Chapter 1

Superposition Principle for the Relative Index

1.1 Collar Spaces

The superposition principle outlined in the introduction deals with modifications (surgeries) of elliptic operators performed on two disjoint closed subsets, say A and B, of the manifold M on which these operators are defined and says that the index increments $\triangle_A$ and $\triangle_B$ resulting from these surgeries are independent (i.e., are just summed if both surgeries are carried out simultaneously). In this chapter, we prove the superposition principle for the relative index in general form as it was stated in [55–60].

In fact, there are only two things important for the superposition principle to hold:

1. The two "regions" where the operators are to be modified should be *separated* from each other.
2. The operators in question should respect the separated subsets, i.e., be *local* in some sense.

The notion of collar spaces discussed in this section takes care of the first thing, and the second thing—the relevant class of operators—will be introduced in the next section.

Collar spaces were originally defined in [55–57,59]. These spaces are Hilbert spaces equipped with the structure of a module over the unital algebra $C^\infty([-1,1])$ of smooth functions on the closed interval $[-1,1]$. We will use the Fréchet topology defined on this algebra by the seminorms

$$\|f\|_k = \max_{t\in[-1,1]} |f^{(k)}(t)|\,.$$

Let us give a formal definition.

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Definition 1.1. Let H be a separable Hilbert space, and let a continuous action of the algebra $C^\infty([-1,1])$ on H be defined. (We assume that the function $f(x) \equiv 1$ acts as the identity operator on H.) Then we say that H is a *collar space*.

Let us produce some examples of collar spaces.

Example 1.2. Apparently the most trivial example of a collar space is the Hilbert space $H = L^2([-1,1],W)$ whose elements are measurable functions $f\colon [-1,1] \longrightarrow W$, where W is a separable Hilbert space and

$$\|f\|_H = \left\{ \int_{-1}^{1} \|f(t)\|_W^2 \, dt \right\}^{1/2} < \infty.$$

The action of the algebra $C^\infty([-1,1])$ on H is defined as pointwise multiplication.

Example 1.3. Let M be a smooth complete Riemannian manifold, and let a smooth mapping

$$\chi : M \longrightarrow [-1,1] \tag{1.1}$$

be given such that the set $\chi^{-1}(-1,1)$ is bounded. The Sobolev spaces $H^s(M)$ (defined with the use of powers of the Beltrami–Laplace operator and of the Riemannian volume measure) can naturally be equipped with an action of the algebra $C^\infty([-1,1])$ as follows. Every function $\varphi \in C^\infty([-1,1])$ acts on $H^s(M)$ as the operator of multiplication by the function $\varphi \circ \chi \in C^\infty(M)$. Thus, $H^s(M)$ is a collar space in a natural way.

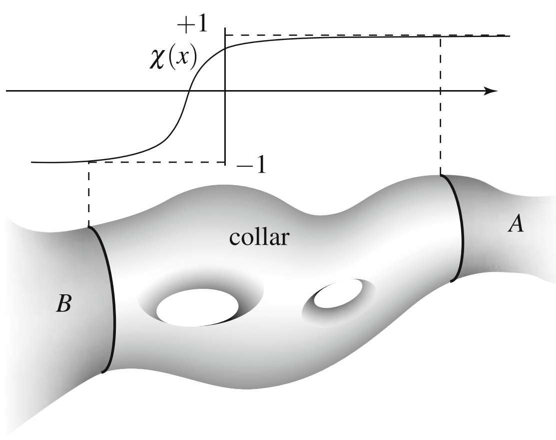

Figure 1.1: A collar space

Let us visualize what happens in this example as well as in more general examples where M is not necessarily a compact C^∞ manifold (see Fig. 1.1). The closed subsets

$A = \chi^{-1}(\{1\})$ and $B = \chi^{-1}(\{-1\})$ play the same role as the sets denoted by the same letters in the classical example given in the Introduction (cf. Fig. 0.1); that is where the surgeries will be applied. The remaining part of M, that is, the *set*

$$U = M \setminus (A \cup B) = \chi^{-1}\{(-1,1)\}$$

separating A *and* B, will be referred to as the *collar*. That is where the operators in question should be "local" for the index increments $\triangle_A$ and $\triangle_B$ to be independent indeed.

Of course, the locality that is actually needed to prove superposition theorems for the relative index is much weaker than the usual locality on M. It is not with respect to, say, the algebra $C_0^\infty(M)$ but rather with respect to the action of $C^\infty([-1,1])$, and even for this action, not in the sense that the commutators with the action of the elements of the algebra are compact but in some weaker sense (see the next section). So far, let us define some notions related to locality with respect to the action of $C^\infty([-1,1])$. These are in fact pretty much standard. We start from the notion of support.

Definition 1.4. Let H be a collar space, and let $h \in H$. The *support* of h is the minimal closed subset $K \subset [-1,1]$ such that the following property holds:

$$\varphi \in C^\infty([-1,1]) \text{ and } \operatorname{supp} \varphi \cap K = \varnothing \Longrightarrow \varphi h = 0.$$

The support of the element h will be denoted by $\operatorname{supp} h$.

Alternatively, one can write

$$\operatorname{supp} h = \bigcap_{\substack{\varphi \in C^\infty([-1,1]) \\ \varphi h = 0}} \varphi^{-1}(\{0\}).$$

This is easily seen to be equivalent to Definition 1.4.

One can readily describe the support of elements of collar spaces in Examples 1.2 and 1.3. In Example 1.2, elements $h \in H$ are (Hilbert space-valued) functions on $[-1,1]$, and $\operatorname{supp} h$ coincides with the support of h in the ordinary sense. In Example 1.3, elements $h \in H^s(M)$ are functions (or distributions) on M, and one has

$$\operatorname{supp} h = \chi(\operatorname{supp}_M h),$$

where $\operatorname{supp}_M h$ is the support of h as a function on M. (Note that $\chi(\operatorname{supp}_M h)$ is necessarily closed.)

The following trivial assertion shows that the notion of support introduced in Definition 1.4 has all natural properties of the ordinary support.

Proposition 1.5. *Let H be a collar space, and let $h, h_1, h_2, \ldots \in H$. Then*

(i) $\operatorname{supp} h$ *is closed.*

(ii) $\operatorname{supp} h = \varnothing$ *if and only if* $h = 0$.

(iii) *One has* $\operatorname{supp}(h_1 + h_2) \subset \operatorname{supp} h_1 \cup \operatorname{supp} h_2$.

(iv) *One has* $\operatorname{supp}(\varphi h) \subset \operatorname{supp}\varphi \cap \operatorname{supp} h$ *for any* $\varphi \in C^\infty([-1,1])$.

(v) *If* $h_n \to h$ *in the space* H (*weakly or strongly*), *then*

$$\operatorname{supp} h \subset \bigcap_k \overline{\bigcup_{n\ge k} \operatorname{supp} h_n},$$

where the bar stands for closure.

Proof. We omit the proof, which is fairly standard. □

For an arbitrary subset $F \subset [-1,1]$, let $\overset{\circ}{H}_F \subset H$ be the linear manifold of elements $h \in H$ such that $\operatorname{supp} h \subset F$. If $F \subset [-1,1]$ is closed, then $\overset{\circ}{H}_F$ is a closed subspace of H. Indeed, if $h_n \to h$ and $\operatorname{supp} h_n \subset F$, then

$$\operatorname{supp} h \subset \bigcap_k \overline{\bigcup_{n\ge k} \operatorname{supp} h_n} \subset F$$

by Proposition 1.5, (v). For arbitrary $F \subset [-1,1]$, the linear manifold $\overset{\circ}{H}_F$ is not necessarily closed, and we define

$$H_F = \overline{\overset{\circ}{H}_F}.$$

Proposition 1.6. *The space* H_F *is invariant under the action of* $C^\infty([-1,1])$ *and hence itself is a collar space.*

Proof is obvious. □

We point out that $H_F \ne H_{\bar F}$ in general. A counterexample can readily be indicated in Example 1.3, where $H^s(M)_{(-1,1)}$ consists of all functions $u \in H^s(M)$ supported in the collar, while $H^s(M)_{[-1,1]}$ coincides with the entire $H^s(M)$.

Finally, we need one technical result.

Proposition 1.7. *Let* $F_1,\dots,F_m \subset [-1,1]$ *be subsets such that their closures* $\overline{F}_1,\dots,\overline{F}_m$ *are pairwise disjoint. Then one has the direct sum decomposition*

$$H_{F_1\cup\dots\cup F_m} = H_{F_1} \oplus \dots \oplus H_{F_m}. \tag{1.2}$$

Proof. Let $V_1\dots,V_m$ be disjoint open neighborhoods of $F_1,\dots,F_m$, respectively, and let $e_1,\dots,e_m \in C^\infty([-1,1])$ be a partition of unity on $\cup F_j$ subordinate to the cover $\cup V_j \supset \cup F_j$. If $u_j \in H_{F_j}$, $j = 1,\dots,m$, then $\sum u_j \in H_{F_1\cup\dots\cup F_m}$. Moreover, if $\sum u_j = 0$, then

$$u_k = e_k \sum_{j=1}^m u_j = 0, \qquad k = 1,\dots,m.$$

Finally, if $u \in H_{F_1\cup\dots\cup F_m}$, then $e_j u \in F_j$, $j = 1,\dots,m$, and $\sum u_j = u$. □

Remark 1.8. 1. For the direct sum decomposition (1.2) to hold, it does not suffice to require that the sets F_j themselves are pairwise disjoint.

2. The sum on the right-hand side in (1.2) need not be orthogonal.

1.2 Proper Operators and Fredholm Operators

Supports. If an operator A on a function space can be written as an integral operator,

$$[Au](x) = \int A(x,y)u(y)\,dy \tag{1.3}$$

with some (possibly, distributional) kernel $A(x,y)$, then one can speak of the *support* (and *singular support*) of the kernel of A. In fact, a (somewhat weaker) notion of support of the kernel can be defined even if we do not consider the kernel itself. Namely, the support of the kernel can be reconstructed from how the operator transforms the supports of functions on which it acts. This reconstruction can be used as a definition of the support of the kernel even if the kernel itself is not necessarily defined.

Let us use this idea to define the support of an operator $A : H_1 \longrightarrow H_2$ acting between collar spaces. Since the supports of *elements* of collar spaces are defined as subsets of the interval $[-1,1]$, it is natural that the supports of *operators* acting on such spaces will be defined as subsets of the Cartesian product $[-1,1]\times[-1,1]$. First, note that an arbitrary subset $K \subset [-1,1]\times[-1,1]$ can be viewed as the graph of a set-valued mapping, which will be denoted by the same letter,

$$K\colon [-1,1] \longrightarrow 2^{[-1,1]}.$$

Namely, for each $t \in [-1,1]$ we define $K(t)$ to be the set (see Fig. 1.2)

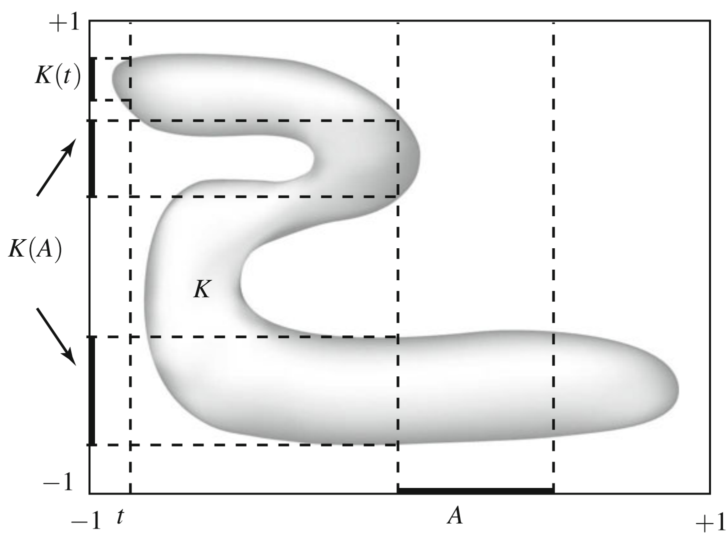

Figure 1.2: A subset $K \subset [-1,1]\times[-1,1]$ as a set-valued mapping

$$K(t) = \{\tau \in [-1,1] \colon (t,\tau) \in K\}, \tag{1.4}$$

where the images $K(t)$ of a point $t \in [-1,1]$ and $K(A)$ of a subset $A \subset [-1,1]$ are shown.

Now we are in a position to give the desired definition.

Definition 1.9. Let H_1 and H_2 be collar spaces, and let $A \colon H_1 \longrightarrow H_2$ be a bounded linear operator. One says that A *is supported in a closed subset* $K \subset [-1,1] \times [-1,1]$ if

$$\operatorname{supp} Au \subset K(\operatorname{supp} u) \tag{1.5}$$

for any element $u \in H$. The minimum closed subset $K \subset [-1,1] \times [-1,1]$ with this property is denoted by $\operatorname{supp} A$ and referred to as the *support* of A.

We need yet to prove that there exists a minimum closed subset with this property. The intersection of all such closed sets K is a natural candidate, but the proof is not completely obvious, because in general one only has $(\bigcap K_\alpha)(X) \subset \bigcap K_\alpha(X)$, while the opposite inclusion fails. The desired assertion is valid nevertheless.

Lemma 1.10. *Let $\{K_\alpha\}$ be an arbitrary family of closed subsets of $[-1,1] \times [-1,1]$ satisfying* (1.5)*. Then the intersection $K = \bigcap_\alpha K_\alpha$ satisfies* (1.5) *as well.*

Proof. Let $u \in H_1$, and let $\tau \notin K(\operatorname{supp} u)$. We claim that $\tau \notin \operatorname{supp} Au$. Indeed, for each α, let U_α be the set of $t \in [-1,1]$ such that $(t,\tau) \notin K_\alpha$. The set U_α is obviously open. Next, $\operatorname{supp} u \subset \bigcup_\alpha U_\alpha$, because it follows from the relation $\tau \notin K(\operatorname{supp} u)$ that for any $t \in \operatorname{supp} u$ one has $(t,\tau) \notin K$ and hence there exists at least one index α such that $(t,\tau) \notin K_\alpha$. Now take a finite subcover of $\operatorname{supp} u$ by U_α, and let $\{e_\alpha\} \subset C^\infty([-1,1])$ be a smooth partition of unity on $\operatorname{supp} u$ subordinate to this subcover. Then

$$Au = \sum_\alpha A(e_\alpha u), \qquad \tau \notin K_\alpha(U_\alpha) \supset \operatorname{supp}(A(e_\alpha u)) \quad \text{for any } \alpha,$$

and hence $\tau \notin \operatorname{supp} Au$ by Proposition 1.5, (iii). □

The supports of operators in collar spaces enjoy many natural properties. For example, the following assertion holds.

Proposition 1.11. *Let H_1, H_2, and H_3 be collar spaces, and let $A \colon H_1 \longrightarrow H_2$ and $B \colon H_2 \longrightarrow H_3$ be some operators. Then $\operatorname{supp}(BA) \subset \operatorname{supp} B \circ \operatorname{supp} A$, where the composition on the right-hand side is understood in the sense of set-valued mappings* (*see Fig.* 1.3.)

Proper operators. The operators for which the index superposition principle will be proved have some special properties related to their supports. Prior to stating the corresponding definition, let us informally explain what these properties are and where they stem from. The operators usually considered in index theory on compact manifolds have the property known as *locality*: their commutators with the operators of multiplication by

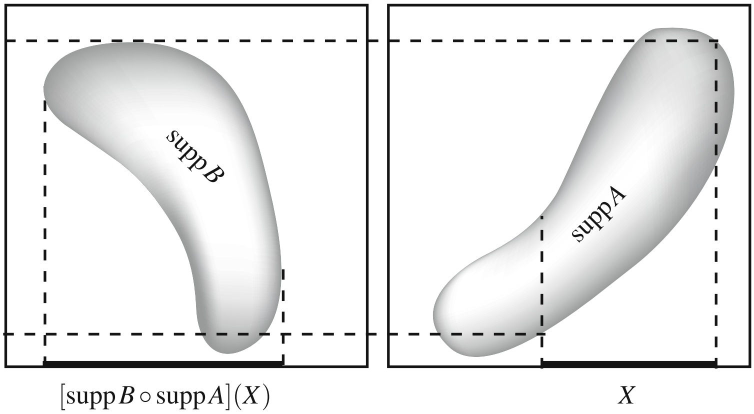

Figure 1.3: Composition of supports

continuous functions are compact. If A is such an operator and $\sum \varphi_j^2(x) = 1$ is a partition of unity on the underlying manifold M, then

$$A = \sum_j \varphi_j \circ A \circ \varphi_j \quad \text{modulo compact operators.}$$

By taking an appropriately fine partition of unity, we can ensure that the kernel of the operator on the right-hand side is supported in an arbitrarily small neighborhood of the diagonal

$$\Delta_M = \{(x,x) \colon x \in M\} \subset M.$$

This operator has the same index as A.

The locality property itself is not really important for the index superposition theorem in collar spaces to be true. (And it even does not hold, say, in applications to Fourier integral operators.) All we need is that we can make the support of our operator to be localized in an arbitrarily small neighborhood of the diagonal, without changing the index of the operator. To avoid making additional assumptions, we will consider families of operators continuously depending on a parameter $\delta > 0$ whose supports shrink to the diagonal $\Delta = \{(t,t) | t \in [-1,1]\}$ as $\delta \to 0$. If all operators in such a family are Fredholm, then the index is independent of δ. In specific applications, one is usually given a single operator rather than a family; however, as a rule, one can readily include this operator in a family with the shrinking support property.

Let us proceed to a rigorous definition. By Δ_ε we denote the ε-neighborhood of the diagonal Δ, i.e., the set of pairs $(t,\tau) \in [-1,1] \times [-1,1]$ such that $|t - \tau| < \varepsilon$.

Definition 1.12. Let H_1 and H_2 be collar spaces. A family of bounded linear operators

$$A_\delta : H_1 \longrightarrow H_2$$

depending on the parameter $\delta > 0$ is called a *proper operator* if the following conditions are satisfied:

(i) the operator A_δ depends on δ continuously in the operator norm.

(ii) For every $\varepsilon > 0$, there exists a $\delta_0 > 0$ such that, for all $\delta < \delta_0$, one has

$$\operatorname{supp} A_\delta \subset \Delta_\varepsilon.$$

Remark 1.13. One can restate condition (ii) as follows: for $\delta < \delta_0$, the support of $A_\delta u$ is contained in the ε-neighborhood of the support of u for an arbitrary element $u \in H_1$.

Example 1.14. Let us return to Example 1.3, assuming for simplicity that M is a smooth *compact* manifold (which actually does not affect our conclusions). Pseudodifferential operators (ΨDO) on Sobolev spaces on M can be included in operator families satisfying the conditions of Definition 1.12 as follows. Consider an lth-order ΨDO,

$$P\colon H^s(M) \longrightarrow H^{s-l}(M),$$

with kernel $k_P(x,y)$, $x,y \in M$. Take a cutoff function $\varphi_\delta(x,y)$ on $M \times M$ such that φ_δ continuously depends on δ, is equal to 1 in the δ-neighborhood of the diagonal, and is zero outside the 2δ-neighborhood. The ΨDO P_δ with kernel $\varphi_\delta(x,y)k_P(x,y)$ has the same symbol as P and coincides with P provided that δ is sufficiently large. The family P_δ obviously satisfies the desired conditions.

Remark 1.15. The set of proper operators from a collar space H into itself is an algebra. Indeed, it follows from the triangle inequality, Proposition 1.11, and Definition 1.12 that the sum and product of proper operators are again proper operators.

Now we can introduce the class of Fredholm operators to be dealt with in what follows. Recall that an *almost inverse* of a Fredholm operator

$$A\colon H \longrightarrow G$$

is an operator

$$B\colon G \longrightarrow H$$

such that

$$AB = 1 + K_1, \qquad BA = 1 + K_2,$$

where K_1 and K_2 are compact operators on G and H, respectively. An almost inverse of A will be denoted by $A^{[-1]}$; note that the almost inverse is by no means unique, even though the notation suggests otherwise.

Definition 1.16. A *collar Fredholm operator* (a *c-Fredholm operator*) between collar spaces H and G is a proper operator

$$D_\delta\colon H \longrightarrow G$$

with the following properties:

1. The operator is Fredholm for every $\delta > 0$.
2. For every $\delta > 0$, there exists an almost inverse $D_\delta^{[-1]}$ of the operator D_δ such that the family formed by these almost inverses is again a proper operator.

Example 1.17. Let us again return to Example 1.3. If A is an elliptic pseudodifferential operator on a smooth compact manifold M equipped with a smooth function $\chi : M \longrightarrow [-1,1]$ and $A^{[-1]}$ is an almost inverse of A, then both operators can be made proper operators in Sobolev spaces on M by the technique used in Example 1.14. Thus, A can be interpreted as a c-Fredholm operator. Note that all elements A_δ of the corresponding family represent the stable homotopy class $[A] \in \operatorname{Ell}(M)$ of the original elliptic operator A; that is why it is expedient not to distinguish between A and the family A_δ unless a misunderstanding is possible. As a rule, we omit the parameter δ in the notation of a proper (in particular, c-Fredholm) operator.

Remark 1.18. The class of c-Fredholm operators is wider than that of Atiyah's general elliptic operators (see Definition 0.6). One relevant example is given by Fourier integral operators (see Section 0.3.4).

1.3 Superposition Principle

In this section, we prove Theorem 0.10. To this end, we need the notion of surgery in collar spaces. Surgery for Atiyah's general elliptic operators was defined in Section 0.2.2 of the Introduction, while surgery in collar spaces was temporarily left to the reader's imagination in Section 0.2.3, and now we describe it in detail.

Surgery of collar spaces and surgery diagrams. Let H and G be collar spaces, let $Q \subset [-1,1]$ be a closed subset, and let $F = [-1,1] \setminus Q$ be the complement of Q. Next, let

$$j : H_F \longrightarrow G_F$$

be an isomorphism of collar spaces[1] (not necessarily norm-preserving).

Definition 1.19. The triple (H, G, j) is called a *surgery of collar spaces on the subset Q*. In this situation, we also say that H and G *coincide* on F, or that G is a *modification of H on Q* (and vice versa) or *is obtained from H by surgery on Q*. Furthermore, we write

$$H \overset{F}{=} G \quad \text{and} \quad H \overset{Q}{\longleftrightarrow} G.$$

This notation omits the isomorphism j for brevity, but one should have in mind that the specific form of the isomorphism is important in the definition: different isomorphisms correspond to different surgeries. The mere existence of an isomorphism is not of much value anyway, because there always exists one provided that H_F and G_F are of the same dimension.

[1] Thus, j commutes with the action of $C^\infty([-1,1])$.

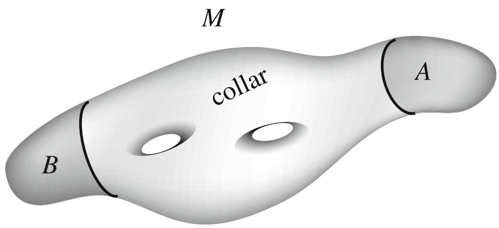

Figure 1.4: The original manifold M

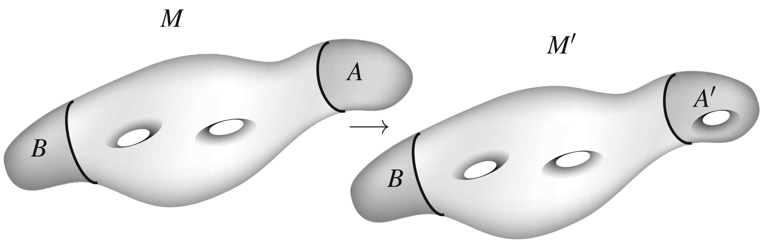

Figure 1.5: Surgery on the set $\{1\}$

Example 1.20. Consider the smooth closed manifold M shown in Fig. 1.4. Let a smooth function $\chi\colon M \longrightarrow [-1,1]$ be given, making the Sobolev spaces $H^s(M)$ collar spaces as in Example 1.3. We assume that

$$\chi^{-1}(\{1\}) = A, \qquad \chi^{-1}(\{1\}) = B,$$

so that the collar is the lighter area, just as in Fig. 1.1.

Now let us cut away the part A and paste some different part A', thus obtaining a new smooth closed manifold M' (see Fig. 1.5). The function $\chi|_{M\setminus A}$ extends to be a smooth function on M' if we set $\chi|_{A'} = 1$; thus, the Sobolev spaces $H^s(M')$ are collar spaces as well.

One obviously has

$$H^s(M) \overset{[-1,1)}{=} H^s(M'), \qquad \text{or} \qquad H^s(M) \overset{\{1\}}{\longleftrightarrow} H^s(M'),$$

the corresponding mapping

$$j\colon H^s_{[-1,1)}(M) \longrightarrow H^s_{[-1,1)}(M')$$

being defined as follows:

$$[jf](x) = f(x), \quad x \in M \setminus A, \qquad [jf](x) = 0, \quad x \in A', \qquad \text{for } f \in C_0^\infty(M \setminus A).$$

(The mapping j extends to the entire $H^s_{[-1,1)}(M)$ uniquely by continuity.)

In what follows, most often surgeries on the sets $\{1\}$ and $\{-1\}$ are used. To simplify the notation, we omit the braces and write $H \overset{1}{\longleftrightarrow} G$ rather than $H \overset{\{1\}}{\longleftrightarrow} G$ and accordingly $H \overset{-1}{\longleftrightarrow} G$ rather than $H \overset{\{-1\}}{\longleftrightarrow} G$.

Now we introduce the very important notion of *surgery diagrams*. Let H_i, $i = 1,\dots,4$, be collar spaces, and let

$$Q_{12}, Q_{13}, Q_{24}, Q_{34} \in [-1,1]$$

be closed subsets, whose complements will be denoted by

$$F_{kl} = [-1,1] \setminus Q_{kl}.$$

Assume that

$$H_1 \overset{Q_{12}}{\longleftrightarrow} H_2, \quad H_1 \overset{Q_{13}}{\longleftrightarrow} H_3, \quad H_1 \overset{Q_{23}}{\longleftrightarrow} H_2, \quad H_1 \overset{Q_{24}}{\longleftrightarrow} H_2;$$

i.e., H_2 is obtained from H_1 by surgery on Q_{12}, etc., with the underlying isomorphisms

$$\begin{aligned} j_{12}\colon H_{1F_{12}} &\longrightarrow H_{2F_{12}}, & j_{13}\colon H_{1F_{13}} &\longrightarrow H_{3F_{13}}, \\ j_{23}\colon H_{2F_{23}} &\longrightarrow H_{3F_{23}}, & j_{24}\colon H_{2F_{24}} &\longrightarrow H_{4F_{24}}. \end{aligned}$$

Then we can draw the *surgery diagram*

$$\begin{tikzcd} H_1 \arrow[r, leftrightarrow, "Q_{12}"] \arrow[d, leftrightarrow, "Q_{13}"'] & H_2 \arrow[d, leftrightarrow, "Q_{24}"] \\ H_3 \arrow[r, leftrightarrow, "Q_{34}"'] & H_4 \end{tikzcd}$$

combining all the four surgeries. We say that this surgery diagram *commutes* if the underlying isomorphisms j_{kl} form a commutative diagram

$$\begin{tikzcd} H_{1F} \arrow[r, "j_{12}"] \arrow[d, "j_{13}"'] & H_{2F} \arrow[d, "j_{24}"] \\ H_{3F} \arrow[r, "j_{34}"'] & H_{4F}, \end{tikzcd}$$

where

$$F = F_{12} \cap F_{13} \cap F_{24} \cap F_{34} \equiv [-1,1] \setminus \big(Q_{12} \cup Q_{13} \cup Q_{24} \cup Q_{34}\big).$$

(We denote the restrictions of the isomorphisms j_{kl} to subspaces by the same symbol as the homomorphisms themselves.)

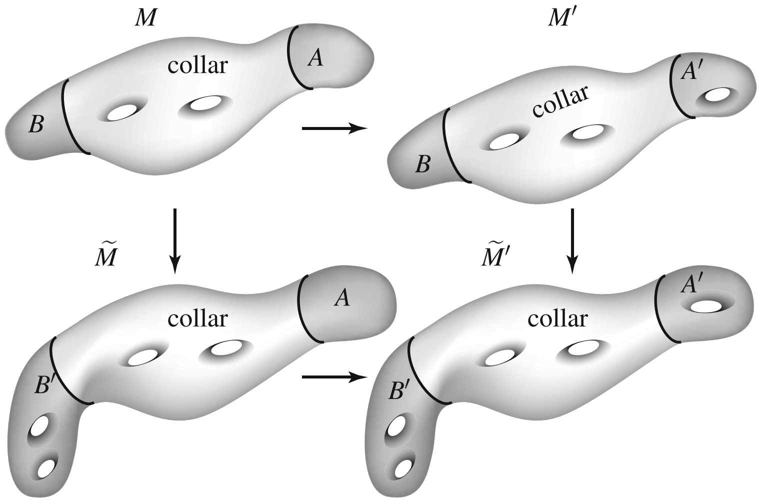

Figure 1.6: Surgery diagram for a manifold M

Example 1.21. Consider the surgery diagram for manifolds shown in Fig. 1.6. With obvious notation, this surgery diagram can be written as

$$\begin{array}{ccc} M & \xleftrightarrow{A} & M' \\ {\scriptstyle B}\updownarrow & & \updownarrow{\scriptstyle B} \\ \widetilde{M} & \xleftrightarrow[A]{} & \widetilde{M}'. \end{array}$$

If the Sobolev spaces on each of the four manifolds occurring in this diagram are equipped with the structure of collar spaces in the same way as in Example 1.20, then from this surgery diagram for manifolds we obtain the following surgery diagram for Sobolev spaces:

$$\begin{array}{ccc} H^s(M) & \xleftrightarrow{1} & H^s(M') \\ {\scriptstyle -1}\updownarrow & & \updownarrow{\scriptstyle -1} \\ H^s(\widetilde{M}) & \xleftrightarrow[1]{} & H^s(\widetilde{M}'). \end{array}$$

The latter diagram obviously commutes, because all underlying isomorphisms of Sobolev spaces on the collar are just the identity homomorphism.

Remark. In practical applications, just as in the above example, one usually has $Q_{12} = Q_{34}$ and $Q_{13} = Q_{24}$.

Remark. We have only introduced the simplest kind of surgery diagrams—squares. We will not need more general ones in this book.

Surgery of c-Fredholm operators and surgery diagrams. Now consider the notion of surgery for c-Fredholm operators.

Let H_i and G_i, $i = 1,2$, be collar spaces, let $Q \subset [-1,1]$ be a closed subset, and assume that

$$H_1 \overset{Q}{\longleftrightarrow} H_2, \qquad G_1 \overset{Q}{\longleftrightarrow} G_2. \tag{1.6}$$

Next, let

$$A_1 \colon H_1 \longrightarrow G_1, \qquad A_2 \colon H_2 \longrightarrow G_2$$

be c-Fredholm operators.

Definition 1.22. One says that A_1 and A_2 are obtained from each other by *surgery* on Q (or *coincide on* $[-1,1] \setminus Q$) if for any closed subset $K \subset [-1,1] \setminus Q$ there exists a $\delta_0 > 0$ such that

$$A_{1\delta}u = A_{2\delta}u \qquad \text{for all } u \in H_1 \text{ with } \operatorname{supp} u \subset K \tag{1.7}$$

for all $\delta < \delta_0$. In this case, we write

$$A_1 \overset{Q}{\longleftrightarrow} A_2, \quad \text{or} \quad A_1 \overset{F}{=} A_2, \quad F = [-1,1] \setminus Q.$$

Remark. Of course, there is some abuse of notation in (1.7); more carefully, one should write

$$j_G(A_{1\delta}u) = A_{2\delta}(j_H u),$$

where j_H and j_G are the isomorphisms underlying the collar space surgeries (1.6). This is well defined. Indeed, $\operatorname{supp} u \subset K \subset [-1,1] \setminus Q$ and, for sufficiently small δ, one has $\operatorname{supp} A_{1\delta}u \subset [-1,1] \setminus Q$, because A_1 is a proper operator and the distance between the disjoint compact sets K and Q is strictly positive.

Let us return to Example 1.21. Suppose that we have mth-order elliptic differential operators P, P', $\widetilde{P}$, and $\widetilde{P}'$ on M, M', $\widetilde{M}$, and $\widetilde{M}'$, respectively, such that

$$P = P' \text{ on } M \setminus A, \quad P = \widetilde{P} \text{ on } M \setminus B, \quad P' = \widetilde{P}' \text{ on } M' \setminus B, \quad \widetilde{P} = \widetilde{P}' \text{ on } \widetilde{M} \setminus A.$$

Then, treating the operator P as a proper (and c-Fredholm) operator acting in the collar spaces $H^s(M) \longrightarrow H^{s-m}(M)$, etc., we obtain the surgeries

$$P \overset{1}{\longleftrightarrow} P', \quad P \overset{-1}{\longleftrightarrow} \widetilde{P}, \quad \widetilde{P} \overset{1}{\longleftrightarrow} \widetilde{P}', \quad P' \overset{-1}{\longleftrightarrow} \widetilde{P}'.$$

Let us combine these surgeries into the *surgery diagram*

$$\begin{array}{ccc} P & \overset{1}{\longleftrightarrow} & P' \\ {\scriptstyle -1}\updownarrow & & \updownarrow{\scriptstyle -1} \\ \widetilde{P} & \underset{1}{\longleftrightarrow} & \widetilde{P}'. \end{array} \tag{1.8}$$

Such a surgery diagram of c-Fredholm operators is said to *commute* if the two underlying surgery diagrams of collar spaces commute. This is obviously the case in this example, because these underlying diagrams are

$$\begin{tikzcd} H^s(M) \arrow[r, leftrightarrow, "1"] \arrow[d, leftrightarrow, "-1"'] & H^s(M') \arrow[d, leftrightarrow, "-1"] \\ H^s(\widetilde{M}) \arrow[r, leftrightarrow, "1"'] & H^s(\widetilde{M}'), \end{tikzcd} \qquad \begin{tikzcd} H^{s-m}(M) \arrow[r, leftrightarrow, "1"] \arrow[d, leftrightarrow, "-1"'] & H^{s-m}(M') \arrow[d, leftrightarrow, "-1"] \\ H^{s-m}(\widetilde{M}) \arrow[r, leftrightarrow, "1"'] & H^{s-m}(\widetilde{M}'), \end{tikzcd}$$

and they clearly commute, as was explained above.

Now if P, P', $\widetilde{P}$, and $\widetilde{P}'$ are elliptic *pseudo*differential operators and we only know that

$$\begin{gathered} \sigma(P) = \sigma(P') \text{ over } M \setminus A, \quad \sigma(P) = \sigma(\widetilde{P}) \text{ over } M \setminus B, \\ \sigma(P') = \sigma(\widetilde{P}') \text{ over } M' \setminus B, \quad \sigma(\widetilde{P}) = \sigma(\widetilde{P}') \text{ over } \widetilde{M} \setminus A, \end{gathered}$$

where $\sigma(D)$ stands for the principal symbol of a pseudodifferential operator D, then a few more technicalities are needed. First, we act as in Example 1.17 to make these operators c-Fredholm (essentially by modifying them by appropriate families of compact operators). Next, we need to ensure that diagram (1.8) holds—and commutes. To this end, we take a smooth partition of unity

$$e^2_{-1\delta}(t) + e^2_{0\delta}(t) + e^2_{1\delta}(t) = 1$$

on $[-1,1]$ smoothly depending on the parameter $\delta > 0$ and such that

$$e_{0\delta} = \begin{cases} 1, & |t| \le 1-\delta, \\ 0, & |t| \ge 1-\frac{\delta}{2}, \end{cases} \qquad \operatorname{supp} e_{-1} \subset \left[-1, -1+\frac{\delta}{2}\right], \qquad \operatorname{supp} e_1 \subset \left[1, 1-\frac{\delta}{2}\right].$$

Then we can replace the operators P, P', $\widetilde{P}$, and $\widetilde{P}'$ by

$$\begin{gathered} e_{-1\delta} \circ P \circ e_{-1\delta} + e_{0\delta} \circ P \circ e_{0\delta} + e_{1\delta} \circ P \circ e_{1\delta}, \\ e_{-1\delta} \circ P \circ e_{-1\delta} + e_{0\delta} \circ P \circ e_{0\delta} + e_{1\delta} \circ P' \circ e_{1\delta}, \\ e_{-1\delta} \circ \widetilde{P} \circ e_{-1\delta} + e_{0\delta} \circ P \circ e_{0\delta} + e_{1\delta} \circ P \circ e_{1\delta}, \\ e_{-1\delta} \circ \widetilde{P} \circ e_{-1\delta} + e_{0\delta} \circ P \circ e_{0\delta} + e_{1\delta} \circ P' \circ e_{1\delta}, \end{gathered}$$

respectively. The new operators differ from the original ones by families of compact operators, and it is an easy exercise to check that now we have the commutative diagram (1.8).

Proof of the superposition principle. Now we are in a position to prove Theorem 0.10. First, now that we have introduced c-Fredholm operators and surgery diagrams for such operators, we can restate the theorem without resorting to intuition for understanding any notions. The theorem states that if there is a commutative surgery diagram

$$\begin{tikzcd} D \arrow[r, leftrightarrow, "-1"] \arrow[d, leftrightarrow, "1"'] & D_- \arrow[d, leftrightarrow, "1"] \\ D_+ \arrow[r, leftrightarrow, "-1"'] & D_\pm \end{tikzcd} \tag{1.9}$$

of c-Fredholm operators, then

$$\operatorname{ind}(D) - \operatorname{ind}(D_-) = \operatorname{ind}(D_+) - \operatorname{ind}(D_\pm).$$

To prove Theorem 0.10, we need some lemmas.

Lemma 1.23. *Let $A_1 : H_1 \longrightarrow H_2$ and $A_2 : H_1 \longrightarrow H_2$ be c-Fredholm operators between collar spaces H_1 and H_2, and suppose that A_1 and A_2 coincide on some open set $F \subset [0,1]$, $A_1 \overset{F}{=} A_2$. Then their arbitrary proper almost inverses differ on F by compact operators; more precisely,*

$$A_1^{[-1]} \overset{F}{=} A_2^{[-1]} + K,$$

where K is a compact proper operator.

Proof. By the definition of an almost inverse operator, we have

$$A_1 A_1^{[-1]} = 1 + K_1, \qquad A_2^{[-1]} A_2 = 1 + K_2 \tag{1.10}$$

with some compact proper operators K_1 and K_2. Let us multiply the first relation by $A_2^{[-1]}$ on the left; we obtain

$$A_2^{[-1]} A_1 A_1^{[-1]} = A_2^{[-1]} + A_2^{[-1]} K_1.$$

Next, let us multiply the second relation in (1.10) by $A_1^{[-1]}$ on the right; we obtain

$$A_2^{[-1]} A_1 A_1^{[-1]} \overset{F}{=} A_2^{[-1]} A_2 A_1^{[-1]} = A_1^{[-1]} + K_2 A_1^{[-1]}.$$

By combining the last two relations, we see that

$$A_2^{[-1]} \overset{F}{=} A_1^{[-1]} + \{K_2 A_1^{[-1]} - A_2^{[-1]} K_1\},$$

which completes the proof, because the expression in braces is a compact operator. □

Lemma 1.24. *Let $A_1 : H_1 \longrightarrow H_2$ and $A_2 : H_1 \longrightarrow H_2$ be proper operators between collar spaces H_1 and H_2. Suppose that*

$$A_1 \overset{F_j}{=} A_2, \qquad j \in \mathscr{J}, \tag{1.11}$$

for some open sets F_j, $j \in \mathscr{J}$, forming an open cover of the interval $[-1,1]$,

$$\bigcup_{j \in \mathscr{J}} F_j = [-1,1].$$

Then there exists a $\delta_0 > 0$ *such that* $A_{1\delta} = A_{2\delta}$ *for* $\delta < \delta_0$. *Likewise, if relation* (1.11) *holds modulo compact proper operators, then* $A_{1\delta} = A_{2\delta}$ *modulo compact proper operators for* $\delta < \delta_0$.

Proof. We can assume without loss in generality that the cover is finite, $\mathscr{J} = \{1,\dots,N\}$. Let $\{e_j\}_{j=1}^N$ be a smooth partition of unity on $[-1,1]$ subordinate to the cover $\{F_j\}_{j=1}^N$. Then, for each $j = 1,\dots,N$, $\operatorname{supp} e_j$ is a closed subset of F_j, and by Definition 1.22 there exists a $\delta_j > 0$ such that

$$A_{1\delta}u = A_{2\delta}u \qquad \text{for } \operatorname{supp} u \subset \operatorname{supp} e_j \text{ and } \delta < \delta_j.$$

Set

$$\delta_0 = \min_{j=1,\dots,N} \delta_j > 0.$$

Then for $\delta < \delta_0$ one has

$$A_{1\delta}u = \sum_{j=1}^N A_{1\delta}(e_j u) = \sum_{j=1}^N A_{2\delta}(e_j u) = A_{2\delta}u.$$

The case of equality modulo compact proper operators can be considered in a completely similar way. □

Remark. Note a useful corollary (which we however do not need in the proof of the theorem): the difference of two arbitrary proper almost inverses of a c-Fredholm operator A is a compact proper operator.

Lemma 1.25. *Assume that all four operators D, D_-, D_+, and $D_\pm$ act in the same pair of spaces. Then Theorem* 0.10 *is true.*

Proof. Under the assumptions of the lemma, we can represent the difference of the right- and left-hand sides of the desired relation in the form of the index of a product of four operators as follows:

$$\operatorname{ind} D - \operatorname{ind} D_- - \operatorname{ind} D_+ + \operatorname{ind} D_\pm = \operatorname{ind}\bigl(DD_-^{[-1]}D_\pm D_+^{[-1]}\bigr).$$

However, we have

$$DD_-^{[-1]}D_\pm D_+^{[-1]} \overset{(-1,1]}{\equiv} DD^{[-1]}D_+D_+^{[-1]} \equiv 1$$

(where the symbol $\equiv$ is used to denote equality modulo compact proper operators) and further

$$DD_-^{[-1]}D_\pm D_+^{[-1]} \overset{[-1,1)}{\equiv} DD_-^{[-1]}D_-D^{[-1]} \equiv DD^{[-1]} \equiv 1.$$

By Lemma 1.24, we obtain

$$DD_-^{[-1]}D_\pm D_+^{[-1]} \equiv 1,$$

and hence the index of this operator is zero. □

Thus, we have proved Theorem 0.10 for the special case in which all the operators act in the same pair of spaces. Let us proceed to the general case, where the underlying surgery diagrams for collar spaces are nontrivial. Here the key role is played by a special direct sum decomposition that exists for an arbitrary collar space and which is described in the following two technical lemmas.

Lemma 1.26. *Let H be a collar space. Define the following subspaces of H:*

$$H^{(0)} = H_{(-1,1)}, \quad H^{(-)} = H_{[-1,-1/2)} \ominus H_{(-1,-1/2)}, \quad H^{(+)} = H_{(1/2,1]} \ominus H_{(1/2,1)}, \tag{1.12}$$

where $L \ominus L'$ stands for an arbitrary complement (not necessarily the orthogonal complement) of a closed subspace L' of a Hilbert space L. Then one has the direct sum decompositions

$$\begin{gathered} H = H^{(-)} \oplus H^{(0)} \oplus H^{(+)} = H^{(-)} \oplus H_{(-1,1]} = H_{[-1,1)} \oplus H^{(+)}, \\ H_{[-1,1)} = H^{(-)} \oplus H^{(0)}, \qquad H_{(-1,1]} = H^{(0)} \oplus H^{(+)} \end{gathered} \tag{1.13}$$

of Hilbert spaces. (These decompositions are in general neither orthogonal, nor do they respect the action of the algebra $C^\infty([-1,1])$.)

Proof. (i) Let us prove that $H = H^{(-)} \oplus H_{(-1,1]}$.

Let $u \in H$. Take a partition of unity $e_1(t) + e_2(t) = 1$ in $C^\infty([-1,1])$ such that $e_1(t) = 0$ for $t > -3/4$ and $e_2(t) = 0$ for $t < -7/8$. Then $u = e_1u + e_2u$, where $e_1u \in H_{[-1,-1/2)}$ and $e_2u \in H_{(-1,1]}$. Let $e_1u = v + w$, where $v \in H^{(-)}$ and $w \in H_{(-1,-1/2)}$. Then we finally obtain

$$u = u_1 + u_2, \qquad u_1 = v \in H^{(-)}, \quad u_2 = w + e_2u \in H_{(-1,1]}, \tag{1.14}$$

and moreover,

$$\|u_1\| + \|u_2\| \le C\,\|u\|$$

with some constant C.

It remains to prove that the decomposition (1.14) is unique, i.e., $H^{(-)} \cap H_{(-1,1]} = \{0\}$. Let $u \in H^{(-)} \cap H_{(-1,1]}$. Then, in particular, $u \in H_{[-1,-1/2)}$ and $u \in H_{(-1,1]}$. We claim that then $u \in H_{(-1,-1/2)}$ and hence $u = 0$, because $H_{(-1,-1/2)}$ is a complement of $H^{(-)}$. We again write $u = e_1u + e_2u$, where e_1 and e_2 are the same as above. Since $u \in H_{[-1,-1/2)}$, it follows that $u = \lim v_n$, where $v_n \in H$ and $\operatorname{supp} v_n \subset [-1,-1/2)$; then $e_2u = \lim e_2v_n$, $\operatorname{supp} e_2v_n \subset (-1,-1/2)$, and hence $e_2u \in H_{(-1,-1/2)}$. Now consider e_1u. Since $u \in H_{(-1,1]}$, it follows that $u = \lim w_n$, where $w_n \in H$ and $\operatorname{supp} w_n \subset (-1,1]$; then $e_1u = \lim e_1w_n$, $\operatorname{supp} e_1w_n \subset (-1,-1/2)$, and hence $e_1u \in H_{(-1,-1/2)}$.

(ii) In a similar way, one proves that $H_{(-1,1]} = H^{(0)} \oplus H^{(+)}$.

(iii) It follows from (i) and (ii) that $H = H^{(-)} \oplus H^{(0)} \oplus H^{(+)}$.

(iv) The remaining relations in (1.13) can be proved in a similar way. □

Lemma 1.27. *Let us equip the collar space H with the modified action of the algebra $C^\infty([-1,1])$ defined via the original action of this algebra by the formula*

$$\varphi * h = (\varphi \circ \theta)h, \tag{1.15}$$

where $\theta\colon [-1,1] \to [-1,1]$ *is a monotone smooth function such that*

$$\theta(t) = \begin{cases} -1, & t \in [-1,-1/2], \\ 1, & \theta \in [1/2,1]. \end{cases}$$

The decompositions (1.13) *respect the action* (1.15); *if we denote the space* H *equipped with the action* (1.15) *by* $\widetilde{H}$, *then*

$$H^{(-)} \subset \widetilde{H}_{\{-1\}}, \qquad H^{(+)} \subset \widetilde{H}_{\{1\}}, \qquad \widetilde{H}_{[-1,1)} \subset H_{[-1,1)}, \qquad \widetilde{H}_{(-1,1]} \subset H_{(-1,1]}. \tag{1.16}$$

Proof. The assertion of the lemma becomes clear if we notice that the action (1.15) is the multiplication by $\varphi(-1)$ on $H_{[-1,-1/2)}$ and by $\varphi(1)$ on $H_{(1/2,1]}$. □

Now we can return to the proof of Theorem 0.10. Suppose that the c-Fredholm operators in the theorem are

$$D\colon H \longrightarrow G, \quad D_-\colon H_- \longrightarrow G_-, \quad D_+\colon H_+ \longrightarrow G_+, \quad D_\pm\colon H_\pm \longrightarrow G_\pm\,.$$

The collar spaces in which these operators act form the commutative surgery diagrams

$$\begin{array}{ccc} H & \xleftrightarrow{-1} & H_- \\ {\scriptstyle 1}\updownarrow & & \updownarrow{\scriptstyle 1} \\ H_+ & \xleftrightarrow[-1]{} & H_\pm, \end{array} \qquad \begin{array}{ccc} G & \xleftrightarrow{-1} & G_- \\ {\scriptstyle 1}\updownarrow & & \updownarrow{\scriptstyle 1} \\ G_+ & \xleftrightarrow[-1]{} & G_\pm. \end{array} \tag{1.17}$$

Consider, for example, the first of these diagrams. We have the underlying isomorphisms

$$\begin{aligned} &j_-\colon H_{(-1,1]} \longrightarrow H_{-(-1,1]}, \quad j_+\colon H_{[-1,1)} \longrightarrow H_{+[-1,1)}, \\ &j_\mp\colon H_{+(-1,1]} \longrightarrow H_{\pm(-1,1]}, \quad j_\pm\colon H_{-[-1,1)} \longrightarrow H_{\pm[-1,1)}, \end{aligned} \tag{1.18}$$

which form the commutative surgery diagram

$$\begin{array}{ccc} H_{(-1,1)} & \xrightarrow{j_-} & H_{-(-1,1)} \\ {\scriptstyle j_+}\downarrow & & \downarrow{\scriptstyle j_\pm} \\ H_{+(-1,1)} & \xrightarrow[j_\mp]{} & H_{\pm(-1,1)} \end{array} \tag{1.19}$$

of collar spaces.

Lemma 1.28. *The isomorphisms* (1.18) *can be extended to isomorphisms*

$$\begin{aligned} &\widetilde{j}_-\colon H \longrightarrow H_-, \quad \widetilde{j}_+\colon H \longrightarrow H_+, \\ &\widetilde{j}_\mp\colon H_+ \longrightarrow H_\pm, \quad \widetilde{j}_\pm\colon H_- \longrightarrow H_\pm \end{aligned} \tag{1.20}$$

of Hilbert spaces such that the diagram

$$\begin{array}{ccc} H & \xrightarrow{\widetilde{j}_-} & H_- \\ {\scriptstyle \widetilde{j}_+}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{j}_\pm} \\ H_+ & \xrightarrow[\widetilde{j}_\mp]{} & H_\pm \end{array} \tag{1.21}$$

commutes and the mappings (1.20) *themselves commute with the action* (1.15) *of the algebra* $C^\infty([-1,1])$.

Proof. We use Lemma 1.26 and seek the desired diagram (1.21) in the form

$$\begin{array}{ccc} H^{(-)} \oplus H^{(0)} \oplus H^{(+)} & \xrightarrow{\widetilde{j}_-} & H_-^{(-)} \oplus H_-^{(0)} \oplus H_-^{(+)} \\ {\scriptstyle \widetilde{j}_+}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{j}_\pm} \\ H_+^{(-)} \oplus H_+^{(0)} \oplus H_+^{(+)} & \xrightarrow[\widetilde{j}_\mp]{} & H_\pm^{(-)} \oplus H_\pm^{(0)} \oplus H_\pm^{(+)}, \end{array} \tag{1.22}$$

where at the vertices we have used the decompositions (1.13) and all four isomorphisms in the diagram should respect the direct sum structure (i.e., be block diagonal). (To achieve this, we use the freedom in the choice of subspaces with $(-)$ and $(+)$ subscripts.) Thus, we should actually construct three separate commutative diagrams,

$$\begin{array}{ccc} H^{(0)} & \xrightarrow{j_-} & H_-^{(0)} \\ {\scriptstyle j_+}\big\downarrow & & \big\downarrow{\scriptstyle j_\pm} \\ H_+^{(0)} & \xrightarrow[j_\mp]{} & H_\pm^{(0)}, \end{array} \quad \begin{array}{ccc} H^{(-)} & \xrightarrow{\widetilde{j}_-} & H_-^{(-)} \\ {\scriptstyle j_+}\big\downarrow & & \big\downarrow{\scriptstyle j_\pm} \\ H_+^{(-)} & \xrightarrow[\widetilde{j}_\mp]{} & H_\pm^{(-)}, \end{array} \quad \begin{array}{ccc} H^{(+)} & \xrightarrow{j_-} & H_-^{(+)} \\ {\scriptstyle \widetilde{j}_+}\big\downarrow & & \big\downarrow{\scriptstyle \widetilde{j}_\pm} \\ H_+^{(+)} & \xrightarrow[j_\mp]{} & H_\pm^{(+)}, \end{array} \tag{1.23}$$

where the tilde is placed over yet unknown mappings (see below). The first of these diagrams is already known; it just expresses the fact that the first surgery diagram in (1.17) commutes.

Let us construct the second diagram in (1.23). We can assume without loss of generality that all spaces in it are infinite-dimensional. (If this is not the case, then we can add an infinite-dimensional Hilbert space (say, l^2) as a direct summand to these spaces and the identity operators on l^2 to the original operators.) We choose the complements $H^{(-)}$ and $H_-^{(-)}$ (see the statement of Lemma 1.26) arbitrarily. The vertical mappings in this diagram are known (because

$$j_+ : H_{[-1,1)} \equiv H^{(-)} \oplus H^{(0)} \longrightarrow H_{+[-1,1)} \equiv H_+^{(-)} \oplus H_+^{(0)}$$

is given, and likewise for $j_\pm$), and this uniquely determines the spaces

$$H_+^{(-)} = j_+\big(H^{(-)}\big), \qquad H_\pm^{(-)} = j_\pm\big(H_-^{(-)}\big).$$

(One can readily verify that they are complements of the appropriate subspaces as required in Lemma 1.26.) It remains to determine the horizontal arrows $\widetilde{j}_-$ and $\widetilde{j}_\mp$. For the first arrow, one can take an arbitrary isomorphism (which always exists between separable Hilbert spaces), and then one should take

$$\widetilde{j}_\mp = j_\pm \circ \widetilde{j}_- \circ j_+^{-1}$$

to make the diagram commute.

The construction of the third diagram in (1.23) is completely similar, with vertical and horizontal arrows exchanging their roles.

The resulting isomorphisms commute with the action (1.15), which follows from the first two embeddings in (1.16).

The proof of the lemma is complete. □

Now we see that the four collar spaces (with respect to the action (1.15)) in the first surgery diagram in (1.17) are isomorphic and hence can be identified with each other. The same is true for the collar spaces in the second diagram in (1.17). The operators D, D_-, D_+, and $D_\pm$ remain c-Fredholm with respect to the new action of $C^\infty([-1,1])$; they all act in the same pair of spaces and satisfy the surgery diagram (1.9) in the sense of the new action. (The last assertion follows from the third and fourth embeddings in (1.16).) Thus, we have reduced the analysis to the case covered by Lemma 1.25. The proof of Theorem 0.10 is complete. □

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