Preface

The work by G.W. Hill on the motion of the lunar perigee, published in Acta Mathematica in 1886, has led to the general name of 'Hill's equation' for the linear second-order ordinary differential equation with periodic coefficients. Already in 1883, G. Floquet laid the foundation of the theory of periodic linear ordinary differential equation systems, and Lyapunov in his 1907 treatise on the stability of motion recognised the fundamental importance of this theory in the context of H. Poincaré's work. Hill's equation gained further eminence with the advent of quantum mechanics in the mid-1920s. The time-independent Schrödinger equation in one spatial dimension with a periodic potential, which may be used to describe the effective electrostatic action on an electron of the regular arrangement of atomic nuclei in a crystal, is of Hill type. Here the spectral parameter of the equation has a direct physical interpretation as the total energy of an electron, and the characteristic pattern of alternating intervals of stability and instability of the equation corresponds to regions of admissible and forbidden energies which, depending on the position of the system's Fermi energy, give an explanation of why some crystals are transparent insulators while others are shiny conductors. Moreover, impurities which perturb the perfect periodic symmetry can lead to additional discrete energy levels in the forbidden regions, thus creating the semiconductors which have had such a pervasive impact on the development of the electronic technology shaping our lives today.

The mathematical study of ordinary differential equation systems with periodic symmetry and of their spectral properties has produced a body of knowledge and a collection of techniques which have been collated in monographs such as W. Magnus and S. Winkler's 1966 book on Hill's equation and, with stronger emphasis on the spectral theory, M. S. P. Eastham's 1973 book which has become a classic on the subject. These spectral properties are also treated as a special case of the general spectral theory of ordinary differential operators in Weidmann's 1987 lecture notes and similar works.

Our motivation for writing the present volume on a subject of such venerable history has arisen from a variety of considerations. Firstly, there has been ongoing progress in the study of periodic differential systems over the past decades, leading to new developments as well as some changes in perspective. For example, the analysis of the higher-dimensional lattice-periodic Schrödinger operator has spun off as a separate subject with its own very specific challenges and connections to other areas such as number theory; see Y. Karpeshina's 1997 monograph [106]. Secondly, there has been a growing interest in the Schrödinger operator's relativistic brother, the Dirac operator. Although some of their spectral properties are very different and many useful techniques for the one-dimensional Schrödinger, or more generally the Sturm-Liouville, operator do not carry over to the Dirac case, due for example to the lower unboundedness of the latter's spectrum, the two differential expressions have much in common when expressed in the form of a Hamiltonian system. From this point of view, the Dirac equation is even the more natural object, whereas in the case of the Sturm-Liouville equation some complications arise from the fact that the weight matrix multiplying the spectral parameter is singular. Finally, the use of oscillation properties of solutions of differential equations has proved to be an effective tool in the study of the spectral properties of the associated differential operators, beginning from C.F. Sturm's observations in the 1830s and H. Prüfer's more powerful reformulation in 1926, the great potential of which was first recognised in J. Weidmann's 1971 paper on oscillation methods for differential equation systems. In the present book, we make systematic use of this technique from an early stage onwards, thus demonstrating that it is not just a trick to count eigenvalues in some specific situations, but that oscillation plays a fundamental and ubiquitous role in the spectral analysis of ordinary differential equation systems of Sturm-Liouville and Dirac type. In fact, it is the close interplay between linear differential equation theory, non-linear oscillation properties and the more abstract theory of linear operators that gives this subject its particular flavour.

In this book we endeavour to give a detailed overview of the techniques and results of the analysis of systems of ordinary differential equations with periodic coefficients, with particular emphasis on the spectral theory of Sturm-Liouville and Dirac operators. It was our aim to provide an introductory text easily accessible to the beginning postgraduate, assuming only elementary knowledge of mathematical analysis, the theory of ordinary differential equations and, in the later parts, of self-adjoint operators in Hilbert space. We decided to include a complete treatment of the singular boundary-value problems on the real line and half-line, following H. Weyl's approach, in as far as it applies to the periodic equation and its perturbations; this includes a number of results which are generally well-known among experts, but for which it is not easy to pinpoint a straightforward proof in the existing textbook literature.

As is natural for a subject of such venerable history, we could not include everything. Periodic equations have given rise to a number of special functions, for example Mathieu and Lamé functions, which have been studied in great detail elsewhere. We therefore feel that in a time when software providing symbolic and numerical computation with these and other special functions is readily available, a circumstantial treatment here would be out of place and consequently we merely refer to the older literature as required. Some aspects of inverse spectral problem as well as extensions such as almost periodic equations or analysis on regular trees are treated only briefly in end-of-chapter notes with literature references. We do, however, include the spectral analysis of perturbed periodic problems, both as a modern development of practical significance and as an instructive application of much of the general theory of the periodic problem explained in this book.

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