_\alpha, f) \sim (u_\beta, f')$ if and only if $u_\alpha = u_\beta$ and $\theta_{\alpha\beta}(u_\alpha)(f) = f'$.

Given bundle gluing data as above, we can form a new bundle from the same data by replacing the fibre F by the group G, and taking the action of the structural group G on the fibre G to be the usual left multiplication. This creates the associated 'principal G-bundle' P_G. Notice that since we can multiply G by itself from both left and right, the bundle P_G admits a global right G-action. (The left action was 'used up' by the construction process.)

Given the principal G-bundle $G \to P_G \to B$ we can recover the original bundle (up to bundle equivalence) via the *associated bundle construction*. This involves quotienting the product $P_G \times F$ by the equivalence relation $\sim$ defined by $(p, f) \sim (pg^{-1}, g \cdot f)$ for any $g \in G$. Thus the original bundle is associated to the principal bundle P_G by the construction $(P_G \times F)/\sim$.

More generally, given a manifold X on which G acts, we can form an X-bundle associated to P_G in the same way, as $(P_G \times X)/\sim$, where $(p, x) \sim (pg^{-1}, g \cdot x)$. In this way we can create many new bundles sharing the 'same' gluing information.

Consider a manifold M^n. To say that M is orientable is equivalent to saying that the tangent bundle has structure group $\mathrm{SO}(n)$, or equivalently that TM has an associated principal $\mathrm{SO}(n)$-bundle $P_{SO(n)}$.

The manifold M is said to have a *spin structure* if it is orientable and the principal $\mathrm{SO}(n)$-bundle associated to the tangent bundle $P_{SO(n)} \to M$ can be lifted to a principal $\mathrm{Spin}(n)$-bundle $P_{Spin(n)} \to M$ via a map $\xi : P_{Spin(n)} \to P_{SO(n)}$, such that ξ restricts to a double covering map on each fibre, and which is equivariant in the sense that $\xi(pg) = \xi(p)\pi(g)$, where $p \in P_{Spin(n)}$, $g \in \mathrm{Spin}(n)$ and $\pi : \mathrm{Spin}(n) \to \mathrm{SO}(n)$ is the standard double covering homomorphism.

We might therefore think of the existence of a spin structure for M in terms of M satisfying an enhanced orientability condition.

We can characterise both the orientability and spin conditions topologically using the Stiefel-Whitney classes $w_1 \in H^1(M; \mathbb{Z}_2)$ and $w_2 \in H^2(M; \mathbb{Z}_2)$. These provide a useful criterion for checking the existence of spin structures. A manifold M is orientable if and only if $w_1 = 0 \in H^1(M; \mathbb{Z}_2)$, and M is spin if and only if both $w_1 = 0$ and $w_2 = 0$. (These facts are far from obvious. They can be established by studying cohomology exact sequences which arise from certain fibrations - see [LM; §II.1].)

Note that spin structures (that is principal Spin-bundles $\xi : P_{Spin(n)} \to P_{SO(n)}$ for given $P_{SO(n)}$) are not in general unique. In fact the number of spin structures turns out to be in one-to-one correspondence with the elements of $H^1(P_{SO(n)}; \mathbb{Z}_2)$ for which the restriction to the fibre of $P_{SO(n)}$ is non-zero. Assuming X is connected, the spin structures on a bundle E over X (not necessarily the tangent bundle) are indexed by $H^1(X;\mathbb{Z}_2)$. (See [LM; page 81] for details.)

There are other characterisations of spin structures. For example in dimensions at least 5, a manifold is spin if and only if every compact orientable embedded surface has trivial normal bundle. (It suffices to consider only embedded 2-spheres if the manifold is simply-connected.) For comparison, the manifold is orientable if the restriction of the normal bundle to any embedded circle is trivial (which is equivalent to the restriction of the tangent bundle being trivial since S^1 is parallelisable).

We can extend the notion of spin structure naturally to any orientable vector bundle by lifting (if possible) the corresponding principal SO(n)-bundle to a principal Spin(n)-bundle. Similar characterisations apply in terms of the vanishing of Stiefel-Whitney classes, and also in terms of pull-back bundles over embeddings of orientable surfaces in the base being trivial.

Many of the most obvious examples of manifolds admit spin structures. For example all spheres S^n admit spin structures. This is easy to see for $n \geq 3$ as then $H^1(S^n;\mathbb{Z}_2)$ and $H^2(S^n;\mathbb{Z}_2)$ are both zero. All Lie groups are spin. This can be seen from the fact that the tangent bundle of every Lie group is trivial, which forces the vanishing of all Stiefel-Whitney classes. Among the projective spaces $\mathbb{R}P^n$ is spin if and only if $n \equiv 3$ mod 4, $\mathbb{C}P^n$ is spin if and only if n is odd, and $\mathbb{H}P^n$ is spin for all n. Note that any product of spin manifolds is again spin, but this is not in general true for bundles. For example there is a unique non-trivial S^3-bundle over S^2 which is known to be non-spin, despite both S^2 and S^3 being spin.

§2.4 *Spinor bundles*

Consider the Clifford algebra $Cl_n = T_n/I_n$. If we transform $\mathbb{R}^n$ by an element of SO(n), this induces a natural transformation of Cl_n since it transforms the tensor algebra T_n whilst preserving the inner product and hence the ideal I_n. Thus we obtain a group homomorphism (representation) $\gamma_n : \mathrm{SO}(n) \to \mathrm{Aut}(Cl_n)$.

Let E be the total space of an orientable n-plane vector bundle. There is an associated principal bundle $P_{SO(n)}(E)$. To this principal bundle we can associate a bundle with fibre Cl_n using the representation γ_n above:

$$P_{SO(n)}(E) \times_{\gamma_n} Cl_n := P_{SO(n)}(E) \times Cl_n / \sim$$

where $(p,\sigma) \sim (pg^{-1}, \gamma_n(g)(\sigma))$ for $g \in \mathrm{SO}(n)$ and $\sigma \in Cl_n$. This is the *Clifford bundle* associated to E. We will denote this bundle of Clifford algebras by $Cl(E)$. (Remember that if we forget its multiplicative structure, Cl_n is just a vector space and therefore $Cl(E)$ is a vector bundle - albeit a vector bundle of a special kind.) Notice that Cl_n has $\mathbb{R}^n$ as a vector subspace, and that the action γ_n restricted to $\mathbb{R}^n$ is just the usual left action of SO(n) on $\mathbb{R}^n$. Thus we see that E itself is a sub-bundle of $Cl(E)$. In particular this means that we can regard sections of E as sections of $Cl(E)$, i.e. $\Gamma(E) \subset \Gamma(Cl(E))$.

If E is non-trivial, we will not be able to canonically identify individual fibres of $Cl(E)$ with Cl_n. However within each fibre we have a well-defined multiplication between elements (as γ_n preserves multiplication in Cl_n). As a consequence we see that $\Gamma(Cl(E))$ *has the structure of an algebra.*

Now suppose that E is the total space of an n-plane vector bundle with spin structure, for example the tangent bundle of a spin manifold. In this case we are able to define very special vector bundles associated to E called *spinor bundles.*

Consider the associated principal spin bundle $P_{Spin(n)}$, and take any Clifford module V. Now $\mathrm{Spin}(n) \subset Cl_n$, and so the action of the Clifford algebra on V, $\rho : Cl_n \times V \to V$, restricts to give an action of $\mathrm{Spin}(n)$ on V. We can then use this action to form new associated bundles $P_{Spin(n)} \times_\rho V$. Such bundles are called *spinor bundles.* It is not difficult to see that the bundle $P_{Spin(n)} \times_\rho V$ is a bundle of modules over the bundle of algebras $Cl(E)$, in the sense that each fibre in the spinor bundle is a module over the corresponding algebra fibre in $Cl(E)$. (This point is explained in detail in [LM; page 97].)

The following fact will be fundamental in the definition of Dirac operators:

Theorem 2.4.1. *Let $S(E)$ be a spinor bundle associated to E (either real or complex). Then the space of sections $\Gamma(S(E))$ is a left module over the space of sections $\Gamma(Cl(E))$.*

Note that the sections of spinor bundles are usually referred to as *spinors.*

Looking at the table in §2.1 we see, for example, that the *real* Clifford algebras in dimensions 1 and 5 have the structure of a complex algebra. By the periodicity phenomenon, this is true more generally in dimensions congruent to 1 and 5 modulo 8. Consequently the irreducible modules for these algebras are naturally complex vector spaces. Since complex multiplication clearly commutes with the Clifford action on these modules, we see that the corresponding irreducible (real!) spinor bundles actually have the structure of complex vector bundles, and that Clifford multiplication is complex linear. Consequently the space of (real!) spinors is naturally a complex vector space (of infinite dimension). We can make similar statements about quaternionic structures for spinor bundles and spaces of spinors in dimensions 2, 3 and 4 modulo 8.

Typically we will be interested in irreducible spinor bundles, that is, spinor bundles for which V is an irreducible Clifford module. As we saw in §2.1, in many dimensions there is a unique irreducible Clifford module. Thus in these cases we have a unique irreducible spinor bundle for each principal spin bundle. In the other cases, there are two such bundles. In our discussion of spinor bundles we have so far been implicitly assuming that V is a real Clifford module, since the spin groups arise naturally in a real Clifford algebra context. However we can also work with complex modules V for the complex Clifford algebra $\mathbb{C}l_n = Cl_n \otimes \mathbb{C}$. Recall that there is a unique irreducible module for the complex Clifford algebra in all even dimensions, and therefore corresponding to this we have a unique irreducible complex spinor bundle for each principal spin bundle in these dimensions. We will denote such bundles by $\mathbb{S}$. In the next chapter we will see that these play an important role in the theory of positive scalar curvature.

As a simple example of a *reducible* spinor bundle, we can consider Cl_n as a left module over itself. Thus we get a spinor bundle $P_{Spin(n)} \times_\rho Cl_n$ where ρ here is the left action of Cl_n on itself restricted to $\mathrm{Spin}(n)$. Note that this is not the same as the bundle $Cl(E)$. In fact

$$Cl(E) = P_{Spin(n)} \times_{Ad} Cl_n,$$

where $Ad : \mathrm{Spin}(n) \to \mathrm{Aut}(Cl_n)$ denotes the adjoint action $Ad(g)(\sigma) = g\sigma g^{-1}$.

Just as for Clifford algebras and Clifford modules, Clifford and spinor bundles are subject to splittings. As $\mathrm{Spin}(n) \subset Cl_n^0$, we see that the canonical action of $\mathrm{Spin}(n)$ on Cl_n respects the splitting $Cl_n = Cl_n^0 \oplus Cl_n^1$, as does the adjoint action. It follows (from the observation about the adjoint action) that we obtain a corresponding bundle splitting $Cl(E) = Cl(E)^0 \oplus Cl(E)^1$. In the case of splittings by the real volume element in dimensions 0 and 3 mod 4, and by the complex volume element in all even dimensions, we obtain corresponding bundle splittings by observing that the volume element in the Clifford algebra gives a global volume section in $Cl(E)$ respectively $\mathbb{C}l(E)$. Consequently in these dimensions we have $Cl(E) = Cl(E)^+ \oplus Cl(E)^-$, $S(E) = S(E)^+ \oplus S(E)^-$, and similarly in the complex case. Sections of S^+ (respectively S^-) are referred to as positive (respectively negative) spinors. In dimensions 0 mod 4 in the real case (and in all even dimensions in the complex case), fibrewise Clifford multiplication yields the following maps:

$$Cl^0(E) \times S^{\pm}(E) \to S^{\pm}(E);$$

$$Cl^1(E) \times S^{\pm}(E) \to S^{\mp}(E).$$

In particular this means that multiplying positive (respectively negative) spinors by sections of $Cl^1(E)$ produces negative (respectively positive) spinors in these dimensions.

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