Chapter 2

Oscillations

2.1 Introduction

The Floquet theory of Chapter 1 gives an overview of the global growth properties of the solutions of periodic systems. For the purposes of spectral analysis of formally symmetric systems, the oscillations or rotations of the real-valued solutions are also of similar importance. The tool for studying oscillations in Sturm-Liouville and Dirac systems is the Prüfer transform, which is introduced in section 2.2 and then used to analyse the boundary-value problems with separated boundary conditions on the period interval in section 2.3. When conjoined with the results on the periodic and semi-periodic boundary-value problems, this leads to the observation that the oscillations of the solutions of periodic systems have a linear growth asymptotic. The growth rate is a continuous, monotone increasing function of the real spectral parameter. It is known as the rotation number and is connected, by the physical interpretation of the equations, to the quasimomentum and the integrated density of states. In the special case of Hill's equation, the oscillation properties can be equivalently studied by counting zeros of solutions.

We shall continue to use the notation of the general 2×2 system (1.5.1), assuming the general hypotheses on the coefficient matrices B and W. However, from section 2.3 onwards, we shall often make the more specific assumption that this system is either a Sturm-Liouville or a Dirac system. The underlying reason for this is that the general hypotheses on the matrix-valued function W allow it to be singular, as in the case of the Sturm-Liouville system. This leads to specific effects, such as the results in section 2.5, and also means that some proofs, e.g. that of Theorem 2.3.4, employ different techniques for the Sturm-Liouville system and for the Dirac system, and will not extend in a straightforward way to the fully general system (1.5.1).

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2.2 The Prüfer transform

Consider now the 2×2 linear periodic system (1.5.1) under the general hypotheses made on the functions B and W in section 1.5, and with spectral parameter $\lambda \in \mathbb{R}$. This makes A an $\mathbb{R}^{2\times 2}$-valued function, and it is therefore sufficient to study the $\mathbb{R}^2$-valued solutions of the system; indeed, a $\mathbb{C}^2$-valued solution can be understood as a complex linear combination of two $\mathbb{R}^2$-valued solutions.

By the uniqueness theorem for solutions of initial-value problems, no non-trivial solution of (1.5.1) takes the value $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, and so so every non-trivial solution traces out a trajectory in the punctured plane $\mathbb{R}^2 \setminus \{0\}$. Therefore, for any solution $u : X \to \mathbb{R}^2 \setminus \{0\}$, there are locally absolutely continuous functions $\rho : X \to (0, \infty)$ and $\theta : X \to \mathbb{R}$ such that

$$u = \rho \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}. \tag{2.2.1}$$

Clearly $\rho = \sqrt{u_1^2 + u_2^2}$ is uniquely determined, and θ is unique up to an additive constant integer multiple of 2π; ρ is called the *Prüfer radius*, and θ the *Prüfer angle*, of the solution u.

Differentiating (2.2.1) and using (1.5.1), we find that

$$\rho' \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix} + \rho\theta' \begin{pmatrix} \cos\theta \\ -\sin\theta \end{pmatrix} = \rho J(B + \lambda W) \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix},$$

and hence we obtain the set of differential equations for the Prüfer variables

$$\theta' = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}^T (B + \lambda W) \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}, \tag{2.2.2}$$

$$(\log\rho)' = \begin{pmatrix} -\cos\theta \\ \sin\theta \end{pmatrix}^T (B + \lambda W) \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}. \tag{2.2.3}$$

The right-hand sides of both equations are quadratic trigonometric polynomials in θ and do not refer to ρ. Consequently, the equation for ρ can be solved by a simple quadrature once θ is known, and the equation for θ is a single non-linear first-order differential equation (known as the *Prüfer equation*) which is essentially equivalent to the linear 2×2 system. We have not so far indicated the dependence of ρ and θ on λ, but we will do so in the next section.

Specifically for the Sturm-Liouville equation (1.5.2), the Prüfer transformation takes the form

$$y = \rho\sin\theta, \qquad py' = \rho\cos\theta, \tag{2.2.4}$$

and the Prüfer equation becomes

$$\theta' = \frac{1}{p}\cos^2\theta + (\lambda w - q)\sin^2\theta, \tag{2.2.5}$$

with the Prüfer radius given by

$$(\log \rho)' = \frac{1}{2}\left(\frac{1}{p} + q - \lambda w\right)\sin 2\theta, \tag{2.2.6}$$

and hence

$$\rho(x) = \rho(x_0)\exp\left(\int_{x_0}^{x} \frac{1}{2}\left(\frac{1}{p} + q - \lambda w\right)\sin 2\theta\right) \qquad (x, x_0 \in X).$$

For the Dirac system, the Prüfer equation is

$$\theta' = \lambda - q + p_1 \cos 2\theta - p_2 \sin 2\theta, \tag{2.2.7}$$

and the Prüfer radius satisfies

$$(\log \rho)' = p_1 \sin 2\theta + p_2 \cos 2\theta, \tag{2.2.8}$$

and therefore can be expressed as

$$\rho(x) = \rho(x_0)\exp\left(\int_{x_0}^{x} (p_1 \sin 2\theta + p_2 \cos 2\theta)\right) \qquad (x, x_0 \in X).$$

Analysis of the Prüfer equation is a powerful tool, because the rotation of the solutions in the punctured phase plane (corresponding to oscillations of solutions around 0 in the case of the Sturm-Liouville equation) turns out to encode much of the spectral properties of the system. However, the Prüfer variables in their original form do not always lead to a differential equation with a direction field which can be interpreted easily. Therefore the applicability and power of the Prüfer method is much enhanced by observing that the phase plane can be subjected to a linear transformation before introducing polar coordinates, with coefficients depending on the spectral parameter λ and on the independent variable of the system. If this linear transformation fixes a direction (typically the u_1 direction) and hence introduces no overall rotation, the resulting generalised Prüfer angle ϕ will have the same asymptotics as the original one, but may satisfy a crucially simplified differential equation.

This equation will generally differ from the particular forms (2.2.5) and (2.2.7), but its right-hand side will always be a homogeneous quadratic trigonometric polynomial of the angle variable, being of the form

$$\phi' = a \cos^2 \phi + b \sin\phi \cos\phi + c \sin^2 \phi, \tag{2.2.9}$$

with locally integrable, real-valued functions a, b, c. We call (2.2.9) a *Prüfer-type equation.*

Note that the Prüfer-type equation is equivalent to the generalised Riccati equation; indeed, setting $\zeta = \tan\phi$ gives

$$\zeta' = a + b\zeta + c\zeta^2.$$

However, the solutions of the latter equation have moving singularities, which makes it less convenient to study than equation (2.2.9), whose solutions are locally bounded. Moreover, for the purposes of spectral analysis, the long-range growth of the Prüfer angle or a related variable is usually of greater interest than the local information captured in the Riccati variable ζ.

The transition between Prüfer-type equations is described in the following theorem.

Theorem 2.2.1 (Kepler transformation).
Let $X \subset \mathbb{R}$ be an interval, $\phi, f, g : X \to \mathbb{R}$ locally absolutely continuous functions with $f > 0$. Then, with a suitable choice of branches of arctan*,*

$$\tilde{\phi} := \arctan(f(\tan\phi + g)) \tag{2.2.10}$$

is locally absolutely continuous and satisfies

$$\tilde{\phi} \in [(n - \tfrac{1}{2})\pi, (n + \tfrac{1}{2})\pi] \qquad \Leftrightarrow \qquad \phi \in [(n - \tfrac{1}{2})\pi, (n + \tfrac{1}{2})\pi] \tag{2.2.11}$$

for all $n \in \mathbb{Z}$. Moreover,

$$\tilde{\phi}' = (\log f)' \sin\tilde{\phi}\,\cos\tilde{\phi} + f g' \cos^2\tilde{\phi} + \frac{f\phi'}{\cos^2\phi + f^2(\sin\phi + g\cos\phi)^2}. \tag{2.2.12}$$

Proof. We write (2.2.10) as $\tan\tilde{\phi} = f(\tan\phi + g)$. Then (2.2.11) follows immediately, and (2.2.12) follows on differentiation. □

We note that, when ϕ satisfies a Prüfer-type equation (2.2.9), then (2.2.12) takes the same form when (2.2.10) is used in the last term. The Kepler transformation (2.2.10) typically arises when (2.2.1) is modified to

$$u = \rho \begin{pmatrix} f_1 \sin\tilde{\theta} \\ f_2 \cos\tilde{\theta} \end{pmatrix}$$

with f_1 and f_2 (both > 0) at our choice. Then the relation between $\tilde{\theta}$ and θ is $\tan\tilde{\theta} = (f_2/f_1)\tan\theta$.

As an example, with the choice of coefficients $p = w = 1$, the Prüfer equation (2.2.5) for the Sturm-Liouville equation is a relatively complicated function of θ and, even if $q = 0$, does not readily reveal the solutions of the equation, which can be solved explicitly in this simple case. A Kepler transformation with $f = \sqrt{\lambda}$, $g = 0$ gives the generalised Prüfer angle $\tilde{\theta} = \arctan(\sqrt{\lambda}\tan\theta)$ with the differential equation

$$\tilde{\theta}' = \frac{\sqrt{\lambda}(\cos^2\theta + (\lambda - q)\sin^2\theta)}{\cos^2\theta + \lambda\sin^2\theta} = \sqrt{\lambda} - q\frac{\sin^2\tilde{\theta}}{\sqrt{\lambda}},$$

which in the case $q = 0$ clearly shows linear growth with slope $\sqrt{\lambda}$ for $\tilde{\theta}$ and hence, up to an error globally bounded by π, for θ as well. For non-zero q, this Prüfer equation is a much better starting point for the derivation of asymptotics than the original Prüfer equation (2.2.5).

In this example, the modifying functions f and g are constant. A more sophisticated use of the Kepler transformation with non-constant coefficient functions appears in the proof of Theorem 2.4.3 below, and this technique will play a central role in Chapter 3.

2.3 The boundary-value problem with separated boundary conditions

The usefulness of the Prüfer transformation for spectral analysis is rooted in the fact that the Prüfer angle depends monotonically on the spectral parameter. This is a consequence of the following basic result on first-order differential inequalitites, due to Chaplygin and Peano.

Theorem 2.3.1. *Let $X \subset \mathbb{R}$, and let $f : X \times \mathbb{R} \to \mathbb{R}$ be locally integrable with respect to the first variable and satisfy the Lipschitz condition*

$$|f(x,y) - f(x,z)| \leq K(x)\,|y-z|$$

with locally integrable $K : X \to [0,\infty)$. Let $x_0 \in X$ and let ϕ and ψ satisfy the differential inequalities

$$\phi'(x) \leq f(x,\phi(x)), \qquad \psi'(x) \geq f(x,\psi(x))$$

in X, with $\phi(x_0) \leq \psi(x_0)$. Then

(a) $\phi(x) \leq \psi(x)$ $(x \in X, x \geq x_0)$;

(b) *if $\phi(x_1) = \psi(x_1)$ for some $x_1 \in X$ and $x_1 > x_0$, then $\phi(x) = \psi(x)$ throughout $[x_0, x_1]$.*

Proof. Let $\delta := \psi - \phi \in AC_{\text{loc}}(X)$. Then $\delta(x_0) \geq 0$ and

$$\delta'(x) = f(x,\psi(x)) - f(x,\phi(x)) \geq -K(x)\,|\delta(x)|. \tag{2.3.1}$$

Now assume on the contrary that there is $x' > x_0$ such that $\delta(x') < 0$ and set $x'' := \sup\{x \in [x_0, x') \mid \delta(x) \geq 0\}$. Since δ is continuous, $\delta(x'') = 0$ and $\delta < 0$ on (x'', x'). Therefore, integrating (2.3.1), we have

$$\log \frac{\delta(x')}{\delta(x)} \geq \int_x^{x'} K \qquad (x \in (x'', x')),$$

and hence

$$\delta(x') \geq \lim_{x \to x''} \delta(x) \exp\left(\int_x^{x'} K\right) = 0.$$

This contradiction shows that there is no such x', and part (a) of the theorem follows.

To prove part (b), we again integrate (2.3.1), this time over (x, x_1) with $x \in (x_0, x_1)$. Since now $\delta \geq 0$, this gives

$$\log \frac{\delta(x_1)}{\delta(x)} \geq -\int_x^{x_1} K,$$

and hence $\delta(x) \leq \delta(x_1) \exp\left(\int_x^{x_1} K\right) = 0$. Thus also $\delta \leq 0$ in $[x_0, x_1]$, and part (b) follows. □

Application of this theorem to the differential equation (2.2.2) for the Prüfer angle gives rise to the following statement, which is a generalisation of the classical Sturm Comparison Theorem.

Corollary 2.3.2 (Sturm Comparison)**.** *Let $B_1, B_2, W_1, W_2 : X \to \mathbb{R}^{2\times 2}$ satisfy the general hypotheses, let $\lambda_1, \lambda_2 \in \mathbb{R}$ and assume that*

$$B_1 + \lambda_1 W_1 - B_2 - \lambda_2 W_2 \geq 0$$

in the sense of positive semidefinite matrices. Let θ_1, θ_2 be Prüfer angles of solutions of

$$u = J(B_1 + \lambda_1 W_1)\, u, \qquad v = J(B_2 + \lambda_2 W_2)\, v,$$

respectively, and $\theta_1(x_0) \geq \theta_2(x_0)$ for some $x_0 \in X$. Then $\theta_1(x) \geq \theta_2(x)$ for all $x \in X$, $x \geq x_0$.

This observation proves very useful when comparing differential equations of the type (1.5.1) and in the treatment of perturbations. The most immediate consequence of Theorem 2.3.1 is the monotonic dependence of the Prüfer angle on the spectral parameter. We write (2.2.2) as $\theta'(x) = F(x, \theta, \lambda)$ and consider two different values λ' and λ'' ($\lambda' < \lambda''$) of λ. Since W is positive semidefinite in (2.2.2), $F(x, \theta, \lambda') \leq F(x, \theta, \lambda'')$, and this leads to the following basic property of θ.

Theorem 2.3.3. *Let $\theta(\cdot, \lambda)$ be the Prüfer angle of a real-valued solution of* (1.5.1) *with $\theta(x_0, \lambda) = \theta_0 \in \mathbb{R}$ and θ_0 is independent of λ. Then, for any fixed $x > x_0$, $\theta(x, \lambda)$ is a strictly increasing continuous function of λ.*

Proof. The continuity follows from (2.2.2) and Theorem 1.2.1 (b). Next, we take $\phi(x) = \theta(x, \lambda')$, $\psi(x) = \theta(x, \lambda'')$ and $f(x, y)$ as $F(x, \theta, \lambda'')$ in Theorem 2.3.1. Then part (a) shows that θ is an increasing function of λ. To show that it is strictly increasing, suppose on the contrary that $\theta(x_1, \lambda') = \theta(x_1, \lambda'')$ for some $x_1 > x_0$. Then, by part (b) of Theorem 2.3.1, $\theta(\cdot, \lambda') = \theta(\cdot, \lambda'')$ on $[x_0, x_1]$. Then again, by (2.2.2),

$$0 = (\lambda' - \lambda'') \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}^T W \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

on $[x_0, x_1]$. This implies that $Wu = 0$ for the non-trivial solution $u = \rho \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$, contradicting the general assumption made on W at the beginning of section 1.5. □

The properties of the Prüfer angle provide direct insight into the spectral properties of a class of boundary-value problems for the differential system on a finite interval which, for our purposes, we take to be the period interval $[0, a]$. We also take $\alpha \in [0, \pi)$, $\beta \in (0, \pi]$ and consider the *separated boundary conditions* which for the system (1.5.1) take the general form

$$u_1(0)\cos\alpha - u_2(0)\sin\alpha = 0, \qquad u_1(a)\cos\beta - u_2(a)\sin\beta = 0. \tag{2.3.2}$$

In the case of the Sturm-Liouville equation (1.5.2), these conditions become

$$y(0)\cos\alpha - (py')(0)\sin\alpha = 0, \qquad y(a)\cos\beta - (py')(a)\sin\beta = 0;$$

here the special case $\alpha = 0$ is called the *Dirichlet boundary condition*, and the case $\alpha = \pi/2$ the *Neumann boundary condition*. At the end-point a, these names are used for $\beta = \pi$ and $\beta = \pi/2$, respectively.

Values of the spectral parameter λ for which there is a non-trivial solution, called an *eigenfunction*, of (1.5.1) satisfying the boundary conditions (2.3.2) are called *eigenvalues* of the boundary-value problem with separated boundary conditions. All such eigenvalues are real; as in the proof of Proposition 1.8.1, the boundary conditions ensure that the boundary terms arising from an integration by parts vanish. Also, the orthogonality property (1.8.4) again holds. The eigenvalues of the boundary-value problems for the Sturm-Liouville equation with $(\alpha, \beta) = (\pi/2, \pi/2)$ and with $(\alpha, \beta) = (0, \pi)$ are briefly referred to as *Neumann eigenvalues* and *Dirichlet eigenvalues*, respectively.

Theorem 2.3.4. (a) *The Sturm-Liouville boundary-value problem on* $[0, a]$ *with separated boundary conditions has infinitely many real eigenvalues*

$$\lambda_0 < \lambda_1 < \lambda_2 < \dots$$

with $\lim_{n\to\infty} \lambda_n = \infty$. *Any real-valued eigenfunction for the* n*-th eigenvalue has* n *zeros in the open interval* $(0, a)$.

(b) *The boundary-value problem for the Dirac system on* $[0, a]$ *with separated boundary conditions has infinitely many eigenvalues*

$$\dots < \lambda_{-2} < \lambda_{-1} < \lambda_0 < \lambda_1 < \lambda_2 < \dots$$

with $\lim_{n\to\pm\infty} \lambda_n = \pm\infty$.

In both (a) *and* (b), *the Prüfer angle of any real-valued eigenfunction for* λ_n *satisfies* $\theta(a, \lambda_n) - \theta(0, \lambda_n) = n\pi + \beta - \alpha$.

Proof. (a) For each $\lambda \in \mathbb{R}$, let $\theta(\cdot, \lambda)$ be the Prüfer angle of the solution of the initial-value problem

$$-(py')' + qy = \lambda wy, \qquad \begin{pmatrix} y \\ py' \end{pmatrix}(0, \lambda) = \begin{pmatrix} \sin\alpha \\ \cos\alpha \end{pmatrix},$$

with $\theta(0, \lambda) = \alpha$. By (2.2.4), λ is an eigenvalue if and only if $\theta(a, \lambda) = \beta \pmod{\pi}$. From Theorem 2.3.3, $\theta(a, \cdot)$ is a continuous, strictly increasing function, and so its range is an interval of extent determined by the limits $\lim\limits_{\lambda \to \pm\infty} \theta(a, \lambda)$.

For $\lambda \to -\infty$, we observe first that $\theta(0, \lambda) = \alpha \geq 0$ and that, by (2.2.5), $\theta'(x, \lambda) = \frac{1}{p} > 0$ whenever $\theta(x, \lambda) = 0 \pmod{\pi}$, showing that $\theta(\cdot, \lambda)$ can only increase through the values $n\pi$, $n \in \mathbb{Z}$. Hence $\theta(\cdot, \lambda) \geq 0$, and by monotonicity the limit

$$\theta_{-\infty}(x) := \lim_{\lambda \to -\infty} \theta(x, \lambda) \geq 0$$

exists. We wish to show that the limit is zero for all $x > 0$. For $\lambda \neq 0$, the Prüfer equation (2.2.5) gives

$$\frac{\theta(a, \lambda) - \alpha}{\lambda} = \int_0^a \left(\frac{1}{\lambda} \Big(\frac{1}{p} \cos^2\theta - q \sin^2\theta \Big) + w \sin^2\theta \right). \tag{2.3.3}$$

Now $\theta(a, \lambda) < \theta(a, 0)$ when $\lambda < 0$, by Theorem 2.3.3, and so the left-hand side of (2.3.3) tends to zero as $\lambda \to -\infty$. On the right-hand side, as $\lambda \to -\infty$, the integrand tends to $w \sin^2\theta_{-\infty}$ a.e., and it is bounded by $\frac{1}{p} + |q| + w \in L^1[0, a]$ independently of $\lambda \leq -1$. Hence, by Lebesgue's dominated convergence theorem,

$$0 = \int_0^a w \sin^2\theta_{-\infty},$$

and so $\theta_{-\infty} = 0 \pmod{\pi}$ a.e. Next, for $\lambda < 0$ and $x \geq t$, (2.2.5) gives

$$\theta(x, \lambda) - \theta(t, \lambda) \leq \int_t^x \left(\frac{1}{p} \cos^2\theta - q \sin^2\theta \right).$$

Hence, in the limit $\lambda \to -\infty$,

$$\theta_{-\infty}(x) - \theta_{-\infty}(t) \leq \int_t^x \frac{1}{p}. \tag{2.3.4}$$

Since $\theta_{-\infty}(a) = \alpha$, (2.3.4) (with $t = a$) shows that $\theta_{-\infty}(x)$ cannot take the value π for x near to a, and hence $\theta_{-\infty}(x) = 0$ for such x. Using (2.3.3) to repeat this argument shows that $\theta_{-\infty}(x)$ can never jump to π, and hence it is zero a.e. as required.

Next, for $\lambda \to \infty$, we show that $\theta(a, \lambda) \to \infty$. We assume on the contrary that $\theta(a, \cdot)$ is bounded; then so is $\theta(x, \cdot) \leq \theta(a, \cdot) + \pi$, and again $\theta_\infty(x) := \lim\limits_{\lambda \to \infty} \theta(x, \lambda)$

exists for all $x \in [0, a]$. Sending $\lambda \to \infty$ in (2.3.3), we find as before that $\theta_\infty = 0 \pmod{\pi}$ a.e. On the other hand, for $\lambda > 0$ and $x \geq t$ we have

$$\theta(x, \lambda) - \theta(t, \lambda) \geq \int_t^x \left(\frac{1}{p} \cos^2 \theta - q \sin^2 \theta \right).$$

Hence, in the limit $\lambda \to \infty$,

$$\theta_\infty(x) - \theta_\infty(t) \geq \int_t^x \frac{1}{p} > 0,$$

which shows that θ_∞ is strictly increasing. Since θ_∞ is an integer multiple of π, this contradicts the assumed boundedness of $\theta(a, \cdot)$.

Thus altogether we have shown that the range of $\theta(a, \cdot)$ is $(0, \infty)$. The eigenvalues appear as the pre-images of $\beta + \pi \mathbb{N}_0 \subset (0, \infty)$. The Prüfer angle of the eigenfunction for λ_n will cross $0 \pmod{\pi}$ exactly n times, giving the n zeros of the eigenfunction.

(b) As in (a), let $\theta(\cdot, \lambda)$ be the Prüfer angle of a solution, with $\theta(0, \lambda) = \alpha$. From the Prüfer equation (2.2.7) for the Dirac system, we have

$$|\theta'(\cdot, \lambda) - \lambda| \leq |q| + |p_1| + |p_2|, \tag{2.3.5}$$

and so the total variation of $\theta(x, \lambda) - \lambda x$ ($x \in [0, a]$) is bounded by $C := \|q\|_1 + \|p_1\|_1 + \|p_2\|_1$. In particular,

$$\lambda a - C + \alpha \leq \theta(a, \lambda) \leq \lambda a + C + \alpha.$$

Consequently, the continuous, strictly monotone increasing function $\theta(a, \cdot)$ has range $\mathbb{R}$. The eigenvalues arise as the pre-images of $\beta + \pi \mathbb{Z}$ and can be numbered so that $\theta(a, \lambda_n) = \beta + n\pi$. □

The proof of Theorem 2.3.4 also leads to the following method of counting eigenvalues in an interval.

Theorem 2.3.5 (Relative Oscillation Theorem). *Let $N(\lambda', \lambda'']$ be the number of eigenvalues in the interval $(\lambda', \lambda''] \subset \mathbb{R}$ of a Sturm-Liouville or Dirac boundary-value problem on $[0, a]$ with separated boundary conditions* (2.3.2). *Let $\theta(\cdot, \lambda)$ be the solution of the associated Prüfer equation with $\theta(0, \lambda) = \alpha$. Then*

$$\frac{1}{\pi}(\theta(a, \lambda'') - \theta(a, \lambda')) - 1 \leq N(\lambda', \lambda''] \leq \frac{1}{\pi}(\theta(a, \lambda'') - \theta(a, \lambda')) + 1.$$

Proof. The number of points λ in the interval $(\lambda', \lambda'']$ where $\theta(b, \lambda) = \beta \pmod{\pi}$ is at least $[\frac{1}{\pi}(\theta(a, \lambda'') - \theta(a, \lambda'))]$ and at most $[\frac{1}{\pi}(\theta(a, \lambda'') - \theta(a, \lambda'))] + 1$; here $[x]$ denotes the greatest integer not exceeding x. □

2.4 The rotation number

For the periodic systems, the existence of Floquet solutions allows us to obtain asymptotic information on the growth of Prüfer angles. We first observe that the asymptotics of the Prüfer angle are uniform throughout each instability interval, and are closely related to its counting index. The key idea is to use the eigenvalues of a boundary-value problem with separated boundary conditions on the period interval as tags for the instability intervals. We call the closed intervals of which $\mathbb{R} \setminus \mathcal{S}$ is composed *closed instability intervals*, by a slight abuse of terminology. A closed instability interval either consists of a single coexistence point, or it is the closure of an instability interval.

Theorem 2.4.1. *Let* $(\lambda_n)_{n\in\mathcal{J}}$ *be the ordered set of eigenvalues for the boundary-value problem on* $[0, a]$ *for the Sturm-Liouville or Dirac system with Neumann boundary conditions* (2.3.2), *in which* $\alpha = \beta = \pi/2$. *Then there is a monotonic enumeration* $(\overline{I}_n)_{n\in\mathcal{J}}$ *of the closed instability intervals such that* $\lambda_n \in \overline{I}_n$ $(n \in \mathcal{J})$.

For every $\lambda \in \overline{I}_n$, *the Prüfer angle of any real-valued solution of the system with spectral parameter* λ *satisfies*

$$\theta(x, \lambda) = \frac{n\pi x}{a} + O_{\text{unif}}(1) \qquad (x \to \infty). \tag{2.4.1}$$

The O_{unif} *term is bounded uniformly with respect to* $\lambda \in \overline{I}_n$.

As shown in Theorem 2.3.4, $\mathcal{J} = \mathbb{N}_0$ for the Sturm-Liouville system and $\mathcal{J} = \mathbb{Z}$ for the Dirac system.

Proof. By the last statement of Theorem 2.3.4, the Prüfer angle for an eigenfunction for λ_n satisfies

$$\theta_n(a) - \theta_n(0) = n\pi \qquad (n \in \mathcal{J}). \tag{2.4.2}$$

Now the eigenfunction is a multiple of the first column in the canonical fundamental matrix Φ and, in view of the Neumann boundary condition at 0, the monodromy matrix has lower left entry $M_{21}(\lambda_n) = 0$. Hence the product of its diagonal elements is equal to 1. By (1.6.2), it follows that $|D(\lambda_n)| \geq 2$, and we also have $\operatorname{sgn} D(\lambda_n) = (-1)^n$. The alternating sign shows that λ_n and λ_{n+1} lie in distinct closed instability intervals. Therefore the interval $[\lambda_n, \lambda_{n+1}]$ contains at least one stability interval.

Because of the continuity of D, there are points

$$\Lambda_p, \Lambda_s \in [\lambda_n, \lambda_{n+1}] \tag{2.4.3}$$

such that $D(\Lambda_p) = 2$ and $D(\Lambda_s) = -2$. Let Θ_p and Θ_s be Prüfer angles of the (respectively, periodic and semi-periodic) Floquet solutions for Λ_p and Λ_s with initial values $\Theta_p, \Theta_s \in [\frac{\pi}{2}, \frac{5\pi}{2})$.

From (2.4.2) and the periodicity of the system, it follows that

$$\theta_n(x) = \frac{n\pi x}{a} + O(1), \qquad \theta_{n+1}(x) = \frac{(n+1)\pi x}{a} + O(1) \qquad (x \to \infty), \tag{2.4.4}$$

where the $O(1)$ term represents the variation of the angle in each period interval; for the Sturm-Liouville system it is bounded by π, as the Prüfer angle cannot decrease through $\mathbb{Z}\pi$, for the Dirac system it is bounded uniformly in $[\lambda_n, \lambda_{n+1}]$ because of the integrability of the coefficients, as in (2.3.5).

Since Θ_p and Θ_s are a-periodic modulo π, a similar argument gives

$$\Theta_p(x) = \frac{m_p \pi x}{a} + O(1), \qquad \Theta_s(x) = \frac{m_s \pi x}{a} + O(1) \qquad (x \to \infty) \tag{2.4.5}$$

for some $m_p \in 2\mathbb{Z}, m_s \in 2\mathbb{Z}+1$. The $O(1)$ terms are uniformly bounded as above.

Now Theorem 2.3.3 implies, on the basis of (2.4.3) and the initial values for the Prüfer angles, that

$$\Theta_p(x), \Theta_s(x) \in [\theta_n(x), \theta_{n+1}(x) + 2\pi] \qquad (x \geq 0).$$

Comparing the asymptotics (2.4.4) and (2.4.5), this shows that $m_p, m_s \in \{n, n+1\}$. Specifically, if n is even, then $m_p = n$, $m_s = n+1$; if n is odd, then $m_s = n$, $m_p = n+1$.

The above reasoning holds true for all $\Lambda_p, \Lambda_s \in [\lambda_n, \lambda_{n+1}]$ such that $D(\Lambda_p) = 2$ and $D(\Lambda_s) = -2$. Hence, by Theorem 2.3.3, all Λ_p are either greater or less than all Λ_s, which will contradict the properties of Hill's discriminant in Theorem 1.6.1 unless there is only one Λ_p and one Λ_s with these properties. Thus we have shown that $[\lambda_n, \lambda_{n+1}]$ contains exactly one stability interval.

At the two end-points of $\overline{I}_n$ we thus have periodic or semi-periodic solutions with Prüfer angles satisfying (2.4.1). By Theorem 2.3.3, this asymptotic holds for all solutions for all $\lambda \in \overline{I}_n$. □

The analysis in the proof of Theorem 2.4.1 reveals the following fundamental properties of the periodic and semi-periodic eigenvalues of Sturm-Liouville and Dirac systems.

Theorem 2.4.2. *The periodic eigenvalues $(\lambda_n)_{n \in \mathcal{J}}$ and the semi-periodic eigenvalues $(\mu_n)_{n \in \mathcal{J}}$ of the Sturm-Liouville ($\mathcal{J} = \mathbb{N}_0$) or Dirac ($\mathcal{J} = \mathbb{Z}$) system on $[0, a]$ form interlacing sequences*

$$(\cdots \leq)\, \lambda_0 < \mu_0 \leq \mu_1 < \lambda_1 \leq \lambda_2 < \mu_2 \leq \mu_3 < \lambda_3 \leq \lambda_4 < \ldots \tag{2.4.6}$$

The Prüfer angle of a real-valued eigenfunction for a periodic or semi-periodic eigenvalue λ satisfies

$$\theta(a, \lambda) = \theta(0, \lambda) + \begin{cases} 2(m+1)\pi & \text{if } \lambda \in \{\lambda_{2m+1}, \lambda_{2m+2}\}, \\ (2m+1)\pi & \text{if } \lambda \in \{\mu_{2m}, \mu_{2m+1}\}, \end{cases} \tag{2.4.7}$$

for any suitable integer m. Moreover,

$$\mathcal{S} = (\cdots \cup)\, (\lambda_0, \mu_0) \cup (\mu_1, \lambda_1) \cup (\lambda_2, \mu_2) \cup \ldots \tag{2.4.8}$$

and $\overline{I}_n = [\lambda_{n-1}, \lambda_n]$ for even n, $\overline{I}_n = [\mu_{n-1}, \mu_n]$ for odd n.

Theorem 2.4.1 shows that, for $\lambda \in \overline{I}_n$ and the Prüfer angle of any solution, the limit

$$k(\lambda) := \lim_{x\to\infty} \frac{\theta(x,\lambda)\,a}{x} = n\pi$$

exists. The next theorem will show that this function k extends to the stability intervals. It is a non-decreasing continuous function called the *rotation number*.

Theorem 2.4.3. *Let $\lambda \in \mathcal{S}$ and let D be Hill's discriminant of a Sturm-Liouville or Dirac system. Then there is $k(\lambda) \in \mathbb{R}$ such that the Prüfer angle $\theta(x,\lambda)$ of any solution of the system with spectral parameter λ satisfies*

$$\theta(x,\lambda) = \frac{k(\lambda)\,x}{a} + O(1) \qquad (x \to \infty) \tag{2.4.9}$$

and $D(\lambda) = 2\cos k(\lambda)$.

Proof. Since $|D(\lambda)| < 2$, we are in Case 3 of the classification in section 1.4. The Floquet multipliers are $e^{\pm i\nu(\lambda)}$ with $\nu \in \mathbb{R} \setminus \pi\mathbb{Z}$, and the corresponding complex-valued Floquet solutions u and $\overline{u}$ are linearly independent with

$$u(x+a,\lambda) = e^{i\nu(\lambda)x/a}u(x,\lambda). \tag{2.4.10}$$

We will show that $k(\lambda)$ is related to $\nu(\lambda)$.

We begin by considering the components u_1 and u_2 of u. First, they have no zeros because of the linear independence of u and $\overline{u}$. Also, their arguments are nowhere equal $\mod \pi$, because otherwise we could multiply u by a suitable complex constant c to make both components of cu real at that point, again contradicting the linear independence of u and $\overline{u}$. Hence, either $\arg u_1 < \arg u_2 < \pi + \arg u_1$, or $-\pi + \arg u_1 < \arg u_2 < \arg u_1$. Thus we can write

$$u_1 = R_1 e^{i\phi_1}, \quad u_2 = \pm i R_2 e^{i\phi_2}, \tag{2.4.11}$$

where $R_j (> 0)$ and ϕ_j are real-valued absolutely continuous functions with

$$|\phi_1 - \phi_2| < \frac{\pi}{2}. \tag{2.4.12}$$

Without loss of generality we take the plus sign in (2.4.11).

Now let v be any real-valued, non-trivial solution of the differential system with Prüfer angle θ. Thus $v = c(u+\overline{u}) + id(u-\overline{u})$ with real constants c and d, not both zero. By (2.2.1) and (2.4.11),

$$\tan\theta = \frac{R_1}{R_2}\frac{-c\cos\phi_1 + d\sin\phi_1}{c\sin\phi_2 + d\cos\phi_2} = \frac{R_1}{R_2}\frac{\sin(\phi_1-\gamma)}{\cos(\phi_2-\gamma)}$$

for some $\gamma \in (\frac{\pi}{2}, \frac{\pi}{2}]$. Hence

$$\tan\theta = \frac{R_1}{R_2}\cos(\phi_1-\phi_2)\{\tan(\phi_2-\gamma) + \tan(\phi_1-\phi_2)\}.$$

By (2.4.12) $\cos(\phi_1-\phi_2)>0$ and $\tan(\phi_1-\phi_2)$ is bounded. Then, by Theorem 2.2.1, θ and $\phi_2-\gamma$ take the values in $(\mathbb{Z}+\frac{1}{2})\pi$ at the same points, and consequently their difference $\theta-\phi_2+\gamma$ is globally bounded.

From (2.4.10) and (2.4.11), R_1 and R_2 are a-periodic and

$$\phi_2(x+a,\lambda)=\phi_2(x,\lambda)+\nu(\lambda).$$

Thus the function $\phi_2(x,\lambda)-\nu(\lambda)x/a$ is a-periodic and continuous, and therefore globally bounded. By what we have just proved $\theta(x,\lambda)-\nu(x,\lambda)x/a$ has the same property and (2.4.9) follows with $k=\nu$. The relation $D=2\cos k$ follows from (1.4.6) □

In addition to describing the rate of oscillation of solutions of the periodic equation, the rotation number for a one-dimensional Schrödinger equation or Dirac system has a further physical interpretation. Let u be a Floquet solution for spectral parameter $\lambda\in\mathcal{S}$, so that $u(a)=e^{\mu(\lambda)}\,u(0)$, and consider $v(x):=e^{-i\kappa x}\,u(x)$ with a parameter $\kappa\in\mathbb{R}$. Then v will be a-periodic if $\mu(\lambda)=i\kappa a$, which in the light of Theorem 2.4.3 is equivalent to choosing

$$\kappa=\frac{k(\lambda)}{a}.$$

In the case of a one-dimensional Schrödinger equation, the first component v_1 satisfies the modified equation

$$\left(-i\frac{d}{dx}+\kappa\right)^2 y(x)+q(x)\,y(x)=\lambda\,y(x).$$

In the case of the Dirac system, v satisfies

$$\sigma_2\left(-i\frac{d}{dx}+\kappa\right)v(x)+p_1\sigma_3\,v(x)+p_2\sigma_1\,v(x)+q(x)\,v(x)=\lambda\,v(x).$$

In both cases the quantum mechanical momentum operator $-i\frac{d}{dx}$ is replaced with $-i\frac{d}{dx}+\kappa$. Instead of a twisted periodic boundary-value problem on $[0,a]$ for the original periodic equation, one can therefore equivalently consider a periodic boundary-value problem for the modified equation, and this point of view is preferred in physics. Due to this interpretation, κ is called *quasi-momentum.*

The rotation number plays the following further role in quantum mechanics. Consider the boundary-value problem for the a-periodic Sturm-Liouville or Dirac system with separated boundary conditions on an interval $[0,x]$. Let $\theta(\cdot,\lambda)$ be the Prüfer angle for spectral parameter λ such that $\theta(0,\lambda)=\alpha$, where α is the parameter of the boundary condition at 0. By Theorem 2.3.5 and the formulae (2.4.1) and (2.4.9), the asymptotic number of eigenvalues in an interval $(\lambda',\lambda'']$ is given by

$$\frac{1}{\pi}(\theta(\lambda'',x)-\theta(\lambda',x))\sim\frac{k(\lambda'')-k(\lambda')}{\pi a}\,x\qquad(x\to\infty).$$

Thus the number of eigenvalues (corresponding to physical bound states) grows asymptotically linear with the length of the interval, and it makes sense to interpret

$$\frac{k(\lambda)}{\pi a}$$

as the *integrated density of states.*

Theorem 2.4.3 and Theorem 1.6.2 yield a formula for the derivative of the rotation number in the stability intervals of Hill's equation and the Dirac system. Indeed, the identity $D(\lambda) = 2\cos k(\lambda)$ implies that

$$D'(\lambda) = -2k'(\lambda)\, \sin k(\lambda) \tag{2.4.13}$$

and, bearing in mind that $k(\lambda)$ plays the role of $\nu(\lambda)$ in (1.6.13), we find that

$$k'(\lambda) = \frac{1}{i\, \det Y(0)} \int_0^a |y|^2\, w, \tag{2.4.14}$$

where y is a complex Floquet solution for $\lambda \in \mathcal{S}$ and Y is the fundamental matrix formed from y and $\overline{y}$. (For the Dirac system, read $w = 1$.)

Equation (2.4.13) shows that the derivative of the rotation number diverges at the points of transition from stability to instability. Thus k is continuously differentiable on $\mathcal{S}$ and constant on each component interval of $\mathcal{I}$, but it is not a continuously differentiable function throughout. For Hill's equation, the derivative of the rotation number is bounded below in the stability intervals as follows.

Theorem 2.4.4. *The rotation number k of Hill's equation* (1.5.2) *satisfies*

$$(k^2)'(\lambda) \geq \left(\int_0^a \sqrt{\frac{w}{p}} \right)^2 \qquad (\lambda \in \mathcal{S}).$$

Proof. Let $\lambda \in \mathcal{S}$ and y a corresponding (complex-valued) Floquet solution of (1.5.2). Then we can write

$$y(x) = |y|(x)\, e^{i\phi_1(x)} \qquad (x \in \mathbb{R}) \tag{2.4.15}$$

as in (2.4.11) and calculate the rotation number from a Prüfer angle θ of the solution $\operatorname{Re} y$ according to (2.4.9). As shown in the proof of Theorem 2.4.3, $\theta - \phi_1$ is universally bounded, and so

$$k(\lambda) = \lim_{x\to\infty} \frac{\theta(x)\, a}{x} = \lim_{x\to\infty} \frac{\phi_1(x)\, a}{x}. \tag{2.4.16}$$

Now differentiating (2.4.15) gives

$$y'(x) = (|y|'(x) + i\phi_1'(x)\, |y|(x))\, e^{i\phi_1(x)},$$

and hence the (constant) Wronskian of the fundamental matrix formed from y and $\overline{y}$ is

$$\det Y = y\, p\overline{y}' - py'\,\overline{y} = -2i\, p\phi_1'\, |y|^2.$$

Writing ϕ_1 as the integral over its derivative, we hence conclude from (2.4.16) that

$$k(\lambda) = \lim_{x\to\infty} \frac{a}{x} \int_0^x \frac{i\,\det Y(0)}{2p\,|y|^2} = i\,\det Y(0) \int_0^a \frac{1}{2p\,|y|^2}.$$

Combining this with (2.4.14), the Wronskian cancels out and we obtain

$$\begin{aligned}(k^2)'(\lambda) &= 2\,k(\lambda)\,k'(\lambda)\\ &= \int_0^a \frac{1}{|y|^2\,p} \int_0^a |y|^2\,w \geq \left(\int_0^a \sqrt{\frac{w}{p}}\right)^2\end{aligned}$$

by the Cauchy-Schwarz inequality (note that the integral on the right always exists, since w and $1/p$ are assumed integrable over $[0,a]$). □

The proof of Theorem 2.4.1 set out from the observation that the eigenvalues of a particular boundary-value problem on $[0,a]$ with separated boundary conditions lie in closed instability intervals. This holds true for all boundary-value problems where the boundary conditions at the two end-points are the same, even if the period interval is shifted by an offset τ to $[\tau,\tau+a]$, as the following result shows. Clearly the eigenvalues of the periodic and semi-periodic boundary-value problems, and consequently the positions of the stability intervals of which they are end-points, remain unchanged under such a shift.

Theorem 2.4.5. (a) *Let $\tau \in [0,a)$. Then the n-th eigenvalue of the boundary-value problem for the Sturm-Liouville or Dirac system on $[\tau,\tau+a]$ with boundary conditions* (2.3.2)*, in which $\alpha = \beta \in (0,\pi)$, lies in the closed instability interval $\overline{I}_n$.*

In the Dirichlet case where $\alpha = 0$ and $\beta = \pi$, the n-th eigenvalue lies in the closed instability interval $\overline{I}_{n+1}$.

(b) *Let $\lambda \in \overline{I}_n$, $n \in \mathcal{J}\setminus\{0\}$, and let $\alpha = \beta \in (0,\pi)$. Then there is $\tau \in [0,a)$ such that λ is the n-th eigenvalue of the boundary-value problem for the Sturm-Liouville or Dirac system on $[\tau,\tau+a]$ with boundary conditions* (2.3.2)*.*

The same holds true in the Dirichlet case $\alpha = 0$ and $\beta = \pi$, except that λ is the $(n-1)$-th eigenvalue.

Proof. (a) Let u be a real-valued eigenfunction, and define another solution v of the system by $v(x) = u(x+a)$. Then, by (2.3.2) referred to $[\tau,\tau+a]$,

$$u_1(\tau)\cos\alpha - u_2(\tau)\sin\alpha = 0$$

and

$$v_1(\tau)\cos\alpha - v_2(\tau)\sin\alpha = 0.$$

Hence the Wronskian of u and v is zero at τ, making u and v linearly dependent. Thus $u(x+a) = cu(x)$ with a real constant c. If now $|c| = 1$, u is either periodic or semi-periodic. If however $|c| \neq 1$, then u is exponentially unbounded at ∞ or $-\infty$, and then the periodic equation is not stable. Either way, the eigenvalue lies in a closed instability interval.

The statement about the eigenvalue numbering follows from the comparison of the growth of the Prüfer angle of u, obtained in Theorem 2.3.4, with the asymptotic (2.4.1).

(b) Since $\lambda \in \overline{I}_n$, there is a real-valued Floquet solution u satisfying (1.3.5). As g is periodic and the exponential factor is positive, the Prüfer angle of u satisfies $\theta(x+a,\lambda) = n\pi + \theta(x,\lambda)$, from (2.4.1). Since $n \neq 0$, we have in particular either $\theta(a,\lambda) \geq \pi + \theta(0,\lambda)$ or $\theta(a,\lambda) \leq -\pi + \theta(0,\lambda)$. The Intermediate Value Theorem then ensures the existence of a point $\tau \in [0,a)$ such that $\theta(\tau,\lambda) = \alpha \pmod{\pi}$ as required. □

The theorem can also be expressed as follows. Let $\Lambda_{\tau,n}$ denote the n-th eigenvalue in the above boundary-value problems on $[\tau, \tau+a]$. Then the range of $\Lambda_{\tau,n}$, as a function of τ, is $\overline{I}_{n+1}$ in the Dirichlet case, and $\overline{I}_n$ otherwise.

2.5 Zeros of solutions of Hill's equation

The Sturm-Liouville equation has the special property that the zeros of real-valued solutions reflect the growth of the corresponding Prüfer angle. Indeed, it is clear from the Prüfer transformation formulae (2.2.4) that these zeros occur exactly at the points where the Prüfer angle vanishes modulo π, and the Prüfer equation (2.2.5) shows that the angle is strictly increasing at such points. Hence the Prüfer angle increases by exactly π between any two adjacent zeros of a solution.

Bearing this in mind, Theorem 2.4.2 provides the following information about the number of zeros of eigenfunctions of the boundary-value problems with periodic or semi-periodic boundary conditions. (Theorem 2.3.4 (a) has the analogous statement for the problem with separated boundary conditions.)

Theorem 2.5.1. (a) *Any eigenfunction for the periodic eigenvalue λ_0 has no zeros in $[0,a]$. Any eigenfunction for the periodic eigenvalues λ_{2m+1} and λ_{2m+2} has $2(m+1)$ zeros in $[0,a)$ $(m \in \mathbb{N}_0)$.*

(b) *Any eigenfunction for the semi-periodic eigenvalues μ_{2m} and μ_{2m+1} has $2m+1$ zeros in $[0,a)$ $(m \in \mathbb{N}_0)$.*

For general real-valued solutions of Hill's equation on $\mathbb{R}$ we can draw the following conclusions.

Theorem 2.5.2. *Let λ_0 be the smallest periodic eigenvalue. For $\lambda \in \mathbb{R}$, let y be a non-trivial real-valued solution of (1.5.2). If $\lambda > \lambda_0$, then y has infinitely many zeros. If $\lambda \leq \lambda_0$, then y has at most one zero.*

Proof. As $\lambda_0 = \inf \mathcal{S}$, we know that the rotation number $k(\lambda) > 0$ if $\lambda > \lambda_0$, and the first statement follows from the asymptotics (2.4.1), (2.4.9).

If $\lambda < \lambda_0$, assume there are points $x_1 < x_2$ such that $y(x_1) = y(x_2) = 0$. The Prüfer angle θ of y can be normalised so that $\theta(x_1) = 0$; then $\theta(x_2) = n\pi$ with some $n \in \mathbb{N}$. Let θ_0 be the Prüfer angle of a periodic eigenfunction for λ_0; then by Theorem 2.4.2, θ_0 is a-periodic and (without loss of generality) takes values in $(0, \pi)$. By the intermediate value theorem, there is a point $x_0 \in (x_1, x_2)$ such that $\theta(x_0) = \theta_0(x_0)$. But also $\theta(x_2) > \theta_0(x_2)$, contradicting Theorem 2.3.3. □

The statements of Theorem 2.5.2 are sometimes expressed by saying that Hill's equation is *oscillatory* if $\lambda > \lambda_0$, and *disconjugate* if $\lambda < \lambda_0$.

Studying the zeros of the periodic eigenfunctions gives rise to the following upper bounds to the periodic eigenvalues at the lower ends of stability intervals.

Theorem 2.5.3. *The smallest periodic eigenvalue of Hill's equation satisfies*

$$\lambda_0 \leq \left(\int_0^a q\right) \Big/ \left(\int_0^a w\right),$$

with equality if and only if $q = \lambda_0 w$ a.e.

Proof. By Theorem 2.5.1 (a), the periodic eigenfunction y for eigenvalue λ_0 has no zeros and can therefore be taken to be positive. Then, by (1.5.2),

$$(p(\log y)')' = q - \lambda_0 w - \frac{(py')^2}{py^2}. \tag{2.5.1}$$

The left-hand side is the derivative of an a-periodic function, and so integration over a period interval gives

$$\int_0^a q - \lambda_0 \int_0^a w = \int_0^a \frac{(py')^2}{py^2} \geq 0,$$

and hence the stated estimate.

Equality implies that $py' = 0$, and thus $p(\log y)' = 0$ throughout, and the last statement follows in view of (2.5.1). □

Theorem 2.5.4. *Let $m \in \mathbb{N}_0$ and assume that*

$$\int_0^a q(x) e^{2\pi r x i/a}\, dx = 0 \qquad (r \in \{0, \ldots, 2m\}). \tag{2.5.2}$$

Then the periodic eigenvalue λ_{2m} of Hill's equation satisfies

$$\lambda_{2m} \leq \frac{\sup p}{\inf w} (2m\pi/a)^2,$$

with equality if and only if $q = 0$ and p, w are constant a.e.

Proof. Let y be a real-valued eigenfunction for λ_{2m}. By Theorem 2.5.1 (a), y has $2m$ zeros $x_1, \dots x_{2m} \in [0, a)$. We introduce the function

$$f(x) := \prod_{r=1}^{2m} \sin(\pi(x - x_r)/a)$$

which is a-periodic and has a finite Fourier expansion of the form

$$\sum_{r=-m}^{m} c_r \exp(2\pi rxi/a).$$

Hence

$$\int_0^a f'^2 \leq (2\pi m/a)^2 \int_0^a f^2. \tag{2.5.3}$$

Also, by (2.5.2),

$$\int_0^a q\, f^2 = 0. \tag{2.5.4}$$

The function y/f (extended by continuity at each x_r) is a-periodic and (without loss of generality) positive. Then, with $g := \log(y/f)$, we can write $y = fe^g$. Multiplication of Hill's equation (1.5.2) by fe^{-g} and integration over the period interval gives

$$\int_0^a (q - \lambda_{2m} w)\, f^2 = \int_0^a f\, e^{-g}\, (py')' = f\, e^{-g} (py')\Big|_0^a + \int_0^a p\, (fg')^2 - \int_0^a p\, f'^2. \tag{2.5.5}$$

The boundary terms cancel by periodicity and, using (2.5.4), we find that

$$\lambda_{2m} \leq \int_0^a p\, f'^2 \Big/ \int_0^a w\, f^2,$$

and the stated estimate follows by (2.5.3).

In the case of equality, it follows that p and w must be constant in view of the last integral estimates. Moreover, from (2.5.5), $g' = 0$ and hence g is constant. Also to make (2.5.3) sharp, f must be a linear combination of $e^{\pm 2m\pi xi/a}$ only. This means that $y(x) = \sin(2m\pi x/a + \phi)$ with a constant $\phi \in \mathbb{R}$. Using this information along with $\lambda_{2m} = p(2m\pi/a)^2/w$ in Hill's equation gives $q = 0$ a.e. □

There is also a corresponding result to Theorem 2.5.4 for the semi-periodic eigenvalue μ_{2m+1} in which $2m$ is replaced by $2m + 1$.

2.6 The upper end-points of the stability intervals

The eigenvalues λ_{2m} and μ_{2m+1} in the previous section are the lower end-points of the stability intervals. In this section we consider the upper end-point $\lambda^{(n)}$ of the

n-th stability interval. Thus $\lambda^{(n)}$ is μ_{n-1} or λ_{n-1} according as n is odd or even. As a further application of Theorem 2.5.1, we give a lower bound for $\lambda^{(n)}$. In the following theorem, we write

$$A = \int_0^a p^{-1} \tag{2.6.1}$$

and

$$M = \sup(pw) \quad \text{in } [0, a].$$

Theorem 2.6.1. *Let*

$$\int_0^a q_- \geq -4n^2/A, \tag{2.6.2}$$

where $q_-(x) = \min\{q(x), 0\}$ *and* n *is a positive integer. Then*

$$\lambda^{(n)} \geq M^{-1}(n\pi/A)^2 \left(1 + \frac{A}{4n^2} \int_0^a q_-\right). \tag{2.6.3}$$

Equality holds in (2.6.3) *only when* $q = 0$ *and* $pw = M$ *a.e.*

Proof. Let y be the eigenfunction corresponding to $\lambda^{(n)}$, so that y is a-periodic or a-semi-periodic as the case may be. Let $x_0, \ldots, x_n$ be $n+1$ consecutive zeros of y in the order $x_0 < x_1 < \cdots < x_n$. By Theorem 2.5.1, we have

$$x_n = x_0 + a. \tag{2.6.4}$$

In Hill's equation (1.5.2), we write $\lambda = \lambda^{(n)}$ and then multiply by y and integrate over $[x_r, x_{r+1}]$. After an integration by parts, this gives

$$\lambda^{(n)} \int_{x_r}^{x_{r+1}} y^2 w = \int_{x_r}^{x_{r+1}} (py'^2 + qy^2). \tag{2.6.5}$$

Also,

$$2y(x) = \int_{x_r}^{x} y' - \int_{x}^{x_{r+1}} y'$$

giving

$$2|y(x)| \leq \int_{x_r}^{x_{r+1}} |y'|$$

for all x in $[0, a]$. Hence, by the Cauchy-Schwarz inequality,

$$y^2(x) \leq \tfrac{1}{4}(x_{r+1} - x_r) \int_{x_r}^{x_{r+1}} y'^2. \tag{2.6.6}$$

Let us for the moment consider the special case when $p(x) = 1$, so that $A = a$ in (2.6.1). Then (2.6.5), (2.6.6) and the inequality $q \geq q_-$ give

$$\lambda^{(n)}/\rho_r \geq (x_{r+1} - x_r)^{-1} + \tfrac{1}{4} \int_{x_r}^{x_{r+1}} q_-, \tag{2.6.7}$$

where

$$\rho_r = (x_{r+1} - x_r) \int_{x_r}^{x_{r+1}} y'^2 \bigg/ \int_{x_r}^{x_{r+1}} y^2 w. \tag{2.6.8}$$

We now sum (2.6.7) for $r = 0, \ldots, n-1$ and use the harmonic mean/arithmetic mean inequality

$$\sum_{r=0}^{n-1} (x_{r+1} - x_r)^{-1} \geq n^2 \left(\sum_{r=0}^{n-1} (x_{r+1} - x_r) \right)^{-1}.$$

Then (2.6.7) gives

$$\lambda^{(n)} \sum_{r=0}^{n-1} \rho_r^{-1} \geq n^2 a^{-1} + \tfrac{1}{4} \int_0^a q_-, \tag{2.6.9}$$

where we have used (2.6.4) and the periodicity of q_-. By (2.6.3) (with $A = a$), the right-hand side of (2.6.9) is non-negative, and hence $\lambda^{(n)} \geq 0$.

At this point, we require the Wirtinger inequality which states that

$$\int_c^d |f'|^2 \geq \frac{\pi}{(d-c)^2} \int_c^d |f|^2 \tag{2.6.10}$$

for any f which is $AC[c, d]$ with $f(c) = f(d) = 0$. Applying this inequality in (2.6.8), along with $w \leq M$, we have

$$\rho_r \geq M^{-1} \pi^2 (x_{r+1} - x_r)^{-1}. \tag{2.6.11}$$

Then (2.6.9) yields (2.6.3) with $A = a$.

To deal with general $p(x)$, we make the change of variable

$$u = \int_0^x p^{-1},$$

so that (1.5.2) becomes

$$-d^2 Y(u)/du^2 + Q(u)Y(u) = \lambda W(u) Y(u), \tag{2.6.12}$$

where $Y(u) = y(x)$, $Q(u) = p(x)q(x)$ and $W(u) = p(x)w(x)$. We then apply the result just proved for $p(x) = 1$ to (2.6.12), and (2.6.3) follows immediately.

To identify when equality occurs in (2.6.3), we note first that, following the use of the Cauchy-Schwarz inequality, strict inequality occurs in (2.6.6) since y' is not a constant. Hence, comparing (2.6.5) and (2.6.7), equality holds in (2.6.7) only if $q = 0$ a.e. Also, equality can only hold in (2.6.11) when $w = M$ a.e., and this gives rise to the case $pw = M$ a.e. for general p. □

In the particular case when $p = w = 1$ and $n = 1$, (2.6.3) becomes

$$\mu_0 \geq \frac{\pi^2}{a^2}\left(1 + \tfrac{a}{4}\int_0^a q_-\right) \tag{2.6.13}$$

with equality only when $q = 0$ a.e.

We next give an example to show that the number 4 which appears in (2.6.3) and (2.6.13) is best possible, that is, it cannot be replaced by a larger constant. Let δ be a real number such that $0 < \delta < a/2$ and let ψ be a real-valued function with a continuous second derivative in $[0, a]$ such that

$$\psi(x) = \begin{cases} x & (0 \leq x \leq \frac{a}{2} - \delta), \\ a - x & (\frac{a}{2} + \delta \leq x \leq a), \end{cases} \tag{2.6.14}$$

and $\psi > 0$, $\psi'' < 0$ in $(\frac{a}{2} - \delta, \frac{a}{2} + \delta)$. Now define

$$q = \psi''/\psi$$

in $(\frac{a}{2} - \delta, \frac{a}{2} + \delta)$ and $q = 0$ elsewhere in $[0, a]$. Then

$$\psi'' - q\psi = 0$$

in $[0, a]$. Thus ψ satisfies (1.5.2) with $\lambda = 0$, $p = w = 1$ and the above q. By (2.6.14), ψ is also a-semi-periodic and, since ψ has only one zero in $[0, a)$, it follows from Theorem 2.5.1 (b) that ψ is an eigenfunction corresponding to either μ_0 or μ_1. Hence one of μ_0 and μ_1 is zero and, in either case,

$$\mu_0 \leq 0. \tag{2.6.15}$$

Next,

$$\begin{aligned}\int_0^a q_- &= \int_{\frac{a}{2}-\delta}^{\frac{a}{2}+\delta} \psi''/\psi = [\psi'\psi]\Big|_{\frac{a}{2}-\delta}^{\frac{a}{2}+\delta} + \int_{\frac{a}{2}-\delta}^{\frac{a}{2}+\delta} (\psi'/\psi)^2 \\ &> (\tfrac{a}{2} - \delta)^{-1}\left(\psi'(\tfrac{a}{2} + \delta) - \psi'(\tfrac{a}{2} - \delta)\right) \\ &= -4(a - 2\delta)^{-1}.\end{aligned}$$

Hence

$$1 + \frac{a}{4(1 - 2\delta/a)^{-1}}\int_0^a q_- > 0.$$

It now follows from (2.6.15) that (2.6.13) does not hold with 4 replaced by the larger constant $4(1 - 2\delta/a)^{-1}$. Since δ can be arbitrarily small, the best-possible nature of 4 in (2.6.13) is demonstrated.

2.7 A step-function example

The ordering (2.4.6) of the periodic and semi-periodic eigenvalues λ_n and μ_n is most simply illustrated by the well-known example of Hill's equation where $p = w = 1$ and $q = 0$. Here we have $\lambda_0 = 0$ and, for $m \geq 0$,

$$\begin{aligned}\lambda_{2m+1} = \lambda_{2m+2} &= 4(m+1)^2\pi^2/a^2,\\ \mu_{2m} = \mu_{2m+1} &= (2m+1)^2\pi^2/a^2.\end{aligned}$$

For a more substantial example of (2.4.6), we take $p = 1,\ q = 0$ and w to be a two-valued step-function

$$w(x) = \begin{cases} w_1 & (0 \leq x < a_1),\\ w_2 & (a_1 \leq x < a),\end{cases}$$

where $w_1 \neq w_2$ and with some a_1 in $(0, a)$. Now (1.5.2) is simply

$$y'' + \lambda w y = 0. \tag{2.7.1}$$

We again have the first periodic eigenvalue $\lambda_0 = 0$. For $\lambda > 0$, (2.7.1) is soluble in terms of sines and cosines on either side of the interface point a_1. Then matching of a solution and its first derivative at the interface gives a solution on the whole of $(0, a)$. In particular, we do this for the solutions ϕ_1 and ϕ_2 forming the canonical fundamental system, i.e., with initial values

$$\phi_1(0,\lambda) = 1, \quad \phi_1'(0,\lambda) = 0; \quad \phi_2(0,\lambda) = 0, \quad \phi_2'(0,\lambda) = 1.$$

We write

$$\nu = \sqrt{\lambda}, \quad A_1 = a_1\sqrt{w_1}, \quad A_2 = (a - a_1)\sqrt{w_2}, \quad \sigma = \sqrt{w_2/w_1}, \tag{2.7.2}$$

and we note that $\sigma \neq 1$. Then, omitting the straightforward details of the calculations, we find that

$$\begin{aligned}\phi_1(a,\lambda) &= \cos\nu A_1 \cos\nu A_2 - \frac{1}{\sigma}\sin\nu A_1 \sin\nu A_2,\\ \phi_2'(a,\lambda) &= \cos\nu A_1 \cos\nu A_2 - \sigma\sin\nu A_1 \sin\nu A_2.\end{aligned}$$

Hence Hill's discriminant is, by (1.5.6),

$$\begin{aligned}D(\lambda) &:= \phi_1(a,\lambda) + \phi_2'(a,\lambda)\\ &= 2\cos\nu A_1 \cos\nu A_2 - \left(\sigma + \frac{1}{\sigma}\right)\sin\nu A_1 \sin\nu A_2\\ &= \cos\nu I + \cos\nu J + \frac{1}{2}\left(\sigma + \frac{1}{\sigma}\right)(\cos\nu I - \cos\nu J)\end{aligned} \tag{2.7.3}$$

where

$$I = A_1 + A_2, \quad J = |A_1 - A_2|. \tag{2.7.4}$$

Thus the equations $D(\lambda) = \pm 2$ for the λ_n and μ_n become

$$(\sigma + 1)^2 \cos \nu I - (\sigma - 1)^2 \cos \nu J = \pm 4\sigma. \tag{2.7.5}$$

Explicit solutions of (2.7.5) can be found when there is a suitable connection between I and J, and here we consider the case where

$$I = 2J. \tag{2.7.6}$$

Thus, by (2.7.4), either $A_1 = 3A_2$ or $A_2 = 3A_1$. We take the former, and then (2.7.2) gives

$$\sigma = \frac{1}{3} a_1/(a - a_1). \tag{2.7.7}$$

By (2.7.6), (2.7.5) becomes

$$2\cos^2\theta - \frac{(\sigma - 1)^2}{(\sigma + 1)^2} \cos\theta - 1 \mp \frac{4\nu}{(\sigma + 1)^2} = 0, \tag{2.7.8}$$

where

$$\theta = \nu J \tag{2.7.9}$$

and the $\mp$ produce the λ_n and μ_n respectively.

(a) *The periodic eigenvalues* λ_n. Taking the minus alternative in (2.7.8), we can factorize to obtain

$$(\cos\theta - 1)(\cos\theta - \cos\alpha) = 0,$$

where

$$\alpha = \cos^{-1}\left(-\frac{1}{2} - \frac{2\sigma}{(\sigma + 1)^2}\right) \quad \left(\frac{\pi}{2} < \alpha < \pi\right). \tag{2.7.10}$$

The first factor gives

$$\nu J = 2(m + 1)\pi \quad (m \in \mathbb{N}_0).$$

These values yield double eigenvalues because, by (2.7.3) and (2.7.6), $D'(\lambda)$ is zero at these values. The second factor gives

$$\nu J = \begin{cases} 2m\pi + \alpha \\ 2(m + 1)\pi - \alpha \end{cases} \qquad (m \in \mathbb{N}_0).$$

Thus, altogether, we obtain the periodic eigenvalues $\lambda_0 = 0$ and, for $m \geq 0$,

$$\left.\begin{aligned} &\lambda_{4m+1} = (2m\pi + \alpha)^2/J^2, \quad \lambda_{4m+2} = (2(m + 1)\pi - \alpha)^2/J^2, \\ &\lambda_{4m+3} = \lambda_{4m+4} = 4(m + 1)^2\pi^2/J^2. \end{aligned}\right\} \tag{2.7.11}$$

(b) *The semi-periodic eigenvalues* μ_n. Taking the plus alternative in (2.7.8), we write the equation as

$$\cos^2\theta - \tau\cos\theta - \tau = 0,$$

where

$$\tau = \tfrac{1}{2}(\sigma-1)^2/(\sigma+1)^2 \tag{2.7.12}$$

and we note that $0 < \tau < \frac{1}{2}$. Thus $\cos\theta = \cos\beta$ or $\cos\theta = \cos\gamma$, where

$$\beta = \cos^{-1}\left(\tfrac{1}{2}\left(\tau + \sqrt{\tau^2+4\tau}\right)\right), \qquad \gamma = \cos^{-1}\left(\tfrac{1}{2}\left(\tau - \sqrt{\tau^2+4\tau}\right)\right) \tag{2.7.13}$$

and

$$0 < \beta < \gamma < \pi. \tag{2.7.14}$$

Hence, recalling (2.7.9), we obtain

$$\nu J = \begin{cases} 2m\pi + \beta \\ 2(m+1)\pi - \beta \end{cases} \qquad (m \in \mathbb{N}_0)$$

and similarly with γ in place of β. Thus, by (2.7.14), we obtain the semi-periodic eigenvalues

$$\left.\begin{array}{ll} \mu_{4m} = (2m\pi+\beta)^2/J^2, & \mu_{4m+1} = (2m\pi+\gamma)^2/J^2, \\ \mu_{4m+2} = (2(m+1)\pi-\gamma)^2/J^2, & \mu_{4m+3} = (2(m+1)\pi-\beta)^2/J^2. \end{array}\right\} \tag{2.7.15}$$

The ordering (2.4.6) can now be seen in the values of the λ_n and μ_n in (a) and (b) provided that $\gamma < \alpha$. Thus, by (2.7.10) and (2.7.13), we have to show that

$$1 + \frac{4\sigma}{(\sigma+1)^2} > \sqrt{\tau^2+4\tau} - \tau,$$

i.e.,

$$2 - 2\tau > \sqrt{\tau^2+4\tau} - \tau,$$

by (2.7.12). Now, by (2.7.12) again, $\tau < \frac{1}{2}$ giving

$$\sqrt{\tau^2+4\tau} + \tau < 2,$$

and we are done.

A simple choice of the parameters in this example is

$$w_1 = 9, \quad w_2 = 1, \quad a_1 = \tfrac{a}{2}. \tag{2.7.16}$$

It is easy to check that $J = a$ in (2.7.4) and that (2.7.6) holds. Then (2.7.7) and (2.7.12) give

$$\sigma = \tfrac{1}{3}, \quad \tau = \tfrac{1}{8}$$

and finally (2.7.10) and (2.7.13) give

$$\alpha = \cos^{-1}(-\tfrac{7}{8}), \quad \beta = \cos^{-1}(\tfrac{1+\sqrt{33}}{16}), \quad \gamma = \cos^{-1}(\tfrac{1-\sqrt{33}}{16}).$$

2.8 Even coefficients

In the case where p, q and w are even functions in (1.5.2), there are four eigenvalue problems, each with separated boundary conditions, whose eigenvalues and eigenfunctions are usefully related to those for the periodic and semi-periodic problems over $[0, a]$. These four problems comprise (1.5.2) on the half-interval $[0, \frac{1}{2}a]$ together with (in turn) the boundary conditions:

$$\begin{array}{ll} \text{(I)} & y'(0) = y'(\tfrac{1}{2}a) = 0 \\ \text{(II)} & y(0) = y(\tfrac{1}{2}a) = 0 \\ \text{(III)} & y'(0) = y(\tfrac{1}{2}a) = 0 \\ \text{(IV)} & y(0) = y'(\tfrac{1}{2}a) = 0. \end{array}$$

We denote the eigenvalues and eigenfunctions in these problems by $\lambda_n^{\mathrm{I}}, \psi_n^{\mathrm{I}}$ etc, and their relation to the periodic λ_n, ψ_n and semi-periodic μ_n, ξ_n is given in the two propositions which follow. For the proofs, we recall the basic oscillation property that the n-th eigenfunction (such as ψ_n^{I} etc) in problems with separated boundary conditions has exactly n zeros in the open interval in question, here $(0, \frac{1}{2}a)$ (see Theorem 2.3.4 (a)).

Theorem 2.8.1. (a) *For $n \geq 1$, λ_n^{I} and $\lambda_{n-1}^{\mathrm{II}}$ are the same as λ_{2n-1} and λ_{2n} (but not necessarily in that order). Also, ψ_n^{I} is even and ψ_{n-1}^{II} is odd. For $n = 0$, $\lambda_0^{\mathrm{I}} = \lambda_0$ and ψ_0^{I} is even.*

(b) *For $n \geq 0$, λ_n^{III} and λ_n^{IV} are the same as μ_{2n} and μ_{2n+1} (but not necessarily in that order). Also, ψ_n^{III} is even and ψ_n^{IV} is odd.*

Proof. (a) Since p, q and w are all even, $y(-x)$ is also a solution of (1.5.2) when $y(x)$ is. For Problem I, the initial condition $y'(0) = 0$ means that ψ_n^{I} is even:

$$\psi_n^{\mathrm{I}}(-x) = \psi_n^{\mathrm{I}}(x),$$

these two solutions of (1.5.2) having the same initial values at $x = 0$. The other condition $y'(\frac{1}{2}a) = 0$ then shows that the derivative of ψ_n^{I} is zero at both $-\frac{1}{2}a$ and $\frac{1}{2}a$. Thus ψ_n^{I} has period a. Further, ψ_n^{I} has exactly n zeros in the open interval $(0, \frac{1}{2}a)$ and hence exactly $2n$ zeros in $[-\frac{1}{2}a, \frac{1}{2}a)$. Hence, by Theorem 2.5.1 (a), λ_n^{I} is either λ_{2n-1} or λ_{2n}. The rest of part (a) is proved similarly, where we note that ψ_{n-1}^{II} also has $2n$ zeros in $[-\frac{1}{2}a, \frac{1}{2}a)$: $2(n-1)$ of them in $(-\frac{1}{2}a, 0) \cup (0, \frac{1}{2}a)$ and the other 2 at $-\frac{1}{2}a$ and 0.

The proof of part (b) is again similar, making use of Theorem 2.5.1 (b). □

This theorem will be used in a further discussion of the Mathieu equation in section 3.7 but, meanwhile, a more explicit illustration of it is provided by two examples.

Example 1. Let $p = 1$, $q = 0$ and let w be the step-function

$$w(x) = \begin{cases} 9 & (0 \le x < \frac{1}{4}a, \quad \frac{3}{4}a \le x < a), \\ 1 & (\frac{1}{4}a \le x < \frac{3}{4}a), \end{cases} \tag{2.8.1}$$

a-periodically extended to obtain an even function on $\mathbb{R}$. This example is the same as in (2.7.16) with x shifted through $\frac{1}{4}a$. This does not affect the values of the periodic and semi-periodic eigenvalues found above in (2.7.11) and (2.7.15).

Referring in particular to Proposition 2.8.1 (b), a simple but careful calculation shows that, for (2.8.1),

$$(\mu_{8m}, \mu_{8m+1}) = (\lambda_{4m}^{\mathrm{III}}, \lambda_{4m}^{\mathrm{IV}}), \qquad (\mu_{8m+2}, \mu_{8m+3}) = (\lambda_{4m+1}^{\mathrm{III}}, \lambda_{4m+1}^{\mathrm{IV}}),$$

$$(\mu_{8m+4}, \mu_{8m+5}) = (\lambda_{4m+2}^{\mathrm{IV}}, \lambda_{4m+2}^{\mathrm{III}}), \quad (\mu_{8m+6}, \mu_{8m+7}) = (\lambda_{4m+3}^{\mathrm{IV}}, \lambda_{4m+3}^{\mathrm{III}})$$

as ordered pairs. Thus sometimes III comes before IV and sometimes vice versa.

Example 2. Let $p = w = 1$ and

$$q(x) = c\cos 2x - 2\alpha^2 \cos 4x$$

where c and α are non-zero constants. Then (1.5.2) has the special feature that the transformation

$$y = z\exp(\alpha\cos 2x)$$

produces a z-equation in which the argument $4x$ does not appear. Thus

$$z''(x) - 4\alpha\sin 2x\, z'(x) + (\lambda + 2\alpha^2 - (c + 4\alpha)\cos 2x)\, z(x) = 0.$$

It is now simple to verify the following explicit solutions of the z-equation with the corresponding values of λ. In (i) "even" comes before "odd" and, in (ii), "odd" comes before "even".

(i) Let $c = -8\alpha$. Then there are solutions

$$\cos x, \quad \lambda = 1 - 4\alpha - 2\alpha^2,$$
$$\sin x, \quad \lambda = 1 + 4\alpha - 2\alpha^2.$$

By Theorem 2.5.1 (b), these eigenvalues are μ_0 and μ_1.

(ii) Let $c = -12\alpha$. Then there are solutions

$$\sin 2x, \quad \lambda = 4 - 2\alpha^2,$$
$$\cos 2x + \left(1 - \sqrt{1 + 16\alpha^2}\right)/4\alpha, \quad \lambda = 2 + 2\sqrt{1 + 16\alpha^2} - 2\alpha^2.$$

By Theorem 2.5.1 (a), these eigenvalues are λ_1 and λ_2.

2.9 Comparison of eigenvalues

In this section we compare the eigenvalues in two separate boundary-value problems. For the Sturm-Liouville case (1.5.2), the coefficients p, q, w and p_1, q_1, w_1 in the two problems satisfy

$$p_1 \geq p, \quad q_1 \geq q, \quad 0 < w_1 \leq w \tag{2.9.1}$$

in $[0, a]$. For the Dirac case (1.5.4),

$$q_1 \geq q \tag{2.9.2}$$

but the p_1 and p_2 coefficients remain the same in the two problems. It is a question of how the corresponding eigenvalues compare when the coefficients are related as in (2.9.1) and (2.9.2).

We consider first the easiest situation where the boundary conditions are separated as in (2.3.2), and we can also add the further distinguishing features

$$0 \leq \alpha_1 \leq \alpha \quad \pi \geq \beta_1 \geq \beta \tag{2.9.3}$$

between the two problems.

Theorem 2.9.1. (a) *For the Sturm-Liouville problem on* $[0, a]$ *with separated boundary conditions, let* (2.9.1) *and* (2.9.3) *hold. Then the eigenvalues* λ_n *and* $\lambda_{1,n}$ *in the two problems satisfy*

1. *if* $w_1 = w$ *a.e., then* $\lambda_{1,n} \geq \lambda_n$ *for* $n \geq 0$*;*
2. *otherwise,* $\lambda_{1,n} \geq \lambda_n$ *provided that* $\lambda_n \geq 0$.

(b) *For the Dirac problem on* $[0, a]$ *with separated boundary conditions, let* (2.9.2) *and* (2.9.3) *hold. Then* $\lambda_{1,n} \geq \lambda_n$ *for* $n \in \mathbb{Z}$.

Proof. For both (a) and (b), let $\theta(x, \lambda)$ and $\theta_1(x, \lambda)$ be the Prüfer angles for the two problems being compared, respectively. Then

$$\theta_1(a, \lambda) = \alpha_1 \leq \alpha = \theta(a, \lambda).$$

Then, under the stated conditions, we can apply Theorem 2.3.1 to the Prüfer equations (2.2.5) and (2.2.7) for the two problems, to obtain $\theta_1(x, \lambda) \leq \theta(x, \lambda)$ in $[0, a]$ and, in particular,

$$\theta_1(a, \lambda_n) \leq \theta(a, \lambda_n) = \beta + n\pi$$

as in Theorem 2.3.4. Thus $\theta_1(a, \lambda_n) \leq \beta_1 + n\pi = \theta_1(a, \lambda_{1,n})$, giving $\lambda_{1,n} \geq \lambda_n$ by the monotonicity of θ_1 as a function of λ (see Theorem 2.3.3). □

Moving on to the boundary conditions (1.8.1), the situation becomes less simple because we no longer have the initial condition $\theta(0, \lambda) = \alpha$, and Theorem

2.3.1 cannot be applied. There is however an alternative method for the Sturm-Liouville case which is based on the Dirichlet integral

$$J(f,g) := \int_0^a (pf'\overline{g}' + qf\overline{g}) \tag{2.9.4}$$

defined for $AC[0,a]$ functions f and g. We first develop the properties of $J(f,g)$ which we need, bearing in mind that the boundary conditions under consideration now are (1.8.2), to include the periodic, semi-periodic and ω-twisted cases.

If, in addition, g' is $AC[0,a]$, an integration by parts in (2.9.4) gives

$$J(f,g) = -\int_0^a f\{(p\overline{g}')' - q\overline{g}\} + [pf\overline{g}']\Big|_0^a. \tag{2.9.5}$$

If f satisfies the first, and g the second, boundary condition in (1.8.2), the integrated terms cancel out since $|\omega| = 1$. In particular, if g is an eigenfunction ψ_n for (1.5.2) and (1.8.2) corresponding to the eigenvalue λ_n, (2.9.5) gives

$$J(f,\psi_n) = \lambda_n c_n, \tag{2.9.6}$$

where c_n denotes the Fourier coefficient

$$c_n := \int_0^a f(x)\overline{\psi}_n(x)w(x)\,dx \tag{2.9.7}$$

and we recall that λ_n is real, by Proposition 1.8.1. If now ψ_n is taken to be normalised and we also recall the orthogonality property (1.8.4), then a special case of (2.9.6) is

$$J(\psi_m,\psi_n) = \begin{cases} \lambda_n & (m = n), \\ 0 & (m \neq n). \end{cases} \tag{2.9.8}$$

Theorem 2.9.2. *Let f be $AC[0,a]$ and satisfy the first boundary condition in* (1.8.2)*. Then, with the Fourier coefficients c_n defined as above,*

$$\sum_{n=0}^{\infty} \lambda_n |c_n|^2 \leq J(f,f). \tag{2.9.9}$$

Proof. We suppose first that $q \geq 0$. Then, by (2.9.4), $J(g,g) \geq 0$ and, in particular,

$$J\Big(f - \sum_{n=0}^{N} c_n\psi_n, f - \sum_{n=0}^{N} c_n\psi_n\Big) \geq 0$$

where $N \in \mathbb{N}$, On 'multiplying out' the left-hand side here, we obtain

$$J(f,f) - \sum_{n=0}^{N} c_n J(\psi_n, f) - \sum_{n=0}^{N} \overline{c}_n J(f,\psi_n) + \sum_{n=0}^{N} \lambda_n c_n \overline{c}_n \geq 0,$$

where we have used (2.9.8) to deduce the last summation. Since $J(\psi_n, f) = \overline{J(f, \psi_n)}$, we obtain

$$\sum_{n=0}^{N} \lambda_n |c_n|^2 \le J(f, f),$$

on using (2.9.6). Then (2.9.9) follows on letting $N \to \infty$.

To prove (2.9.9) without the assumption that $q \ge 0$, let K be a constant which is sufficiently large to make $q + Kw \ge 0$ in $[0, a]$. Then (1.5.2) can be written as

$$-(py')' + Qy = \Lambda wy,$$

where $\Lambda = \lambda + K$ and $Q = q + Kw$. Since $Q \ge 0$, the above proof gives

$$\sum_{n=0}^{\infty} (\lambda_n + K)|c_n|^2 \le \int_0^a \Big(p|f'|^2 + (q + Kw)|f|^2 \Big).$$

The terms involving K on each side are equal on account of the Parseval formula

$$\sum_{n=0}^{\infty} |c_n|^2 = \int_0^{\infty} |f|^2 w \tag{2.9.10}$$

(for which we refer forward to (4.2.8)). Now (2.9.9) follows in the general case. □

We are now in a position to prove the result which corresponds to Theorem 2.9.1(a).

Theorem 2.9.3. *For the Sturm-Liouville problem on* $[0, a]$ *with the boundary condition* (1.8.2), *let* (2.9.1) *hold. Then parts* (i) *and* (ii) *of Theorem* 2.9.1 (a) *continue to hold.*

Proof. Let $\psi_{1,n}$ denote the normalised eigenfunctions corresponding to $\lambda_{1,n}$ and let $J_1(f, g)$ be the Dirichlet integral (2.9.4) with p and q replaced by p_1 and q_1. By (2.9.4) we have

$$J_1(f, f) \ge J(f, f). \tag{2.9.11}$$

Now consider the choice

$$f = \gamma_0 \psi_{1,0} + \cdots + \gamma_m \psi_{1,m}, \tag{2.9.12}$$

where the constants γ_i are chosen so that

$$|\gamma_0|^2 + \cdots + |\gamma_m|^2 = 1 \tag{2.9.13}$$

and the first m Fourier coefficients of f are zero, that is,

$$c_n = 0 \qquad (0 \le n \le m - 1) \tag{2.9.14}$$

with the c_n defined as in (2.9.7). Thus, by (2.9.7) and (2.9.12), we have m homogeneous linear algebraic equations (2.9.14) to be satisfied by the $m+1$ numbers $\gamma_0, \ldots, \gamma_m$, and such numbers aways exist satisfying the normalising condition (2.9.13). We note that (2.9.13) and the orthogonality of the $\psi_{1,n}$ give

$$\int_0^a |f|^2 w_1 = 1. \tag{2.9.15}$$

By (2.9.9) and (2.9.14),

$$J(f,f) \geq \sum_m^\infty \lambda_n |c_n|^2 \geq \lambda_m \sum_m^\infty |c_n|^2 = \lambda_m \int_0^a |f|^2 w \tag{2.9.16}$$

by the Parseval formula (2.9.10). Also, by (2.9.8) applied to J_1,

$$J_1(f,f) = \lambda_{1,0}|\gamma_0|^2 + \cdots + \lambda_{1,m}|\gamma_m|^2 \leq \lambda_{1,m}(|\gamma_0|^2 + \cdots + |\gamma_m|^2).$$

Hence, by (2.9.11), (2.9.13), (2.9.15) and (2.9.16), we have

$$\lambda_{1,m} \geq \lambda_m \int_0^a |f|^2 w \geq \lambda_m \int_0^a |f|^2 w_1 = \lambda_m$$

provided that either $w = w_1$ or $\lambda_m \geq 0$ (or both). Thus $\lambda_{1,m} \geq \lambda_m$, as required. □

2.10 Least eigenvalues

There is an important consequence of (2.9.9) concerning the least eigenvalue λ_0. From (2.9.9) and (2.9.10), we have

$$J(f,f) \geq \lambda_0 \sum_{n=0}^\infty |c_n|^2 = \lambda_0 \int_0^a |f|^2 w,$$

and the inequality is strict unless $c_n = 0$ for all n such that $\lambda_n > \lambda_0$, that is, unless f is an eigenfunction corresponding to λ_0. Thus

$$\lambda_0 = \min \left(J(f,f) \Big/ \int_0^a |f|^2 w \right), \tag{2.10.1}$$

the minimum being taken over all non-trivial f in $AC[0,a]$ which satisfy the first boundary condition $f(a) = \omega f(0)$ in (1.8.2).

In this section we make two applications of (2.10.1) to derive inequalities for λ_0 in terms of the coefficients in (1.5.2). In the first theorem we write $\lambda_0(\omega)$ to emphasise that the boundary conditions are (1.8.2).

Theorem 2.10.1. *Let $\lambda_0(w)$ be the least eigenvalue in the boundary-value problem* (1.5.2)*,* (1.8.2)*. Then*

$$\lambda_0(1) \le \lambda_0(\omega) \le \lambda_0(1) + \Big(\frac{\arg\omega}{a}\Big)^2 \frac{\sup p}{\inf w}. \tag{2.10.2}$$

Proof. Here $\lambda_0(1)$ is the least periodic eigenvalue over $[0,a]$, and the left-hand inequality follows from (1.8.5) since $\lambda_0(1)$ is the least point of the stability set S. To prove the right-hand inequality in (2.10.2), let ψ_0 be the (real-valued) eigenfunction corresponding to $\lambda_0(1)$, and define

$$f(x) = \psi_0(x)\omega^{x/a}.$$

Then f satisfies (1.8.2). Also $|f| = |\psi_0|$ and

$$\begin{aligned} J(f,f) &= \int_0^a \Big(p|\psi_0' + (a^{-1}\log\omega)\psi_0|^2 + q\psi_0^2\Big) \\ &= J(\psi_0,\psi_0) + \Big(\frac{\arg\omega}{a}\Big)^2 \int_0^a p\psi_0^2 \end{aligned}$$

since $|\omega| = 1$. Using this in (2.10.1), we obtain

$$\lambda_0(\omega) \le \lambda_0(1) + \Big(\frac{\arg\omega}{a}\Big)^2 \int_0^a p\psi_0^2 \Big/ \int_0^a \psi_0^2 w, \tag{2.10.3}$$

and the right-hand inequality in (2.10.2) follows. □

Since $\lambda_0(-1)$ is the first semi-periodic eigenvalue over $[0,a]$, we obtain the following corollary by taking $\omega = -1$ in (2.10.2).

Corollary 2.10.2. *The length of the first stability interval does not exceed*

$$(\pi^2 \sup p)/(a^2 \inf w).$$

Another consequence of (2.10.3) is that, when $q(x) = 0$,

$$0 \le \lambda_0(\omega) \le \Big(\frac{\arg\omega}{a}\Big)^2 \int_0^a p \Big/ \int_0^a w.$$

This follows from (2.10.3) since now $\lambda_0(1) = 0$ and $\psi_0(x) = a^{-\frac{1}{2}}$.

The next theorem concerns the least periodic eigenvalue $\lambda_0(1)$, and it complements Theorem 2.5.3 by providing a lower estimate for $\lambda_0(1)$. We define the constant d by

$$d = \int_0^a q \Big/ \int_0^a w. \tag{2.10.4}$$

Theorem 2.10.3. *Let* $L = \inf_{[0,a]}(pw)$. *Then*

$$d - \frac{1}{16L}\left(\int_0^a |q - dw|\right)^2 \leq \lambda_0(1) \leq d, \tag{2.10.5}$$

and the number 16 *is best possible.*

Proof. The right-hand inequality was proved in Theorem 2.5.3. To prove the left-hand inequality, let f and g be real valued, a-periodic $AC[0,a]$ functions. Then, by (2.9.4)

$$J(f,f) = \int_0^a \Big(p(f' + gf)^2 - 2pgf'f + (q - pg^2)f^2\Big). \tag{2.10.6}$$

But

$$\int_0^a 2pgf'f = -\int_0^a (pg)'f^2$$

on integrating by parts and using the periodicity of pgf^2. Hence (2.10.6) gives

$$\begin{aligned} J(f,f) &\geq \int_0^a ((pg)' + q - pg^2)f^2 \\ &\geq \Big(\inf_{[0,a]}((pg)' + q - pg^2)/w\Big)\int_0^a f^2 w. \end{aligned}$$

It now follows from (2.10.1) that

$$\lambda_0(1) \geq \inf_{[0,a]}((pg)' + q - pg^2)/w. \tag{2.10.7}$$

We have now to make a choice for g, and we choose $g = -(g_1 - k)/p$, where

$$g_1(x) = \int_0^x (q - dw) \tag{2.10.8}$$

and k is a constant yet to be specified. By (2.10.4), g_1 has period a and hence (2.10.7) gives

$$\lambda_0(1) \geq d + \inf_{[0,a]}(-pg^2/w) \geq d - \frac{1}{L}\sup_{[0,a]}|g_1 - k|^2. \tag{2.10.9}$$

Now the choice $k = 0$ would yield (2.10.5) but without the factor 16. Thus, to prove (2.10.5) as stated, we make a less obvious choice, and we proceed as follows. Since g_1 is continuous and a-periodic, there are points X and Y at which g_1 attains its infimum and supremum respectively, and such that $X \leq Y < X + a$. We now choose k so that

$$g_1(Y) - k = -(g_1(X) - k),$$

that is

$$k = \tfrac{1}{2}\,(g_1(X) + g_1(Y)).$$

Then

$$\sup_{[0,a]} |g_1 - k| = g_1(Y) - k = \tfrac{1}{2}\,(g_1(Y) - g_1(X)). \tag{2.10.10}$$

Writing $q_1 = q - dw$, we have from (2.10.8)

$$g_1(Y) - g_1(X) = \int_X^Y q_1 \le \int_X^Y |q_1|. \tag{2.10.11}$$

Also, since g_1 has period a,

$$\int_X^Y q_1 = -\int_Y^{X+a} q_1.$$

Hence again

$$g_1(Y) - g_1(X) \le \int_Y^{X+a} |q_1|. \tag{2.10.12}$$

Adding (2.10.11) and (2.10.12), we obtain

$$2\,(g_1(Y) - g_1(X)) \le \int_X^{X+a} |q_1| = \int_0^a |q_1|.$$

Now (2.10.5) follows from this and (2.10.9) and (2.10.10).

Finally, to show that 16 is best possible, we give an example in which $p = w = 1$. We introduce constants δ and η, where $\delta > 0$ and $0 < \eta < 1$, to be chosen later. Let $\alpha = 2\delta + 2$ and consider

$$q(x) = \begin{cases} -1 & (0 \le x \le \delta), \\ 1 & (\delta + 1 < x \le 2\delta + 1), \end{cases}$$

while $q(x) = 0$ elsewhere in $(\delta, 2\delta + 2)$. Then, first of all, $d = 0$ in (2.10.4) and (2.10.5). Next, in (2.10.1), we take f to be the function whose graph in the plane $\mathbb{R}^2$ consists of straight line segments joining the points $(0,1)$, $(\delta, 1)$, $(\delta + 1, 1 - \eta)$, $(2\delta + 1, 1 - \eta)$ and $(2\delta + 2, 1)$. A calculation gives

$$\lambda_0 \Big/ \Big(\int_0^a |q| \Big)^2 \le \frac{\tfrac{1}{4}\,(2\eta^2/\delta^2 - (2\eta - \eta^2)/\delta)}{\delta(2 + 2\eta + \eta^2) + 2(1 - \eta + \tfrac{1}{3}\eta^2)}. \tag{2.10.13}$$

We minimise the numerator on the right-hand side, considered as a quadratic in $1/\delta$, by choosing $\delta = 4\eta/(2 - \eta)$. Then the right-hand side of (2.10.13) becomes

$$\frac{-\tfrac{1}{32}(2 - \eta)^2}{\delta(2 + 2\eta + \eta^2) + 2(1 - \eta + \tfrac{1}{3}\eta^2)}. \tag{2.10.14}$$

If η were zero, (2.10.14) would have the value $-1/16$. Hence, by choice of $\eta \in (0,1)$ and therefore of $\delta > 0$, we can make (2.10.14) arbitrarily close to $-1/16$, as required. $\square$

2.11 Chapter notes

§**2.2** The Prüfer transformation appeared first in [144]. Rofe-Beketov has an extension of this idea to higher-order systems, see [156].

The Kepler transformation [161] is named after Kepler's method of calculating the area of an ellipse sector by deforming the ellipse into a circle.

§**2.3** We refer to Walter [192] for the theory of differential inequalities. The Lipschitz condition in Theorem 2.3.1 can be relaxed to the one-sided condition

$$f(x,y) - f(x,z) \leq K(x)(y-z) \quad (y \geq z).$$

The generalised Sturm comparison theorem, Corollary 2.3.2, was given by Weidmann [194, Theorem 16.1].

§**2.4** The rotation number theory has also been developed for more general spectral problems involving almost periodic potentials by Johnson and Moser [103], and for the p-Laplacian by Zhang [209]. In the latter case however, there can be additional periodic eigenvalues lying between those arising from the rotation number method; see Binding and Rynne [14].

The quasi-momentum extends to an analytic function on $\mathbb{C} \setminus \overline{\mathcal{I}}$, giving a conformal mapping between this set and the slitted complex plane with symmetric vertical cuts of different heights at $\operatorname{Re} z = n\pi$ with non-zero integer n. The complex quasi-momentum plays an important role in the inverse theory of the periodic equation [128]. Details can be found in [116].

The Sturm-Liouville part of Theorem 2.4.2 has a long history: Coddington and Levinson [28, Chapter 8, Theorem 3.1], Ince [97, §10.8], Magnus and Winkler [127, Chapter 2], Titchmarsh [183, §21.4]. These results go back to Lyapunov (1899) in the case of $q = 0$– Starzhinskii [173]– and to Hamel [71] and Haupt [80]; see also Kramers [118].

In the Sturm-Liouville case, the interlacing property (2.4.6) also holds for more general coupled boundary conditions [52, 112]. An early reference is Birkhoff [15].

For the general system (1.5.1) with W positive definite, (2.4.6) is due to Harris [77, Theorem 2.6] with a proof based on the discriminant $D(\lambda)$; see also Yakubovich and Starzhinskii [203, Chapter VIII, §6].

The inequality in Theorem 2.4.4 was shown for the one-dimensional periodic Schrödinger operator ($w = p = 1$) by Moser [137].

The Dirac part of Theorem 2.4.5 is new here. For the Sturm-Liouville part, see also [48, Theorem 3.1.3] and [90]. An early result related to the rotation number is in Wallach [191, Theorem II].

§**2.5** The first part of Theorem 2.5.2 is in [127, Chapter 4]. Our proof here appears to be new. Another proof is in [48, Theorem 3.2.1]. The second part of Theorem 2.5.2 is due to Moore [136]; see also Swanson [179, Chapter 2, §11].

Theorem 2.5.3 is due to Borg [17]. The proof here follows Ungar [189]. Given the properties of $J(f, f)$ in sections 2.9 and 2.10 below, another proof of the theorem simply takes $f = 1$ in (2.10.1).

Theorem 2.5.3 is due to Blumenson [16]; see also [48, §3.3].

§**2.6** Putnam [147] and [148]. Other inequalities due to Putnam are in [145].

For the Wirtinger inequality (2.6.10), of which there are several proofs, see [72, §7.7]. One proof uses our (2.10.1) as applied to the Dirichlet boundary-value problem on $[c, d]$.

The best possible nature of the number 4 was first proved by van Kampen and Wintner [190]. The proof given here is based on Borg [18].

§**2.7** This step-function example was considered by Hochstadt [89] in relation to Theorem 2.4.2 and coexistence. It was also considered by Brown and Eastham [24] in the context of an eigenvalue-inducing interface problem. See also Pipes [142, section III] and, for a related inverse problem, Efendiev and Orudzhev [54].

Another Meissner-type equation $y'' + (\lambda - q)y = 0$, with a three-valued step-function q, was discussed by Yoshitomo [205] where it was shown *inter alia* that the first and second instability intervals are non-vanishing. Other, more specialised papers are [60] and [210] by Gan and Zhang.

§**2.8** The four eigenvalue problems were related to the periodic and semi-periodic problems by Hochstadt [85]; see also [48, Theorem 1.3.4].

With coefficients as in Example 2, (1.5.2) is known as the Whittaker-Hill equation [5, p.145, footnote]. The transformation from y to z goes back to Ince [96] and, for a recent use, see Djakov and Mityagin [37, 38] and Hemery and Veselov [82].

§**2.9** The form $J(f, f)$ appears to have been first applied to the ω-twisted problem by Strutt [176, 177]. Our treatment of $J(f, f)$ in Theorems 2.9.2 and 2.9.3 follows that of Titchmarsh [183, §§14.1-14.7].

§**2.10** Theorem 2.10.1 is due to Eastham [45], [48, Theorem 5.5.3], and a corresponding result for the Schrödinger equation in two or more dimensions is in [46] and [48, Theorem 6.9.3]. Further, more detailed estimates for $\lambda_0(\omega)$ are in Atkinson [6] and Heil [81, Theorem 6.9.3], especially in relation to the weight function w.

Theorem 2.10.3 is due to Kato [107] in the case $p = w = 1$, and it improves on an earlier result of Wintner [202] which had 1 in place of 16 in (2.10.5). Further inequalities of the type (2.10.5) are given by Müller-Pfeiffer [139] and [140, pp. 120-126]. In particular, when $p = w = 1$ and $q \geq 0$, $\pi^2 d/(\pi^2 + a^2 d) \leq \lambda_0(1) \leq d$.

The example showing that 16 is best possible in (2.10.5) is due to Eastham [47].

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