examining students and applicants for staff positions, giving expert advice on treatises and inventions submitted to the Academy, supervising the printing and distribution of its publications. When the new President Brevern needed to reform the Academy's budget in 1740, the committee he charged with giving their expertise consisted of Goldbach, Schumacher, Euler and Krafft; Goldbach's final report seems to give more weight to scientific than to purely financial considerations.[45]

In comparison with this wide-ranging activity in the Academy's administration and on its educational and technical sidelines, the record of strictly scientific collaboration between Euler and Goldbach is, as has already been stated, meager. The few surviving notes from 1732–37 (nº 20–24 below) bear on individual problems in elementary and differential geometry, but no systematic investigation emerges. In 1738–39, the summation of number-theoretically defined series again surfaces in a series of missives (nº 25–32) which are not easy to interpret out of their original context; some very interesting and far-reaching speculations about zeta-type series can, however, be tentatively identified and at least partially reconstructed in them. As established by A.A. Kiselëv in his notes to the 1965 edition, Goldbach triggered here a research project that was to lead Euler towards important results in analytic number theory.[46] At least one of Goldbach's results in this area impressed Euler so much that he included it – attributed in a very explicit way, which was unusual – as the starting point for one of his papers.[47] Euler particularly admired the fact that the peculiar type of series considered here for the first time admitted neither what could properly called a general term nor a "law of continuation" (i. e., a recursion formula), yet could be summed explicitly.

Even taking all of this into account, the extant evidence from the 1730s does not by itself show that a close, friendly relationship between Euler and Goldbach had evolved on both the professional and personal levels. This is, however, clearly proven by the fact that the correspondence continued and grew in intensity when Euler left the Petersburg Academy and there was no necessity – indeed no functionally defined point – in pursuing it.[48]

45 *Materialy*, t. VI, p. 504–505, 528–532. Goldbach's report is also reproduced as Appendix A2 in the Yushkevich / Kopelevich biography (1994), p. 180–182.

46 See the present editors' notes to letters nº 25, 26 and 28–30.

47 E. 72, *Variae observationes circa series infinitas* (presented in April 1737 but printed only in 1744) quotes as *Theorema* 1 Goldbach's summation of the series $\frac{1}{3}+\frac{1}{7}+\frac{1}{8}+\frac{1}{15}+\frac{1}{24}+\frac{1}{26}+\frac{1}{35}+\ldots$, where the denominators are all the proper powers of natural numbers diminished by 1 (O. I/14, p. 217). Euler also brought this calculation, which Goldbach had communicated without any response to Daniel Bernoulli as early as 1729 (cf. *Correspondance*, t. II, p. 296), to his correspondents' attention in several letters, again emphasising Goldbach's original feat (R 594 to C.G. Ehler, R 202 to Johann I Bernoulli: O. IVA/2, p. 165).
Strangely, Goldbach later quoted to Euler an erroneous version of this old result, insisting even when Euler had tactfully pointed out the mistake (see below nº 73–75).

48 The official correspondence about publications, appointments, expert opinions, errands and so on that Euler exchanged with the Petersburg Academy as a (very active) foreign member during his entire stay at Berlin was addressed to its officers, mainly Schumacher and Müller.

1.2.4. A Russian-Prussian axis

Euler's move to Berlin was motivated – to use the terms of social science – both by "push" and "pull" factors. The political instability which threatened the Petersburg Academy's status has already been mentioned;[49] the risks that the extreme climate and constant overwork posed for Euler's health were compounded – as we know from a later letter to Müller[50] – by the ever-present danger of disastrous fires in a city built almost entirely of wood.

On the Prussian side, the re-establishment of the Berlin Academy sadly neglected by his father was among the first priorities Frederick II set himself when he succeeded to the throne in 1740: within a couple of weeks he had started negotiations to lure top-flight men of letters to Berlin. When Voltaire preferred to keep his independence and Wolff cautiously chose a traditional university post at Halle, it became all the more important to attract the rising stars among the scientists of the day: Maupertuis, who had just "flattened the Earth" by his spectacular expedition "to the Pole", and Euler, who was by then already considered the world's greatest research mathematician.[51] In February 1741, an improved salary offer by the Prussian ambassador Mardefeld induced Euler to ask for his discharge, which was granted only after some diplomatic tug-of-war: Goldbach's pleas to President Brevern and Vice-Chancellor Ostermann finally overcame Schumacher's procrastinating resistance.

Although Euler was warmly received by several of Prussia's intellectual leaders and even by Frederick himself, who personally welcomed him in a short letter written from an encampment in Silesia, the war severely hampered the establishment of the restored academy; in the end it could be officially opened only in January 1746, and its set-up and finances remained shaky for a long time.[52] Goldbach, who was familiar with the circumstances and acquainted with many of the protagonists of social life in Berlin, was the ideal correspondence partner to receive confidential first-hand reports and sometimes give advice.

Indeed, the frequent and often extensive letters Euler sends to Goldbach are a main source not only for learning about his professional activities, but also his personal and social life in the 1740s. Euler's growing family, his living conditions

49 Euler himself emphasised this aspect in the short autobiographical sketch he dictated to his eldest son in 1767: he decided to leave Russia "when things began to look rather awkward under the ensuing regency" ("nachdem [. . .] es bey der darauffolgenden Regentschafft ziemlich mißlich auszusehen anfieng": quoted according to Fellmann 1995, p. 13).

50 Cf. JW 1, p. 226.

51 "le plus grand Algébriste de l'Europe" is the term Frederick's envoy, Axel Freiherr von Mardefeld, uses in a letter to the king written on June 17th, 1741 (cf. Winter 1957, p. 18), in which he announces Euler's imminent departure for Berlin, telling him that the Russian court regrets the loss – and warning him at the same time that Euler's physical appearance does not give a favourable impression.

52 See below nº 40, 43, 54, 66, 72; cf. also Harnack 1900, vol. I, p. 245–331; Winter 1957, p. 12–48; O. IVA/6 (correspondence between Euler and Frederick, in particular Winter's introduction, p. 280–288).

and his financial success are mentioned as well as his encounters with high-ranking members of the nobility and the administrative élite, with scholarly and scientific colleagues, with artists and men of letters orbiting around the court. A detailed account of Euler's journey to Berlin on the Baltic (nº 38) is followed, nine years later, by the report on another of his rare trips (nº 146), when he meets his Basel relatives at Frankfurt on the Main in order to take his widowed mother back to Berlin to live with him. Goldbach shows his interest by enquiring after Euler's health, his domestic circumstances, common acquaintances, developments in Berlin society and the studies of his godson Johann Albrecht.

Even more space in the letters is given over to the exchange of academic and scientific news – mostly on Euler's side: as has already been remarked, Goldbach is very reticent about his own work.[53] Although Euler's duties and benefits from his ongoing membership in the Petersburg Academy are mainly dealt with in his official correspondence with Schumacher, later also with Teplov and Müller, he continues to ask Goldbach for his help with financial claims. And the Russian Academy's *arbiter elegantiarum* is still his main authority in matters of sophisticated style: whenever a poetic device for a prize paper or a commemorative medal is needed, he requests Goldbach's advice.

On Goldbach's side, the main incentive for continuing with the correspondence is the desire to keep in touch with developments in Western Europe: he asks for news of the prize contests at Paris and Berlin, enquires about the progress of Euler's and his colleagues' publications and enlists his help in obtaining scientific and literary items from German bookshops.[54]

But of course the central point of interest for most of those reading the Goldbach-Euler correspondence today is the copious information on research in mathematics (in particular number theory), on the questions often suggested by Goldbach and on their development in Euler's fertile mind. Since this will be systematically discussed in Section 2 of this Introduction, a terse list of the principal topics discussed in the period from 1741 to 1756 – in more or less chronological order – will have to suffice here:[55]

- representation of numbers by quadratic – and higher-order – forms (in particular, Fermat's Four Squares Theorem): this is the only field which remains constantly in view during all the 35-year exchange
- integration of rational and algebraic functions: via partial fraction decomposition, this leads to a discussion of complex zeros and the Fundamental Theorem of Algebra, which also involves Nicolaus I Bernoulli; later on there are traces of Euler's discovery of the addition theorem for elliptic integrals

53 Some exceptions are mentioned in note 11 above.

54 The systematically ordered Subject Index (*infra* p. 1201–1205) permits to locate the letters in which these issues are mentioned.

55 The places where some given topic appears in the correspondence can be identified by consulting the Subject Index (p. 1201–1205).

- the connection between complex powers and trigonometric functions, “Euler’s formula”, trigonometric (“Fourier”) series
- series of zeta and “multi-zeta” type, infinite products and continued fractions, conditional convergence, divergence (with some hints at what later came to be called summation methods), orders of infinity
- the determination of algebraic curves characterised by geometrical conditions (in particular, a tricky challenge problem in catoptrics publicly proposed by Euler in 1745).

Besides these ongoing research projects, there is also a wealth of incidental remarks from many areas of mathematics: here we find paragraphs on primes represented by polynomials or with a given difference, algebraic differential equations and their singular solutions, perfect numbers and the factoring of Fermat numbers, divisor sums and partitions (leading to the “Pentagonal Number Theorem”, one of Euler’s most surprising results), the additive representation of integers by primes (with the conjecture that is Goldbach’s one claim to lasting fame), the reordering of divergent series, transcendental numbers and their numerical approximation, an interpolation of the factorials connected with asymptotics for the harmonic series, general terms and sums for series defined by recursion, the number of dissections of a polygon into triangles (here “Catalan numbers” and their generating function make their first appearance), rational parametrisation of algebraic equations (and solvable cases of degree 5 and 6), an investigation of the general properties of solids enclosed by plane faces (which yields the earliest non-trivial discoveries in combinatorial topology), “multiply polygonal” numbers, Euler’s attack on “Fermat’s Last Theorem” (where he made the first step beyond Fermat himself by presenting a proof for the case $n = 3$, which however has an important gap), and so on ... almost to infinity.

Among the topics in natural science raised in the correspondence we find Euler’s theories of gravity and magnetism, his attempt to produce achromatic lenses, planetary motion (with a curious digression that leads from the three-body problem towards a kind of epistemological “anthropic principle”), the discussion on the correct measure of “action” that is preserved in collisions (an important source of confusion and dissent in 18th-century mechanics), astronomical tables, comets and eclipses – not to mention such little gems by the wayside as an enquiry into the composition of erythrocytes (with an early foray into fractal geometry). However, Euler is aware that Goldbach does not take as much interest in these fields as in his research in pure mathematics, and mostly limits his remarks to short hints.

Goldbach’s role in the development of Euler’s scientific work – more specifically of his number-theoretical research – is often underrated. Of course the mathematical achievements in Goldbach’s few published works pale compared with the huge body of results that Euler obtained. But at the beginning of his career, Euler attacked problems that were “fashionable” at the time and had already been studied by his contemporaries, principally the Bernoullis – and Goldbach. In his first letter to Goldbach, Euler reported on his results on interpolation of sequences, an area of mathematics in which Goldbach was active. Later other topics to which Goldbach had contributed came up: Fermat's observation on triangular numbers that are fourth powers, the Riccati equation, the integration of binomial differentials. In the context of this last problem Euler attempted to compute the integral $\int \frac{dx}{\sqrt{x^4-1}}$, but remarked in n° 11 that he could not do it "even by admitting logarithms". This certainly was one of the reasons why he was electrified when he saw Fagnano's account of "elliptic integrals", and started working out the addition formula in greater generality.

Even more important for defining Euler's mathematical interests was Goldbach's fascination with number-theoretic problems. Goldbach's innocent question whether Euler knew of Fermat's claim that all numbers of the form $2^{2^n}+1$ are prime eventually made Euler study everything by Fermat he could lay his hands on. Euler's contemporaries, first of all the Bernoullis, remained indifferent to this aspect of Euler's research, leaving Goldbach as virtually the only person with whom Euler could discuss such topics[56] until, towards the end of Euler's life and after Goldbach's death, Lagrange entered the stage.

1.2.5. The final years

In the decade from Euler's move to Berlin in June 1741 to the end of 1750, Goldbach and he wrote frequently to each other, on average every two months on each side; many of both correspondents' letters are long and substantial. However, towards the end of this period Goldbach's messages became rarer and shorter, and the scientific content dwindled: as he explained to Euler in 1751, "the attention required for such speculations is ever more fading away, by a progression that strongly converges to nothing". Goldbach was by now in his sixties and gradually retired from his professional activities; if he had earlier dreamed of a vigorous return, undistracted by other obligations, to his mathematical pastimes, his creativity and his concentration span now drained away fast. In spite of Euler's

56 Among the few exceptions we mention Euler's correspondents Ehler, Naudé, Krafft, Winsheim and Segner: Some of the topics in Euler's correspondence with Ehler were the "Chinese Remainder Theorem" (R 581: April 8th, 1735), Fermat's Little Theorem and the primality of Fermat numbers (R 584: June 1735), triangular numbers that are squares (R 592: August 27th, 1736), and the summation of the series $1+d+d^3+d^6+\dots$ (R 594: February 11th, 1737). With Naudé, Euler discussed squares among the triangular numbers as well as partitions (R 1903–1904: August to September 1740). In the Krafft correspondence, two letters from 1744 (R 1282, 1283) deal with series involving figurate numbers, one from 1746 (R 1288) poses a problem on sums of divisors; in 1747 Krafft is looking for an analogue to perfect numbers among negative integers and marvels at Euler's recursion formula for sums of divisors (R 1293, 1294); a letter from 1750 (R 1303) mentions divisibility properties of the numbers 2^p-2 and 2^p-1, and another one (R 1304) asks Euler about his result that integers which are sums of two squares in a unique way are prime (the Krafft correspondence has been summarised in JW 3, p. 134–176, but many of Euler's letters are apparently lost). In the autumn and winter of 1748, Euler discussed Mersenne primes and factors of 2^p-1 with Winsheim (R 2813–2816). And in 1757, he pointed out errors in Segner's attempted proof of Fermat's Last Theorem (R 2494–2497).

encouragement, there are no new suggestions from his side, and his analysis of the questions Euler raises becomes ever more laborious and repetitive. Euler, who has always exhibited great patience with Goldbach's (relative) shortcomings, continues to assess his efforts kindly and explain at length what has not been correctly understood; but after a few years his part of the dialogue also trails off.

Moreover there are also weighty external reasons for this fade-out: after Maupertuis' flight from Berlin in 1753, in the aftermath of the König affair, Euler was burdened with official chores as acting president of the Berlin Academy (without receiving either the title or the salary raise that should have gone with the task). And in August 1756, after a decade of strained quiet, the struggle between Prussia and Austria for supremacy in Central Europe again erupted into a war that was to go on for seven years, and this time Russia joined the hostilities against Frederick. In October 1760, a Russian army even occupied Berlin for a short time, pillaging Euler's estate at nearby Charlottenburg.

Apparently Goldbach's position in the Tsarina's Foreign Department precluded the continuation of his correspondence with Prussia, although the postal service between Petersburg and Berlin went on more or less undisturbed (in fact Euler regularly exchanged letters, parcels and bills with Müller and several other colleagues in Russia all during the war). The only note Euler was able to send to Goldbach between June 1756 and June 1762 was a letter of recommendation delivered personally by F.U.Th. Aepinus, who had been called to the chair of physics at the Petersburg Academy.[57]

But even when normal means of communication were reopened after "this violent thunderstorm",[58] the tone of the correspondence was changed: the messages exchanged in the two years that remained of Goldbach's life – eight letters from Berlin, just four very short notes from Petersburg – are centered on personal news, touching on mathematics only in a desultory way. They contain reports on Euler's family, congratulations on Goldbach's career and – increasingly – expressions of concern, advice and good wishes for his declining health. Euler's last letter, written eight months before Goldbach's death, ended up not among the Goldbach papers, but in G.F. Müller's files at Petersburg; apparently by then the Academy's Secretary took charge of his terminally ill friend's correspondence. Müller also informed Euler of Goldbach's death and described to him in detail his last days, the funeral rites and the arrangements for the disposal of his financial and literary estate.[59]

Let us close this summary of our protagonists' relationship, which for most of the time of their acquaintance had been restricted to written expression, with a remark on form and content: The style of the letters – the only testimony of their

57 This note was accompanied by a letter in which Johann Albrecht Euler also recommended himself to his godfather's goodwill; see nº 184, 184ª.

58 Cf. nº 185. After Peter III succeeded to the Russian throne and made a separate peace with Prussia in May 1762, Euler immediately profited from the occasion to resume contact with his former patron after a five-year hiatus – incidentally mentioning that he was hoping for compensation of his losses during the Russian army's occupation of Berlin.

59 Cf. JW 1, p. 253–258.

dealings that remains – is easy to misinterpret as distanced, since conventions have changed so much since Goldbach's lifetime. The tone employed by both correspondents always remains very courtly: they address each other in not only respectful but seemingly stilted terms and close their letters by elaborate, almost uniformly phrased salutations. But although the expression of their mutual regard seems much more formal than is customary in recent times or, indeed, in contemporary letters from other countries, the personal warmth Goldbach and Euler felt for each other should not be overlooked. Both partners profit from every opportunity to congratulate each other on their professional, social and economic success; Goldbach seldom neglects to send his sincere regards to Euler's wife, his felicitations on his children's birth and his condolences when one of his relatives dies. When the bookseller Spener erroneously charges Goldbach for a book Euler has had sent to him as a gift, Euler is literally inconsolable in his fear that this misunderstanding might damage their relationship; when Euler learns about an illness affecting his elder friend, he is sincerely concerned and tries to counsel him. This tone of affectionate, on rare occasions even jocular, comradeship is hardly ever perceptible in Euler's other correspondence. There should be no doubt that Euler, the greatest mathematician of his time, and Goldbach, the worldly-wise "dilettante", regarded each other as tried and valued friends for a lifetime.

1.3. The Euler-Goldbach correspondence: chronology and statistics

The main remaining testimony of the relationship between Leonhard Euler and Christian Goldbach sketched in the last section is their correspondence. This consists of 196 extant letters, 102 written by Euler and 94 by Goldbach.[60] In the table on the next two pages, three periods of intense correspondence and two gaps when almost no letters were exchanged can be distinguished:

- October 1729 to January 1732, letters 1–19: Goldbach is living in Moscow with the court of his former pupil Tsar Peter II; Euler is launching his career at the Petersburg Academy. Taking up many of the topics Goldbach has already been discussing with their colleague Daniel Bernoulli – mainly in the theory of series, integral calculus and elementary number theory – Euler works at impressing Goldbach with the advances he has made in his research. Goldbach has found a partner who shares his interests in pure mathematics and can be expected to tackle the most difficult problems successfully.

60 In most cases, the autograph manuscript of the letter as it was actually sent is preserved among the documentary estate of the recipient. For most of Goldbach's letters from the periods 1730–31 and 1742–1749, we also have the (often partial) copies that he entered in his "correspondence book". A fuller description of the manuscripts is given in Section 3.1 below; see also the Synoptic Table of the correspondence, p. 1141–1146.

- February 1732 to June 1741, letters 20–37: Goldbach and Euler now both live in Petersburg and regularly meet at the Academy, where they collaborate in various roles as researchers, scientific experts, educators and administrators. Most of their communication takes place directly; only a few occasional messages document their interaction in this period. Euler still relies on Goldbach's patronage to support his career, but his outstanding talent makes him the rising star of Russia's small academic community, while Goldbach is withdrawing, in the same period, from public visibility as a scientist.
- July 1741 to July 1756, letters 38–183: Euler has moved to Berlin and is now supplying the lion's share of research for *two* academies; Goldbach – by now in his fifties – occupies a senior position in Russian civil service, commuting between Moscow and Petersburg.[61] The correspondence with Euler is now the only link to Western Europe he has left and his only hold on his passion for mathematics. Up to the late 1740s, both partners write frequently, alternating regularly; besides the scientific topics, they discuss many events in their academic and personal life and help each other with errands. Towards the end of this period, the correspondence thins out both in quantity and content; Goldbach is gradually retiring from his charges, and Euler is busy with many other tasks and new networks.
- August 1756 to May 1762, letter 184: Prussia has unleashed the third and most obstinate war for Silesia; this time Russia is seriously involved. Euler's correspondence with Petersburg is restricted to official business with the Academy, of which he is still an active foreign member. Only once is he able to entrust personal messages for Goldbach from himself and his son Johann Albrecht, Goldbach's godson, to a colleague who is travelling to Petersburg.
- June 1762 to November 1764, letters 185–196: Immediately after the conclusion of a peace treaty between Prussia and the new Tsar Peter III, Euler resumes the correspondence, but Goldbach is now in his seventies and his health is failing. Euler's effort to reawaken his interest in mathematical ideas is not very successful; the notes sent on both sides are now mainly limited to civilities and friendly advice. Apparently Goldbach, who was terminally ill by then, did not take note of Euler's last letter; later Müller reported to Euler on their old friend's death and funeral.

61 A word on the service provided by the Royal Prussian post may be in place here. According to a description with regard to a slightly earlier period (Matthias 1812, vol. 1, p. 325–328), the riding mail couriers for Russia left Berlin every Tuesday and Saturday evening at 6 p.m., going to Moscow by way of Königsberg, Riga and Petersburg; on the same weekdays, mail from Russia arrived at Berlin by noon.
Indeed most of Euler's letters from Berlin and of those Goldbach sent from Petersburg are dated from these two days of the week; when Goldbach was in Moscow, he often wrote on Monday or Thursday. The sequence of letters in our correspondence strongly suggests that letters between Berlin and Petersburg usually took two weeks to arrive (see, e. g., n° 39, note 1, and n° 107, note 1); so the time lapse until a reply arrived was expected to be approximately one month (at least in summer).

The table on the next double page shows the chronological progress of the Euler-Goldbach correspondence, month by month, and the number of letters written on both sides each year. On the right-hand page some important events for the two correspondents and their environment are listed in order to correlate the sequence of letters with the course of their personal and professional lives.

Year						Letters							E	G	n°	lg.
	J	F	M	A	M	J	J	A	S	O	N	D				
1725																
1726																
1727																
1728																
1729										E		G	1	1	1–2	Latin
1730	E				G	EG	EG	E		GE	GE		6	5	3–13	
1731											GE	G	1	2	14–16	
1732	EGE						[E]						3	1	17–20	
1733																
1734																
1735			[E]							G			1	1	21–22	
1736		G												1	23	
1737							E						1		24	
1738										G				1	25	
1739											GEGE	EGE	4	3	26–32	
1740			G					EG					1	2	33–35	French
1741				E		G		EG	EE		G	E	5	3	36–43	German
1742		EGG	EE	G	E	GE	G	E		GE		GEG	8	8	44–59	
1743	EE	GGE	G	E	GE	G	EG	E	G	E		G	8	8	60–75	
1744	E		G	E		G	EG	G	E	G	E		5	5	76–85	
1745	G	E			G	E	G	E	G	E	GGE	G	5	7	86–97	
1746	E	EG	G	E	GGE	E	GE	G	E	G	E		8	7	98–112	
1747				EG	E	G	E	G	EG	E	G	E	6	5	113–123	
1748	G	E		G	E	GE	G	E	G	E			5	5	124–133	
1749		G	EG	GEE		G	E						4	4	134–141	
1750			G	E		E	G	EE		G	E		5	3	142–149	
1751						G	EG	G	E	G		E	3	4	150–156	
1752					GE	E	[G]	E		GE	G	E	5	4	157–165	
1753			G	E		G		E					2	2	166–169	
1754																
1755				G	E			GE				GG	2	4	170–175	
1756	EG	E	GG	E	G	E							4	4	176–183	
1757				E									1		184	
1758																
1759																
1760																
1761																
1762						E			E	G	E		3	1	158–188	
1763										EEGG	E	E	4	2	189–194	
1764	G		E										1	1	195–196	
1765																
1766																

Year	Goldbach	Euler	other relevant events
1725	arrives in Pb (July)		AcPb opens (Sep)
1726		negotiates with AcPb	
1727	supervisor of Peter II	arrives in Pb (May)	Peter II Tsar
1728	in Moscow with court		troubles at AcPb
1729			Start of *Commentarii*
1730			Anna Ioannovna Tsarina
1731		professor of physics	
1732	to Pb (Jan)		
1733		professor of mathematics	Keyserling AcPb President
1734		marries (Jan), * J.A. Euler	Korff AcPb President
1735		seriously ill (Jan)	
1736			
1737	AcPb "deputy President"		
1738		loses sight in right eye (Aug)	
1739			
1740		* Karl Euler	Frederick II King (May)
1741		moves to Berlin (June/July)	Elizabeth I Tsarina (Nov)
1742	to Foreign Office (Mar)		end of 1st Silesian war (June)
1743	to Pb (Jan)	* Christoph Euler	*Société Litteraire* Berlin
1744	to Moscow (Feb)	* Charlotte Euler	AcBe statute proclaimed (Jan)
1745	in Pb (Jan)	† Paul Euler (Mar)	end of 2nd Silesian war (Dec)
1746	receives estate in Livonia		Maupertuis AcBe President
1747		Fellow of Royal Society	
1748			† Johann I Bernoulli (Jan)
1749	to Moscow (Jan)	meets Frederick II	
1750	to Pb (Jan)	takes his mother to Berlin	
1751	buys house in Pb		
1752			Maupertuis-König affair
1753	to Moscow (Jan)	buys estate at Lietzow	
1754	to Pb (Mar)	*de facto* president of AcBe	
1755		member of AcSci Paris	
1756			3rd Silesian war (Aug)
1757			
1758			
1759			† Maupertuis (July)
1760	Privy Councillor	J.A. Euler marries	Berlin occupied by Russians
1761		travels to Halle with Karl	
1762			Catherine II Tsarina (July)
1763		negotiates return to AcPb	end of 3rd Silesian war (Feb)
1764	dies (Nov 20)		
1765			
1766		returns to Pb (July)	

2. Main subjects of the correspondence

This introduction is written for the benefit of a 21st-century reader who would like to get a first overview of the questions Euler and Goldbach discussed in their correspondence. For this reason we have used modern notions and modern notation throughout the present section. This change of notation with respect to the sources is, in our view, harmless when we write, e. g., x^2 instead of xx, or replace the ratio p between the circumference and the diameter of a circle by π, a notation that came into common use during Euler's lifetime.[1]

The use of our notations for factorials (Euler called them "Wallis's hypergeometric series") and binomial coefficients is slightly more problematic, because they are accompanied by a whole body of formulae and relations that were not as familiar to Euler's contemporaries as they are to us.[2]

The modern index notation (a_n) for sequences was also not available to 18th-century mathematicians; instead they used two-line "tables" linking index numbers and the corresponding terms. Thus, e. g., the array

[index:]	1	2	3	4	5
[term:]	1	2	6	24	120

refers to the sequence $a_n = n!$ of factorials.

In the 17th century, Wallis, Huygens and their contemporaries had solved "quadrature" problems without systematic notions of calculus or an established notation for integration. The general idea of an algorithm for solving problems of that kind and the $\int$ sign were devised by Leibniz, and in the 1690s Euler's teacher Johann Bernoulli coined the word "integral"; but since the notion of a *definite* integral $\int_a^b f(x)\,dx$ was lacking, 18th-century mathematicians still had to fix the range of integration by words. In Euler's later work, the gradual transition towards a formalised notation for the definite integral can be studied.

As will be discussed below, the understanding of the terms "irrational" and "transcendental" in the writings of Euler and Goldbach also differs greatly from their modern meaning.

In the following we will present the main topics that show up in the correspondence between Euler and Goldbach. Most of these topics belong to one of the

1 Euler's textbooks were actually instrumental in disseminating this sign and the concept of π as a number, which had been introduced in 1706 by the Welsh Newtonian William Jones (see Mattmüller 2008, p. 42).

2 In E. 421, Euler used the notation $[\frac{m}{n}]$ for his interpolation of the sequence of factorials at $x = \frac{m}{n}$; in E. 652 he introduced the symbol $\Delta : n = 1 \cdot 2 \cdots n$, and in E. 768 he employed the notation $\Phi : m$. The symbol $n!$ is due to Kramp (1808); Gauss used Πn (as did Riemann, by the way), and Legendre (1811) introduced the notation $\Gamma(s)$ still in use (with a shift of the argument: $\Gamma(n) = (n-1)!$). For the binomial coefficient that we denote by $\binom{\alpha}{\nu}$, Euler used, in E. 575 and E. 584, the symbol $[\frac{\alpha}{\nu}]$; in E. 663, E. 709, E. 722, E. 726, E. 747 and E. 768, he employed several other notations such as $(\frac{n}{q})$.

"three A's": algebra, analysis and arithmetic. The main difficulty in classifying Euler's work comes from the subsequent development: Euler probably did not consider his summation of the series $\sum \frac{1}{n^2}$ as being related to number theory, and neither did the Bernoullis;[3] modern readers, on the other hand, recognise this result as an example of the theory of special values of zeta functions and L-series, which today is considered a part of number theory.

Even the topics that we have listed under the title "analysis" have a number-theoretical touch: the technique of interpolation is used nowadays for defining p-adic analogs of classical functions such as the Gamma function and L-series (where, by the way, Bernoulli numbers play a central role); Euler used the Riccati equation for deriving continued fraction expansions of e, and elliptic integrals are today mainly studied in connection with elliptic curves.

Despite these problems we could not bring ourselves to throw out the classification into number theory on the one hand and analysis on the other; to accommodate some of the more problematic cases we have added a section on analytic tools in number theory.

2.1. Number theory

Number theory in Euler's times was an area of mathematics that dealt with properties of whole numbers and with problems that were more or less inherited from Pythagoras, Euclid and Diophantus: prime numbers, perfect[4] and amicable[5] numbers, square and polygonal[6] numbers, Diophantine equations, and similar topics. Fermat regularly tried to interest mathematicians such as Huygens, Pascal and Brouncker in number-theoretic problems, but his efforts – like those of Euler a century later – were largely in vain.

2.1.1. Fermat's legacy

Euler was brought into contact with Fermat's problems by Goldbach's first letter, when Goldbach asked him whether he knew of Fermat's claim that all numbers of the form $2^{2^n} + 1$ are prime. Subsequently Euler started reading Fermat (see

3 All the same, Jacob (I) Bernoulli gave to his 1689 paper on the problem of evaluating sums of the form $\sum_{n=1}^{\infty} n^{-k}$ the title "*Arithmetic* theses on infinite series and their finite sum".

4 A number n is called perfect if it equals the sum of its proper divisors; for example, $6 = 1 + 2 + 3$ is perfect.

5 Two numbers m and n are called amicable if $s(n) = m$ and $s(m) = n$, where $s(n)$ denotes the sum of the proper divisors of n; the classical example of a pair of amicable numbers is $m = 220$ and $n = 284$.

6 Polygonal numbers, sometimes also called figurate numbers, generalise the notion of square numbers; they count the dots that can be arranged in a pattern of nested regular polygons. The n-gonal numbers $p_k^{(n)}$ form a sequence with first term $p_1^{(n)} = 1$ and second difference equal to $n - 2$; explicitly, they are given by $p_k^{(n)} = \frac{1}{2}((n-2)k^2 - (n-4)k)$.

result	Fermat	Euler-Goldbach letters
$2^{2^n}+1$ is prime	Fermat 1679, p. 162	n° 2–4, 7, 8, 52
2^p-1 is only prime if p is prime	Fermat 1679, p. 177	n° 15
$q \mid 2^p-1$ implies $q=2kp+1$	Fermat 1679, p. 176	n° 165
$4n-1 \neq x^2+y^2$ for $x, y \in \mathbb{Q}$	Fermat 1891, t. II, p. 202–205	n° 11
$8n+7 \neq x^2+y^2+z^2$ for $x,\ y,\ z\ \in\ \mathbb{Q}$	Fermat 1891, t. II, p. 66	n° 11
Fermat's Little Theorem	Fermat 1679, p. 161	n° 15, 47
$p \nmid x^2+y^2$ for primes $p=4n-1$ and coprime integers x, y	Fermat 1679, p. 162	n° 15, 47, 73, 74
Two Squares Theorem	Fermat 1891, t. II, p. 203, 213, 221–222	n° 87, 115, 138
Four Squares Theorem	Wallis 1693, p. 857	n° 5, 74, 115, 127, 138, 140, 141, 147
$8m+3=x^2+y^2+z^2$ and the Triangular Number Theorem	Wallis 1693, p. 857	n° 74, 114, 125
Polygonal Number Theorem	Wallis 1658, p. 857	n° 11
There is no right-angled triangle in numbers whose area is a square	Frénicle 1676, p. 100	
The only triangular number that is a fourth power is 1	Wallis 1693, p. 858	n° 7
Solvability of "Pell's Equation"	Fermat 1679, p. 190	n° 9, 169, 173
Fermat's Last Theorem ($n=3$)	Fermat 1679, p. 193	n° 125, 169, 171
Multiply Polygonal Numbers	Fermat 1679, p. 168	n° 167

Table 2.1: Fermat's problems in the Euler-Goldbach correspondence

infra p. 39) and investigated almost all the problems that Fermat had left. Among his first five papers on number theory (E. 26, E. 29, E. 36, E. 54, E. 98), only E. 36 on amicable numbers does not refer to work done by Fermat (although Fermat, like many of his contemporaries such as Descartes and Frénicle, had also studied amicable numbers).

Euler rediscovered the power of Fermat's main technique, infinite descent, and found out how to use it for proving the following results that Fermat had obtained by the same method:

- Certain Diophantine equations, such as $x^4 \pm y^4 = z^2$, do not have any non-trivial solutions in integers: see, e. g., E. 98, in which Euler referred to Frénicle's *Traité des triangles rectangles en nombres* (1676): this contains a proof (believed to be akin to Fermat's) of his result that there is no right-angled triangle with integral sides whose area is a square. This problem is equivalent to solving $x^4 - y^4 = z^2$ in non-zero integers.[7] In E. 98, Euler presented his own version of Frénicle's proof and applied the same technique to equations such as $x^4 + y^4 = z^2$.

- The only fourth power among the triangular numbers $T_n = \frac{1}{2}n(n+1)$ for $n \geq 1$ is $T_1 = 1$. The corresponding problem for squares in the sequence of triangular numbers led Euler to investigating "Pell's equation" (incidentally establishing this erroneous attribution).
- All odd prime divisors of integers of the form $x^2 + y^2$ have the form $4n + 1$; Euler gave several proofs of this fact, and with a little help from Euler, Goldbach found one, too (see nº 73). For the corresponding results on prime divisors of the forms $x^2 \pm 2y^2$, Euler used a proof by descent (see E. 256 for $x^2 + 2y^2$ and E. 449 for $x^2 - 2y^2$). Euler's idea became the basis of the first proof of the quadratic reciprocity law by Gauss,[8] who replaced descent by induction.
- Every prime $p = 4n + 1$ is the sum of two squares: see, e. g., E. 228. Euler later tried to prove the Four Squares Theorem along the same lines.

In fact, Euler worked out proofs for most of Fermat's claims – with some notable exceptions:

- the Four Squares Theorem, according to which every positive integer is the sum of at most four integral squares: Euler could reduce the claim to representing prime numbers as sum of four squares and was also able to prove that positive integers are sums of four squares of *rational* numbers, but it was Lagrange who first succeeded in fully proving the theorem.[9]
- the solvability of "Pell's Equation": for positive non-square integers A, the equation $Ay^2 + 1 = x^2$ has infinitely many solutions in integers. This was also first proved by Lagrange, after Euler had translated Brouncker's method into the language of continued fractions.
- the Triangular Numbers Theorem: every positive integer is the sum of at most three triangular numbers $\frac{1}{2}n(n+1)$. The observation that this is equivalent to representing numbers of the form $8m + 3$ as sums of three integral squares was known to Fermat and Euler. The "Three Squares Theorem" was proved by Gauss[10] and figures prominently in his diary.[11]

7 Catherine Goldstein (1995) has devoted an important monograph to this theorem, the history of its reception and various "reconstructions" of Fermat's ideas.

8 Cf. Gauss, *Disquisitiones* (1801), art. 107–114.

9 Euler came very close to a complete proof, however: see Lemmermeyer 2010.

10 For a later version of his proof, cf. Gauss, *Disquisitiones* (1801), art. 266–293.

11 In his early years, Gauss kept a diary in which he entered his main discoveries (see Gauss 2005). In entry nº 18 from July 10th, 1796, he encoded the result that every positive number is a sum of three triangular numbers as "EUREKA! num $= \Delta + \Delta + \Delta$".

– the general Polygonal Number Theorem, according to which every positive integer is the sum of at most three triangular numbers, four squares, five pentagonal numbers etc., was derived by Cauchy (1815) from Gauss's Three Squares Theorem just mentioned.
– "Fermat's Last Theorem", the unsolvability of $x^n + y^n = z^n$ in non-zero integers for exponents $n > 2$, was finally proved by Wiles and Taylor in the 1990s.

Although Euler's first reaction on learning of Fermat's statement concerning the primality of $2^{2^n} + 1$ was rather reserved, in his next letter to Goldbach (n° 5) he admitted that he had become interested and had started reading Fermat's works. At that time Euler had access to Fermat's *Varia Opera* edited in 1679, to Wallis's *Commercium Epistolicum*[12] first edited in 1658, which contains several letters by Fermat, and to Frénicle's 1676 *Traité*, which contains Fermat's proof by descent that there is no triangle in integers whose area is a square. André Weil suggests (Weil 1984, ch. III, § IV) that Euler read Fermat's *Observations on Diophantus*, which had been edited by his son Samuel in 1670, only in 1748; in fact, in his letter n° 125 to Goldbach, which was written in February 1748, Euler mentioned for the first time what later would be called Fermat's Last Theorem, a result that Fermat did not mention in his correspondence and which only appeared in the commentary to Diophantus.[13]

Fermat numbers and Mersenne numbers

Prime numbers of a special form have a tendency to show up in most investigations on amicable or perfect numbers. Euclid already proved (*Elements* IX.36) that if $2^p - 1$ is a prime number, then $2^{p-1}(2^p - 1)$ is perfect, i. e., this number equals the sum of its proper divisors. In two notes dating probably from the late 1740s but published only in 1849, Euler proved the converse and established the generality of Euclid's construction: all even perfect numbers are of the form $2^{p-1}(2^p - 1)$ where p and $2^p - 1$ are prime.[14]

One of the classical problems that Fermat mentioned several times in his correspondence was that of finding an explicit formula that would yield prime numbers greater than any given number. Fermat believed he had found a solution to this problem: he conjectured that all numbers of the form $F_n = 2^{2^n} + 1$ are prime. Goldbach asked Euler in n° 3 whether he knew Fermat's claim, and explained in n° 8 that all Fermat numbers are pairwise coprime. Euler, in n° 7, looked at the more general problem of the primality of numbers of the form $(2p)^{2^k} + 1$ and observed that these are not always prime. In n° 52, Euler indicated the divisor 641 of the fifth Fermat number.

12 Euler refers to this in E. 26.

13 In the spring of 1757, J.A. Segner discussed his attempt to prove Fermat's Last Theorem with Euler in several letters (R 2494–2497, to appear in O. IVA/8).

14 See E. 798, § 8, and E. 792, § 106–109.

Mersenne numbers first show up in n° 5, where Euler remarks that some believe all numbers of the form $M_p = 2^p - 1$ to be prime,[15] whereas $23 \mid M_{11}$, $47 \mid M_{23}$ and $223 \mid M_{37}$. In n° 15, he observed that $q \mid M_p$ for $p = 11$, 23, 83 and $q = 2p+1$. More generally, Euler claimed that $2^n - 1$ is divisible by $n+1$ whenever $n+1$ is a prime number; this is a special case of "Fermat's Little Theorem" that $p \mid a^{p-1} - 1$ for primes p and numbers a coprime to p, which Euler proved in n° 47. Euler later generalised this result to $m \mid a^{\varphi(m)} - 1$ for arbitrary coprime numbers a and m, where $\varphi(m)$ denotes the number of integers $1 \leq k < m$ coprime to m.[16] The number of known perfect numbers, and in particular the primality of $2^{31} - 1$, is discussed in n° 162–165.

Sums of squares and polygonal numbers

The problem of representing numbers as sums of two squares has a long history. Apparently Diophantus already knew how the representations of 65 as a sum of two squares can be computed from the representations of its prime factors 5 and 13, and the idea underlying the "product formula" $(a^2+b^2)(c^2+d^2) = (ac-bd)^2 + (ad+bc)^2$ was also known to Indian mathematicians such as Brahmagupta. Girard and Bachet studied the problem of writing numbers as sums of two squares, but it was Fermat who finally succeeded in proving the fundamental result that primes $p = 4n+1$ can be written uniquely as a sum of two squares.[17]

In n° 6, Goldbach conjectured that the smallest nontrivial divisor of a number of the form $a^{2^x} + 1$ has the same form $n^{2^x} + 1$, and stated this could easily be proved for $x = 1$. In his reply, Euler politely remarked that the claim is true if one allows 1 as the smallest divisor, and then observed that "even when these smallest divisors do not exceed a square by 1, possibly all of them are sums of two squares". In n° 11, Euler observed that a number of the form $4x+3$ is not the sum of two rational squares,[18] and that a number of the form $8x+7$ is not the sum of three rational squares; both observations also occur in letters by Fermat to which Euler did not have access, and both claims can be proved by elementary congruence arguments. In n° 47, Euler stated that a sum of two coprime squares $a^2 + b^2$ is never divisible by any number of the form $4n-1$ and acknowledged that this theorem was due to Fermat. Euler also sketched a plan of how to attack

15 He did not give a reference for this statement and admitted in n° 7 that he did not remember where he had seen it; but in fact the error is present in several textbooks that were widely diffused in Euler's time (cf. n° 5, note 4).

16 Cf. E. 271, *Theorema* 11 (O. I/2, p. 554–555).

17 Fermat gave a rough sketch of his proposed proof in a letter to Carcavi written in August 1659; this letter was published much later under the title *Relations des nouvelles decouvertes en la science des nombres* (see Fermat's *Œuvres*, t. II, p. 431–436).

18 Euler observed in the introduction to E. 556 that the equation $x^2 + y^2 = 3z^2$ is not solvable in rational integers, and then studied, more generally, the solvability of $fx^2 + gy^2 = hz^2$. This investigation was completed by Legendre, who showed that the necessary solvability conditions given by Euler are actually sufficient.

Fermat's Two Squares Theorem. This result showed up again in nº 52 and 87, and Euler finally proved the Two Squares Theorem in nº 115.

Like Bachet, Fermat believed that the content of the Four Squares Theorem, which claims that every positive integer is a sum of at most four integral squares, was known to Diophantus. Bachet first stated it explicitly and tested it empirically for small numbers. Fermat claimed to have found a proof of this result in the late 1650s.

Euler's and Goldbach's attempts at proving the Four Squares Theorem are documented in many letters of their correspondence. Already in nº 5, Euler mentioned this "elegant theorem" due to Fermat. Eighteen years later he gave, in nº 127, the product formula for sums of four squares, a result which is used in most proofs of the Four Squares Theorem.

In the same letter (see also nº 147), Euler suggested proving results like this using generating functions: write

$$(1 + x^1 + x^3 + x^6 + x^{10} + x^{15} + \ldots)^3 = \sum a_n x^n, \tag{2.1}$$

where the exponents on the left-hand side are the triangular numbers;[19] then a_n is the number of ways in which n can be written as a sum of at most three triangular numbers, and the desired theorem is equivalent to the claim that $a_n > 0$ for all $n \geq 1$. Euler presented this idea in E. 565, and in E. 586 he explained the same idea for sums of squares and general polygonal numbers.

Jacobi later gave a proof of the Four Squares Theorem,[20] adopting Euler's idea by observing that

$$\Big(\sum_{k=-\infty}^{\infty} q^{k^2} \Big)^4 = \sum_{n=0}^{\infty} r_4(n) q^n,$$

where $r_4(n)$ is the number of representations of n as a sum of at most four squares; using analytic properties of the "theta function" $\sum q^{k^2}$ on the left-hand side,[21] he found the formula $r_4(n) = 8 \sum_{d|n, 4\nmid d} d$ for $n \geq 1$ (for odd numbers n, we have $r_4(n) = 8\sigma(n)$, where $\sigma(n)$ denotes the sum of the divisors of n). Legendre, by the way, found the analogous result that the number of representations of $n \geq 1$ as a sum of four triangular numbers is equal to $\sigma(2n+1)$. The corresponding numbers for sums of three triangular numbers and for sums of three squares are connected to class numbers of certain binary quadratic forms.

In nº 138 Euler wrote that he could *almost* prove the Four Squares Theorem: the only piece of the puzzle that was still missing was the lemma that if ab and a

19 Euler's efforts at finding a closed form of the series (2.1) are mentioned in his letter R 594 to Ehler from February 1737 (see Smirnov 1963, p. 381–382). Daniel Bernoulli, in his letter R 145 dated April 14th, 1742 (to be published in O. IVA/3), asked Euler whether his method of summation also applied to sums of the form $a + a^4 + a^9 + a^{16} + \ldots$.

20 See Jacobi 1829 (*Gesammelte Werke*, Bd. I, p. 239).

21 In modern terms, this theta function is a modular form of weight $\frac{1}{2}$ with respect to $\Gamma_0(4)$.

are sums of four squares, then so is b. And in n° 141, he actually almost succeeded, coming very close to a proof of the missing lemma.[22]

Euler observed twice, in n° 74 and 125, that Fermat's claim that every number is the sum of at most three triangular numbers is equivalent to $8n + 3$ being a sum of three squares. In n° 144, Euler formulated a conjecture that happens to be equivalent to this result: every odd number $2n+1$ can be written as a sum of four squares in such a way that $2n + 1 = p^2 + q^2 + r^2 + s^2$ and $p + q + r + s = 1$.

Diophantine equations

One of the first Diophantine problems ever studied was that of finding Pythagorean triples, that is, positive integers x, y and z with $x^2 + y^2 = z^2$. Proclus credits the Pythagoreans with an infinite family of solutions,[23] and Euclid already knew all of them. Diophantus later studied much deeper problems, all of which asked for solutions in rational numbers. In the centuries that followed, mathematicians in the Islamic world solved Diophantine problems such as $x^4 + y^2 = z^2$ and $x^2 + y^2 = z^4$, and already claimed that $x^4 + y^4 = z^4$ is impossible in integers. Fermat studied nontrivial Diophantine equations (such as the "Pell equation" $Ax^2 + 1 = y^2$ or the Bachet-Mordell equation $y^2 + 2 = x^3$) over the integers.[24]

The correspondence between Euler and Goldbach regularly touched upon Diophantine equations of varying difficulty. The first such topic was brought up by Goldbach in n° 6, where he claimed that no triangular number increased by 4 is an eighth or tenth power. In his reply Euler mentioned Fermat's theorem that no triangular number > 1 is a fourth power. Goldbach answered in n° 8 he had even proved (in Goldbach 1724) that no triangular number > 1 could be a square, which Euler showed to be false in the next letter n° 9. There he also mentioned that he could not yet solve the analogous problem of making values of cubic polynomials squares; in fact, equations $y^2 = f(x)$ for cubic polynomials in general describe curves of genus 1 and have a much richer structure than conics, where f is quadratic.

After having studied triangular numbers that are squares Euler immediately asked the more general question how ax^2+bx+c can be made to be a square.[25] This led Euler to the problem of solving equations of the type $ax^2 + 1 = y^2$ in integers, "which had once been discussed between Wallis and Fermat". Euler referred to a method in Wallis's *Algebra*, which he credited to the English mathematician John Pell but which is actually due to Viscount William Brouncker, the first President

22 See Lemmermeyer 2010 for a detailed discussion of this episode.

23 Cf. Dickson (1919–1923), vol. 2, p. 165.

24 Bachet knew how to obtain additional rational solutions of an equation $y^2 = x^3 + k$ from a known solution; his method is equivalent to what we call the "tangent method". Fermat claimed that the only positive integral solution of $y^2 = x^3 - 2$ is $(x, y) = (3, 5)$, which Euler tried to prove in his *Algebra* (E. 388, § 188, 193: O. I/1, p. 429–432) by using algebraic numbers of the form $a + b\sqrt{-2}$.

25 Triangular numbers that are squares correspond to solutions of the equation $2x^2 + 2x = y^2$.

problem	Euler-Goldbach letters
$\frac{1}{2}n(n+1)+4 \neq a^8$	n° 6
$\frac{1}{2}n(n+1)+10 \neq a^8$	n° 6
$\frac{1}{2}n(n+1) = x^4$	n° 7, 8, 9
$a(a-1)(a-2) = 6b^2$	n° 9
$a(a-1)(a-2) = 3b(b+1)$	n° 9
$2z^4 \pm 2 = x^2$	n° 119, 121
$x^2+y^2+z^2-2x-2y-2z+1=0$	n° 129
$xy(x+y) = a$	n° 139
$xyz(x+y+z) = a$	n° 139
"Pell's equation"	n° 9, 169, 173
"Fermat's Last Theorem"	n° 125, 169, 171
$p^2+eq^2 = a^2+b^2$	n° 168–173, 178–181, 187, 195

Table 2.2: Diophantine problems discussed in the Euler-Goldbach correspondence

of the Royal Society.[26] Brouncker's method consisted in what we call computing a cycle of reduced forms; Euler later realised that this algorithm is equivalent to computing the continued fraction expansion of $\sqrt{a}$, and in n° 169 he mentioned that he had found the solutions of $nx^2+1=y^2$ for $n=61$ and $n=109$ within a few minutes using his "new method" (see also n° 173).

One of Fermat's favourite problems was the investigation of "multiply polygonal" numbers. This problem first occurred in a book attributed to Diophantus,

26 Cf. n° 9, note 7.
The "Pell equation" has a long history: Archimedes used integral solutions of $x^2-3y^2=1$ and $x^2-3y^2=-2$ for approximating $\sqrt{3}$ in his calculation of π, and his famous cattle problem involves several similar equations. Indian mathematicians such as Brahmagupta and Bhaskara developed methods for solving the equation in integers.
In 1657, Fermat challenged mathematicians in Europe, and especially Wallis and Brouncker in England, to show that the equation $Ax^2+1=y^2$ has solutions in integers for all positive non-square integers A. Brouncker found a method for solving these equations similar to the Indian method, and Fermat claimed to have a proof that the equation always has a nontrivial solution.
In his proof that $x^2-Ay^2=1$ is always solvable for non-square positive numbers A, Lagrange mentioned Wallis, but not Brouncker. Legendre did not use the expression "Pell's equation", and Gauss remarked in his *Disquisitiones* (1801) that it was incorrect to credit Pell with this result. Dirichlet, who had studied the *Disquisitiones* inside out, must have known that naming the equation after Pell was incorrect. In fact, he always talked about "the indeterminate equation" $t^2-Du^2=1$; in Dirichlet 1842 he even called it "Fermat's equation"; only in 1846 did he use the phrase "the well-known Pell equation". The first textbooks on number theory and Diophantine equations which started appearing in the middle of the 19th century all used the term "Pell's equation", as did Kummer and Kronecker in their publications. Wilhelm Berkhan, for example, wrote that "the English mathematician J. Pell († 1685)" succeeded in solving the equation $ax^2+1=y^2$ (Berkhan 1856, p. 121).

large parts of which are lost.[27] In a report sent to Carcavi for Huygens,[28] Fermat mentioned the following two problems:

– Given a number, to find in how many ways it is polygonal.
– To find a number which is polygonal in a given number of ways, and to find the smallest such number.

Euler reported on his efforts in this direction in nº 167.

In nº 125, Euler remarked that he had found in Fermat's work the claim that for exponents $n > 2$ the equation $x^n + y^n = z^n$ is impossible in non-zero numbers, and lamented the fact that Fermat's proof was lost. In nº 169 he wrote that he had a proof for the exponents $n = 3$ and $n = 4$, and in nº 171 he mentioned, in connection with the case $n = 5$, the fact that primes dividing $a^5 + b^5$ must have the form $10n + 1$.

In nº 139, Euler remarked that he had studied the solvability of the Diophantine equation $xy(x + y) = a$ and observed that this equation has rational solutions whenever $a = pq(pm^3 \pm qn^3)$ for integers p, q, m, n. For the higher-dimensional equation $xyz(x+y+z) = a$ Euler wrote down a parametric solution which he had found "after taking a lot of trouble", and in fact even with the tools of algebraic geometry available today[29] it is a difficult problem to derive Euler's fantastic parametrisation of certain rational points on the surface defined by this equation.

2.1.2. Quadratic forms and quadratic residues

It seems that the investigation of binary quadratic forms has its origin in Diophantine problems coming from geometry: the form x^2+y^2 is related to the Pythagorean Theorem, and the question which numbers occur as the sum $a + b$ of the two smaller sides in a right-angled triangle with integral sides a, b and c leads to the form $x^2 - 2y^2$: in fact, by the "main theorem on right-angled triangles" we have $a = m^2 - n^2$ and $b = 2mn$, hence $a + b = (m + n)^2 - 2n^2$. It is therefore no surprise that Fermat studied the forms $x^2 + y^2$ and $x^2 - 2y^2$ and their possible prime divisors very early on.

Euler mentioned his first observation in this direction in nº 40: he claimed that odd prime divisors of $x^2 - 2y^2$ all have the form $8n \pm 1$, and presented analogous results for the forms $x^2 - my^2$ with $m = 3$ and $m = 5$. A year later, in nº 54, Euler had extended his calculations to many other values of m and stated that the prime divisors of $x^2 - my^2$ coprime to $2m$ all lie in certain residue classes modulo $4m$. This statement can be seen as the essential part of the quadratic reciprocity law: for if p is an odd prime coprime to m which divides $x^2 - my^2$ for coprime values of x and y, this means that $x^2 \equiv my^2 \bmod p$, so these are exactly the

27 See Heath 1885, p. 247–259.

28 *Relation des nouvelles découvertes en la science des nombres*, August(?) 1659: published in Fermat, *Œuvres*, t. II, p. 431–436, and in Huygens, *Œuvres*, t. II, p. 458–462.

29 Cf. Elkies 2009.

primes for which the Legendre symbol[30] $\left(\frac{m}{p}\right) = +1$. Euler's observation means that $\left(\frac{m}{p}\right) = \left(\frac{m}{q}\right)$ for prime numbers $p \equiv q \bmod 4m$, which is one possible formulation of the quadratic reciprocity law.[31]

In n° 166, Goldbach claimed that for prime $p = 4n + 1$ and for every divisor $d \mid n$, the prime p can be written as $p = x^2 + dy^2$. Euler answered in n° 167 that he already knew this was true – at least if one allowed rational values for x and y – but was unable to prove it. For several years Goldbach and Euler continued to discuss whether primes $p = 4ef + 1$ can be rationally represented by forms $x^2 + ey^2$ (see n° 168–173, 179, 182). This question was answered in a satisfactory manner only by Gauss's theory of binary quadratic forms, and in particular his principal genus theorem.[32]

Goldbach's remark in n° 39 that numbers of the form $(3m + 2)n^2 + 3$ cannot be squares led Euler, in n° 40, to present a result he had known for a long time, namely that numbers of the form $4mn - m - n$ for positive values of m and n cannot be squares. Since the equation $a^2 = 4mn - m - n$ is equivalent to $4a^2 + 1 = (4m - 1)(4n - 1)$, this follows from the fact that sums of two coprime squares cannot be divisible by numbers of the form $4m - 1$, which Euler proved in n° 47.

Euler's result that $a^2 \neq 4mn - m - n$ for positive integers m, n is discussed and generalised in several more of the Goldbach letters, starting with n° 49. In December 1742 (n° 57), Goldbach attempted a proof, but Euler pointed out in his reply n° 60 that this was not valid. In the letters n° 65–72, each of Goldbach's attempts of closing the gap was proved to be incorrect by Euler, until Goldbach eventually found a full proof (via Fermat's descent) in n° 73. The discussion continued until February 1745 (n° 79–87).

2.1.3. Goldbach's conjectures

Goldbach was very fond of studying additive problems involving prime numbers. In n° 51 he put forward the conjecture that every number > 2 is the sum of three primes (where 1 is counted as a prime). Euler replied in n° 52 that this would

30 For odd primes p and a not a multiple of p, the Legendre symbol $\left(\frac{a}{p}\right)$ has the value ± 1 determined by the congruence $a^{(p-1)/2} \equiv \left(\frac{a}{p}\right) \bmod p$.

31 The standard formulation of the quadratic reciprocity law, $\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{(p-1)(q-1)/4}$ for odd primes p and q, is due to Legendre. For a description of Euler's achievement, see Edwards 1983.

32 Cf., e. g., Lemmermeyer 2007.

follow from Goldbach's earlier (but not documented) conjecture that every even number is a sum of two primes.[33]

In n° 57 Goldbach asked for a characterisation of twin primes. In n° 73, he observed that the values attained by the polynomial $f(x) = x^2 + 19x - 19$ are very often prime, but at once added the remark that every non-constant polynomial with integral coefficients must also yield composite values (this result is again mentioned by Euler much later in n° 163). Goldbach's question whether $f(2^m)$ is always prime was answered negatively by Euler in n° 74. Euler's famous prime-producing polynomial $f(x) = x^2 - x + 41$ is not mentioned in the correspondence with Goldbach, but can be found in a published note addressed in 1772 to Johann III Bernoulli.[34]

Goldbach and Euler formulated several incidental questions about the representation of numbers which were more or less disregarded by subsequent research. Some of these conjectures that still remained open when the Euler-Goldbach correspondence was last published in 1965 have by now been settled:

33 Both forms of the Goldbach conjecture – the "ternary" and the stronger "binary" one – appeared in print for the first time in the Cambridge professor Edward Waring's *Meditationes Algebraicae* (1770). At the end of this work, which deals mainly with algebraic equations, Waring adds some paragraphs on number theory. In particular, he remarks: "It seemed appropriate to add here two or three properties of prime numbers", of which the first reads: "Every even number consists of two prime numbers, and every odd number is either prime or consists of three prime numbers, etc." ("Omnis par numerus constat e duobus primis numeris, & omnis impar numerus vel est primus numerus, vel constat e tribus primis numeris, &c.": *op. cit.*, p. 217). The third and last of the additions is, by the way, the observation that for prime p, $(p-1)!+1$ is divisible by p, also published here for the first time and attributed to Waring's student John Wilson.
Waring's discovery is generally assumed to have been independent of Goldbach; however, this is somewhat called into doubt by the fact that, a few pages earlier in the same work, Waring mentions two other results by "D^s. Goldbatch" (as he calls him). Indeed he could have learnt of these results from the short paper that Goldbach had published in the 1724 supplement of *Acta Eruditorum*; but it is also conceivable that a common colleague told Waring about them (probably not Goldbach himself: Waring graduated only in 1757 when Goldbach was no longer active), and in this case Waring might also have heard of Goldbach's conjecture from the same source.
A weaker variant of the ternary Goldbach conjecture had already been stated by Descartes: "But any even number also arises from one or two or three primes" ("Sed et omnis numerus par fit ex uno vel duobus vel tribus primis"). However the manuscript containing this sentence was only published in 1908 (cf. *Varia Mathematica*: Descartes, *Œuvres*, t. X, p. 298).
Lagrange, at the very end of his 1775 *Recherches d'Arithmétique*, observed that "by induction" (i. e., on an observational basis) he had formed another similar conjecture: every prime of the form $4n-1$ is the sum of a prime of the form $4n+1$ and the double of a prime of the same form (like Euler and Goldbach, Lagrange considered 1 to be a prime number in this connection).

34 See E. 461: O. I/3, p. 337.
Goldbach's and Euler's polynomials gained additional interest through a 1912 publication by Frobenius, who showed that they owe their existence to class number phenomena: the prime-producing property of Euler's polynomial $f(x) = x^2 - x + 41$, for example, is related to the fact that the class number of the binary quadratic forms (or, equivalently, the quadratic field) with discriminant -163 is equal to 1.

– Goldbach's claim that the Diophantine equations

$$n^2 + n + 8 = 2a^8 \qquad \text{and} \qquad n^2 + n + 8 = 2a^{10}$$

do not have solutions in positive integers (see n° 6) is correct.

– Goldbach's conjecture that every number $4n + 3$ can be represented as

$$2a^2 + 4b^2 + c^2 + 2 = 2(a + 1)^2 + 4B^2 + 2C^2$$

(see n° 130) is false.

– Goldbach's claim in n° 132 that if a is a positive integer not divisible by 4 with $a^2 < 8m + 7$, then the equation $8m + 7 = a^2 + b^2 + c^2 + d^2$ has an integral solution, follows from the Three Squares Theorem.

Other problems are still open, the most famous being of course the one universally known as the Goldbach conjecture – at least in its binary form: Every even integer > 2 is the sum of two primes. A complete proof of the weaker "ternary" version, according to which every integer > 1 is the sum of at most three primes, has recently been announced.

Euler was right in refusing to rise to Goldbach's bait: neither the 18th nor the 19th century had any tools that permitted to tackle the question. Substantial progress became possible only much later, within the framework of analytic number theory. Among the important contributions to the Goldbach problem(s), we list the following:

1922 Hardy and Littlewood (1923) show that, assuming the Generalised Riemann Hypothesis, every sufficiently large odd integer is the sum of three primes.

1930 Shnirel'man (1930) proves the existence of a number S with the property that every integer > 1 is the sum of at most S primes.

1937 Vinogradov (for a presentation in English see Rexrode 1966) shows that every sufficiently large odd integer is the sum of three primes.

1966 Chen (see Chen 1973) proves that every sufficiently large even integer is the sum of a prime and a number with at most two prime factors.

1995 Ramaré (1995) proves that every integer > 1 is the sum of at most seven primes.

2012 Terence Tao (see Tao 2014) proves that every integer > 1 is a sum of at most five primes.

2013 Harald Helfgott announces that he has completed the proof of the ternary Goldbach conjecture by closing the remaining "size gap" (previously it was known that every odd natural number ≥ 7 smaller than $8 \cdot 10^{26}$ or larger than $2 \cdot 10^{1346}$ is the sum of three prime numbers). As a corollary, every integer > 1 is a sum of at most four primes.

A conjecture made in n° 164, according to which every odd number can be written in the form $2a^2+p$ for primes p, has also received quite some attention, e. g., from Hardy and Littlewood.

Somewhat less known is a beautiful observation due to Goldbach in letter n° 55: let S be the set of all integers a such that $4a^2+1$ is prime; then for every $c \in S$ there are $a, b \in S$ such that $c = a+b$. This would imply in particular that there is always a prime of the form $4a^2+1$ between $4n^2+1$ and $4\cdot(2n)^2+1$, a conjecture reminiscent of "Bertrand's postulate" stating that there is a prime number between n and $2n$ – a theorem which can be proved quite easily.

2.2. Analytic tools in number theory

Many analytic tools used by number theorists have their origin in Euler's work: zeta functions, L-series, theta functions, Lambert series, summation formulae, the dilogarithm and multi-zeta values can all be traced back to Euler.

2.2.1. Zeta functions

The problem of finding a finite expression for the sum $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\ldots$ of the inverse squares had received a lot of attention at the hands of Leibniz and the Bernoullis,[35] after Pietro Mengoli had first posed the question.[36] Jacob Bernoulli had already found in his 1692 *Positiones arithmeticae de seriebus infinitis* (*Pars*

35 In a letter to Johann Bernoulli dated November 1696, Leibniz asked for the value of the sum $\frac{1}{1}+\frac{1}{4}+\frac{1}{9}+$ etc.. Leibniz already suggested using calculus for the evaluation of this sum, and he and Jacob Bernoulli came up with the formula

$$1-\frac{1}{4}+\frac{1}{9}-\frac{1}{16}\pm\ldots=\int_0^1\frac{\log(1+x)}{x}\,dx.$$

The same approach was later taken by Euler when he expressed $\zeta(2)$ with the help of the dilogarithm $\mathrm{Li}_2(x)$ in n° 54. We remark that the substitution $x = 1-\mathrm{e}^{-t}$ shows that

$$\zeta(2)=-\int_0^1\frac{\log(1-x)}{x}\,dx=\int_0^\infty\frac{t\,dt}{\mathrm{e}^t-1}.$$

Although this formula does not seem to have played a major role in Euler's summation of $\zeta(2)$, its generalisation

$$\int_0^\infty\frac{x^{s-1}}{\mathrm{e}^x-1}\,dx=\Gamma(s)\zeta(s),$$

which is an easy consequence of Euler's definition of the zeta function via integrals, occurs in an 1823 paper by Abel as well as in Riemann's famous memoir from 1859 (Riemann 1860) on prime numbers and the zeta function, where it is used to extend the zeta function to the whole complex plane.

36 See Mengoli 1650, *Praefatio*, p. [ix]. For an excellent account of the early history of the zeta function, see Schuppener 1994.

altera, Prop. XXIV) that

$$\sum_{k=1}^{\infty}(2k-1)^m = (2^m-1)\sum_{k=1}^{\infty}(2k)^m.$$

Euler was eventually able to show that the sum of the inverse squares equals $\pi^2/6$. The principal tool used for this result is the product representation of the sine function:

$$\frac{\sin \pi x}{\pi x} = \prod_{n=1}^{\infty}\Big(1-\frac{x^2}{n^2}\Big). \tag{2.2}$$

Euler immediately observed that the same technique allowed him to compute sums of the form[37]

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}$$

for any even integer s. In fact, he found the formula

$$\zeta(2k) = \sum_{n=1}^{\infty} n^{-2k} = (-1)^{k-1}\frac{(2\pi)^{2k}}{2(2k)!}B_{2k}, \tag{2.3}$$

where k denotes an integer > 1, and where the B_k are Bernoulli numbers.[38] Throughout his life, Euler tried without success to find a similar formula for odd integers $s > 1$. By working with the limits $1^n x - 2^n x^2 + 3^n x^3 \mp \ldots$ as $x \to 1$ he could "evaluate" the zeta function at negative integers,[39] and used his results for guessing the correct functional equation of $\zeta(s)$.[40] Euler also found the product decomposition of the zeta function in the form

$$\zeta(s) = \frac{2^s \cdot 3^s \cdot 5^s \cdot 7^s \cdot 11^s \cdots}{(2^s-1)(3^s-1)(5^s-1)(7^s-1)(11^s-1)\cdots},$$

and used it to deduce the fact that the sum $\sum \frac{1}{p}$ of the inverse primes diverges; actually he even found the correct asymptotic behaviour of $\sum_{p\le x}\frac{1}{p}$, which he

37 The notation $\zeta(s)$ was introduced by Riemann in 1859.

38 The convention for the Bernoulli numbers used here and everywhere in this volume is the one suggested by Jacob Bernoulli in *Ars Conjectandi* (1713). In a table for the sums of powers (p. 97–98) he implicitly defines B_m as the coefficient of n in the development of $\sum_{k=1}^{n} k^m$ as a polynomial (of degree $m+1$) in n. This yields the sequence $B_0 = 1$, $B_1 = \frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_3 = 0$, $B_4 = -\frac{1}{30}$, $B_5 = 0$, $B_6 = \frac{1}{42}$, …, which Goldbach mentions in a letter from 1738 (see n° 25, note 2) and which Euler also used in many papers, starting with the recursion formula given in E. 25, § 2 (O. I/14, p. 42–44). See Faber's introduction in O. I/16.2, p. XVI–XXXIX.

39 The fact that Euler's limit coincides with the analytic continuation of the zeta function is not obvious at all.

40 A rigorous analysis of Euler's work on the functional equation that takes account of 19th-century developments can be found in Landau 1906.

formulated as the claim that the sum of the reciprocals of all primes is "so to speak the logarithm of the harmonic series" (i. e., asymptotic to $\log\log x$ as $x \to \infty$).[41]

In his correspondence with Goldbach, these results played only a minor role. In n° 52, Euler considered integrals of the form $\int \frac{x^{m-1} - x^{m-n-1}}{1-x^n}\,dx$ and their connection with the famous expansion[42]

$$\pi \cot \pi x = \frac{1}{x} - \frac{1}{1-x} + \frac{1}{1+x} - \frac{1}{2-x} + \frac{1}{2+x} - \ldots$$

of the cotangent function, which follows easily from (2.2) by taking the logarithmic derivative. These integrals later showed up in Dirichlet's explicit evaluation of his L-series in his proof of the class number formula. Euler derived Leibniz's series for $\frac{\pi}{4}$ as well as his own formula $\frac{\pi^2}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$ and a wealth of similar results from this source.

Euler introduced the function $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$, nowadays called the dilogarithm $\mathrm{Li}_2(x)$, in n° 54 in connection with finding a numerical approximation of $\zeta(2)$, and in E. 20 he proved the functional equation

$$\mathrm{Li}_2(x) + \mathrm{Li}_2(1-x) = -\log x \log(1-x) + \zeta(2)$$

of the dilogarithm.

Expressions that are nowadays called multi-zeta values first show up in Goldbach's letter n° 57; see also n° 59 and 61.

2.2.2. The Pentagonal Number Theorem

In a letter to Euler dated August 29th, 1740, Philippe Naudé (the Younger) asked Euler in how many ways a number n can be written as a sum of positive integers. In his answer written on September 12th (23rd),[43] Euler explained that if we denote this "partition number" by $p(n)$, then

$$\prod_{n=1}^{\infty} \frac{1}{1-x^n} = \sum_{n=0}^{\infty} p(n)\, x^n$$

with $p(0) = 1$. Euler mentioned the identity

$$\prod_{n=1}^{\infty} (1-x^n) = 1 - x - x^2 + x^5 + x^7 - x^{12} - x^{15} + \ldots = \sum_{n=-\infty}^{\infty} (-1)^n x^{\frac{n(3n+1)}{2}} \qquad (2.4)$$

over and over again in his correspondence.[44]

41 See the last theorem of E. 72 (O. I/14, p. 242–244); cf. *infra* n° 163, note 4.

42 Cf. Aigner / Ziegler 2010, p. 149–154.

43 See R 1903–1904: Euler's letter has been edited in Smirnov 1963, p. 179–206.

44 Equation (2.4) is called the Pentagonal Number Theorem since the exponents on the right-hand side are "pentagonal numbers".

Euler's identity (2.4) is closely related to the functional equations of Dedekind's eta function

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \qquad q = \mathrm{e}^{2\pi i z}$$

and Jacobi's theta function[45]

$$\theta_3(v, \tau) = \sum_{n=-\infty}^{\infty} x^{n^2} \mathrm{e}^{2n\pi i v}, \qquad x = \mathrm{e}^{\pi i \tau}.$$

In addition, (2.4) is a special case of Jacobi's triple product identity for θ_3: setting $x = q^{3/2}$ and $z = q^{1/2}$ in

$$\prod_{n=1}^{\infty} (1 - x^{2n})(1 + x^{2n-1} z)(1 + x^{2n-1} z^{-1}) = \sum_{n=-\infty}^{\infty} x^{n^2} z^n$$

yields Euler's identity (2.4).

Gauss observed[46] that Euler's series (2.1), whose exponents are the triangular numbers, admits a similar product expansion:

$$1 + x + x^3 + x^6 + x^{10} + \ldots = \frac{1 - x^2}{1 - x} \cdot \frac{1 - x^4}{1 - x^3} \cdot \frac{1 - x^6}{1 - x^5} \cdot \frac{1 - x^8}{1 - x^7} \cdots .$$

These remarks show that what must have been – at first, at least – little more than a curious identity for Euler became an important tool in modern analytic number theory. Euler did see, however, the connection with classical number theory: in n° 113 (see also n° 114 and 115), Euler finds the recursion

$$\sigma(n) = \sigma(n-1) + \sigma(n-2) - \sigma(n-5) - \sigma(n-7) + \sigma(n-12) + \sigma(n-15) - \ldots$$

for the function $\sigma(n)$, which denotes the sum of all divisors of the number n. The connection with the Pentagonal Number Theorem is presented in n° 144, along with the remark that Euler had now found a proof; this proof was published in E. 244, another two in E. 541.[47]

See Euler's letters R 236, R 238 to Nicolaus I Bernoulli (September 1st, 1742; November 10th, 1742: O. IVA/2, p. 518–519, 555–560) and to Goldbach (n° 74, October 15th, 1743; see also n° 75, 76, 102), Daniel Bernoulli's reply R 140 (January 28th, 1741: to be published in O. IVA/3), as well as R 6 to Adami (February 14th, 1749) and R 23, R 25 to d'Alembert (December 30th, 1747; February 15th, 1748: O. IVA/5, p. 275, 281). Euler published these results in E. 101 and E. 158.

45 The eta and the theta functions are important tools in the theory of elliptic functions (see, e. g., Dedekind 1877).

46 This relation is a special case of a more general identity proved in Gauss 1811, the article in which Gauss determined the sign of quadratic Gauss sums and applied it to give his fourth proof of the quadratic reciprocity law. Gauss remarked that "the equality between two technical expressions [. . .] certainly is highly remarkable". For a modern proof of Gauss's identity and the derivation of the sign of quadratic Gauss sums see Shanks 1958.

47 For a thorough study of the Pentagonal Number Theorem, see Bell 2010.

2.2.3. Bernoulli numbers

The numbers that Euler named after Jacob Bernoulli first showed up in connection with explicit formulae for the power sums $\sum_{a=1}^{n} a^k$. They occur over and over again in Euler's work, for example in the explicit evaluation of $\zeta(2n)$ for integers $n \geq 1$ (see (2.3)), or in the summation formula named after Euler and MacLaurin.[48]

This summation formula is a precise version of rules for approximating integrals, such as the trapezoid rule, which says that

$$\int_0^n f(t)\,dt \approx \frac{1}{2}f(0) + f(1) + f(2) + \ldots + f(n-1) + \frac{1}{2}f(n),$$

or Simpson's rule. Euler wanted to turn this around and find a method for summing series using integrals:

$$f(0) + f(1) + \ldots + f(n) \approx \int_0^n f(t)\,dt + \frac{f(0)+f(n)}{2}.$$

By adding higher correction terms he found, neglecting remainder terms or convergence criteria, that[49]

$$\sum_{k=0}^{n} f(k) = \int_0^n f(t)\,dt + \frac{f(0)+f(n)}{2} + \sum_{k=2}^{\infty} \frac{B_k}{k!}\Big(f^{(k-1)}(n) - f^{(k-1)}(0)\Big).$$

Since the higher derivatives vanish when f is a polynomial, Euler could derive the summation formulae for kth powers that had already been obtained by Jacob Bernoulli (see n° 54 as well as E. 352 and E. 746); it is this connection which made Euler call the numbers B_k Bernoulli numbers in the first place.

Euler presented his summation formula in E. 25, and showed how to derive it in E. 47.[50] He gave the generating function for Bernoulli numbers in E. 130.

Goldbach mentioned that he had found a formula for Bernoulli numbers in n° 25. Euler used his summation formula in n° 64 for giving a (divergent) expression for $\zeta(3)$ in terms of Bernoulli numbers, and for computing an approximation of π in n° 66; a related formula involving $\zeta(2)$ can be found in n° 68.

48 Schuppener (1994, p. 73–89) casts some doubt on the statement – found among others in Faber's introduction to O. I/16.2 (p. VIII, note 2) – that MacLaurin discovered the summation formula independently, and points out the possibility that he could have learned about Euler's result from his correspondence with Stirling, to whom Euler had sent the formula in June 1736, several years before the publication of MacLaurin's *Treatise on Fluxions* in 1742. Schuppener also remarks that the summation formula published in 1827 by Poisson is just a modification of the Euler-MacLaurin formula, and that what today is known as the Poisson summation formula first occurs in some writings of Gauss that were published only in 1900.

49 An excellent introduction to the Euler-MacLaurin summation formula can be found in Edwards 1974, ch. 6.

50 Schuppener 1994 analyses the contributions of Euler, MacLaurin, Poisson etc. concerning summation formulae in careful detail.
Paraphrases and commentaries on many of Euler's early papers on series can also be found in Hofmann 1959, Ferraro 1998, Varadarajan 2006, Sandifer 2007b and Ferraro 2008.

2.3. Algebra: roots of polynomials and transcendence

Algebra in Euler's times basically consisted of studying properties of roots of polynomials: their existence, their representation by radicals, and their approximation by real (or complex) numbers. The entire second part of Euler's *Introduction to Algebra* consists of "indeterminate analysis", that is, the theory of Diophantine equations.

2.3.1. Squaring the circle

The notions of irrationality and transcendence of numbers developed in a rather tortuous way; before we come to the contributions by Goldbach and Euler, we will therefore briefly sketch some relevant stages of their history.

Euclid distinguished carefully between numbers, which were proper multiples of the unit such as 2, 3, 4, ...,[51] and magnitudes, such as line segments or areas of simple figures. Ratios of line segments, for example the ratio between the diagonal and the side of a square, can be commensurable or not; in the first case, their ratio can be expressed as a ratio of (natural) numbers, and in the second case this is impossible. Euclid investigated different types of incommensurable ratios in his Book X; magnitudes whose ratio is equal to a ratio of numbers were called commensurable.[52] Among the incommensurable numbers, Euclid distinguished those whose square is rational: such ratios were called enunciable.[53]

Daniel Bernoulli, Goldbach and Euler often used the word *irrational* in Euclid's original sense: in his *Introductio*, for example, Euler remarked that logarithms of rational numbers to a rational base cannot be "irrational", since if $\log_a b = \sqrt{n}$, then $a^{\sqrt{n}} = b$, which he claimed was impossible for rational numbers a, b.[54] In the summary of a paper from 1758, Euler observed that all irrational quantities which arise from the extraction of (square) roots can be constructed geometrically.[55] By the time he wrote his *Algebra*, he also regarded cube roots of noncubes as irrational

51 The unit 1 was not considered as a number (let alone a prime number) by Euclid. In a letter to Wallis dated February 18th, 1657/8, Brouncker writes: "For that 1 is not a number in the opinion of some, every one knows; but they all doe know as well, that it is a number in the opinion of others." (see Wallis 1658, Letter XX).

52 The Greek word is σύμμετρος (*symmetros*) and means that the magnitudes in question have a common measure.

53 The Greek word is ῥητός (*rhetos*); its opposite was called ἄλογος (*alogos*), which was translated into Latin as *irrationalis* by the translator Gerard of Cremona (ca. 1114–1187) and as *surdus* by Leonardo da Pisa (Fibonacci). For more on the usage of the terms "surd" and "irrational number" see Tropfke 1930, vol. 2, p. 94–95.

54 *Introductio in analysin infinitorum*, T. I (E. 101), § 105 (O. I/8, p. 108).

55 "Incommensurabilitas autem in se spectata non obstaret, quominus ratio diametri ad peripheriam geometrice assignari posset, cum quadrati diagonalis ad latus quoque sit incommensurabilis atque in genere omnes quantitates irrationales, quae ab extractione radicum oriuntur, geometrice construi possint": E. 275, *Summarium* from *Novi Commentarii* VIII (O. I/15, p. 1).

numbers.[56] However, Euler was aware that the sort of irrationality exhibited by the rectification of the circle is of a qualitatively different kind:

> "However, the circumference of a circle must be considered to belong to a much higher class of irrationals, which can only be reached by repeating the extraction of roots an infinite number of times; thus it is also impossible to do more geometrically than to approximate the true ratio of circumference and diameter more and more closely."[57]

Numbers that are neither rational nor irrational were occasionally called transcendental,[58] but the definitions and terminology that we use today were only firmly established by Lambert in the 1760s.

Beyond the facts obtained by the Greeks (Theaetetus is credited by Plato with proving the irrationality of $\sqrt{n}$ for all non-square numbers n), definite results on irrationality were few and far between: let us just mention in passing the reflections of Leonardo da Pisa (Fibonacci) in the 13th, Stifel in the 16th, Stevin and Fermat[59] in the 17th century.

Thus Goldbach was one of the first who actually asked for *proofs* of irrationality:[60] In a letter to Goldbach dated April 28th, 1729, Daniel Bernoulli had claimed that the numbers $\log \frac{p+q}{p}$ $(0 < q < p)$ "not only cannot be expressed as rational numbers, but neither as radicals or irrational numbers." In his reply, Goldbach wondered about the reasons supporting this assertion, but Bernoulli could not deliver; he even admitted the possibility that future mathematicians might find such an expression, which would immediately lead to the quadrature of the hyperbola and perhaps that of the circle. A few letters later, Bernoulli went on to ask whether Goldbach could specify a number that could be proved not to be a root of any degree of a rational number. In his letter from October 20th, 1729, Goldbach

56 The title of Chapter 15 is "On cube roots and the irrational numbers arising from them" ("Von den Cubikwurzeln und den hiedurch entstehenden Irrationalzahlen").

57 "Verum peripheria circuli ad genus irrationalium longe sublimius referenda videtur, ad quod demum radicis extractione infinities repetita pertingere liceat, unde etiam geometrice plus praestari non potest, quam ut vera peripheriae ad diametrum ratio continue propius exprimatur." (In the *Summarium* of E. 275 this sentence immediately follows the one quoted above).

58 In 1685, Wallis had called for the invention of new numbers beyond rationals, surds, or roots of polynomial equations.
Euler used the term "transcendental" more often with respect to "quantities" (i. e., functions) than for individual numbers; he never seems to have given a precise definition. In the introduction to E. 71, Euler remarked that "irrational and transcendental quantities, among which are logarithms, circular arcs, and the lengths of other curves, are frequently expressed by infinite series".

59 See Fermat's letter to Roberval dated September 16th, 1636 (*Œuvres complètes*, t. II, p. 59–63).

60 Indeed Euler, just as most of his contemporaries, regularly asserted that numbers such as π, e, or certain logarithms are not rational, but never even attempted to prove these claims.

came up with the number $\sum_{k=1}^{\infty} 10^{-2^{k-1}} = 0.1101000100000001\ldots$; Bernoulli acknowledged that this has the desired property, since none of its powers can have a periodic decimal expansion.[61]

In 1720, Goldbach had proved the formula

$$H_f = \sum_{k=1}^{\infty} \frac{f}{k(k+f)} = 1 + \frac{1}{2} + \ldots + \frac{1}{f} \tag{2.5}$$

for positive integers f; this had, however, already been known to Pietro Mengoli (see Mengoli 1650) and to Jacob Bernoulli. In his letter to Daniel Bernoulli from August 18th, 1729, Goldbach mentioned this result and claimed that the sum $\sum_{k=1}^{\infty} \frac{f}{k(k+f)}$ is irrational for nonintegral rational values of f.[62]

In the early 18th century, many mathematicians were convinced that π is not rational, and perhaps not even irrational in the sense of belonging to the classes of irrational ratios studied by Euclid. In n° 2, his first letter to Euler, Goldbach wrote "that the quadrature of the circle cannot be effected by rational numbers". This claim was not novel: there is even an ambiguous statement in Aristotle's *Physics* (VII, part 4) which some interpret as a claim that the circumference and the diameter of a circle are not commensurable. The Hindu mathematician Nilakantha Somayaji (ca. AD 1500) is credited with the question of why we give rational approximations to π instead of its exact value, and also with the answer that this is because the ratio of the circumference and the diameter cannot be expressed as a ratio of two integers.[63] In 1656, Wallis claimed that π is neither a fraction nor a surd; Grégoire de Saint-Vincent (1647) made the first attempt at *proving* that π is not rational, but his proof contains errors that were criticised, among others, by Huygens.[64]

Lagny, whose paper Goldbach mentions in letter n° 8, conjectured (in geometric language) that $\tan \pi x$ is not rational for any rational number x with $0 < x < \frac{1}{2}$ except for $\frac{1}{4}$; Lambert later proved the much deeper fact that $\tan x$ is irrational for all rational values of x with $\tan x \neq 0$, a fact that implies the irrationality of π since $\tan \frac{\pi}{4} = 1$ is rational.[65]

Goldbach had proved early on that square roots of numbers divisible by 3 but not by 9 cannot be rational (see n° 10, note 5), and in n° 39 he extended this to numbers of the form $(3m+2)n^2 + 3$. Similar results had been known for a long time, but Goldbach's ongoing investigation shows that he was trying hard to go beyond simply *asserting* irrationality.

From his work on the Riccati differential equation, which is mentioned for the first time in letter n° 15, Euler had deduced the simple continued fraction

61 Cf. Fuß, *Correspondance*, t. II, p. 301, 306, 310, 318–319, 326, 329.

62 *Correspondance*, t. II, p. 312–313.

63 See Brezinski 1991, p. 84, and Ramasubramanian 2011.

64 See Baron 1969, p. 230.

65 Lambert 1768; for a modern exposition of Lambert's proof see Petrie 2009.

expansions[66] of e and $(e^2 - 1)/2$. Since these expansions are neither finite nor periodic, it follows from results later proved by Lagrange that e and e^2 are irrational numbers.[67]

In nº 69, Goldbach alluded to his construction of irrational numbers in his discussion with Daniel Bernoulli. Euler answered in nº 70 that this idea was known to him, and explained why rational numbers have periodic decimal expansions. In E. 565, Euler observed that putting $x = \frac{1}{10}$ in the series (2.1), in which the exponents are triangular numbers, produces the number $1.101001000100001\ldots$, which cannot be rational since the number of zeros placed between the units increases continually.

2.3.2. The Fundamental Theorem of Algebra

After the Italians had shown how to solve cubic and quartic equations, it was natural to look at equations of degree 5 and higher. In nº 18, Goldbach mentions a solution of the quintic $x^5 + \frac{5}{2}mx + D = 0$ under suitable conditions. Much later, in nº 165, Euler studies an idea of de Moivre's for investigating the roots of certain quintics. He proposes writing the roots of a polynomial of degree n as a linear combination of nth roots, but – as is now well known – this does not work for general polynomials of degree ≥ 5.

The fact that polynomial equations have as many roots as the highest occurring exponent indicates had already been asserted by Girard (1629) and in Descartes' *Géometrie* (1637). One version of the Fundamental Theorem of Algebra claims that every polynomial with real coefficients can be written as a product of real linear and quadratic factors. In this form, the theorem is used for integrating rational functions via their partial fraction decomposition. In nº 58, Euler mentions his correspondence with Nicolaus I Bernoulli on this topic and explains, as an example, how to factor the polynomial $x^4 - 4x^3 + 2x^2 + 4x + 4$. In nº 64 Euler reports that he can prove the Fundamental Theorem for polynomials of degree < 6.[68]

66 Actually, a continued fraction expansion for $e - 1$ had already been obtained by Roger Cotes (1714); cf. Gowing 1983, p. 24–26, 147–148, and Fowler 1992, p. 361–363.

67 Euler never claimed to have proved the irrationality of e. In E. 71, he remarked that rational numbers can be transformed into a *finite* simple continued fraction (all numerators equal to 1), and that "irrational and transcendental" numbers have infinite simple continued fractions. This implies that infinite simple continued fractions must be irrational in the modern sense if it can be shown that *every* simple continued fraction expansion of a rational number is finite. This in turn would follow from the fact that the simple continued fraction expansion of a number is essentially unique – but Euler neither states nor proves this. On the other hand he seems to have been convinced that e cannot be expressed as a rational number when he wrote in E. 71 (translation taken from Wyman 1985): "These continued fractions converge so fast that it is an easy matter to find the values of e and $\sqrt{e}$ as closely as you please."

68 An attempt to deal with the general case was presented by d'Alembert in 1746 (printed in 1748), but neither this nor Euler's argument in E. 170 are nowadays regarded as complete; the first proofs now accepted as rigorous are due to Gauss and Argand. For a thorough discussion of the origins of the Fundamental Theorem of Algebra and its early proofs, see Gilain 1991.

Approximations of roots of algebraic equations come up in Euler's letters n° 19 and n° 80, and in his last letter n° 196 he mentions Lambert's series for the largest root of a special trinomial.

2.4. Analysis

For properly understanding the breadth of Euler's contribution to analysis one has to recall that calculus had just two generations before been invented by Newton and Leibniz and extended immensely at the hands of Jacob and Johann Bernoulli. After Euler's "algebraic analysis", the *Sturm und Drang* period continued at the hands of Lagrange (calculus of variations), Legendre (elliptic integrals), Fourier (trigonometric series) and others, and ended only when Cauchy's work shifted the focus of mathematicians' interest to the new area of complex analysis.

2.4.1. Interpolation

The term "interpolation" is nowadays attached to the process of finding the value of a function at some point lying between two tabulated (or otherwise known) values. Wallis used the word interpolation for describing a different problem: Given a sequence a_1, a_2, a_3, …, find "the value between the first and the second term". In other words, he was looking for a (preferably simple) formula for a_n that would make sense even for non-integral values of n, in particular for $n = \frac{3}{2}$.

In 1651, Wallis began looking into the problem of "squaring the circle". He wanted to compute what we would write as[69] $\int_0^1 \sqrt{1-x^2}\,dx$, and looked at the more general quantities

$$\frac{1}{A_{p,n}} = \int_0^1 (1 - x^{1/p})^n\,dx$$

where the case he was interested in was $A_{\frac{1}{2},\frac{1}{2}}$. By evaluating special cases, Wallis was led to conjecture

$$A_{p,n} = \binom{n+p}{n}.$$

Manipulating series and guessing patterns, he eventually reached the conclusion that

$$A_{\frac{1}{2},\frac{1}{2}} = \frac{3\cdot 3}{2\cdot 4}\cdot\frac{5\cdot 5}{4\cdot 6}\cdot\frac{7\cdot 7}{6\cdot 8}\cdots = \frac{4}{\pi}. \tag{2.6}$$

Substituting this result into the formula $A_{p,n} = \binom{n+p}{n} = \frac{(n+p)!}{n!\,p!}$ yields

$$\binom{1}{\frac{1}{2}} = \frac{1}{(\frac{1}{2})!\,(\frac{1}{2})!} = \frac{4}{\pi},$$

69 Wallis did not use integral signs, indices as we know them or a particular notation for our binomial coefficients.

that is,

$$\left(\frac{1}{2}\right)! = \frac{\sqrt{\pi}}{2}.$$

Fermat did not think highly of Wallis's *Arithmetica Infinitorum* (1656), mainly because he did not accept Wallis's use of induction (in the early-modern sense of "empirical mathematics") as a method of proof. Fermat could not do a lot with Wallis's interpolation problem, just as Wallis had little respect for Fermat's number-theoretical problems.[70]

Other mathematicians, however, found Wallis's work on interpolation highly inspiring. Newton, in his famous "second letter" to Oldenburg dated October 24th, 1676, explained how Wallis's method led him to the binomial theorem. Stirling, a student of Newton's, attacked the problem of interpolating Wallis's "hypergeometric sequence" 1, 2, 6, 24, 120, ...: actually he interpolated the sequence $\log n!$ numerically – in the modern sense of the word – and from $(\frac{1}{2})! \approx 0.8862269\ldots$ guessed correctly[71] that $(\frac{1}{2})! = \frac{\sqrt{\pi}}{2}$.

The problem of interpolating sequences was taken up by Goldbach in 1722.[72] He claimed that he could give the value of the intermediate terms of *any* sequence, at least as an infinite series;[73] given a sequence (a_n), it is easily proved by induction[74] that

$$a_n = \sum_{k=0}^{\infty} \binom{n-1}{k} \Delta^k a_1$$

for all $n \in \mathbb{N}$; here $\Delta^0 a_1 = a_1$, $\Delta^1 a_1 = a_2 - a_1$, $\Delta^2 a_1 = (a_3 - a_2) - (a_2 - a_1) = a_3 - 2a_2 + a_1$ etc. But, once the binomials have been interpolated, the expression on the right-hand side is defined for all real numbers $n \geq 1$. Goldbach mentioned this result in a letter to Daniel Bernoulli, who replied that the intermediate terms should be given by a "finite expression" rather than by infinite series.[75]

Euler's contemporaries who were working on the problem of interpolating sequences were all convinced that certain "natural" sequences such as $n!$ could be assigned values for non-integral values of n as well, and apparently most of them were also convinced that any natural method of assigning values to such expressions would give the same result.[76] In fact, in his letter n° 91 Euler writes:

70 See in particular Wallis's letter to Kenelm Digby: Fermat, *Œuvres*, t. III, p. 427–457.

71 For Stirling's well-known approximation of $\log n!$, see Stirling 1730, Prop. XXVIII.

72 See his letter to Nicolaus II Bernoulli dated January 2nd, 1722 (*Correspondance*, t. II, p. 128).

73 His results were printed in 1732, in a paper which he had sent to Nicolaus II Bernoulli in January 1728 and to Daniel Bernoulli in November 1728.

74 Goldbach's formula is mentioned in Perron's book on continued fractions as being "well-known" (Perron 1913, § 32). A similar technique was used by Hasse for extending the zeta function analytically to the entire complex plane (see Hasse 1930, in particular the remark at the end).

75 Letter to Goldbach dated Jan. 30th, 1729: *Correspondance*, t. II, p. 247–248.

76 For the general context of their methods, cf. Hofmann 1959, Ferraro 2008.

> "I am confident that one and the same series can never arise by developing two finite expressions that are really different."

Euler thought that any interesting series arises by developing some finite "analytic expression", and conjectured that this cannot be done in two essentially different ways, thus allowing the solution of the interpolation problem by finding this finite expression. On the other hand, Euler realised that there are infinitely many ways of interpolating sequences: in a paper from 1749, he writes:

> "even if all the terms of a series that correspond to integral indices are determined, one can define the intermediate ones, which have fractional indices, in an infinite variety of ways so that the interpolation of this series continues to be indeterminate."[77]

In fact, if (a_n) is a given sequence and $a_n = f(n)$ for some finite expression f, then we also have $a_n = f(n) + h(n)$ for any function h that vanishes at the integers, e. g., $a_n = f(n) + \sin \pi n$ for all $n \in \mathbb{N}$.

An even more serious problem arose when Euler, in E. 190, investigated the series

$$s_a(x) = \frac{1-x}{1-a} + \frac{(1-x)(a-x)}{a-a^3} + \frac{(1-x)(a-x)(a^2-x)}{a^3-a^6} + \dots,$$

which has the property $s_a(a^n) = n$; in other words, it takes the same values at $x = a^n$ as the function $\log_a(x)$. On the other hand, we have

$$s_a(0) = \frac{1}{1-a} + \frac{1}{1-a^2} + \frac{1}{1-a^3} + \frac{1}{1-a^4} + \dots, \tag{2.7}$$

and this clearly implies that $s_a(x) \neq \log_a(x)$. Thus $s_a(x)$ interpolates the same sequence as $\log_a(x)$, yet the two functions are different![78]

Euler observed that for numbers $a > 1$, the series (2.7) yields a finite value; however, this

> "cannot be expressed either in rational or in irrational numbers. It appears therefore especially worth the effort that mathematicians investigate the nature of that transcendental quantity by which its sum is expressed."[79]

77 "etsi omnes seriei termini, qui indicibus integris respondent, sunt determinati, intermedios tamen, qui indices habent fractos, infinitis variis modis definire licet, ita ut interpolatio istius seriei maneat indeterminata": E. 189, § 3 (O. I/14, p. 466).

78 Cf. Gautschi 2008; Bell 2010, p. 323–326.

79 "tamen neque numeris rationalibus neque irrationalibus exprimi potest. Quocirca ea imprimis digna videtur, ut Geometrae naturam illius quantitatis transcendentis investigent, qua eius summa exprimatur": E. 190, § 28 (O. I/14, p. 538).

The function $-s_a(x)$, with $a = \frac{1}{q}$, can be interpreted as a q-analog of the logarithm; the values $s_a(0)$ for integers $a > 1$ are known to be irrational, but their transcendence still seems to be open.[80]

As another side remark, Euler stated the result

$$\frac{1}{a-1} + \frac{1}{a^2-1} + \frac{1}{a^3-1} + \frac{1}{a^4-1} + \ldots = \sum_{n\geq 1} \tau(n) a^{-n},$$

where $\tau(n)$ denotes the number of divisors of a number n. Lambert observed the same fact,[81] which is why series of the form $\sum b_n \dfrac{x^n}{1-x^n}$ are called Lambert series. Dirichlet[82] used Lambert series to derive the average behaviour of the function $\tau(n)$. We note in passing that the identity

$$-\log \prod_{n=1}^{\infty} (1-x^n) = \sum_{n=1}^{\infty} \log \frac{1}{1-x^n} = \sum_{n=1}^{\infty} \frac{1}{n} \frac{x^n}{1-x^n}$$

(the series involved converge for $|x| < 1$) relates the product in the Pentagonal Number Theorem to a Lambert series.[83]

Euler's reason for starting a correspondence with Goldbach was his desire to communicate his own results to someone who had already worked on the interpolation problem for a long time; it was Daniel Bernoulli who suggested Goldbach's name to Euler. In n° 1, Euler presented his interpolation of the sequence of factorials, and in particular showed how to express $(\frac{1}{2})!$ using the quadrature of the circle. He also gave an interpolation of the harmonic series: using the substitution $\frac{1}{n} = \int_0^1 x^{n-1}\,dx$, he found

$$H_n = \sum_{k=1}^{n} \frac{1}{k} = \int_0^1 (1 + x + x^2 + \ldots + x^{n-1})\,dx = \int_0^1 \frac{1-x^n}{1-x}\,dx$$

for all integers $n \geq 1$; observe that the right-hand side has a well-defined meaning for all real exponents $n \geq 0$.[84] By writing $\dfrac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ as a geometric series, the integral can be transformed into the power series

$$\begin{aligned} H_n &= \int_0^1 (1-x^n) \sum_{k=1}^{\infty} x^{k-1}\,dx = \int_0^1 \sum_{k=1}^{\infty} (x^{k-1} - x^{k-1+n})\,dx \\ &= \sum_{k=0}^{\infty} \Big(\frac{1}{k} - \frac{1}{k+n}\Big) = \sum_{k=1}^{\infty} \frac{n}{k(k+n)}, \end{aligned}$$

80 Cf. Erdős 1948 for $a = 2$, Borwein 1991 and Koelink / van Assche 2009 for all integers $a > 1$.

81 See Lambert 1771.

82 Cf. Dirichlet 1838.

83 Cf. again Bell 2010.

84 Dirichlet later used the same trick to compute the values of his L-series at $s = 1$.

thus giving (2.5), a formula that Goldbach had already found directly and which Euler later published in E. 613. Although Goldbach's and Euler's formulae are equivalent from a modern point of view, their contemporaries much preferred Euler's formulation since definite integrals (quadratures of bounded areas) were seen as "finite" objects, whereas power series or infinite products were regarded as expressions that could not be determined exactly, since an infinite number of arithmetical operations was involved.

In the long run, Euler's method of interpolating sequences using integrals turned out to be more powerful than Goldbach's idea of using power series. Euler's interpolation of the sequence of factorials presented in n° 1 is equivalent with the modern definition of the Gamma function as an integral; although Euler at first only regarded "finite analytic expressions" as functions, he later accepted the interpolated sequence of factorials as a genuine function, and in n° 80 Euler already discussed the derivative of the Gamma function.

Some less well-known mathematicians also studied interpolation problems: in letter n° 112, Euler mentions a question by the lawyer and amateur mathematician Jacob Adami[85] concerning the interpolation of the terms of the power series expansion of $\tan x$.

2.4.2. Riccati's differential equation

Wallis's product formula (2.6) for π is related (although not in an obvious way) to a continued fraction expansion for $\frac{4}{\pi}$ found by Brouncker at about the same time. Brouncker claimed that

$$\frac{4}{\pi} = 1 + \cfrac{1^2}{2 + \cfrac{3^2}{2 + \cfrac{5^2}{2 + \ldots}}} \tag{2.8}$$

but did not supply a proof. In E. 123, Euler compared the continued fraction expansion of $\int_0^1 \frac{dx}{1+x} = \log 2$ to Brouncker's formula (2.8) for $\int_0^1 \frac{dx}{1+x^2} = \frac{\pi}{4}$, and then more generally studied the continued fraction expansions of the integrals $\int_0^1 \frac{dx}{1+x^k}$.

When Euler computed the continued fraction expansion of e, he observed a pattern that led him into investigating continued fractions

$$[a_0, a_1, a_2, a_3, \ldots] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots}}} \tag{2.9}$$

in which $a_1, a_2, \ldots$ form an arithmetic sequence (see E. 71). He managed to find an expression for the nth convergent of such a continued fraction, and wrote the limit

85 Cf. Adami's letter R 2 to Euler dated August 16th, 1746.

for $n \to \infty$ as a quotient of power series.[86] These power series satisfy a certain differential equation, which, after a few simple transformations, takes a form that was by then already well known and had been much investigated: that of Riccati's equation. This allowed Euler to compute the values of such continued fractions from solutions of the Riccati differential equation

$$q' = nr^{n-2} - q^2, \tag{2.10}$$

where q is a function of r. Solving this equation in the case $n = 2$ gave Euler a continued fraction expansion of

$$\frac{e^{2/a}+1}{e^{2/a}-1} = \frac{e^{1/a}+e^{-1/a}}{e^{1/a}-e^{-1/a}}$$

which today is usually written in the form

$$\tanh z = \cfrac{z}{1+\cfrac{z^2}{3+\cfrac{z^2}{5+\cfrac{z^2}{7+\cdots}}}}\,.$$

Substituting $z \to iz$ in this equation shows that[87]

$$\tan z = \cfrac{z}{1-\cfrac{z^2}{3-\cfrac{z^2}{5-\cfrac{z^2}{7-\cdots}}}}\,.$$

These expansions were the basis of Lambert's proof that e^z and $\tan z$ are irrational for all nonzero rational values of z; since $\tan\frac{\pi}{4} = 1$, this implies the irrationality of π.[88]

86 From a modern point of view, Euler's proof in E. 71 is not rigorous (for a beautiful survey see Havil 2012, ch. 3). Euler later gave other derivations of the continued fraction expansion: for a discussion of Euler's results see, e. g., Khrushchev 2006. The continued fraction expansion of e can also be determined from Hermite's proof that e is transcendental (see Cohn 2006).

87 Euler derived the continued fraction expansions of $\tan x$ and $\cot x$ from that of $\frac{e^x+1}{e^x-1}$ in E. 594, E. 595, and E. 750. The first extensive investigation of the hyperbolic functions and their analogy with trigonometric functions was published by Lambert in 1768.

88 Legendre observed that Lambert's proof even shows that π^2 is irrational, as follows easily by assuming that $\pi = \sqrt{m}$ for some rational m and substituting $z = \sqrt{m}$ into the continued fraction expansion of $z \tan z$. Siegel (1929, p. 231) was able to show that all continued fractions of the form (2.9), in which the a_i form an arithmetic sequence, are transcendental. In his proof, Siegel used the fact that the logarithmic derivative of a solution of the Bessel differential equation satisfies a Riccati equation.

Differential equations of the type

$$y' = p + qy + ry^2,$$

where p, q and r are functions of the independent variable x, are called Riccati equations. Jacopo Riccati had started studying special cases of this equation around 1714, motivated by the example

$$nx^2 - ny^2 + x^2y' = xy$$

mentioned in Gabriele Manfredi's 1707 treatise. Riccati was able to solve certain equations of this type using the method of separation of variables (Johann I Bernoulli immediately claimed this method as his own, which led to a priority dispute).[89]

The differential equation

$$y' = ax^m + by^2x^p, \tag{2.11}$$

which is slightly more general than (2.10), was studied by Nicolaus II Bernoulli: in a letter to Goldbach dated December 6th, 1721, he claimed that the variables in (2.11) can be separated when $m = -\frac{(2n \pm 1)p + 4n}{2n \pm 1}$ for some positive integers m, n, p.[90] Some months earlier, Daniel Bernoulli had informed his brother in a letter sent from Basel to Venice that he had found a similar, if slightly less general result: for the equation

$$y' = ax^m + by^2, \tag{2.12}$$

separation of variables works and an explicit solution depending on "the quadrature of the circle and the hyperbola" can be given when $m = -\frac{4n}{2n \pm 1}$.[91]

Daniel Bernoulli's claim was proved by Liouville in 1841, in the more precise form that (2.12) has a solution in elementary functions (algebraic expressions of x, trigonometric functions, exponential functions and logarithms) if and only if $m = -\frac{4n}{2n \pm 1}$.

Bernoulli had solved (2.12) by repeatedly applying certain transformations[92] for reducing the exponent m; the individual steps look a lot like applying some kind of Euclidean algorithm, which may have given Euler the idea of writing

89 For a detailed history of the Riccati equation, see Bottazzini's account in his introduction to vol. 1 of Daniel Bernoulli's works (Bottazzini 1996), p. 176–188.

90 Cf. *Correspondance*, t. II, p. 119–124.

91 Daniel's solution was published in 1724 in his *Exercitationes quaedam mathematicae* and reprinted in the 1725 *Acta Eruditorum*; the original letter to his brother seems to be lost, but an extract was included in the posthumous publication of Nicolaus' results in t. I of the Petersburg *Commentarii* (1728), the same volume in which Goldbach published his own results on the Riccati equation.

92 These transformations still show up in 20th-century accounts: see, e. g., Kamke 1956, § 4.20.

down a single substitution in the form of a continued fraction which transformed (2.12) into the simpler equation $y' = a + by^2$ in a single step. Euler communicated this observation to Goldbach in his letter n° 15. The integrability of differential equations, in particular of the Riccati equation, was discussed in n° 16 and 17 (see Euler's articles E. 11 and E. 12), as well as in n° 173.

2.4.3. Integrability of binomial differentials and elliptic integrals

The problem of classifying those triples of rational numbers m, n and p for which the differential $x^m(a + bx^n)^p dx$ can be integrated by elementary functions goes back to Newton and Jacob Bernoulli. After having heard from Daniel Bernoulli that Goldbach had found new results in this direction, Euler reported in n° 11 that he believed that the integral $\int \frac{a^2dx}{\sqrt{a^4 - x^4}}$ cannot be expressed in terms of what we call "elementary functions". Euler knew that this integral shows up in the rectification of ellipses and in connection with the lemniscate and the *elastica* (the curve that a thin beam describes when it is bent). In n° 13 Euler stated the conjecture that the differential $\frac{dz}{\sqrt[m]{z^p + z^q}}$ can be integrated algebraically only if either $\frac{p - m}{m(p - q)}$ or $\frac{q - m}{m(q - p)}$ is an integer;[93] this was later proved by Chebyshëv. For more attempts at integrating these differentials see Goldbach's reply in n° 14; in n° 15, Euler already studied the related problem for Riccati's equation, which we have already discussed above. See also n° 43 and 66.

An elliptic integral had already shown up in Euler's letter n° 11, where Euler remarked that he could not compute the integral $\int \frac{a^2dx}{\sqrt{a^4 - x^4}}$ related to the rectification of the lemniscate; in n° 40, Euler dealt with what Legendre later called an "Eulerian integral of the first kind" (now known by the name Beta function). Euler reported on his work on elliptic integrals, which was sparked by Fagnano's results, in n° 158 and 159.[94]

2.4.4. Divergent series

Divergent series were a controversial tool even in Euler's times. Goldbach defended their use in his correspondence with Nicolaus I and Daniel Bernoulli in the early 1720s. Both Goldbach and Euler were convinced that one could assign values to divergent series, and Euler knew how to obtain breathtaking results with this tool, in particular in number theory.[95]

93 Actually, he wrote "*est numerus rationalis*", but the context shows that in 1730 he already understood the problem in the same way it was much later presented in the *Institutiones calculi integralis* (E. 342, § 104).

94 The history of elliptic integrals and elliptic functions has often been described; for a recent account based on Abel's contributions, see Houzel 2004.

95 Hecke later observed that "the precise knowledge of the behaviour of an analytic function in the vicinity of its singular places is a source of arithmetic theorems" (Hecke 1923, § 55, p. 225),

Euler had various methods for assigning values to divergent series. Take, for example, $S = 1 - 2 + 3 - 4 \pm \ldots$; formally, this series is the value of the power series $f(x) = 1 + 2x + 3x^2 + \ldots$ at $x = -1$; since $f(x) = g'(x)$ for $g(x) = 1 + x + x^2 + x^3 + \ldots = \dfrac{1}{1-x}$ we find $f(x) = \dfrac{1}{(1-x)^2}$, and evaluating this power series at $s = -1$ (which lies on the boundary of the domain of convergence of f and g) Euler deduced that $S = \frac{1}{4}$. Using this method Euler was able to evaluate $\zeta(s)$ at the negative integers and to make a correct guess at the functional equation of the zeta function.

Some paradoxes that arise from manipulating divergent series were pointed out by Lambert, who observed[96] that by subtracting the infinite number $-\log 0 = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots = \infty$ from itself one finds

$$\begin{array}{ccccccc} 1 & +\frac{1}{2} & +\frac{1}{3} & +\frac{1}{4} & +\frac{1}{5} & +\frac{1}{6} & +\ldots \\ & -1 & -\frac{1}{2} & -\frac{1}{3} & -\frac{1}{4} & +\frac{1}{5} & +\ldots \\ \hline 1 & -\frac{1}{2} & -\frac{1}{6} & -\frac{1}{12} & -\frac{1}{20} & -\frac{1}{30} & -\ldots \end{array}$$

that is, $\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} + \ldots = 1$. This is a correct equation since it equals $(1 - \frac{1}{2}) + (\frac{1}{2} - \frac{1}{3}) + (\frac{1}{3} - \frac{1}{4}) + \ldots$, which telescopes to 1. On the other hand, if one subtracts the same infinite series in a slightly shifted way,[97] then one obtains

$$\begin{array}{ccccccc} 1 & +\frac{1}{2} & +\frac{1}{3} & +\frac{1}{4} & +\frac{1}{5} & +\frac{1}{6} & +\ldots \\ & -1 & & -\frac{1}{2} & & -\frac{1}{3} & +\ldots \\ \hline 1 & -\frac{1}{2} & +\frac{1}{3} & -\frac{1}{4} & +\frac{1}{5} & -\frac{1}{6} & +\ldots, \end{array}$$

that is, the difference is now $\log 2$. Lambert explained this by observing that the first series is the limit of $-\log(1-a)$, the second one that of $-\log(1-a^2)$, so the difference should have limit

$$\lim_{a\to 1} \log \frac{1-a^2}{1-a} = \lim_{a\to 1} \log(1+a) = \log 2.$$

A related phenomenon concerning integrals such as $\displaystyle\int_0^1 \frac{dx}{\log x} - \int_0^1 \frac{dy}{\log y}$ was discussed in the correspondence between Lagrange and Euler,[98] where they agreed that expressions of the form $\infty - \infty$ are indeterminate. In a similar vein, Goldbach

which at least partially explains why Euler's results obtained with the help of divergent series, while not proved rigorously, often had lasting importance. Famous examples of results that Euler obtained using divergent series include his product formula for the zeta function and the asymptotic behaviour of the sum $\sum \frac{1}{p}$ of inverse primes.

96 Lambert 1771, vol. II, p. 518.

97 Lambert used the expression "sprungsweise".

98 See the letters R 1387, 1388 dated February 10th and March 23rd, 1775: O. IVA/5, p. 502–509.

mentions in n° 63 that if A, B, C are divergent series for which $\frac{C}{A} = \frac{B}{2A}$ is finite, then it does not follow that $B = 2C$.

Euler manipulated divergent series in n° 66, and in n° 91 he reported about his "little dispute" with Nicolaus I Bernoulli, who held the opinion that an expression such as $\sum(-1)^{n+1}n! = 1 - 2 + 6 - 24 + \ldots$ does not have a determinate sum.[99] Euler then explained how to assign a value to this series, and showed that it can be transformed into a continued fraction.[100] In his reply, Goldbach took Euler's side and gave more examples. Euler answered with the remark that he had already been aware of Goldbach's "most ingenious method" of transforming divergent series into convergent ones, and remarked that "no series can be so divergent that its sum could not always be expressed by a convergent series".

2.5. Geometry, topology, combinatorics

In the fields that have been discussed in Sections 2.1–2.4 of this Introduction something akin to a research programme can be discerned in the Euler-Goldbach correspondence: Questions are pursued at some length, diverse formulations and methods of proof are tested, individual problems are put into a more general perspective, some of them resurface after a long time to be viewed from other angles. Moreover there is often a true dialogue: Goldbach raises points of his own, tries his own approaches or prompts Euler by his queries to elaborate and clarify his ideas. It is no exaggeration to state that the exchange of ideas with Goldbach played a substantial role in shaping Euler's achievements in "Fermatian" number theory.

This is not the case for those other areas of mathematics and natural science that will be presented in the next two sections. Here most items are only incidentally mentioned in one or two letters: Goldbach brings up questions he has read about or heard of in conversation, or Euler says in an aside what he is currently working on, without necessarily expecting Goldbach to enter into a discussion. Many of the questions touched on in this way are intriguing or even important by themselves, but – with one exception – no in-depth discussion follows.[101]

Consequently, the topics in geometry and some neighbouring areas that Euler and Goldbach mention in their correspondence will here only be briefly summarised in a bare list:

99 This topic had evolved from the correspondence on Euler's evaluation of $\zeta(2)$; see Baltus 2008 for details from Euler's correspondence with Bernoulli on this topic.

100 One of the earliest entries in Gauss's mathematical diary (Gauss 2005) was noted down on July 10th, 1796: entry no. 7 records a (divergent) continued fraction expression for the divergent series $1 - 2 + 8 - 64 + \ldots$; in no. 58, Gauss generalised this to the series $1 - a + a^3 - a^6 + a^{10} \mp \ldots$, where the exponents are triangular numbers.

101 The exception concerns the challenge problem in differential geometry that Euler posed in his short note E. 79 – see below.

- Goldbach's work on squareable lunes is mentioned in nº 9 from 1730, where Euler also presents his own analytic solution.[102] In nº 53, written in 1742, we find Goldbach's (re)discovery of the fact that the Pythagorean theorem can be generalised to similar areas bounded by arbitrary curves, which leads to the quadrature of some other moon-shaped figures.[103] Euler's own work on circular lunes is contained in E. 73 and E. 423; ultimately the problem boils down to finding all solutions of the equation $\sqrt{m}\sin\alpha = \sin(m\alpha)$ for which $\sin\alpha$ can be constructed with ruler and compass.[104]
- In nº 10, Goldbach tries his hand at a variant of a problem about the rational division of a circle that had been posed by Kepler in *Astronomia Nova* (1609) and solved by Christopher Wren, Newton and, more recently, Jacob Hermann. In this case, Euler did not pursue Goldbach's remarkably up-to-date solution.
- There are three notes of Euler's from the 1730s (nº 20, 21 and 24) in which he raises questions in the geometry of plane curves, using elementary and differential methods to translate geometrical conditions into algebraic equations for the solution curves. The individual problems can be placed in the context of Euler's research on trajectories, pursuit curves and algebraic rectification.[105]
- In nº 22 from 1735, Goldbach presents his proof of a theorem about normally intersecting circles which Euler had proposed the day before; in nº 23, he indicates an equation that links the sides and diagonals of a general quadrilateral. Apparently both these (elementary) problems in plane geometry did not leave a trace in the Petersburg Academy's records or in either correspondent's publications.[106]
- In the 1743 *Nova Acta Eruditorum*, an anonymous mathematician, who seems never to have been identified, published a challenge problem in dif-

102 The problem of squareable "lunes" goes back to Hippocrates' quadrature of the lunes over an isosceles right-angled triangle. The question of which lunes can be squared came up again in a book by the Scottish Newtonian John Craig published in 1718 and was immediately taken up by the Bernoullis and L'Hôpital. The problem was discussed in the correspondence between Goldbach, Nicolaus II and Daniel Bernoulli from 1722 to 1723 (cf. Bottazzini 1996, p. 188–194).

103 When Euler did not comment on this in his reply, Goldbach asked somewhat petulantly whether his statement had been too obvious, and Euler promptly apologised for his oversight (see nº 55, note 10, and nº 56, note 9).

104 The proof that the list of five squareable lunes first indicated by the Swedish mathematician Martin Johan Wallenius in a thesis presented at Åbo in 1766 and confirmed by Euler is indeed complete was only achieved in the first half of the 20th century by the combined efforts of Landau, Chakalov, Chebotarëv and Dorodnov (cf. Scriba 1987, 1988).

105 Cf. nº 20, note 2, nº 21, notes 1–2, and nº 24, note 1.

106 As has been emphasised above in Section 1.3, the few notes from 1732–1738 that have been preserved with the correspondence may actually represent only an undetermined fraction of the communication between Euler and Goldbach during that period. Pending a thorough study of other parts of their manuscript heritage, many questions remain open.

ferential geometry, following in the footsteps of Euler's quarrelsome teacher Johann Bernoulli: Euler (see nº 76–78) drew Goldbach's attention to this, approved Goldbach's attempt at a solution and sketched his own, general one (E. 75). Two years later, Euler issued another, similar (but much harder) challenge: In a short note (E. 79) he asked for the determination of a "continuous" curve (i. e., one described by a single, preferably algebraic expression) that would send a light beam back to its source after two reflections – just as a circle does after a single reflection. This problem – which may also have been posed in order to identify mathematical talent worthy of promotion and patronage and show up those mathematical grandees who could do nothing to solve it – set off a four-year exchange with Goldbach pursued through more than thirty letters (nº 87–109, 129–139). Euler sent hints at his own approaches, a complete draft of one of his papers and several figures, discussed the only solution that was received by the journal, and criticised Goldbach's – in this case mostly futile – attempts to analyse the question.[107]

- In nº 125, Euler sketches a result on quadrilaterals that can be viewed as a generalisation of Pythagoras' theorem or as a polished version of the formula Goldbach had indicated in nº 23: Assume that the diagonals AC and BD of a quadrilateral $ABCD$ are bisected by P and Q; then

$$\overline{AB}^2 + \overline{BC}^2 + \overline{CD}^2 + \overline{DA}^2 = \overline{AC}^2 + \overline{BD}^2 + 4\,\overline{PQ}^2.$$

  With his reply Goldbach sent a proof that now seems to be lost; Euler (in nº 129, published in E. 135) and G.W. Krafft (1750) gave their own proofs.[108]

- As is well known, Euler's correspondence with Goldbach also contains the first appearance of one of his most celebrated insights – one that was to grow into what is now called combinatorial topology. In nº 149 Euler carries over some near-trivial observations on polygons to the three-dimensional case and arrives at the justly famous polyhedron formula $H + S = A + 2$, nowadays written as $V - E + F = 2$, where V (Euler's S), E (A) and F (H) denote the number of vertices (*anguli solidi*), edges (*acies*) and faces (*hedrae*) of a (convex) polyhedron.[109] Alongside this discovery, two other important

107 Except for a short hint in White 2007, p. 308–311, the intriguing questions posed by these challenge problems and their solutions have apparently not been comprehensively studied up to now either from a mathematical or a sociological point of view.

108 See Sandifer 2007c, ch. 6.
The special case where the quadrilateral is a parallelogram and $\overline{PQ} = 0$ had been published by Lagny in 1707. However, all these 18th-century authors were unaware that the general theorem had actually been discovered much earlier by a Spanish Jesuit, José Zaragoza (see nº 125, note 6).

109 See nº 149, notes 5–9.
Among the copious literature on Euler's polyhedron formula, we mention Lakatos' *Proofs and Refutations* (1963/1976), where the formula is used to exemplify the heuristics of mathematics in general, Sandifer 2007c, ch. 2–3, and Richeson's monograph *Euler's Gem* from 2008, which studies many additional aspects of the problem.

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