Chapter 2

The Radon transformation and applications

The Radon transformation and its dual, or their associates, of which each component of the transformation Θ in Theorem 1.1.1 is a special case, connect the analysis of functions in the plane and in the hyperbolic half-plane: when enriched with an automorphy condition, the dual Radon transformation will also set up a correspondence, in Chapter 3, from automorphic distribution theory (in the plane) to automorphic function theory (in the half-plane).

After having recalled the Iwasawa decomposition $G = NAK$ of the group $G = SL(2, \mathbb{R})$, we consider the Radon transformation V from the homogeneous space G/K to the space G/MN, with $M = \{\pm I\}$, and the dual Radon transformation, which acts in the reverse direction. The space G/MN can be regarded as the quotient of $\mathbb{R}^2 \backslash \{0\}$ by the equivalence that identifies (x, ξ) with $(-x, -\xi)$, while the space G/K is just the hyperbolic half-plane Π: consequently, the dual Radon transformation may be considered as a map from even functions in $\mathbb{R}^2$ to functions in Π. Besides, the maps V and V^* have associates, obtained by multiplying them, on the appropriate side, by arbitrary functions, in the spectral-theoretic sense, of the Euler operator in $\mathbb{R}^2$. All norm computations involving $SL(2, \mathbb{R})$-covariant maps from even functions in the plane to functions in Π rely on the results of calculations involving the Radon transformation and its associates. This is in particular the case for the map $h \mapsto f_0$ introduced in Theorem 1.1.1; we shall rely on these again to complete, in Section 2.2, our study of the totally radial Weyl calculus, as initiated in Section 1.3. The rest of the chapter is concerned with a family of bihomogeneous functions $\hom_{\rho,\nu}$ in the plane, the dual Radon transforms of which will play a basic role in our construction, in Chapter 4, of a new class of non-holomorphic modular forms. Splitting such transforms into two terms, we shall obtain a two-parameter family of functions $z \mapsto (\mathrm{Im}\, z)^{\frac{\rho-1}{2}} \chi_{\rho,\nu} \left(\frac{\mathrm{Re}\, z}{\mathrm{Im}\, z} \right)$ in the hyperbolic half-plane: these functions will constitute the starting points of the

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Poincaré series to be introduced there. The functions $\chi_{\rho,\nu}$ are studied with much care in Section 2.3 and the functions in Π just mentioned are expressed in a natural way involving the resolvent of the Laplace operator Δ on Π in Section 2.4.

2.1 The Radon transformation

Consider the transformation $\Theta = (\Theta_0, \Theta_1)\colon h \mapsto (f_0, f_1)$ introduced in Theorem 1.1.1 or, using (1.1.38) and starting, more generally, from a distribution,

$$(\Theta_0\mathfrak{S})(z) = \langle \mathfrak{S}, (x,\xi) \mapsto 2\exp\left(-2\pi\frac{|x-z\xi|^2}{\operatorname{Im} z}\right)\rangle,$$
$$(\Theta_1\mathfrak{S})(z) = \langle \mathfrak{S}, (x,\xi) \mapsto 2\left[\frac{4\pi}{\operatorname{Im} z}|x-z\xi|^2 - 1\right]\exp\left(-2\pi\frac{|x-z\xi|^2}{\operatorname{Im} z}\right)\rangle. \tag{2.1.1}$$

The two functions just introduced are linked by the equation

$$\Theta_1\mathfrak{S} = \Theta_0(2i\pi\mathcal{E}\mathfrak{S}), \tag{2.1.2}$$

as it follows immediately from the fact that the transpose of the operator $2i\pi\mathcal{E}$ is $-2i\pi\mathcal{E}$. As a consequence, identities involving Θ_0 will always have analogues involving Θ_1, which we shall dispense with making explicit unless clarity demands it. We shall also use, consistently and without reference, the fact that the conjugate of the operator $2i\pi\mathcal{E}$ under the symplectic Fourier transformation, or under $\mathcal{G}$, is $-2i\pi\mathcal{E}$.

The map Θ connects even distributions in the plane to pairs of functions in the hyperbolic half-plane Π, and it has many nice properties; only, do not confuse x, the first of the pair of variables (x,ξ) in the plane (the standard notation in pseudo-differential analysis) with the real part of z. First, recall that Θ is covariant under the pair of actions of $SL(2,\mathbb{R})$ on $\mathbb{R}^2$ and on Π, which means that one always has

$$(\Theta(\mathfrak{S}\circ g^{-1}))(z) = (\Theta\mathfrak{S})(g^{-1}.z) \tag{2.1.3}$$

if, given $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in SL(2,\mathbb{R})$, one sets $g(x,\xi) = (ax+b\xi, cx+d\xi)$ and $g.z = \frac{az+b}{cz+d}$. Also, Θ kills all odd functions on $\mathbb{R}^2$, so we may as well restrict it to the space $\mathcal{S}'_{\text{even}}(\mathbb{R}^2)$: another symmetry expresses itself in terms of the transformation $\mathcal{G}$ in (1.1.24), as the pair of identities

$$\Theta_0(\mathcal{G}\mathfrak{S}) = \Theta_0\mathfrak{S}, \quad \Theta_1(\mathcal{G}\mathfrak{S}) = -\Theta_1\mathfrak{S}. \tag{2.1.4}$$

The first one, say, can be seen by remarking that the (even) function $(x,\xi) \mapsto 2\exp\left(-2\pi\frac{|x-z\xi|^2}{\operatorname{Im} z}\right)$ is $\mathcal{G}$-invariant: to see this, it suffices, taking benefit from the covariance property, to verify that the function $2\exp(-2\pi(x^2+\xi^2))$ is $\mathcal{G}$-invariant, which is immediate. Another proof consists (*cf.* what follows (1.1.24)) in remarking, a consequence of (1.1.34), that this function is the symbol of an even-even operator. From the fact that the set $(\phi_z^0)_{z\in\Pi}$ is total in $L^2_{\text{even}}(\mathbb{R})$, one can then see that, when restricted to even $\mathcal{G}$-invariant tempered distributions, Θ_0 becomes one-to-one.

In view of (1.1.41), one always has the identity

$$\Theta(\pi^2\mathcal{E}^2\mathfrak{S}) = \left(\Delta - \frac{1}{4}\right)\Theta\mathfrak{S}. \tag{2.1.5}$$

In other words, if $\mathfrak{S}$ is homogeneous of degree $-1-\nu$ or $-1+\nu$, the function $\Theta\mathfrak{S}$ is a pair of (generalized) eigenfunctions of Δ for the eigenvalue $\frac{1-\nu^2}{4}$.

The covariance property of Θ_0 (or Θ_1), as well as the way it exchanges the operators $\pi^2\mathcal{E}^2$ and $\Delta - \frac{1}{4}$, are shared by a family of transformations, linked to the so-called Radon transformation, which we need to recall in the case of the group $SL(2,\mathbb{R})$: the Radon transformation has been studied by Helgason [17, 18] in a considerable generality. We here follow with a few changes the exposition, in the case of $SL(2,\mathbb{R})$, made in [60, Sec.4], which is more immediately adapted to our needs related to pseudo-differential analysis, besides being of necessity simpler since it deals only with a rank-one case.

We parametrize the generic elements of the subgroups N, A, K entering the Iwasawa decomposition of $G = SL(2,\mathbb{R}) = NAK$ as

$$n = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, \quad a = \begin{pmatrix} e^{\frac{r}{2}} & 0 \\ 0 & e^{-\frac{r}{2}} \end{pmatrix}, \quad k = \begin{pmatrix} \cos\frac{\theta}{2} & \sin\frac{\theta}{2} \\ -\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix}, \tag{2.1.6}$$

where $b \in \mathbb{R}, r \in \mathbb{R}, 0 \le \theta < 4\pi$. Following the normalizations in ([18], ch.II,3), we set $dn = \pi^{-1}db, dk = (4\pi)^{-1}d\theta$. The homogeneous space G/K is identified with the hyperbolic half-plane Π in the usual way, sending gK to $z = g.i$. On the other hand, the space $\Xi = G/MN$, with $M = \{\pm I\}$, is identified with the quotient of $\mathbb{R}^2\backslash\{0\}$ by the equivalence $\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right) \sim \left(\begin{smallmatrix} -x \\ -\xi \end{smallmatrix}\right)$, under the map $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) MN \mapsto \pm\left(\begin{smallmatrix} a \\ c \end{smallmatrix}\right)$: one must be careful, again, not to use in the same formula x to denote the first coordinate of $\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right)$ (or (x,ξ)) in $\mathbb{R}^2$ and the real part of $z = x + iy \in \Pi$. On Π, we use the invariant measure $dm(z) = \frac{dxdy}{y^2}$ and, identifying functions on Ξ with even functions on $\mathbb{R}^2$, we use there the standard Lebesgue measure on the full plane. Let us also recall that the hyperbolic distance d on Π associated to the (squared) line element $ds^2 = \frac{dx^2+dy^2}{y^2}$ is G-invariant, i.e., that $d(g.z, g.z')$ is independent of g, and characterized as such by the equation $\cosh d(i, x+iy) = \frac{1+x^2+y^2}{2y}$. The Radon transformation V from functions f on Π to even functions on $\mathbb{R}^2$ is defined by the equation

$$(Vf)(g.\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right)) = \int_N f((gn).i)dn: \tag{2.1.7}$$

the integral is convergent, yielding a continuous function Vf if, say, $|f(z)| \le C(\cosh d(i,z))^{-\frac{1}{2}-\varepsilon}$ for some $\varepsilon > 0$. Explicitly, completing if $x \neq 0$ the column

$\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right)$ into the matrix $\left(\begin{smallmatrix} x & 0 \\ \xi & x^{-1} \end{smallmatrix}\right)$,

$$(Vf)(\pm\left(\begin{smallmatrix} x \\ \xi \end{smallmatrix}\right)) = \frac{1}{\pi}\int_{-\infty}^{\infty} f\left(\frac{x^2(i+b)}{x\xi(i+b)+1}\right) db, \quad x \neq 0; \tag{2.1.8}$$

the dual Radon transform V^*, the formal adjoint of V, is defined by

$$(V^*h)(g.i) = \int_K h((gk).\left(\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right))dk \tag{2.1.9}$$

or, in coordinates,

$$(V^*h)(x+iy) = \frac{1}{2\pi}\int_0^{2\pi} h\left(\pm\begin{pmatrix} y^{\frac{1}{2}}\cos\frac{\theta}{2} - xy^{-\frac{1}{2}}\sin\frac{\theta}{2} \\ -y^{-\frac{1}{2}}\sin\frac{\theta}{2} \end{pmatrix}\right) d\theta. \tag{2.1.10}$$

We abbreviate the representation $\pi_{i\lambda,0}$, as defined in (1.2.18), as $\pi_{i\lambda}$ — it lies in the principal series of $SL(2,\mathbb{R})$, whereas the representation $\pi_{\tau+1}$ in (1.3.5) lies in the extended projective discrete series of this group — and abbreviate $h_{i\lambda,0}$ (*resp.* $h^\flat_{i\lambda,0}$) as $h_{i\lambda}$ (*resp.* $h^\flat_{i\lambda}$): in the present section, we only interest ourselves in even functions in the plane. Through the dual Radon transformation, the representation $\pi_{i\lambda}$ can be realized in some Hilbert space of functions in Π: we need to make this explicit.

We have already defined the Euler operator $2i\pi\mathcal{E} = x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial\xi} + 1$. It is essentially self-adjoint on $L^2(\mathbb{R}^2)$ (i.e., it admits a unique self-adjoint extension) if given the initial domain $C_0^\infty(\mathbb{R}^2\backslash\{0\})$. This makes it possible to define, in the spectral-theoretic sense, functions of $\mathcal{E}$. We shall need in particular the operator (a scalar when restricted to even functions of a given degree of homogeneity)

$$T = \left(\frac{\pi}{2}\right)^{\frac{1}{2}} \frac{\Gamma(\frac{1}{2} - i\pi\mathcal{E})}{\Gamma(-i\pi\mathcal{E})} = \pi^{-\frac{1}{2}}(-i\pi\mathcal{E})\int_0^\infty t^{-\frac{1}{2}}(1+t)^{-1+i\pi\mathcal{E}}dt : \tag{2.1.11}$$

also, observe that $(t^{2i\pi\mathcal{E}}h)(x,\xi) = th(tx,t\xi)$ for $t>0$.

We now give useful expressions of the transformation TV and its formal adjoint V^*T^*, with the help of the following special case of (1.2.14):

$$h_{i\lambda}(x,\xi) = |\xi|^{-1-i\lambda}h^\flat_{i\lambda}(\frac{x}{\xi}). \tag{2.1.12}$$

As a consequence of (2.1.11), T acts on $(Vf)_{i\lambda}$ as the scalar $(\frac{\pi}{2})^{\frac{1}{2}}\frac{\Gamma(\frac{1+i\lambda}{2})}{\Gamma(\frac{i\lambda}{2})}$, and it then follows from (2.1.12) and (2.1.8) that, assuming that, say, $f \in C_0^\infty(\Pi)$, one has for almost all λ the equation

$$(TVf)^\flat_{i\lambda}(s) = (2\pi)^{-\frac{3}{2}}\frac{\Gamma(\frac{1+i\lambda}{2})}{\Gamma(\frac{i\lambda}{2})}\int_0^\infty t^{i\lambda-2}dt\int_{-\infty}^{\infty} f\left(\frac{s^2(i+b)}{s(i+b)+t^2}\right) db : \tag{2.1.13}$$

performing the change of variable such that

$$z = \frac{s^2(i+b)}{s(i+b)+t^2}, \quad dm(z) = \frac{2dtdb}{t}, \tag{2.1.14}$$

so that $t^2 = \frac{|z-s|^2}{\operatorname{Im} z}$, one gets

$$(TVf)^\flat_{i\lambda}(s) = \frac{1}{2}(2\pi)^{-\frac{3}{2}} \frac{\Gamma(\frac{1+i\lambda}{2})}{\Gamma(\frac{i\lambda}{2})} \int_\Pi \left(\frac{|z-s|^2}{\operatorname{Im} z}\right)^{-\frac{1}{2}-\frac{i\lambda}{2}} f(z)dm(z). \tag{2.1.15}$$

In the reverse direction, we use the second equation (2.1.12) and (2.1.10), obtaining (after one has set $s = -y\text{cotan}\,\frac{\theta}{2} + x$ in the latter formula) that

$$(V^*T^*h_{i\lambda})(z) = (2\pi)^{-\frac{1}{2}} \frac{\Gamma(\frac{1-i\lambda}{2})}{\Gamma(-\frac{i\lambda}{2})} \int_{-\infty}^{\infty} h^\flat_{i\lambda}(s) \left(\frac{|z-s|^2}{\operatorname{Im} z}\right)^{-\frac{1}{2}+\frac{i\lambda}{2}} ds: \tag{2.1.16}$$

note that the integral on the right-hand side is bounded if $h^\flat_{i\lambda} \in L^2(\mathbb{R})$.

From its very definition (2.1.7), the Radon transformation (as well as its dual) is obviously covariant under the two actions of G, on functions defined on Π and on $\mathbb{R}^2\backslash\{0\}$, through the fractional-linear change of complex coordinate and the linear change of real coordinates associated to the same matrix g. On the other hand, all functions, in the spectral-theoretic sense, of the Euler operator commute with the second action. Consequently, the transformations V and V^* preserve their covariance if multiplied on the left (*resp.* on the right) by an "arbitrary" function of $2i\pi\mathcal{E}$. Operators obtained as products of the Radon (*resp.* dual Radon) transformation by a function of the Euler operator on the left (*resp.* right) side will be called *associates* of the Radon or dual Radon transformation. A subclass consists of operators obtained in a comparable way, only replacing the function of the Euler operator by a function of the hyperbolic Laplacian on the other side: as a consequence of the last assertion in Theorem 2.1.2 below, even functions of $2i\pi\mathcal{E}$ can be replaced by appropriate functions of Δ with no change. We now show that the map Θ_0 introduced in (2.1.1) is an associate of the dual Radon transformation: of course, the same will then be true of the map Θ_1 in view of (2.1.2).

Proposition 2.1.1. *One has*

$$\Theta_0 = V^*(2\pi)^{\frac{1}{2}-i\pi\mathcal{E}}\Gamma\left(\frac{1}{2}+i\pi\mathcal{E}\right). \tag{2.1.17}$$

Proof. Starting from the decomposition (1.2.11) and applying the definition (2.1.1) of Θ, we obtain

$$(\Theta_0 h_{i\lambda})(z) = \frac{1}{2\pi}\int_0^\infty t^{i\lambda}\Theta_0((x,\xi)\mapsto h(tx,t\xi))dt$$

$$= \frac{1}{\pi}\int_0^\infty t^{i\lambda}dt\int_{\mathbb{R}^2} h(tx,t\xi)\exp\left(-2\pi\frac{|x-z\xi|^2}{\operatorname{Im} z}\right)dxd\xi$$
$$= \frac{1}{\pi}\int_{\mathbb{R}^2} h(x,\xi)dxd\xi\int_0^\infty t^{i\lambda-2}\exp\left(-2\pi\frac{|x-z\xi|^2}{t^2\operatorname{Im} z}\right)dt. \tag{2.1.18}$$

The integral is easily computed, which leads to the equation

$$(\Theta_0 h_{i\lambda})(z) = (2\pi)^{\frac{i\lambda-3}{2}}\Gamma(\frac{1-i\lambda}{2})\int_{\mathbb{R}^2} h(x,\xi)\left(\frac{|x-z\xi|^2}{\operatorname{Im} z}\right)^{\frac{i\lambda-1}{2}}dxd\xi. \tag{2.1.19}$$

On the other hand, using (1.2.11) again and (2.1.16), we have

$$(V^*T^*h_{i\lambda})(z) = (2\pi)^{-\frac{3}{2}}\frac{\Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{-i\lambda}{2})}\int_{-\infty}^\infty\left(\frac{|z-s|^2}{\operatorname{Im} z}\right)^{-\frac{1}{2}+\frac{i\lambda}{2}}ds\int_0^\infty t^{i\lambda}h(ts,t)dt: \tag{2.1.20}$$

we make the change of variable

$$t=\xi,\quad s=\frac{x}{\xi},\quad dsdt=\xi^{-1}dxd\xi, \tag{2.1.21}$$

and take advantage of the fact that h is assumed to be even to change the domain $\{(x,\xi)\colon \xi>0\}$ to $\mathbb{R}^2$, ending up with the equation

$$(V^*T^*h_{i\lambda})(z) = \frac{1}{2}(2\pi)^{-\frac{3}{2}}\frac{\Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{-i\lambda}{2})}\int_{\mathbb{R}^2} h(x,\xi)\left(\frac{|x-z\xi|^2}{\operatorname{Im} z}\right)^{\frac{i\lambda-1}{2}}dxd\xi. \tag{2.1.22}$$

Comparing it with (2.1.19), we obtain

$$\Theta_0 h_{i\lambda} = 2(2\pi)^{\frac{i\lambda}{2}}\Gamma(-\frac{i\lambda}{2})V^*T^*h_{i\lambda} \tag{2.1.23}$$

or, since $2i\pi\mathcal{E}h_{i\lambda} = -i\lambda h_{i\lambda}$,

$$\Theta_0 = V^*T^*2(2\pi)^{-i\pi\mathcal{E}}\Gamma(i\pi\mathcal{E}): \tag{2.1.24}$$

as $T^* = (\frac{\pi}{2})^{\frac{1}{2}}\frac{\Gamma(\frac{1}{2}+i\pi\mathcal{E})}{\Gamma(i\pi\mathcal{E})}$, this leads to Proposition 2.1.1. □

This proposition explains several facts. First, since, according to (2.1.5), the operator $\pi^2\mathcal{E}^2$ on $\mathbb{R}^2$ on Π transfers under Θ_0 to the operator $\Delta-\frac{1}{4}$, the same is true for the Radon transformation or its dual, whether it has been multiplied on the appropriate side with a function of the Euler operator or not. Next, consider the formal adjoint of Θ_0, defined by the equation

$$(\Theta_0^*f)(x,\xi) = 2\int_\Pi f(z)\exp\left(-2\pi\frac{|x-z\xi|^2}{\operatorname{Im} z}\right)dm(z), \tag{2.1.25}$$

or

$$\Theta_0^* = 2(2\pi)^{i\pi\mathcal{E}}\Gamma(-i\pi\mathcal{E})TV. \tag{2.1.26}$$

As already noticed, the range (the image) of Θ_0^* is $\mathcal{G}$-invariant: also, $\mathcal{G}(i\pi\mathcal{E}) = (-i\pi\mathcal{E})\mathcal{G}$. As a consequence,

$$\mathrm{Ran}(TV) \quad \text{is invariant under the involution} (2\pi)^{-2i\pi\mathcal{E}}\,\frac{\Gamma(i\pi\mathcal{E})}{\Gamma(-i\pi\mathcal{E})}\mathcal{G}. \tag{2.1.27}$$

We now recall (with a better proof) a theorem given in [60, p. 27].

Theorem 2.1.2. *The transformation TV, initially defined on the space of continuous functions on Π with a compact support, extends as an isometry from $L^2(\Pi)$ onto the subspace $\mathrm{Ran}(TV)$ of $L^2_{\text{even}}(\mathbb{R}^2)$ consisting of all functions invariant under the unitary involution $(2\pi)^{-2i\pi\mathcal{E}}\,\frac{\Gamma(i\pi\mathcal{E})}{\Gamma(-i\pi\mathcal{E})}\mathcal{G}$. The operator V^*T^* extends on $\mathrm{Ran}(TV)$ as the inverse of TV, and is zero on the subspace $(\mathrm{Ran}(TV))^\perp$ of $L^2_{\text{even}}(\mathbb{R}^2)$ consisting of all functions changing to their negatives under the same involution. Moreover, the isometry TV intertwines the two actions of G on $L^2(\Pi)$ and $L^2_{\text{even}}(\mathbb{R}^2)$ respectively, and transforms the operator $\Delta - \frac{1}{4}$ on $L^2(\Pi)$ into the operator $\pi^2\mathcal{E}^2$ on $L^2_{\text{even}}(\mathbb{R}^2)$.*

Proof. The isometry property is a very special case of ([18], ch.II,3), but sorting out notation is not that easy. An alternative proof is as follows. From (2.1.15) and (2.1.11), one has

$$\begin{aligned}\|(TVf)^\flat_{i\lambda}\|^2_{L^2(\mathbb{R})} &= \frac{1}{32\pi^3}\left(\frac{\lambda}{2}\tanh\frac{\pi\lambda}{2}\right) \\ \times \int_{\Pi\times\Pi} f(z)\overline{f}(w)dm(z)dm(w) &\int_{-\infty}^{\infty}\left(\frac{|z-s|^2}{\mathrm{Im}\,z}\right)^{-\frac{1}{2}-\frac{i\lambda}{2}}\left(\frac{|w-s|^2}{\mathrm{Im}\,z}\right)^{-\frac{1}{2}+\frac{i\lambda}{2}} ds.\end{aligned} \tag{2.1.28}$$

Now, one has

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\left(\frac{|z-s|^2}{\mathrm{Im}\,z}\right)^{-\frac{1}{2}-\frac{i\lambda}{2}}\left(\frac{|w-s|^2}{\mathrm{Im}\,z}\right)^{-\frac{1}{2}+\frac{i\lambda}{2}} ds = \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\cosh d(z,w)), \tag{2.1.29}$$

a consequence of Plancherel's formula together with [36, p, 401]

$$\pi^{-\frac{1}{2}}\int_{-\infty}^{\infty}\left(\frac{|z-s|^2}{\mathrm{Im}\,z}\right)^{-\frac{1}{2}-\frac{i\lambda}{2}} e^{-2i\pi s\sigma}ds = y^{\frac{1}{2}}e^{-2i\pi\sigma x}\frac{2\pi^{\frac{i\lambda}{2}}}{\Gamma(\frac{1+i\lambda}{2})}|\sigma|^{\frac{i\lambda}{2}}K_{\frac{i\lambda}{2}}(2\pi|\sigma|y) \tag{2.1.30}$$

and [36, p. 413]

$$\begin{aligned}&\int_0^\infty K_{\frac{i\lambda}{2}}(2\pi\sigma\mathrm{Im}\,z)K_{\frac{i\lambda}{2}}(2\pi\sigma\mathrm{Im}\,w)\cos(2\pi\sigma\mathrm{Re}\,(z-w)))d\sigma \\ &\qquad = \frac{1}{8}(\mathrm{Im}\,z\mathrm{Im}\,w)^{-\frac{1}{2}}\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\cosh d(z,w)).\end{aligned} \tag{2.1.31}$$

The isometry property is then a consequence of (1.2.15) and of Mehler's decomposition [36, p. 398] of functions $f \in C_0^\infty(\Pi)$ provided by the pair of formulas

$$f(z) = \int_0^\infty f_\lambda(z) \left(\frac{\pi\lambda}{2} \tanh \frac{\pi\lambda}{2} \right) d\lambda,$$

$$f_\lambda(z) = \frac{1}{4\pi^2} \int_\Pi f(w) \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\cosh d(z,w)) dm(w). \tag{2.1.32}$$

The factor

$$\frac{\pi\lambda}{2} \tanh \frac{\pi\lambda}{2} = \pi \frac{\Gamma\left(\frac{1+i\lambda}{2}\right) \Gamma\left(\frac{1-i\lambda}{2}\right)}{\Gamma\left(\frac{i\lambda}{2}\right) \Gamma\left(\frac{-i\lambda}{2}\right)} \tag{2.1.33}$$

appears repeatedly in connection with Mehler's transformation. That the range of TV is invariant under the involution under consideration has been established before the statement of the theorem; that it is the full subspace of $L^2_{\text{even}}(\mathbb{R}^2)$ characterized by this invariance or, what amounts to the same, that the image is dense, can be obtained by linking this to a property of Θ_0, with the help of Proposition 2.1.1. □

Note that if $\mathcal{H}_{i\lambda}$ denotes the completion of the space of all f_λ ($f \in C_0^\infty(\Pi)$) under the norm such that

$$\begin{aligned} \|f_\lambda\|^2_{\mathcal{H}_{i\lambda}} &= (4\pi^2)^{-2} \int_{\Pi\times\Pi} f(z) \bar{f}(w) \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\cosh d(z,w)) dm(z) dm(w) \\ &= (4\pi^2)^{-1} (f_\lambda | f)_{L^2(\Pi)}, \end{aligned} \tag{2.1.34}$$

one has the identity

$$\|f\|^2_{L^2(\Pi)} = 4\pi^2 \int_0^\infty \|f_\lambda\|^2_{\mathcal{H}_{i\lambda}} \left(\frac{\pi\lambda}{2} \tanh \frac{\pi\lambda}{2} \right) d\lambda. \tag{2.1.35}$$

The following consequence of (2.1.24) and Theorem 2.1.2 was announced in (1.1.43): if a $\mathcal{G}$-invariant function $h \in L^2_{\text{even}}(\mathbb{R}^2)$ is the image under $2i\pi\mathcal{E}$ of some function in $L^2_{\text{even}}(\mathbb{R}^2)$, so that $\Gamma(i\pi\mathcal{E})h \in L^2_{\text{even}}(\mathbb{R}^2)$ too, one has

$$\|\Theta_0 h\|_{L^2(\Pi)} = 2 \|\Gamma(i\pi\mathcal{E}) h\|_{L^2(\mathbb{R}^2)}. \tag{2.1.36}$$

Equation (1.1.44) follows from the preceding one and from (2.1.2).

Restricting the dual Radon transform to K-invariant functions, and using analytic continuation to replace $i\lambda$ by a more general complex number ν, one observes from (2.1.10) that if $h(x,\xi) = (x^2+\xi^2)^{\frac{-1-\nu}{2}}$, one has

$$\begin{aligned} (V^* h)(iy) &= \frac{1}{2\pi} \int_0^{2\pi} (y \cos^2 \frac{\theta}{2} + y^{-1} \sin^2 \frac{\theta}{2})^{\frac{-1-\nu}{2}} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \left[\frac{y+y^{-1}}{2} + \frac{y-y^{-1}}{2} \cos\theta \right]^{\frac{-1-\nu}{2}} d\theta = \mathfrak{P}_{\frac{-1-\nu}{2}} \left(\frac{y+y^{-1}}{2} \right) \end{aligned} \tag{2.1.37}$$

[36, p. 184] or, more generally, using covariance,

$$(V^*h)(z) = \mathfrak{P}_{\frac{-1-\nu}{2}}(\cosh d(i,z)), \tag{2.1.38}$$

where d is the hyperbolic distance in Π.

The study of the restriction of the Radon, or dual Radon, transform and their associates to K-invariant functions is very classical: even more so, it is usually a preparation for the more general theory. All this fits within the so-called theory of Gelfand pairs and spherical function theory [18, 8]. In Section 2.3, we shall consider the way these transformations can be restricted to MA-invariant functions.

Consider the part of the Weyl calculus concerned with operators preserving the parity of functions, in other words the one defined from the consideration of even symbols only. It would be perfectly possible, if hardly advisable in general, to define a variant of this calculus in which symbols would be pairs of functions in Π, the images of the "true" symbol under the map Θ in (2.1.1). We here mention this possibility since, in the automorphic case, such a transfer will make it possible to bypass some technical difficulties inherent in the automorphic Weyl calculus, the source of which will be described in Section 3.4. One of our main interests in pseudo-differential analysis lies in the composition formulas: in view of Theorem 1.2.2, all we have to do is transferring under any associate of the dual Radon transformation the operations obtained from the integral kernels $\chi^{\varepsilon_1,\varepsilon_2;\varepsilon}_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s)$. Actually, since we are only dealing with even symbols, one must take $\delta=\delta_1=\delta_2=0$ with the notation from the theorem just referred to, so that, from (1.2.27), only the two cases in which $\varepsilon_1=\varepsilon_2=\varepsilon=0$ or 1 must be considered.

As will be seen presently, when dealing with homogeneous symbols of given degrees of homogeneity, the operator with integral kernel $\chi^{0,0,0}_{i\lambda_1,i\lambda_2;i\lambda}$ (*respectively* $\chi^{1,1,1}_{i\lambda_1,i\lambda_2;i\lambda}$) will appear, up to scalar factors, as the transfer under any associate of the Radon transformation of the operator of pointwise multiplication (*respectively.* the Poisson bracket) on functions on Π. The simplicity of the result should not lead one to believe that a non-computational proof should exist as well: for, when restricted to pairs of (generalized) eigenfunctions of Δ for specific eigenvalues, a bilinear operator as simple as the pointwise product of functions may have a variety of quite complicated disguises. Given $h\in\mathcal{S}_{\text{even}}(\mathbb{R}^2)$, let us not confuse, in what follows, the function $h_{i\lambda}$ (a function on $\mathbb{R}^2\backslash\{0\}$, homogeneous of degree $-1-i\lambda$) and the function $h^\flat_{i\lambda}$ on the line (to be precise, on the projective completion of the line).

Proposition 2.1.3. *Let $\lambda_1,\lambda_2,\lambda$ be real numbers, and let h_1,h_2 be two even functions in $\mathcal{S}(\mathbb{R}^2)$. One has the identity*

$$(TV((V^*T^*(h_1)_{i\lambda_1}).(V^*T^*(h_2)_{i\lambda_2})))^\flat_{i\lambda}(s) = 2^{-\frac{9}{2}}\pi^{-2} \tag{2.1.39}$$
$$\times \frac{\Gamma\left(\frac{1-i(\lambda+\lambda_1+\lambda_2)}{4}\right)\Gamma\left(\frac{1+i(\lambda-\lambda_1+\lambda_2)}{4}\right)\Gamma\left(\frac{1+i(\lambda+\lambda_1-\lambda_2)}{4}\right)\Gamma\left(\frac{1+i(\lambda-\lambda_1-\lambda_2)}{4}\right)}{\Gamma\left(-\frac{i\lambda_1}{2}\right)\Gamma\left(-\frac{i\lambda_2}{2}\right)\Gamma\left(\frac{i\lambda}{2}\right)}$$
$$\times \int_{\mathbb{R}^2} \chi^{0,0,0}_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s)(h_1)^\flat_{\lambda_1}(s_1)(h_2)^\flat_{\lambda_2}(s_2)ds_1ds_2.$$

Proof. As already noted in (1.2.29), one has the estimate $|(h_1)^\flat_{\lambda_1}(s_1)| \leq C(1+s_1^2)^{-\frac{1}{2}}$ and a similar one relative to $(h_2)^\flat_{\lambda_2}$. Using (2.1.15) and (2.1.16), one can write the left-hand side of the identity to be proved as

$$\frac{1}{2}(2\pi)^{-\frac{5}{2}}\frac{\Gamma\left(\frac{1-i\lambda_1}{2}\right)\Gamma\left(\frac{1-i\lambda_2}{2}\right)\Gamma\left(\frac{1+i\lambda}{2}\right)}{\Gamma\left(-\frac{i\lambda_1}{2}\right)\Gamma\left(-\frac{i\lambda_2}{2}\right)\Gamma\left(\frac{i\lambda}{2}\right)}$$
$$\times \int_{\mathbb{R}^2} A_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s)(h_1)^\flat_{\lambda_1}(s_1)(h_2)^\flat_{\lambda_2}(s_2)ds_1ds_2 \tag{2.1.40}$$

with

$$A_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s)$$
$$= \int_\Pi \left(\frac{|z-s_1|^2}{\operatorname{Im} z}\right)^{-\frac{1}{2}+\frac{i\lambda_1}{2}} \left(\frac{|z-s_2|^2}{\operatorname{Im} z}\right)^{-\frac{1}{2}+\frac{i\lambda_2}{2}} \left(\frac{|z-s|^2}{\operatorname{Im} z}\right)^{-\frac{1}{2}-\frac{i\lambda}{2}} dm(z). \tag{2.1.41}$$

Using the identity

$$A_{i\lambda_1,i\lambda_2;i\lambda}\left(\frac{as_1+b}{cs_1+d}, \frac{as_2+b}{cs_2+d}; \frac{as+b}{cs+d}\right)$$
$$= |cs_1+d_1|^{1-i\lambda_1}|cs_2+d_1|^{1-i\lambda_2}|cs+d|^{1+i\lambda}A_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s), \tag{2.1.42}$$

a consequence of

$$\frac{|z-\frac{as+b}{cs+d}|^2}{\operatorname{Im} z} = (cs+d)^{-2}\frac{|g^{-1}.z-s|^2}{\operatorname{Im}(g^{-1}.z)}, \quad g = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), \tag{2.1.43}$$

and noting that if s_1, s_2, s are the images of $0, 1, \infty$ under the fractional-linear transformation associated to the matrix $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$, then

$$\chi^{0,0,0}_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s) = |d|^{1-i\lambda_1}|c+d|^{1-i\lambda_2}|c|^{1+i\lambda}, \tag{2.1.44}$$

one gets

$$A_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s) = I(i\lambda_1,i\lambda_2;i\lambda)\chi^{0,0,0}_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s) \tag{2.1.45}$$

with

$$I(i\lambda_1,i\lambda_2;i\lambda) = \int_\Pi \left(\frac{|z|^2}{\operatorname{Im} z}\right)^{-\frac{1}{2}+\frac{i\lambda_1}{2}} \left(\frac{|z-1|^2}{\operatorname{Im} z}\right)^{-\frac{1}{2}+\frac{i\lambda_2}{2}} (\operatorname{Im} z)^{\frac{1}{2}+\frac{i\lambda}{2}} dm(z), \tag{2.1.46}$$

a convergent integral. The justification of all that precedes is based on (1.2.29) and on the easily proved estimate

$$\int_{\mathbb{R}^3} |(s_1-s_2)(s_2-s)(s-s_1)|^{-\frac{1}{2}}((1+s_1^2)(1+s_2^2)(1+s^2))^{-\frac{1}{2}}ds_1ds_2ds < \infty. \quad (2.1.47)$$

Using (2.1.30) and the Plancherel formula for the dx-integration, we obtain

$$I(i\lambda_1, i\lambda_2; i\lambda) = \frac{8\pi^{1-\frac{i(\lambda_1+\lambda_2)}{2}}}{\Gamma(\frac{1-i\lambda_1}{2})\Gamma(\frac{1-i\lambda_2}{2})} \int_0^\infty y^{-\frac{1}{2}+\frac{i\lambda}{2}}dy \int_0^\infty \sigma^{-\frac{i(\lambda_1+\lambda_2)}{2}} \cos(2\pi\sigma) \times K_{\frac{i\lambda_1}{2}}(2\pi\sigma y)K_{\frac{i\lambda_2}{2}}(2\pi\sigma y)d\sigma, \quad (2.1.48)$$

where the $d\sigma$-integration has to be carried first. Integrating instead with respect to dy first so as to take advantage of [36, p. 101], one would formally obtain

$$I(i\lambda_1, i\lambda_2; i\lambda) = \frac{\pi^{\frac{1}{2}}}{2\Gamma(\frac{1-i\lambda_1}{2})\Gamma(\frac{1-i\lambda_2}{2})\Gamma(\frac{1+i\lambda}{2})} \times \Gamma\left(\frac{1-i(\lambda+\lambda_1+\lambda_2)}{4}\right) \quad (2.1.49)$$
$$\times \Gamma\left(\frac{1+i(\lambda-\lambda_1+\lambda_2)}{4}\right)\Gamma\left(\frac{1+i(\lambda+\lambda_1-\lambda_2)}{4}\right)\Gamma\left(\frac{1+i(\lambda-\lambda_1-\lambda_2)}{4}\right),$$

and the process can be justified if one first inserts under the right-hand side of (2.1.48) the factor $h(\varepsilon\sigma)$ for some $h \in \mathcal{S}(\mathbb{R})$ with $h(0) = 1$, letting at the end ε go to zero. □

Even though we shall not need this result in our main applications in Chapter 4, let us mention the following analogue of Proposition 2.1.3, in which the Poisson bracket of two smooth functions in Π is defined as

$$\{f_1, f_2\} = y^2\left(-\frac{\partial f_1}{\partial y}\frac{\partial f_2}{\partial x} + \frac{\partial f_1}{\partial x}\frac{\partial f_2}{\partial y}\right). \quad (2.1.50)$$

Proposition 2.1.4. *Under the assumptions of Proposition* 2.1.3*, one has*

$$(TV(\{V^*T^*(h_1)_{i\lambda_1}).(V^*T^*(h_2)_{i\lambda_2}\}))^\flat_{i\lambda}(s) = 2^{-\frac{7}{2}}\pi^{-2} \quad (2.1.51)$$
$$\times \frac{\Gamma\left(\frac{3-i(\lambda+\lambda_1+\lambda_2)}{4}\right)\Gamma\left(\frac{3+i(\lambda-\lambda_1+\lambda_2)}{4}\right)\Gamma\left(\frac{3+i(\lambda+\lambda_1-\lambda_2)}{4}\right)\Gamma\left(\frac{3+i(\lambda-\lambda_1-\lambda_2)}{4}\right)}{\Gamma\left(-\frac{i\lambda_1}{2}\right)\Gamma\left(-\frac{i\lambda_2}{2}\right)\Gamma\left(\frac{i\lambda}{2}\right)}$$
$$\times \int_{\mathbb{R}^2} \chi^{1,1,1}_{i\lambda_1,i\lambda_2;i\lambda}(s_1,s_2;s)(h_1)^\flat_{\lambda_1}(s_1)(h_2)^\flat_{\lambda_2}(s_2)ds_1ds_2.$$

The proof of this proposition, fully similar to that of Proposition 2.1.3, can be found if desired in [60, p. 73].

2.2 Back to the totally radial Weyl calculus

In this section, we examine the exact way in which the map Λ in Theorem 1.3.1 differs from an isometry, and we connect the totally radial calculus, with symbols living on Π, to the so-called Berezin calculus [2]. This ought to please people interested in quantization theory, by which we here mean the development of analogous (covariant) pseudo-differential analyses in which symbols are functions on rather general homogeneous spaces, in particular hermitian symmetric spaces. Even so, this is not yet the "good" symbolic calculus of totally radial operators: as will be seen in Chapter 6, calculations of an arithmetic character demand that symbols should live on the plane rather than the half-plane.

Lemma 2.2.1. *For every function $F(p,q,r)$ on the cone*

$$C = \{(p,q,r)\colon p > (q^2+r^2)^{\frac{1}{2}}\},$$

one has, assuming summability, and recalling that $\omega_n = \frac{2\pi^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}$,

$$\begin{aligned} I\colon &= \int_{\mathbb{R}^n\times\mathbb{R}^n} F\left(\frac{|x|^2+|\xi|^2}{2}, \langle x,\xi\rangle, \frac{|x|^2-|\xi|^2}{2}\right) dx d\xi \\ &= \frac{\omega_n\omega_{n-1}}{2}\int_C F(p,q,r)[p^2-q^2-r^2]^{\frac{n-3}{2}} dpdqdr. \end{aligned} \tag{2.2.1}$$

Proof. Set

$$F(p,q,r) = H(p+r, q, p-r) = H(a,b,c), \tag{2.2.2}$$

so that

$$I = \int_{\mathbb{R}^n\times\mathbb{R}^n} H(|x|^2, \langle x,\xi\rangle, |\xi|^2) dx d\xi. \tag{2.2.3}$$

Given x, there is an x-dependent rotation in ξ-space which transforms $\langle x,\xi\rangle$ to $|x|\xi_1$. Hence, with $\xi = (\xi_1, \xi_*)$,

$$\begin{aligned} I &= \int_{\mathbb{R}^n\times\mathbb{R}^n} H(|x|^2, |x|\xi_1, |\xi|^2) dx d\xi \\ &= \omega_{n-1}\int_{\mathbb{R}^n} dx \int_{-\infty}^{\infty} d\xi_1 \int_0^\infty t^{n-2} H(|x|^2, |x|\xi_1, \xi_1^2+t^2) dt \\ &= \omega_n\omega_{n-1}\int_{-\infty}^{\infty} d\xi_1 \int_0^\infty\int_0^\infty s^{n-1}t^{n-2} H(s^2, s\xi_1, \xi_1^2+t^2) ds dt \\ &= \frac{1}{4}\omega_n\omega_{n-1}\int_{a>0,c>0,|b|<\sqrt{ac}} (ac-b^2)^{\frac{n-3}{2}} H(a,b,c) da db dc, \end{aligned} \tag{2.2.4}$$

which leads to the expression indicated. □

Theorem 2.2.2. *Assuming $n \geq 2$, let $f \in L^2(\Pi)$. One has the identity (in which Λ is the map Λ in Theorem* 1.3.1*)*

$$\|\Lambda f\|^2_{L^2(\mathbb{R}^{2n})} = \frac{2^{1-2n}\pi}{(\Gamma(\frac{n}{2}))^2} \|\Gamma\left(\tfrac{n-1}{2} + i\sqrt{\Delta - \tfrac{1}{4}}\right) f\|^2_{L^2(\Pi)}. \tag{2.2.5}$$

Proof. Since $(p^2 - q^2 - r^2)^{\frac{1}{2}} = s$, one obtains after a straightforward computation of the jacobian

$$\left|\frac{D(p,q,r)}{D(s,x,y)}\right| = \left|\frac{D(p,q,p+r)}{D(s,x,y)}\right| = \left|\frac{D(s\frac{1+|z|^2}{2y}, \frac{sx}{y}, \frac{s}{y})}{D(s,x,y)}\right| = \frac{s^2}{y^2} \tag{2.2.6}$$

the expression

$$[p^2 - q^2 - r^2]^{\frac{n-3}{2}} dp dq dr = s^{n-1} ds \frac{dx dy}{y^2} \tag{2.2.7}$$

in terms of the coordinates $(s, z) = (s, x + iy)$ linked to (p, q, r) by (1.3.28).

If f satisfies the first identity (2.1.32), (1.3.20) can be rewritten as

$$(\theta f)(s, z) = s^{-\frac{1}{2}} \int_0^\infty K_{\frac{i\lambda}{2}}(4\pi s) f_\lambda(z) \left(\frac{\pi\lambda}{2} \tanh \frac{\pi\lambda}{2}\right) d\lambda. \tag{2.2.8}$$

Then, using Lemma 2.2.1, (2.2.7) and (2.1.35), one obtains

$$\begin{aligned} &\|\Lambda f\|^2_{L^2(\mathbb{R}^{2n})} \\ &= 4\pi^2 \frac{\omega_n \omega_{n-1}}{2} \int_0^\infty s^{n-2} ds \int_0^\infty |K_{\frac{i\lambda}{2}}(4\pi s)|^2 \|f_\lambda\|^2_{i\lambda} \left(\frac{\pi\lambda}{2} \tanh \frac{\pi\lambda}{2}\right) d\lambda. \end{aligned} \tag{2.2.9}$$

Now, according to [36, p. 101], and using the duplication formula for the Gamma function,

$$\int_0^\infty s^{n-2} \left[K_{\frac{i\lambda}{2}}(4\pi s)\right]^2 ds = 2^{-2n} \pi^{-n+\frac{3}{2}} \frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})} \Gamma\left(\frac{n-1+i\lambda}{2}\right) \Gamma\left(\frac{n-1-i\lambda}{2}\right). \tag{2.2.10}$$

The theorem follows. □

Theorem 2.2.3. *For every $w \in \Pi$, define on $\mathbb{R}^n$ the radial function*

$$\phi_w(x) = \left(2\mathrm{Im}\,(-\frac{1}{w})\right)^{\frac{n}{4}} \exp\left(\frac{i\pi}{\bar{w}}|x|^2\right), \tag{2.2.11}$$

generalizing (1.1.32). *For every function $f \in L^2(\Pi)$, one has*

$$\begin{aligned} (\phi_w | \mathrm{Op}(\Lambda f) \phi_w)_{L^2(\mathbb{R}^n)} &= 2^{-n-\frac{1}{2}} \int_\Pi f(z) |(\phi_w|\phi_z)|^2 dm(z) \\ &= 2^{-\frac{n}{2}-\frac{1}{2}} \int_\Pi f(z)(1 + \cosh d(z,w))^{-\frac{n}{2}} dm(z). \end{aligned} \tag{2.2.12}$$

Proof. Using the covariance property, it is no loss of generality to assume that $w = i$. One has

$$W(\phi_i, \phi_i)(x, \xi) = 2^n \exp(-2\pi(|x|^2 + |\xi|^2)) = 2^n e^{-4\pi p}$$

in terms of the coordinates p, q, r introduced just after (1.3.23). Using Lemma 2.2.1, then (1.3.28) to express p in terms of s, z, one can write

$$(\phi_i|\mathrm{Op}(\Lambda f)\phi_i) = 2^{n-1}\omega_n\omega_{n-1} \int_0^\infty s^{n-1} ds \times \int_\Pi s^{-\frac{1}{2}} \left(K_{i\sqrt{\Delta-\frac{1}{4}}}(4\pi s) f\right)(z) \exp\left(-4\pi s \frac{1+|z|^2}{2\mathrm{Im}\, z}\right) dm(z) : \quad (2.2.13)$$

we set

$$\delta = \cosh d(i, z) = \frac{1+|z|^2}{2\mathrm{Im}\, z}. \quad (2.2.14)$$

To continue the calculation, we must integrate by parts, letting the self-adjoint operator $K_{i\sqrt{\Delta-\frac{1}{4}}}(4\pi s)$ act on the function $z \mapsto \exp(-4\pi s \cosh d(i, z))$ rather than on f. On functions of $\delta = \cosh d(i, z)$, the operator Δ acts as the ordinary differential operator $(1-\delta^2)\frac{d^2}{d\delta^2} - 2\delta\frac{d}{d\delta}$ and, on the interval $(1, \infty)$, Legendre functions provide generalized eigenfunctions, since

$$\left[(1-\delta^2)\frac{d^2}{d\delta^2} - 2\delta\frac{d}{d\delta}\right] \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\delta) = \frac{1+\lambda^2}{4} \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\delta). \quad (2.2.15)$$

Mehler's inversion formula (2.1.32) then gives the integral decomposition

$$e^{-4\pi s\delta} = \int_0^\infty \psi(\lambda) \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\delta) d\lambda \quad (2.2.16)$$

if

$$\psi(\lambda) = \frac{\lambda}{4} \tanh\frac{\pi\lambda}{2} \int_1^\infty e^{-4\pi s\delta} \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\delta) d\delta = 2^{-\frac{3}{2}}\pi^{-1} \frac{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{i\lambda}{2})\Gamma(\frac{-i\lambda}{2})} s^{-\frac{1}{2}} K_{\frac{i\lambda}{2}}(4\pi s), \quad (2.2.17)$$

where we have used on one hand (2.1.33), on the other hand [36, p. 194] to compute the last integral. Then, the image of the function $e^{-4\pi s \cosh d(i,z)}$ under the operator $K_{i\sqrt{\Delta-\frac{1}{4}}}(4\pi s)$ is

$$K_{i\sqrt{\Delta-\frac{1}{4}}}(4\pi s)\left(e^{-4\pi s\delta}\right) = 2^{-\frac{3}{2}}\pi^{-1} \int_0^\infty \frac{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{i\lambda}{2})\Gamma(\frac{2-i\lambda}{2})} s^{-\frac{1}{2}} \left[K_{\frac{i\lambda}{2}}(4\pi s)\right]^2 \mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\delta) d\lambda \quad (2.2.18)$$

and, from (2.2.13),

$$(\phi_i|\mathrm{Op}(\Lambda f)\phi_i) = 2^{n-1}\omega_n\omega_{n-1}2^{-\frac{3}{2}}\pi^{-1}\int_0^\infty s^{n-2}ds$$
$$\times\int_\Pi f(z)dm(z)\int_0^\infty \frac{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{i\lambda}{2})\Gamma(\frac{-i\lambda}{2})}\left[K_{\frac{i\lambda}{2}}(4\pi s)\right]^2\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\delta)d\lambda. \quad (2.2.19)$$

Now [36, p. 101], one has (*cf.* (2.2.10))

$$\int_0^\infty s^{n-2}\left[K_{\frac{i\lambda}{2}}(4\pi s)\right]^2 ds = 2^{-2n}\pi^{\frac{3}{2}-n}\frac{\Gamma(\frac{n-1}{2})}{\Gamma(\frac{n}{2})}\Gamma(\frac{n-1+i\lambda}{2})\Gamma(\frac{n-1-i\lambda}{2}): \quad (2.2.20)$$

with the help of the last two equations, one obtains the equation

$$(\phi_w|\mathrm{Op}(\Lambda f)\phi_w) = \frac{2^{-n-\frac{1}{2}}}{(\Gamma(\frac{n}{2}))^2}\int_\Pi f(z)dm(z) \quad (2.2.21)$$
$$\times\int_0^\infty \frac{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{i\lambda}{2})\Gamma(\frac{-i\lambda}{2})}\Gamma(\frac{n-1+i\lambda}{2})\Gamma(\frac{n-1-i\lambda}{2})\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\cosh d(z,w))d\lambda.$$

We transform now the right-hand side of (2.2.12), still under the assumption that $w = i$, by decomposing the function $z \mapsto (1+\cosh d(i,z))^{-\frac{n}{2}}$ into generalized eigenfunctions of Δ. Again, Mehler's inversion formula gives the answer. Using first the Gamma integral, next the integral already used in (2.2.17), we obtain

$$\int_1^\infty (1+\delta)^{-\frac{n}{2}}\mathfrak{P}_{-\frac{1}{2}+\frac{i\lambda}{2}}(\delta)d\delta = \frac{(4\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_0^\infty s^{\frac{n}{2}}e^{-4\pi s}ds\int_1^\infty e^{-4\pi s\delta}\mathfrak{P}_{\frac{1}{2}+\frac{i\lambda}{2}}(\delta)d\delta$$
$$= \frac{1}{2^{\frac{1}{2}}\pi}\frac{(4\pi)^{\frac{n}{2}}}{\Gamma(\frac{n}{2})}\int_0^\infty s^{\frac{n-3}{2}}e^{-4\pi s}K_{\frac{i\lambda}{2}}(4\pi s)ds$$
$$= 2^{-\frac{n}{2}+1}\frac{\Gamma\left(\frac{n-1+i\lambda}{2}\right)\Gamma\left(\frac{n-1-i\lambda}{2}\right)}{(\Gamma(\frac{n}{2}))^2}: \quad (2.2.22)$$

at the last point, we have used the integral given in [13], p. 98. Then,

$$(1+\delta)^{-\frac{n}{2}} = \int_0^\infty \psi(\lambda)\mathfrak{P}_{\frac{1}{2}+\frac{i\lambda}{2}}(\delta)d\lambda \quad (2.2.23)$$

with

$$\psi(\lambda) = 2^{-\frac{n}{2}}\frac{\Gamma(\frac{1+i\lambda}{2})\Gamma(\frac{1-i\lambda}{2})}{\Gamma(\frac{i\lambda}{2})\Gamma(\frac{-i\lambda}{2})}\frac{\Gamma\left(\frac{n-1+i\lambda}{2}\right)\Gamma\left(\frac{n-1-i\lambda}{2}\right)}{(\Gamma(\frac{n}{2}))^2}. \quad (2.2.24)$$

This proves the identity of the right-hand sides of (2.2.12) and (2.2.21). One also observes, since

$$\phi_{-z^{-1}}(x) = (2\mathrm{Im}\, z)^{\frac{n}{4}}e^{-i\pi\bar{z}|x|^2}, \quad (2.2.25)$$

that

$$|(\phi_{-z^{-1}}|\phi_i)|^2 = 2^n (\operatorname{Im} z)^{\frac{n}{2}} [(\operatorname{Re} z)^2 + (1 + \operatorname{Im} z)^2]^{-\frac{n}{2}}$$
$$= \left(\frac{1 + \cosh d(i,z)}{2}\right)^{-\frac{n}{2}} = |(\phi_z|\phi_i)|^2. \tag{2.2.26}$$

□

Remark 2.2.a. The present short remark will not be used in all that follows, and addresses itself only to readers interested in quantization theory, in particular in the Berezin calculus [2]. The first equation (2.2.12) can be interpreted as the fact that the function $2^{-n-\frac{1}{2}} f$ coincides with the contravariant symbol of the operator (on functions defined on the half-line) $R\mathrm{Op}(\Lambda f)R^{-1}$, with R as defined in (1.3.1): more precisely, since Berezin considered only complex-type realizations of Hilbert spaces with reproducing kernels, one should consider the conjugate of the last operator under the Laplace transformation defined in (1.3.7).

2.3 The dual Radon transform of bihomogeneous distributions

N.B. This section, in which the function $\chi_{\rho,\nu}$, basic in Chapter 4, is analyzed, has no independent interest: we therefore suggest that the reader should be temporarily satisfied with a look at Proposition 2.3.2 and Proposition 2.3.5. Theorem 2.4.1, in the section to follow, will already give some explanation of our interest in the function $\chi_{\rho,\nu}$.

Theorem 1.1.3 has shown the relevance of homogeneous functions, or distributions on $\mathbb{R}^2$, to modular form theory. It is natural to refine the notion by the consideration of bihomogeneous symbols, considering the variables x, ξ separately. In other words, besides the Euler operator $\mathcal{E} = \frac{1}{2i\pi}\left(x\frac{\partial}{\partial x} + \xi\frac{\partial}{\partial \xi} + 1\right)$, we wish to consider the operator $\mathcal{B} = \frac{1}{4i\pi}\left(x\frac{\partial}{\partial x} - \xi\frac{\partial}{\partial \xi}\right)$. Since the two operators commute, one may consider their joint spectral theory. Of course, the operator $\mathcal{B}$ does not commute with the action of $SL(2,\mathbb{R})$, or $SL(2,\mathbb{Z})$, and it will not be possible to consider (in Chapter 4) modular distributions which would be at the same time generalized eigenfunctions of it. But applying the Poincaré summation process, starting from functions on Π built from bihomogeneous symbols, will lead to a class of automorphic functions with interesting properties.

Here, we still concentrate on the non-arithmetic situation. Note the equation

$$(\mathcal{B}h)(x,\xi) = \frac{1}{2i\pi}\frac{d}{dr}\bigg|_{r=0} h(e^{\frac{r}{2}}x, e^{-\frac{r}{2}}\xi), \tag{2.3.1}$$

which indicates that $\mathcal{B}$ is the infinitesimal operator of the action on symbols of the one-parameter group $A \subset SL(2,\mathbb{R})$ recalled in (2.1.6). In view of the covariance property (1.1.20), the operator $\mathcal{B}$ has an interpretation in the Weyl calculus, expressed by the commutation identity, in which h is an arbitrary symbol in $\mathcal{S}'(\mathbb{R}^2)$,

$$\mathrm{Op}(\mathcal{B}h) = \frac{1}{2}\left[\frac{QP+PQ}{2}, \mathrm{Op}(h)\right], \tag{2.3.2}$$

involving the basic infinitesimal operators Q and P of Heisenberg's representation. By the way, the operator $\mathcal{E}$, too, has an interpretation in the symbolic calculus (not linked to covariance), to wit the general identity

$$\mathrm{Op}(\mathcal{E}h) = P\mathrm{Op}(h)Q - Q\mathrm{Op}(h)P. \tag{2.3.3}$$

Both formulas are easily obtained from (1.2.6).

In view of arithmetic applications, we consider only even functions of x, ξ in the plane, since the dual Radon transformation kills odd functions. As done in [61, Section 18], the consideration of odd functions of x, ξ is necessary if, besides (Maass) non-holomorphic modular forms of usual type, one interests oneself in so-called Maass forms of weight one [4, Section 2.1]. It is for simplicity that we shall consider here only functions separately even with respect to x and ξ. This will force us to restrict our interest, in Chapter 4, to non-holomorphic modular forms of even type under the symmetry $z \mapsto -\bar{z}$: this is not necessary, but it is sufficient for our main purpose there.

Then, joint generalized eigenfunctions of the pair $(\mathcal{E}, \mathcal{B})$, to wit separately even symbols satisfying the pair of equations

$$2i\pi\mathcal{E}h = \nu h, \quad 4i\pi\mathcal{B}h = (\rho - 1)h, \tag{2.3.4}$$

are multiples of the function

$$\mathrm{hom}_{\rho,\nu}(x, \xi) = |x|^{\frac{\rho+\nu-2}{2}} |\xi|^{\frac{\nu-\rho}{2}}. \tag{2.3.5}$$

Theorem 1.2.2, more precisely (1.2.68), has shown how such symbols, with ν on the line $\mathrm{Re}\,\nu = 0$ and ρ on the line $\mathrm{Re}\,\rho = 1$ occur from the decomposition into homogeneous components of a sharp product such as $|x|^{-1-i\lambda_1} \# |\xi|^{-1-i\lambda_2}$.

Our task in the present section is the computation and analysis of the function on Π obtained from the function (2.3.5) by a dual Radon transformation. Transferring under such a transformation the operator $\mathcal{B}$, one will obtain an operator commuting with Δ. Starting from (2.1.8) and using the equation

$$\left(x\frac{\partial}{\partial x} - \xi\frac{\partial}{\partial \xi}\right) \frac{x^2(i+b)}{x\xi(i+b)+1} = 2\frac{x^2(i+b)}{x\xi(i+b)+1}, \tag{2.3.6}$$

one obtains the general identity

$$\mathcal{B}Vf = V\left(\frac{1}{2i\pi}\left(z\frac{\partial}{\partial z} + \bar{z}\frac{\partial}{\partial \bar{z}}\right) f\right). \tag{2.3.7}$$

Again, on Π, the Euler operator $z\frac{\partial}{\partial z} + \overline{z}\frac{\partial}{\partial \overline{z}}$ does not commute with the action of $SL(2,\mathbb{Z})$ by fractional-linear transformations, and this operator does not preserve automorphic functions: in Chapter 4, something will remain from it, however, in an automorphic situation.

From (2.1.10), we obtain

$$(V^*\mathrm{hom}_{\rho,\nu})(x+iy) = \frac{1}{2\pi}\int_0^{2\pi} |y^{-\frac{1}{2}}\sin\frac{\theta}{2}|^{\frac{\nu-1}{2}} |y^{\frac{1}{2}}\cos\frac{\theta}{2} - xy^{-\frac{1}{2}}\sin\frac{\theta}{2}|^{\frac{\rho+\nu-2}{2}} d\theta$$

$$= y^{\frac{\rho-1}{2}} \times \frac{1}{2\pi}\int_0^{2\pi} |\sin\frac{\theta}{2}|^{\frac{\nu-1}{2}} |\cos\frac{\theta}{2} - \frac{x}{y}\sin\frac{\theta}{2}|^{\frac{\rho+\nu-2}{2}} d\theta. \tag{2.3.8}$$

We must thus compute the integral obtained, a function of $\frac{x}{y}$ only. The simplest case is that for which $\rho = 1$, which corresponds to MA-invariant symbols. As will be seen, while simpler, it is often a singular case rather than a special case only: this will be even more apparent in Chapter 4. For the time being, the computation of the integral (2.3.8) is quite simple when $\rho = 1$. Indeed, setting $t = \frac{x}{y}$, we first write it as

$$(V^*\mathrm{hom}_{1,\nu})(x+iy) = \frac{1}{2\pi} 2^{\frac{1-\nu}{2}} \int_0^{2\pi} |\sin\theta - t(1-\cos\theta)|^{\frac{\nu-1}{2}} d\theta : \tag{2.3.9}$$

after a t-dependent translation in the θ-variable, we can change $\sin\theta - t(1-\cos\theta)$ to $\sqrt{1+t^2}\cos\theta$, so that

$$(V^*\mathrm{hom}_{1,\nu})(x+iy) = 2^{\frac{-1-\nu}{2}} \pi^{-1} \int_0^{2\pi} |t - \sqrt{t^2+1}\cos\theta|^{\frac{\nu-1}{2}} d\theta. \tag{2.3.10}$$

Starting from the classical integral representation [36, p. 184] of Legendre functions

$$\mathfrak{P}_{\frac{\nu-1}{2}}(w) = \frac{1}{2\pi}\int_0^{2\pi} [w + \sqrt{w^2-1}\cos\theta]^{\frac{\nu-1}{2}} d\theta \tag{2.3.11}$$

and using the relation

$$e^{-\frac{i\pi(1-\nu)}{4}} + e^{\frac{i\pi(1-\nu)}{4}} = \frac{2\pi}{\Gamma(\frac{1+\nu}{4})\Gamma(\frac{3-\nu}{4})}, \tag{2.3.12}$$

one obtains, setting $t = \frac{x}{y}$ and assuming that $\mathrm{Re}\,\nu > -1$ for convergence,

$$(V^*\mathrm{hom}_{1,\nu})(x+iy) = 2^{\frac{-1-\nu}{2}} \pi^{-1} \Gamma(\frac{1+\nu}{4})\Gamma(\frac{3-\nu}{4}) \left[\mathfrak{P}_{\frac{\nu-1}{2}}(it) + \mathfrak{P}_{\frac{\nu-1}{2}}(-it)\right] : \tag{2.3.13}$$

this is an analytic function of t on the whole real line, since one has [36, p. 153]

$$\mathfrak{P}_{\frac{\nu-1}{2}}(-it) = {}_2F_1\left(\frac{1-\nu}{2}, \frac{1+\nu}{2}; 1; \frac{1+it}{2}\right), \tag{2.3.14}$$

and the hypergeometric function is single-valued when a cut along the real line, from 1 to ∞, has been made in the plane.

We shall spend more time on the general case, in which ρ is arbitrary. From the second equation (2.3.4), the function $V^*\mathrm{hom}_{\rho,\nu}$ on Π must satisfy the transformation rule $V^*\mathrm{hom}_{\rho,\nu}(az) = a^{\frac{\rho-1}{2}} V^*\mathrm{hom}_{\rho,\nu}(z)$ for $a > 0$. From Theorem 2.1.2 and the fact that the operator T there commutes with $\mathcal{E}$, it must also satisfy the equation $\left(\Delta - \frac{1-\nu^2}{4}\right) V^*\mathrm{hom}_{\rho,\nu} = 0$; finally, it must be invariant under the map $z \mapsto -\bar{z}$. One must thus have

$$V^*\mathrm{hom}_{\rho,\nu}(z) = (\mathrm{Im}\, z)^{\frac{\rho-1}{2}} \chi\left(\frac{\mathrm{Re}\, z}{\mathrm{Im}\, z}\right) \tag{2.3.15}$$

for some even function $\chi = \chi(t)$ on the real line, chosen so that the right-hand side of this equation, as a function of z, should lie in the nullspace of $\Delta - \frac{1-\nu^2}{4}$. Temporarily forgetting the parity condition, it is a straightforward matter to verify that this is the case if and only if the function χ satisfies the ordinary differential equation

$$(1+t^2)\chi''(t) + (3-\rho)t\chi'(t) + \left[\frac{1-\nu^2}{4} + \frac{(\rho-1)(\rho-3)}{4}\right]\chi(t) = 0. \tag{2.3.16}$$

We first solve this equation in each of the intervals $]-\infty, 0[$ and $]0, \infty[$. The WKB method shows that, as $t \to \pm\infty$, $\chi(t)$ must be equivalent to a constant times $|t|^{\frac{\mu+\rho-2}{2}}$, with $\mu = \pm\nu$: more precisely, it is so unless the real part of ν is zero. It is then natural to set

$$\chi(t) = \left(\frac{-1-it}{2}\right)_+^{\frac{\rho+\nu-2}{2}} \psi(t), \tag{2.3.17}$$

where we now make our convention regarding powers of complex numbers with non-integral exponents explicit: we shall denote as z^α the complex power of a number z with $\mathrm{Im}\, z > 0$, when the argument is taken in $]0, \pi[$, and as z_+^α the corresponding complex power of z with $z \notin]-\infty, 0]$ when the argument is taken in $]-\pi, \pi[$. Then,

$$(-iz)_+^\alpha = e^{-\frac{i\pi\alpha}{2}} z^\alpha \quad \text{if } \mathrm{Im}\, z > 0. \tag{2.3.18}$$

Unless otherwise stated, the cut made to make the hypergeometric function a single-valued function will always be the interval $[1, \infty[$.

Lemma 2.3.1. *Given $\rho, \nu \in \mathbb{C}$ with $\nu \notin \mathbb{Z}, \rho + \nu \notin 2\mathbb{Z}$, the function*

$$\chi(t) = \left(\frac{-1-it}{2}\right)_+^{\frac{\rho+\nu-2}{2}} {}_2F_1\left(\frac{1-\nu}{2}, \frac{2-\rho-\nu}{2} : 1-\nu; \frac{2}{1+it}\right) \tag{2.3.19}$$

satisfies the equation (2.3.16) *in* $]-\infty, 0[\cup]0, \infty[$.

Proof. Set $\psi(t) = F(s)$ with $s = \frac{2}{1+it}$. The computations which follow are absolutely tedious but straightforward. If χ and ψ are linked by (2.3.17), one has

$$\begin{aligned}\frac{\chi'(t)}{\chi(t)} &= \frac{\psi'(t)}{\psi(t)} + \frac{i(\rho+\nu-2)}{4}s,\\ \frac{\chi''(t)}{\chi(t)} &= \frac{\psi''(t)}{\psi(t)} + \frac{i(\rho+\nu-2)}{2}s\frac{\psi'(t)}{\psi(t)} - \frac{(2-\rho-\nu)(4-\rho-\nu)}{16}s^2. \end{aligned} \tag{2.3.20}$$

Then, (2.3.16) reads

$$\begin{aligned}&(1+t^2)\psi''(t) + \left[\frac{i(\rho+\nu-2)}{2}s(1+t^2) + (3-\rho)t\right]\psi'(t)\\ &+\Big[-\frac{(2-\rho-\nu)(4-\rho-\nu)}{16}s^2(1+t^2)\\ &\quad + \frac{i(3-\rho)(\rho+\nu-2)}{4}st + \frac{1-\nu^2}{4} + \frac{(\rho-1)(\rho-3)}{4}\Big]\psi(t) = 0. \end{aligned} \tag{2.3.21}$$

Now, one has

$$\psi'(t) = -\frac{i}{2}s^2F'(s), \quad \psi''(t) = -\frac{s^3}{2}F'(s) - \frac{s^4}{4}F''(s). \tag{2.3.22}$$

Also,

$$it = \frac{2}{s} - 1, \quad 1+t^2 = \frac{4(s-1)}{s^2}, \tag{2.3.23}$$

and one obtains

$$\begin{aligned}&4(1-s)\left[\frac{s}{2}F' + \frac{s^2}{4}F''\right] + \left[\frac{i(\rho+\nu-2)}{2}\frac{4(s-1)}{s} - i(3-\rho)(\frac{2}{s}-1)\right]\left(-\frac{i}{2}s^2F'\right)\\ &+\Big[-\frac{(2-\rho-\nu)(4-\rho-\nu)}{4}(s-1) + (3-\rho)\frac{\rho+\nu-2}{4}s(\frac{2}{s}-1)\\ &\qquad + \frac{1-\nu^2}{4} + \frac{(\rho-1)(\rho-3)}{4}\Big]F = 0. \end{aligned} \tag{2.3.24}$$

The coefficient of F here reduces to $\frac{(\rho+\nu-2)(1-\nu)}{4}s$. The coefficient of F'' is $s^2(1-s)$, and the coefficient of F' is

$$(\rho+\nu-2)s(s-1) + \frac{\rho-3}{2}(2s-s^2) - 2s(s-1) = (\nu + \frac{\rho-5}{2})s^2 + (1-\nu)s. \tag{2.3.25}$$

The equation for F equivalent to (2.3.21), hence to (2.3.16), is, after we have divided everything by s,

$$s(1-s)F''(s) + [1-\nu - \frac{5-\rho-2\nu}{2}s]F'(s) + \frac{(\rho+\nu-2)(1-\nu)}{4}F(s) = 0: \tag{2.3.26}$$

a solution of it is the function ${}_2F_1\left(\frac{1-\nu}{2}, \frac{2-\rho-\nu}{2}; 1-\nu; s\right)$. This proves the lemma. □

It is useful to make the way χ transforms under the symmetry $t \mapsto -t$ explicit. From [36, p. 47], one obtains if $\mathrm{Re}\, z \neq 0$, with some care about determinations of power functions, the general identity

$$ {}_2F_1(a,b;c;z) = e^{-i\pi b \mathrm{sign}(\mathrm{Im}\, z)} (-z^{-1})_+^b \left(\frac{1-z}{z}\right)_+^{-b} {}_2F_1\left(c-a,b;c;\frac{z}{z-1}\right): \tag{2.3.27} $$

if $z = \frac{2}{1+it}$ with $t \in \mathbb{R}, t \neq 0$, the signs of $\mathrm{Im}\, z$ and of t are the negative of each other, and one has $\frac{z}{z-1} = \frac{2}{1-it}$ so that, starting from the hypergeometric function occurring in the definition of χ, one must read the product of power functions on the right-hand side of (2.3.27) as

$$ \left(\frac{-1-it}{2}\right)_+^{\frac{2-\rho-\nu}{2}} \left(\frac{-1+it}{2}\right)_+^{\frac{-2+\rho+\nu}{2}}. \tag{2.3.28} $$

It follows that

$$ \chi(t) = e^{\frac{i\pi(2-\nu-\rho)}{2}} \chi(-t) \quad \text{if } t > 0. \tag{2.3.29} $$

Remark 2.3.a. In the next proposition, we define the function $\chi_{\rho,\nu}$ as a certain multiple of the function χ in (2.3.19). The normalization is chosen so that one should have simply

$$ \mathfrak{P}_{\frac{\nu-1}{2}}(-it) = \chi_{1,\nu}(t) + \chi_{1,-\nu}(t) \tag{2.3.30} $$

and, more important, that the quantities denoted as $C(\rho,\nu)$ and $I(\rho,\nu)$ in what follows should be odd functions of ν.

Proposition 2.3.2. *Assume that $\nu \notin \mathbb{Z}$ and $\rho \pm \nu \notin 2\mathbb{Z}$, and set*

$$ \chi_{\rho,\nu}(t) = 2^{\nu-1}\pi^{-\frac{1}{2}} \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{2-\rho+\nu}{2})} \times \left(\frac{-1-it}{2}\right)_+^{\frac{\rho+\nu-2}{2}} {}_2F_1\left(\frac{1-\nu}{2}, \frac{2-\rho-\nu}{2} : 1-\nu; \frac{2}{1+it}\right). \tag{2.3.31} $$

This function is analytic in $\mathbb{R}\backslash\{0\}$ and one has for some constant $C > 0$ the inequality

$$ |\chi_{\rho,\nu}(t)| \leq C(1+|t|)^{\frac{\mathrm{Re}\,(\rho+\nu)-2}{2}}, \quad t \neq 0. \tag{2.3.32} $$

It extends as a C^∞ function to each of the two closed intervals $]-\infty, 0]$ and $[0,\infty[$. The negative of the jump at 0 of the first-order derivative is

$$ C(\rho,\nu) = 2^{2-\rho}\pi^{\frac{1}{2}} \frac{\Gamma(\frac{\nu}{2})\Gamma(\frac{2-\nu}{2})}{\Gamma(\frac{2-\rho+\nu}{2})\Gamma(\frac{2-\rho-\nu}{2})\Gamma(\frac{\rho+\nu}{4})\Gamma(\frac{\rho-\nu}{4})}. \tag{2.3.33} $$

For $\mathrm{Re}\,(\rho+\nu) < 0$, one has

$$ I(\rho,\nu) := \int_{-\infty}^{\infty} \chi_{\rho,\nu}(t)dt = \frac{4C(\rho,\nu)}{\nu^2-\rho^2}: \tag{2.3.34} $$

we still denote as $I(\rho,\nu)$ the analytic continuation of this function.

Proof. Let us temporarily denote as $\chi^o_{\rho,\nu}$ the function in (2.3.19), so as not to have to carry the extra coefficient in the first line of (2.3.31) all the time. Similarly, we denote as $C^o(\rho,\nu)$ and $I^o(\rho,\nu)$ the quantities defined in the same way as $C(\rho,\nu)$ and $I(\rho,\nu)$, only with $\chi_{\rho,\nu}$ replaced by $\chi^o_{\rho,\nu}$.

We first consider the case when $\rho-1\notin 2\mathbb{Z}$. We need to analyze the function $\chi_{\rho,\nu}(t)$ as $t\to 0^+$ or 0^-, so that the argument $\frac{2}{1+it}$ of the hypergeometric function goes to 2: to avoid arguments close to the half-line $[1,\infty[$, we use [36, p. 48],

$$(-z)_+^b\,{}_2F_1(a,b;c;z)=\frac{\Gamma(c)\Gamma(b-a)}{\Gamma(b)\Gamma(c-a)}(-z)_+^{b-a}\,{}_2F_1(a,a-c+1;a-b+1;\frac{1}{z})\\ +\frac{\Gamma(c)\Gamma(a-b)}{\Gamma(a)\Gamma(c-b)}\,{}_2F_1(b,b-c+1;b-a+1;\frac{1}{z}). \qquad (2.3.35)$$

This equation, applied with $z=\frac{2}{1+it}$, shows, since in our case $b-a=\frac{1-\rho}{2}$ is assumed to lie outside $\mathbb{Z}$, that the function $\chi_{\rho,\nu}$, while continuous on each of the two intervals $]-\infty,0]$ and $[0,\infty[$, has a discontinuity at 0. It is an easy, but unnecessary matter, to compute the jump there of this function: actually, we shall kill this discontinuity later by considering only the even part of $\chi_{\rho,\nu}$.

It is clear, since the cut along $[1,\infty[$ made to define the hypergeometric function could be moved slightly, that, on each of the two closed intervals under consideration, the function $\chi_{\rho,\nu}$ is actually C^∞. We need to compute the jump of its first derivative at 0. From [36, p. 41], we pick the relations, for $z\notin[0,\infty[$,

$$z^2\frac{d}{dz}\left[(-z)_+^{\frac{2-\rho-\nu}{2}}\,{}_2F_1\left(\frac{1-\nu}{2},\frac{2-\rho-\nu}{2};1-\nu;z\right)\right]\\ =\frac{\rho+\nu-2}{2}(-z)_+^{\frac{4-\rho-\nu}{2}}\,{}_2F_1\left(\frac{1-\nu}{2},\frac{4-\rho-\nu}{2};1-\nu;z\right) \qquad (2.3.36)$$

and

$$z^2\frac{d}{dz}\left[(-z)_+^{\frac{-\rho-\nu}{2}}\,{}_2F_1\left(\frac{1-\nu}{2},\frac{-\rho-\nu}{2};1-\nu;z\right)\right]\\ =\frac{\rho+\nu}{2}(-z)_+^{\frac{2-\rho-\nu}{2}}\,{}_2F_1\left(\frac{1-\nu}{2},\frac{2-\rho-\nu}{2};1-\nu;z\right). \qquad (2.3.37)$$

With $z=\frac{2}{1+it}$, so that $\frac{dz}{dt}=-\frac{i}{2}z^2$, one obtains from these equations the relations

$$\frac{d}{dt}\chi^o_{\rho,\nu}=\frac{i(2-\rho-\nu)}{4}\chi^o_{\rho-2,\nu},\qquad \frac{d}{dt}\chi^o_{\rho+2,\nu}=-\frac{i(\rho+\nu)}{4}\chi^o_{\rho,\nu}(t). \qquad (2.3.38)$$

We then apply the general identity (2.3.35) to the new hypergeometric function. When $z=\frac{2}{1+it}$, only the first term on the right-hand side (the one accompanied by a power of $-z$) has discontinuities at $t=0$: one has

$$\left(\frac{-1-i0^+}{2}\right)_+^{a-b} - \left(\frac{-1-i0^-}{2}\right)_+^{a-b} = 2^{b-a}[e^{-i\pi(a-b)} - e^{i\pi(a-b)}]$$
$$= 2^{1+b-a} \frac{i\pi}{\Gamma(b-a)\Gamma(1+a-b)}. \qquad (2.3.39)$$

It follows on one hand that

$$C^o(\rho,\nu) = \frac{i(\rho+\nu-2)}{4}[\chi^o_{\rho-2,\nu}(0^+) - \chi^o_{\rho-2,\nu}(0^-)] = \frac{2-\rho-\nu}{4} 2^{\frac{5-\rho}{2}}$$
$$\times \frac{\pi}{\Gamma(\frac{3-\rho}{2})\Gamma(\frac{\rho-1}{2})} \frac{\Gamma(1-\nu)\Gamma(\frac{3-\rho}{2})}{\Gamma(\frac{4-\rho-\nu}{2})\Gamma(\frac{1-\nu}{2})} {}_2F_1\left(\frac{1-\nu}{2}, \frac{1+\nu}{2}; \frac{\rho-1}{2}; \frac{1}{2}\right), \qquad (2.3.40)$$

on the other hand that

$$I^o(\rho,\nu) = -\frac{4i}{\rho+\nu}[\chi^o_{\rho+2,\nu}(0^+) - \chi^o_{\rho+2,\nu}(0^-)] \qquad (2.3.41)$$
$$= \frac{4}{\rho+\nu} 2^{\frac{1-\rho}{2}} \frac{\pi}{\Gamma(\frac{-1-\rho}{2})\Gamma(\frac{3+\rho}{2})} \frac{\Gamma(1-\nu)\Gamma(\frac{-1-\rho}{2})}{\Gamma(\frac{-\rho-\nu}{2})\Gamma(\frac{1-\nu}{2})} {}_2F_1\left(\frac{1-\nu}{2}, \frac{1+\nu}{2}; \frac{\rho+3}{2}; \frac{1}{2}\right).$$

Now, one has [36, p. 41]

$${}_2F_1\left(\frac{1-\nu}{2}, \frac{1+\nu}{2}; \gamma; \frac{1}{2}\right) = 2^{1-\gamma}\pi^{\frac{1}{2}} \frac{\Gamma(\gamma)}{\Gamma(\frac{1+2\gamma-\nu}{4})\Gamma(\frac{1+2\gamma+\nu}{4})}. \qquad (2.3.42)$$

Obtaining the ratio $\frac{I(\rho,\nu)}{C(\rho,\nu)} = \frac{I^o(\rho,\nu)}{C^o(\rho,\nu)}$ is just a matter of applying the last three formulas, and simplifying a few factors by means of the functional equation of the function Gamma. To obtain $C(\rho,\nu)$, we must also apply the duplication formula, which leads to the equation

$$C^o(\rho,\nu) = 2^{3-\rho-\nu}\pi \frac{\Gamma(\frac{2-\nu}{2})}{\Gamma(\frac{2-\rho-\nu}{2})\Gamma(\frac{\rho+\nu}{4})\Gamma(\frac{\rho-\nu}{4})}, \qquad (2.3.43)$$

and finally to (2.3.33).

This completes the proof of Proposition 2.3.2 under the extra assumption that $\rho - 1 \notin 2\mathbb{Z}$. The general case follows by a continuity argument: however, since the case when $\rho = 1$ will be very important in the sequel, let us just indicate the differences in a direct proof in this case. Equation (2.3.35) does not apply any more: instead, one has

$$\chi_{1,\nu}(t) = \frac{\Gamma(\nu)\Gamma(1-\nu)}{(\Gamma(\frac{1+\nu}{2})\Gamma(\frac{1-\nu}{2}))^2} \sum_{n\geq 0} \frac{1}{(n!)^2} \frac{\Gamma(\frac{1-\nu}{2}+n)}{\Gamma(\frac{1-\nu}{2})} \frac{\Gamma(\frac{1+\nu}{2}+n)}{\Gamma(\frac{1+\nu}{2})} \qquad (2.3.44)$$
$$\times \left(\frac{1+it}{2}\right)^n \left[\log\left(-\frac{2}{1+it}\right) + 2\frac{\Gamma'(n+1)}{\Gamma(n+1)} - \frac{\Gamma'(\frac{1-\nu}{2}+n)}{\Gamma(\frac{1-\nu}{2}+n)} - \frac{\Gamma'(\frac{1-\nu}{2}-n)}{\Gamma(\frac{1-\nu}{2}-n)}\right];$$

it is understood, there, that the argument $-\frac{2}{1+it}$ of the logarithm is to be taken in the interval $]-\pi,\pi[$. Then, one can write

$$\begin{aligned}
-C(1,\nu) \colon &= \left(\frac{d}{dt}\chi_{1,\nu}\right)(0^+) - \left(\frac{d}{dt}\chi_{1,\nu}\right)(0^-) \\
&= 2i\pi \frac{\Gamma(\nu)\Gamma(1-\nu)}{(\Gamma(\frac{1+\nu}{2})\Gamma(\frac{1-\nu}{2}))^2} \sum_{n\geq 1} \frac{1}{(n)^2} \frac{\Gamma(\frac{1-\nu}{2}+n)}{\Gamma(\frac{1-\nu}{2})} \frac{\Gamma(\frac{1+\nu}{2}+n)}{\Gamma(\frac{1+\nu}{2})} \frac{in}{2^n} \\
&= -2\pi \frac{\Gamma(\nu)\Gamma(1-\nu)}{(\Gamma(\frac{1+\nu}{2})\Gamma(\frac{1-\nu}{2}))^3} \sum_{n\geq 1} \frac{\Gamma(\frac{1-\nu}{2}+n)\Gamma(\frac{1+\nu}{2}+n)}{\Gamma(n)} \frac{2^{-n}}{n!} \\
&= -\pi \frac{1-\nu^2}{4} \frac{\Gamma(\nu)\Gamma(1-\nu)}{(\Gamma(\frac{1+\nu}{2})\Gamma(\frac{1-\nu}{2}))^2} {}_2F_1\left(\frac{3-\nu}{2}, \frac{3+\nu}{2}; 2; \frac{1}{2}\right). \qquad (2.3.45)
\end{aligned}$$

Again, the special value of the hypergeometric function is to be found in [36, p. 40]: it is $\frac{\pi^{\frac{1}{2}}}{\Gamma(\frac{5+\nu}{4})\Gamma(\frac{5-\nu}{4})}$: using this, one obtains the case $\rho = 1$ of (2.3.33).

This was the only place where a special argument was needed when $\rho = 1$. This case will be important in Chapter 4, where its singularity will originate from the fact that the Eisenstein series E_s is undefined for $s = 1$. □

We can now make $V^*\mathrm{hom}_{\rho,\nu}$ explicit: in particular, in view of (2.3.13), it will confirm (2.3.30).

Proposition 2.3.3. *Under the assumptions of Proposition* 2.3.2, *to be completed by* $\mathrm{Re}\,\nu > \max(\mathrm{Re}\,\rho - 2, -\mathrm{Re}\,\rho)$, *one has*

$$\begin{aligned}
(V^*hom_{\rho,\nu})(z) &= (\mathrm{Im}\,z)^{\frac{\rho-1}{2}} \qquad (2.3.46) \\
\times\, 2^{\frac{\rho-\nu}{2}} \pi^{-1} &\frac{\Gamma(\frac{2-\rho+\nu}{2})\Gamma(\frac{\rho+\nu}{4})\Gamma(\frac{4-\rho-\nu}{4})}{\Gamma(\frac{\nu+1}{2})} \left[\chi^{\mathrm{even}}_{\rho,\nu}\left(\frac{\mathrm{Re}\,z}{\mathrm{Im}\,z}\right) + \chi^{\mathrm{even}}_{\rho,-\nu}\left(\frac{\mathrm{Re}\,z}{\mathrm{Im}\,z}\right)\right],
\end{aligned}$$

with

$$\chi^{\mathrm{even}}_{\rho,\nu}(t) = \frac{1}{2}\left[\chi_{\rho,\nu}(t) + \chi_{\rho,\nu}(-t)\right]. \qquad (2.3.47)$$

Proof. The proof of Proposition 2.3.2 shows that the functions $\chi^{\mathrm{even}}_{\rho,\pm\nu}$ are continuous on the real line, even at 0: they have a discontinuity of the first-order derivative there, expressed by the pair of equations

$$\begin{aligned}
\left(\chi^{\mathrm{even}}_{\rho,\nu}\right)'(0^+) - \left(\chi^{\mathrm{even}}_{\rho,\nu}\right)'(0^-) &= -C(\rho,\nu), \\
\left(\chi^{\mathrm{even}}_{\rho,-\nu}\right)'(0^+) - \left(\chi^{\mathrm{even}}_{\rho,-\nu}\right)'(0^-) &= -C(\rho,-\nu) : \qquad (2.3.48)
\end{aligned}$$

since the coefficient $C(\rho,\nu)$ is an odd function of ν, the sum $\chi^{\mathrm{even}}_{\rho,\nu} + \chi^{\mathrm{even}}_{\rho,-\nu}$ is a C^1 function on the line, actually a C^∞ function in view of the differential equation it satisfies on each of the two closed intervals $]-\infty,0]$ and $[0,\infty[$. The function

$y^{\frac{1-\rho}{2}}(V^*\mathrm{hom}_{\rho,\nu})(x+iy)$ must coincide with a multiple of this function. The coefficient is obtained by considering an equivalent as $t = \frac{x}{y} \to \infty$: to obtain this equivalent, we further assume, which is not a loss of generality because of the possibility of analytic continuation of the formula obtained, that $\mathrm{Re}\,\nu > 0$. In this case, it is immediate, from (2.3.8), that

$$\begin{aligned} y^{\frac{1-\rho}{2}}(V^*\mathrm{hom}_{\rho,\nu})(x+iy) &\sim \left|\frac{x}{y}\right|^{\frac{\rho+\nu-2}{2}} \times \frac{1}{2\pi}\int_0^{2\pi} \left|\sin\frac{\theta}{2}\right|^{\nu-1} d\theta \\ &= \pi^{-\frac{1}{2}} \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{\nu+1}{2})} \left|\frac{x}{y}\right|^{\frac{\rho+\nu-2}{2}}. \end{aligned} \tag{2.3.49}$$

On the other hand, (2.3.31), (2.3.47) and the equation

$$e^{\frac{i\pi}{4}(\rho+\nu-2)} + e^{-\frac{i\pi}{4}(\rho+\nu-2)} = \frac{2\pi}{\Gamma(\frac{\rho+\nu}{4})\Gamma(\frac{4-\rho-\nu}{4})} \tag{2.3.50}$$

yield the equivalent, as $|t| \to \infty$,

$$\chi^{\mathrm{even}}_{\rho,\nu}(t) \sim 2^{\frac{\nu-\rho}{2}}\pi^{\frac{1}{2}} \frac{\Gamma(\frac{\nu}{2})}{\Gamma(\frac{2-\rho+\nu}{2})\Gamma(\frac{\rho+\nu}{4})\Gamma(\frac{4-\rho-\nu}{4})} |t|^{\frac{\rho+\nu-2}{2}}. \tag{2.3.51}$$

The proposition follows. □

We need another lemma.

Lemma 2.3.4. *Under the assumptions that $\nu \notin \mathbb{Z}$, $\rho - 1 \notin 2\mathbb{Z}$ and $\rho \pm \nu \notin 2\mathbb{Z}$, one has*

$$(1+t^2)^{\frac{1-\rho}{2}}\chi_{\rho,\nu}(t) = \frac{\Gamma(\frac{2+\rho-\nu}{4})\Gamma(\frac{2+\rho+\nu}{4})}{\Gamma(\frac{4-\rho-\nu}{4})\Gamma(\frac{4-\rho+\nu}{4})}\chi_{2-\rho,\nu}(t). \tag{2.3.52}$$

Proof. We start from [36, p. 47], writing when $\mathrm{Re}\,z \neq 0$ the identity

$$\begin{aligned} {}_2F_1\left(\frac{1-\nu}{2}, \frac{\rho-\nu}{2}; 1-\nu; z\right) &= \exp\left(i\pi\frac{1-\rho}{2}\mathrm{sign}(\mathrm{Im}\,z)\right) \\ &\times (-z)_+^{\frac{1-\rho}{2}} \left(\frac{1-z}{z}\right)_+^{\frac{1-\rho}{2}} {}_2F_1\left(\frac{1-\nu}{2}, \frac{2-\rho-\nu}{2}; 1-\nu; z\right). \end{aligned} \tag{2.3.53}$$

With $z = \frac{2}{1+it}$, one has $(-z)^{-1} = \frac{-1-it}{2}$, $\frac{1-z}{z} = \frac{-1+it}{2}$, and one must read the product of power functions on the right-hand side of (2.3.53) as

$$e^{i\pi\frac{\rho-1}{2}\mathrm{sign}\,t}\left(\frac{-1-it}{2}\right)_+^{\frac{\rho-1}{2}}\left(\frac{-1+it}{2}\right)_+^{\frac{1-\rho}{2}}. \tag{2.3.54}$$

One can then write

$$\begin{aligned}\chi^o_{2-\rho,\nu}(t) &= \left(\frac{-1-it}{2}\right)_+^{\frac{\nu-\rho}{2}} {}_2F_1\left(\frac{1-\nu}{2}, \frac{\rho-\nu}{2}; 1-\nu; \frac{2}{1+it}\right)\\ &= e^{i\pi\frac{\rho-1}{2}\operatorname{sign} t}\left(\frac{-1-it}{2}\right)_+^{\frac{\nu-1}{2}} \left(\frac{-1+it}{2}\right)_+^{\frac{1-\rho}{2}} {}_2F_1\left(\frac{1-\nu}{2}, \frac{2-\rho-\nu}{2}; 1-\nu; \frac{2}{1+it}\right)\\ &= e^{i\pi\frac{\rho-1}{2}\operatorname{sign} t}\left(\frac{1+t^2}{4}\right)^{\frac{1-\rho}{2}} \left(\frac{-1-it}{2}\right)_+^{\frac{\nu+\rho-2}{2}} {}_2F_1\left(\frac{1-\nu}{2}, \frac{2-\rho-\nu}{2}; 1-\nu; \frac{2}{1+it}\right)\\ &= e^{i\pi\frac{\rho-1}{2}\operatorname{sign} t}\left(\frac{1+t^2}{4}\right)^{\frac{1-\rho}{2}} \chi^o_{\rho,\nu}(t). \end{aligned} \tag{2.3.55}$$

Hence,

$$\left(\frac{1+t^2}{4}\right)^{\frac{1-\rho}{2}} \left(\chi^o_{\rho,\nu}\right)^{\text{even}}(t) = \frac{1}{2}\left[e^{i\pi\frac{1-\rho}{2}}\chi^o_{2-\rho,\nu}(t) + e^{i\pi\frac{\rho-1}{2}}\chi^o_{2-\rho,\nu}(-t)\right] \tag{2.3.56}$$

or, using (2.3.29), one has for $t > 0$

$$\begin{aligned}\left(\frac{1+t^2}{4}\right)^{\frac{1-\rho}{2}} \left(\chi^o_{\rho,\nu}\right)^{\text{even}}(t) &= \frac{1}{2}\left[e^{i\pi\frac{1-\rho}{2}} + e^{i\pi\frac{\rho-1}{2}}e^{i\pi\frac{\nu-\rho}{2}}\right]\chi^o_{2-\rho,\nu}(t)\\ &= \frac{\pi e^{i\pi\frac{\nu-\rho}{4}}}{\Gamma(\frac{\rho+\nu}{4})\Gamma(\frac{4-\rho-\nu}{4})}\chi^o_{2-\rho,\nu}(t) : \end{aligned} \tag{2.3.57}$$

using (2.3.29) again,

$$\left(\chi^o_{2-\rho,\nu}\right)^{\text{even}}(t) = \frac{\pi e^{i\pi\frac{\nu-\rho}{4}}}{\Gamma(\frac{2-\rho+\nu}{4})\Gamma(\frac{2+\rho-\nu}{4})}\chi^o_{2-\rho,\nu}(t). \tag{2.3.58}$$

Hence,

$$\left(\frac{1+t^2}{4}\right)^{\frac{1-\rho}{2}} \left(\chi^o_{\rho,\nu}\right)^{\text{even}}(t) = \frac{\Gamma(\frac{2-\rho+\nu}{4})\Gamma(\frac{2+\rho-\nu}{4})}{\Gamma(\frac{\rho+\nu}{4})\Gamma(\frac{4-\rho-\nu}{4})}\left(\chi^o_{2-\rho,\nu}\right)^{\text{even}}(t). \tag{2.3.59}$$

Finally, using the extra coefficient from $\chi^o_{\rho,\nu}$ to $\chi_{\rho,\nu}$ given in (2.3.31), one obtains

$$\frac{\left(\frac{1+t^2}{4}\right)^{\frac{1-\rho}{2}} \left(\chi^o_{\rho,\nu}\right)^{\text{even}}(t)}{\left(\chi^o_{2-\rho,\nu}\right)^{\text{even}}(t)} = 2^{1-\rho}\frac{\Gamma(\frac{2-\rho+\nu}{4})\Gamma(\frac{2+\rho-\nu}{4})}{\Gamma(\frac{\rho+\nu}{4})\Gamma(\frac{4-\rho-\nu}{4})}\,\frac{\Gamma(\frac{\rho+\nu}{2})}{\Gamma(\frac{2-\rho+\nu}{2})}, \tag{2.3.60}$$

which simplifies to (2.3.52) by an application of the duplication formula.

From Proposition 2.3.2, one may note that the product $\Gamma(\frac{4-\rho+\nu}{4})\Gamma(\frac{4-\rho-\nu}{4}) \times C(\rho,\nu)$ is invariant under the map $\rho \mapsto 2-\rho$. It follows from this, and Lemma 2.3.4, that

$$\int_{-\infty}^{\infty} (1+t^2)^{\frac{1-\rho}{2}} \chi_{\rho,\nu}^{\text{even}}(t)dt = \frac{(\nu-\rho)(\nu+\rho)}{(\nu-2+\rho)(\nu+2-\rho)} I(\rho,\nu). \tag{2.3.61}$$

□

To kill the discontinuity of $\chi_{\rho,\nu}$ at the origin, we replace it by its symmetrized version $\chi_{\rho,\nu}^{\text{even}}$ as defined in (2.3.47): note that this does not change the jump of the first-order derivative at 0.

Proposition 2.3.5. *Assume that $\nu \notin \mathbb{Z}$ and that $\rho \pm \nu \notin 2\mathbb{Z}$. One has in the distribution sense*

$$\left[-(1+t^2)\frac{d^2}{dt^2} + (\rho-3)t\frac{d}{dt} - \frac{1-\nu^2}{4} - \frac{(\rho-1)(\rho-3)}{4}\right] \chi_{\rho,\nu}^{\text{even}}(t) = C(\rho,\nu)\delta. \tag{2.3.62}$$

On the other hand, one has in Π the equation

$$\left(\Delta - \frac{1-\nu^2}{4}\right)\left[z \mapsto (\operatorname{Im} z)^{\frac{\rho-1}{2}} \chi_{\rho,\nu}^{\text{even}}\left(\frac{\operatorname{Re} z}{\operatorname{Im} z}\right)\right] = C(\rho,\nu)(\operatorname{Im} z)^{\frac{\rho-1}{2}} \delta_{(0,i\infty)}, \tag{2.3.63}$$

where $\delta_{(0,i\infty)}$ is the measure $\frac{dy}{y}$ on the hyperbolic line from 0 to $i\infty$.

Proof. Since the function $\chi_{\rho,\nu}^{\text{even}}$ is C^∞ in $[0,\infty[$ (up to the boundary), continuous on the line, and since it satisfies in $]0,\infty[$ the differential equation (2.3.16), its image, in the distribution sense, under the operator on the left-hand side of (2.3.62), depends only on the discontinuity at 0 of its first-order derivative. One can then, without modifying the result, replace the complete operator by $-\frac{d^2}{dt^2}$, and χ_{ν}^{even} by the function, linear on $]-\infty,0]$ and on $[0,\infty[$, with the same half-derivatives at 0: this leads to (2.3.62).

The second part of the lemma is a corollary of the first. □

2.4 The symmetries $\nu \mapsto -\nu$ and $\rho \mapsto 2-\rho$

In the function $\text{hom}_{\rho,\nu}$ in the plane defined in (2.3.5), the parameters ρ and ν appear in a clear way: in particular, $-1+\nu$ stands for the global degree of homogeneity, so that the functions $\text{hom}_{\rho,\nu}$ and $h_{\rho,-\nu}$, when $\operatorname{Re}\nu < 0$, could be regarded as "ingoing" and "outgoing" in the sense of scattering theory [34]. However, the $\mathcal{G}$-transform of a function homogeneous of degree $-1+\nu$ is homogeneous of degree $-1-\nu$, and, under the dual Radon transform or any of its associates, a homogeneous distribution and its $\mathcal{G}$-transform have proportional images (*cf.* Theorem 2.1.2). Indeed, Proposition 2.3.3 confirms that, up to multiplication by a scalar, $V^*\text{hom}_{\rho,\nu}$ depends only on the pair (ρ,ν^2).

The function $x+iy \mapsto (C(\rho,\nu))^{-1} y^{\frac{\rho-1}{2}} \chi^{\rm even}_{\rho,\nu}\left(\frac{x}{y}\right)$ is the starting point from which, with the help of a Poincaré series process, we shall build automorphic functions of a new style in Chapter 4. As a consequence of Lemma 2.3.4, it changes to a multiple when ρ is changed to $2-\rho$. On the other hand, it is essential, in view of our applications in Chapter 4, to make a clearcut distinction between the function $(C(\rho,\nu))^{-1} y^{\frac{\rho-1}{2}} \chi^{\rm even}_{\rho,\nu}\left(\frac{x}{y}\right)$ and its transform under the symmetry $\nu \mapsto -\nu$, even though they satisfy the same differential equation (2.3.63). More precisely, in order to consider integral superpositions of these functions with a fixed ρ and $\mathrm{Re}\,\nu < 0$, we need to characterize, when $\mathrm{Re}\,\nu < 0$, the first of these functions within the pair under consideration.

The answer is provided by the resolvent of Δ: as is well-known (say, from spherical function theory, i.e., the reduction to K-invariant functions), the operator Δ is essentially self-adjoint in $L^2(\Pi) = L^2(\Pi, dm)$, where $dm(x+iy) = \frac{dxdy}{y^2}$, if, say, one takes $C_0^\infty(\Pi)$ as its initial domain; it has a purely continuous spectrum, coinciding with the interval $[\frac{1}{4}, \infty[$, so that the resolvent $\nu \mapsto \left(\Delta - \frac{1-\nu^2}{4}\right)^{-1}$ is well-defined for $\mathrm{Re}\,\nu \neq 0$. It is usually made explicit in terms of its integral kernel $k_{\frac{1-\nu}{2}}(z,z')$, a function of $d = d(z,z')$, according to the general (Gelfand's) theory of point-pair invariants: reducing the problem to its special case concerned with K-invariant theory, one obtains explicitly, assuming $\mathrm{Re}\,\nu < 0$,

$$k_{\frac{1-\nu}{2}}(z,z') = \frac{1}{4\pi} \frac{(\Gamma(\frac{1-\nu}{2}))^2}{\Gamma(1-\nu)} (\cosh \frac{d}{2})^{\nu-1} {}_2F_1\left(\frac{1-\nu}{2}, \frac{1-\nu}{2}; 1-\nu; \frac{1}{\cosh^2 \frac{d}{2}}\right). \tag{2.4.1}$$

This can be found in many places, including [32, 55], and could also be derived from (2.4.11) below.

This formula does not lead to tractable integrals when $\left(\Delta - \frac{1-\nu^2}{4}\right)^{-1}$ has to be applied to general (not K-invariant) functions: the proof of the theorem to follow will rely on an alternative construction [60, p. 205] of the resolvent, based of M.Riesz's theory [43] dealing with the solid convex cone in $\mathbb{R}^3$. We shall dispense with giving a priori arguments showing that the resolvent extends to spaces of distributions containing measures such as the one occurring in the next theorem: this will result, instead, from the explicit form (2.4.9) of $\left(\Delta - \frac{1-\nu^2}{4}\right)^{-1}$.

Theorem 2.4.1. *Assume that* $0 < \mathrm{Re}\,\rho < 2$ *and* $\mathrm{Re}\,\nu < 0$. *It is convenient to set*

$$\psi(z) = \frac{\mathrm{Re}\,z}{\mathrm{Im}\,z}. \tag{2.4.2}$$

Recalling Proposition 2.3.5, *denote as* $\delta^{(\rho)}_{(0,i\infty)}$ *the measure in* Π *supported by the hyperbolic line from* 0 *to* $i\infty$, *coinciding with* $y^{\frac{\rho-1}{2}} \frac{dy}{y}$ *in terms of* $y = \mathrm{Im}\,z$. *One has*

$$\left[\left(\Delta - \frac{1-\nu^2}{4}\right)^{-1} \delta^{(\rho)}_{(0,i\infty)}\right](z) = \frac{1}{C(\rho,\nu)} (\operatorname{Im} z)^{\frac{\rho-1}{2}} \left(\chi^{\rm even}_{\rho,\nu} \circ \psi\right)(z), \tag{2.4.3}$$

with $C(\rho,\nu)$ as defined in (2.3.33).

Proof. Let C be the cone in $\mathbb{R}^3$ consisting of points $\eta = (\eta_0, \eta_1, \eta_2)$ with $\eta_0 > 0$ and $\eta_0^2 - \eta_1^2 - \eta_2^2 > 0$, and let $\mathcal{H}$ be the sheet of hyperboloid defined within C by the equation $\eta_0^2 - \eta_1^2 - \eta_2^2 = 1$. It is a very classical fact that $\mathcal{H}$, provided with the (Riemannian) metric which is the restriction to it of the indefinite metric $-d\eta_0^2 + d\eta_1^2 + d\eta_2^2$ in C, is another model of $\Pi = G/K$: the map κ from $\mathcal{H}$ to Π providing the required isometry is defined as $\kappa(\eta) = \frac{\eta_2+i}{\eta_0-\eta_1}$. With

$$\Box = \frac{\partial^2}{\partial\eta_0^2} - \frac{\partial^2}{\partial\eta_1^2} - \frac{\partial^2}{\partial\eta_2^2} \tag{2.4.4}$$

and

$$E = \eta_0 \frac{\partial}{\partial\eta_0} + \eta_1 \frac{\partial}{\partial\eta_1} + \eta_2 \frac{\partial}{\partial\eta_2}, \tag{2.4.5}$$

it is easily verified (this is an extension to hyperboloids of the classical theory of spherical harmonics) that if Ψ is a function in C homogeneous of degree $k \in \mathbb{C}$ satisfying the equation $\Box\Psi = 0$, its restriction to $\mathcal{H}$ satisfies the eigenvalue equation

$$\Delta_{\mathcal{H}} \left(\Psi\big|_{\mathcal{H}}\right) = -k(k+1)\Psi\big|_{\mathcal{H}}, \tag{2.4.6}$$

if one denotes as $\Delta_{\mathcal{H}}$ the operator obtained by transferring under κ the hyperbolic Laplacian Δ on Π.

M.Riesz's theory [43, 44] gives a fundamental solution at 0 of the operator $\Box$ as the convolution by the function (supported in the closure of C)

$$Z_2 = \frac{1}{2\pi} \left(\eta_0^2 - \eta_1^2 - \eta_2^2\right)_{\rm pos}^{-\frac{1}{2}}, \tag{2.4.7}$$

where the subscript indicates that the whole function is to be multiplied by the characteristic function of C. Then, if Ψ is homogeneous of degree $\frac{-5-\nu}{2}$ in C, the function $Z_2 * \Psi$ lies in the nullspace of $\Box$ and is homogeneous of degree $\frac{-1-\nu}{2}$ so that, as a consequence of the equation (2.4.6) taken with $k = \frac{-1-\nu}{2}$, one has

$$\left(\Delta_{\mathcal{H}} - \frac{1-\nu^2}{4}\right) \left((Z_2 * \Psi)\big|_{\mathcal{H}}\right) = \Psi\big|_{\mathcal{H}}. \tag{2.4.8}$$

This provides the following recipe for computing the image of the resolvent $\left(\Delta - \frac{1-\nu^2}{4}\right)^{-1}$ on a function $f \in C_0^\infty(\Pi)$, under the assumption (to be justified presently) that $\operatorname{Re}\nu < 0$: transfer f to the function $f \circ \kappa$ on $\mathcal{H}$, extend the function obtained to a function Ψ on C homogeneous of degree $\frac{-5-\nu}{2}$, restrict the function

$Z_2 * \Psi$ to $\mathcal{H}$, finally compose this restriction with κ^{-1}. Since, for $r > 0$ and $\xi \in \mathcal{H}$, one has $d(r\xi) = r^2 dr \frac{d\xi_1 d\xi_2}{\xi_0}$, one obtains the formula we have been looking for:

$$\left[\left(\Delta - \frac{1-\nu^2}{4}\right)^{-1} f\right](\kappa(\eta)) = \frac{1}{2\pi}\int_0^\infty r^{\frac{-1-\nu}{2}}\,dr \times \int_{\mathcal{H}} [(\eta_0 - r\xi_0)^2 - (\eta_1 - r\xi_1)^2 - (\eta_2 - r\xi_2)^2]_{\rm pos}^{-\frac{1}{2}} f(\kappa(\xi))\frac{d\xi_1 d\xi_2}{\xi_0}. \quad (2.4.9)$$

One easily checks that the measure $\frac{d\xi_1 d\xi_2}{\xi_0}$ coincides with $dm(\kappa(\xi))$, the transfer under κ of the invariant measure dm on Π.

When $\kappa(\eta) = z$ and $\kappa(\xi) = z'$, one has

$$\eta_0\xi_0 - \eta_1\xi_1 - \eta_2\xi_2 = \cosh d(z, z'), \quad (2.4.10)$$

so that the integral kernel of the operator obtained in (2.4.9) is the function

$$(z, z') \mapsto \frac{1}{2\pi}\int_0^{e^{-d(z,z')}} r^{\frac{\nu-1}{2}}(1 - 2r\cosh d(z,z') + r^2)^{-\frac{1}{2}}\,dr = \frac{1}{2\pi^{\frac{1}{2}}}\frac{\Gamma(\frac{1-\nu}{2})}{\Gamma(\frac{2-\nu}{2})} e^{\frac{(\nu-1)d(z,z')}{2}}\, {}_2F_1\left(\frac{1}{2}, \frac{1-\nu}{2}; \frac{2-\nu}{2}; e^{-2d(z,z')}\right). \quad (2.4.11)$$

With the help of two transformations (one of which is quadratic) of the hypergeometric function, one can see [60, p. 206] that this is identical to the integral kernel $k(z, z')$ in (2.4.1): however, the expression (2.4.9) of the operator will be more convenient in the *MA*-invariant case. From any of the two expressions of the integral kernel in (2.4.11), one sees that it is bounded by a constant times $|\log d(z, z')|$ near the diagonal and by a constant times $(\cosh d(z, z'))^{\frac{\mathrm{Re}\,\nu - 1}{2}}$ when $d(z, z') \geq 1$. Since this kernel is K-biinvariant and, in polar geodesic coordinates ρ, θ around $i \in \Pi$ (with $\rho = d(i, z)$), the invariant measure expresses itself as $\sinh \rho d\rho d\theta$, it follows from the most popular criterion regarding L^2-continuity that, provided that $\mathrm{Re}\,\nu < -1$, the operator $\left(\Delta - \frac{1-\nu^2}{4}\right)^{-1}$ defined in (2.4.9) is indeed the resolvent. Now, if one only has $\mathrm{Re}\,\nu < 0$, the criterion just mentioned does not apply but it is immediate from the same estimates regarding the kernel that the operator under consideration is continuous from $L^2(\Pi)$ to the Banach space of bounded continuous functions: as such, it depends analytically on ν, and must still coincide with the resolvent (hence, be a continuous endomorphism of $L^2(\Pi)$).

We now substitute for f the measure $\delta^{(\rho)}_{(0,i\infty)}$. When $\xi_2 = 0$ and $y = \frac{1}{\xi_0 - \xi_1}$, one has $\frac{dy}{y} = \frac{d\xi_1}{\xi_0}$. We must thus, in the preceding integral, replace $f(\kappa(\xi))\frac{d\xi_1 d\xi_2}{\xi_0}$ by $\frac{d\xi_1}{\sqrt{1+\xi_1^2}}$ and set the variable ξ_2 at the value 0, getting as a result (with $\xi_0 = \sqrt{1 + \xi_1^2}$)

$$\ell_{\rho,\nu}(\kappa(\eta)) \colon = \left[\left(\Delta - \frac{1-\nu^2}{4} \right)^{-1} \delta^{(\rho)}_{(0,i\infty)} \right] (\kappa(\eta)) \tag{2.4.12}$$
$$= \frac{1}{2\pi} \int_0^\infty r^{\frac{-1-\nu}{2}} dr \int_{-\infty}^\infty [(\eta_0 - r\xi_0)^2 - (\eta_1 - r\xi_1)^2 - \eta_2^2]_{\text{pos}}^{-\frac12} (\xi_0 - \xi_1)^{\frac{1-\rho}{2}} \frac{d\xi_1}{\xi_0}.$$

Now,

$$(\eta_0 - r\xi_0)^2 - (\eta_1 - r\xi_1)^2 - \eta_2^2 = 1 - 2r(\eta_0\xi_0 - \eta_1\xi_1) + r^2 : \tag{2.4.13}$$

recalling that $1 + \eta_2^2 = \eta_0^2 - \eta_1^2$, consider the matrix $(1+\eta_2)^{-\frac12} \left(\begin{smallmatrix} \eta_0 & \eta_1 \\ \eta_1 & \eta_0 \end{smallmatrix} \right)$, which corresponds to a Lorentz transformation (in $(1+1)$-dimensional spacetime) in the variable (ξ_0, ξ_1), thus preserving the measure $\frac{d\xi_1}{\xi_0}$. Under this transformation, the expression (2.4.13) transforms into $1 - 2r\xi_0\sqrt{1+\eta_2^2} + r^2$, while $\xi_0 - \xi_1$ transforms into $(1+\eta_2^2)^{-\frac12}(\eta_0 - \eta_1)(\xi_0 - \xi_1)$. Hence, setting $z = \kappa(\eta)$, so that $\eta_0 - \eta_1 = (\operatorname{Im} z)^{-1}, \eta_2 = \frac{\operatorname{Re} z}{\operatorname{Im} z} = \psi(z)$, one obtains

$$\ell_{\rho,\nu}(z) = (\operatorname{Im} z)^{\frac{\rho-1}{2}} \left(1 + (\psi(z))^2\right)^{\frac{\rho-1}{4}} \tag{2.4.14}$$
$$\times \frac{1}{2\pi} \int_0^\infty r^{\frac{-1-\nu}{2}} dr \int_{-\infty}^\infty [1 - 2r\xi_0\sqrt{1+(\psi(z))^2} + r^2]_{\text{pos}}^{-\frac12} (\xi_0 - \xi_1)^{\frac{1-\rho}{2}} \frac{d\xi_1}{\xi_0}.$$

This is an even function of $t = \frac{\operatorname{Re} z}{\operatorname{Im} z}$, which can be written, after one has performed the change of variable $r \mapsto [2\xi_0\sqrt{1+t^2}]^{-1} r$, as

$$\ell_{\rho,\nu}(z) = \frac{2^{\frac{\nu-1}{2}}}{2\pi} (\operatorname{Im} z)^{\frac{\rho-1}{2}} (1+t^2)^{\frac{\rho+\nu-2}{4}} \operatorname{Int}_{\rho,\nu}(t), \tag{2.4.15}$$

with

$$\operatorname{Int}_{\rho,\nu}(t) = \int_0^\infty r^{\frac{-\nu-1}{2}} dr \int_{-\infty}^\infty \left[1 - r + \frac{r^2}{4(1+t^2)(1+\xi_1^2)} \right]_{\text{pos}}^{-\frac12} (\xi_0 - \xi_1)^{\frac{1-\rho}{2}} \xi_0^{\frac{\nu-3}{2}} d\xi_1. \tag{2.4.16}$$

As $|t| \to \infty$, the integral goes to

$$\operatorname{Int}_{\rho,\nu}(\infty) = \int_0^1 r^{\frac{-\nu-1}{2}} (1-r)^{-\frac12} dr \int_{-\infty}^\infty (1+\xi_1^2)^{\frac{\nu-3}{4}} (\xi_0 - \xi_1)^{\frac{-1+\rho}{2}} d\xi_1$$
$$= \pi^{\frac12} \frac{\Gamma(\frac{1-\nu}{2})}{\Gamma(\frac{2-\nu}{2})} \times \int_{-\infty}^\infty (\cosh t)^{\frac{\nu-1}{2}} e^{\frac{(1-\rho)t}{2}} dt$$
$$= 2^{\frac{-1-\nu}{2}} \pi^{\frac12} \frac{\Gamma(\frac{\rho-\nu}{4})\Gamma(\frac{2-\rho-\nu}{4})}{\Gamma(\frac{2-\nu}{2})}, \tag{2.4.17}$$

where we have used [36, p. 432] at the end: note that this expression does not change under the symmetry $\rho \mapsto 2-\rho$. One thus has the equivalent, as $\frac{\operatorname{Re} z}{\operatorname{Im} z} \to \infty$,

$$(\operatorname{Im} z)^{\frac{1-\rho}{2}} \ell_{\rho,\nu}(z) \sim \frac{\pi^{-\frac12}}{4} \frac{\Gamma(\frac{\rho-\nu}{4})\Gamma(\frac{2-\rho-\nu}{4})}{\Gamma(\frac{2-\nu}{2})} \left| \frac{\operatorname{Re} z}{\operatorname{Im} z} \right|^{\frac{\rho+\nu-2}{2}}. \tag{2.4.18}$$

Since the image under $\Delta-\frac{1-\nu^2}{4}$ of the function $\ell_{\rho,\nu}$ is zero in the complement of the hyperbolic line from 0 to $i\infty$, it follows from the structure of $\ell_{\rho,\nu}(z)$ as the product of $(\mathrm{Im}\,z)^{\frac{1-\rho}{2}}$ by an even function of $\frac{\mathrm{Re}\,z}{\mathrm{Im}\,z}$ and from (2.3.16) that the function in (2.4.18) must be a linear combination of the functions $\chi^{\mathrm{even}}_{\rho,\nu}(t)$ and $\chi^{\mathrm{even}}_{\rho,-\nu}(t)$: in view of the equivalent of $\chi_{\rho,\nu}(t)$ as $|t|\to\infty$ resulting from (2.3.31), it has to be a multiple of the function $\chi^{\mathrm{even}}_{\rho,\nu}(t)$ only (recall the assumption that $\mathrm{Re}\,\nu<0$). The proof of Theorem 2.4.1 now reduces to proving that the functions $\ell_{\rho,\nu}(t)$ and $\frac{1}{C(\rho,\nu)}\chi^{\mathrm{even}}_{\nu}(t)$ are equivalent as $|t|\to\infty$.

Comparing (2.4.18) to the equivalent (2.3.51) of the second function, using the duplication formula

$$\Gamma(\frac{2-\rho-\nu}{4})\Gamma(\frac{4-\rho-\nu}{4})=(2\pi)^{\frac{1}{2}}2^{\frac{-1+\rho+\nu}{2}}\Gamma(\frac{2-\rho-\nu}{2}) \tag{2.4.19}$$

and the expression (2.3.33) of $C(\rho,\nu)$, we obtain (2.4.3). □

Remark 2.4.a. Even though one has $\left(\Delta-\frac{1-\nu^2}{4}\right)\left((\mathrm{Im}\,z)^{\frac{\rho-1}{2}}\chi_{\rho,\nu}\circ\psi\right)=0$ in the complement of a one-dimensional set, this phenomenon leaves no trace after one has applied to the function $(\mathrm{Im}\,z)^{\frac{\rho-1}{2}}\chi_{\rho,\nu}\circ\psi$ a non-local operator such as the Radon transformation. It would be somewhat misleading to regard the function under consideration as "almost" an eigenfunction of Δ.

Our construction of Poincaré series (of a novel kind) in Chapter 4 is based on the use of the functions $(\mathrm{Im}\,z)^{\frac{\rho-1}{2}}\chi_{\rho,\nu}\circ\psi$, with $\mathrm{Re}\,\nu<0$. These Poincaré series will take the place usually taken by Eisenstein series. In a way similar to that which leads, classically, to so-called incomplete Eisenstein series (this terminology, borrowed from [21, 23], sounds more appropriate than the traditional one of incomplete theta series), we may consider integral superpositions of the functions $\chi_{\rho,\nu}$ for a fixed ρ. What we obtain as a result is a space of images of the measure $\delta^{(\rho)}_{(0,i\infty)}$ (introduced in Theorem 2.4.1) under fairly general functions, in the spectral-theoretic sense, of the Laplacian: an integral transform will make these explicit.

In view of the spectrum of the hyperbolic Laplacian Δ, whether in the free half-plane or in the automorphic situation, a function of Δ is the same as an even function H of the operator $2\sqrt{\Delta-\frac{1}{4}}$ (the factor 2 is of course for convenience only), provided that, in the second case, one interests oneself only in automorphic functions orthogonal to constants (so that the square root should not create a difficulty). Experience, in particular with quantization theory [63, p. 57-59], shows that, as a function of one real variable, it is most of the time the function H, rather than the corresponding function of Δ, that appears simple, or interesting. This may be considered, in view of (2.1.5), as one more argument in favor of using the plane, rather than the half-plane, in the study of Δ.

Theorem 2.4.2. *Let $H = H(\mu)$ be an even holomorphic function in some strip $|\mathrm{Im}\,\mu| < \beta_0$, such that $\int_{\mathrm{Im}\,\mu=\beta} |\mu|^2 |H(\mu)|^2 d\mu < \infty$ for every β with $|\beta| < \beta_0$: set $G(\sigma) = \int_{-\infty}^{\infty} H(\lambda) e^{2i\pi\lambda\sigma} d\lambda$. Assuming $0 < \mathrm{Re}\,\rho < 2$, the image of the measure $\delta_{(0,i\infty)}^{(\rho)}$ under the operator $H\left(2\sqrt{\Delta - \frac{1}{4}}\right)$ is a function ϕ, which can be made explicit in terms of $\sinh(4\pi\tau) = \frac{\mathrm{Re}\,z}{\mathrm{Im}\,z}$ as*

$$\phi(z) = -\frac{1}{4\pi} (\mathrm{Im}\,z)^{\frac{\rho-1}{2}} (\cosh 4\pi\tau)^{\frac{\rho-2}{2}} \int_{|\tau|}^{\infty} G'(\sigma) \mathfrak{P}_{\frac{\rho-2}{2}} \left(\frac{\cosh 4\pi\sigma}{\cosh 4\pi\tau} \right) d\sigma. \tag{2.4.20}$$

Given any number $\beta \in]0, \beta_0[$, one has the identity

$$\begin{aligned} \phi(z) &= -\frac{1}{4i\pi} \int_{\mathrm{Re}\,\nu=-\beta} \nu H(i\nu) \left[\left(\Delta - \frac{1-\nu^2}{4} \right)^{-1} \delta_{(0,i\infty)}^{(\rho)} \right] (z) d\nu \\ &= -\frac{1}{4i\pi} \int_{\mathrm{Re}\,\nu=-\beta} \nu \frac{H(i\nu)}{C(\rho,\nu)} (\mathrm{Im}\,z)^{\frac{\rho-1}{2}} \chi_{\rho,\nu}^{\mathrm{even}} \left(\frac{\mathrm{Re}\,z}{\mathrm{Im}\,z} \right) d\nu. \end{aligned} \tag{2.4.21}$$

In the case when one has $H(\lambda) = K_{\frac{i\lambda}{2}}(\alpha)$ for some $\alpha > 0$, so that $G(\sigma) = 2\pi \exp(-\alpha \cosh(4\pi\sigma))$, one has

$$\phi(z) = \left(\frac{2\alpha}{\pi} \right)^{\frac{1}{2}} |z|^{\frac{\rho-1}{2}} K_{\frac{\rho-1}{2}} \left(\frac{\alpha |z|}{\mathrm{Im}\,z} \right). \tag{2.4.22}$$

Proof. Setting $F = \frac{1}{2i\pi} G'$, one has $\widehat{F}(\mu) = \mu H(\mu)$: it follows from the assumption made about H that $\int_{-\infty}^{\infty} |G(\sigma)|^2 e^{4\pi|\beta|\sigma} d\sigma < \infty$ whenever $|\beta| < \beta_0$, and that the same holds with F substituted for G.

Recall from the proof of Theorem 2.4.1 that the function $\ell_{\rho,\nu}$ is the image of the measure $\delta_{(0,i\infty)}^{(\rho)}$ under $\left(\Delta - \frac{1-\nu^2}{4}\right)^{-1}$ and that, from (2.4.14),

$$\begin{aligned} \ell_{\rho,\nu}(z) &= 2^{-\frac{3}{2}} \pi^{-1} (\mathrm{Im}\,z)^{\frac{\rho-1}{2}} \\ &\times \left(1 + (\psi(z))^2\right)^{\frac{\rho-2}{4}} \int_0^{\infty} r^{\frac{-\nu-2}{2}} \mathrm{Int}_{\rho} \left(\frac{1+r^2}{2r} \left(1 + (\psi(z))^2\right)^{-\frac{1}{2}} \right) dr, \end{aligned} \tag{2.4.23}$$

with

$$\mathrm{Int}_{\rho}(c) = \int_{-\infty}^{\infty} (c - \xi_0)_{\mathrm{pos}}^{-\frac{1}{2}} (\xi_0 - \xi_1)^{\frac{1-\rho}{2}} \frac{d\xi_1}{\xi_0}. \tag{2.4.24}$$

This integral is zero if $c \leq 1$, and we now compute it for $c > 1$. Setting $c = \cosh a$ with $a > 0$ and making the change of variable $\xi_1 = \sinh \eta$, one obtains, using [36, p. 407] at the end,

$$\mathrm{Int}_{\rho}(c) = \int_{-a}^{a} [\cosh a - \cosh \eta]^{-\frac{1}{2}} e^{\frac{(\rho-1)\eta}{2}} d\eta = 2^{\frac{1}{2}} \pi \mathfrak{P}_{\frac{\rho-2}{2}}(\cosh a), \tag{2.4.25}$$

an expression invariant when changing ρ to $2-\rho$.

As told in the statement of the proposition, we set $\frac{\mathrm{Re}\, z}{\mathrm{Im}\, z} = \sinh 4\pi\tau$ for $z \in \Pi$, and we also make the change of variable $r = e^{-4\pi\sigma}$ in this integral. We obtain

$$\ell_{\rho,\nu}(z) = 2\pi(\mathrm{Im}\, z)^{\frac{\rho-1}{2}} (\cosh 4\pi\tau)^{\frac{\rho-2}{2}} \int_{|\tau|}^{\infty} e^{2\pi\nu\sigma} \mathfrak{P}_{\frac{\rho-2}{2}} \left(\frac{\cosh 4\pi\sigma}{\cosh 4\pi\tau} \right) d\sigma. \quad (2.4.26)$$

Since

$$F(\sigma) = \int_{-\infty}^{\infty} \widehat{F}(\lambda) e^{2i\pi\lambda\sigma} d\lambda = \frac{1}{i} \int_{\mathrm{Re}\,\nu=-\beta} \widehat{F}(-i\nu) e^{2\pi\nu\sigma} d\nu, \quad (2.4.27)$$

the function ϕ defined in (2.4.20) can be written as

$$\frac{1}{2i}(\mathrm{Im}\, z)^{\frac{\rho-1}{2}} (\cosh 4\pi\tau)^{\frac{\rho-2}{2}} \int_{|\tau|}^{\infty} \mathfrak{P}_{\frac{\rho-2}{2}} \left(\frac{\cosh 4\pi\sigma}{\cosh 4\pi\tau} \right) d\sigma . \frac{1}{i} \int_{\mathrm{Re}\,\nu=\beta} \widehat{F}(-i\nu) e^{2\pi\nu\sigma} d\nu$$
$$= -\frac{1}{4\pi} \int_{\mathrm{Re}\,\mu=-\beta} \widehat{F}(-i\nu) \ell_{\rho,\nu}(z) d\nu. \quad (2.4.28)$$

This leads to the pair of equations (2.4.21), of which we now consider the first.

When ν moves along the straight line from $-\beta - i\infty$ to $-\beta + i\infty$, the variable $\mu = \frac{1-\nu^2}{4}$ describes a parabola $\mathcal{P}^-$ enclosing the spectrum $[\frac{1}{4}, \infty[$ of Δ in the clockwise sense: denoting as $\mathcal{P}^+$ the negative of the contour that precedes, one transforms the integral under consideration into the integral

$$\frac{1}{2i\pi} \int_{\mathcal{P}^+} H\left(2\sqrt{\mu - \frac{1}{4}} \right) (\mu - \Delta)^{-1} d\mu, \quad (2.4.29)$$

which completes the main part of Theorem 2.4.2, in view of Dunfords's integral representation of the resolvent of an operator.

When $H(\lambda) = K_{\frac{i\lambda}{2}}(\alpha)$, that $G(\sigma) = 2\pi \exp(-\alpha \cosh(4\pi\sigma))$ follows from [36, p. 408]. Then,

$$\begin{aligned} \phi(z) &= 2\pi\alpha(\mathrm{Im}\, z)^{\frac{\rho-1}{2}} (\cosh 4\pi\tau)^{\frac{\rho-2}{2}} \int_{\tau}^{\infty} \sinh(4\pi\sigma) e^{-\alpha\cosh(4\pi\sigma)} \mathfrak{P}_{\frac{\rho-2}{2}} \left(\frac{\cosh 4\pi\sigma}{\cosh 4\pi\tau} \right) d\sigma \\ &= \frac{\alpha}{2} (\mathrm{Im}\, z)^{\frac{\rho-1}{2}} (\cosh 4\pi\tau)^{\frac{\rho}{2}} \int_1^{\infty} e^{-\alpha t \cosh(4\pi\tau)} \mathfrak{P}_{\frac{\rho-2}{2}}(t) dt \\ &= \left(\frac{2\alpha}{\pi} \right)^{\frac{1}{2}} (\mathrm{Im}\, z)^{\frac{\rho-1}{2}} (\cosh 4\pi\tau)^{\frac{\rho-1}{2}} K_{\frac{\rho-1}{2}}(\alpha \cosh(4\pi\tau)) \end{aligned} \quad (2.4.30)$$

according to [36, p. 194]: now, $\cosh(4\pi\tau) = \frac{|z|}{\mathrm{Im}\, z}$, which leads to (2.4.22). □

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