Notions of Positivity and the Geometry of Polynomials
Trends in Mathematics, 17–31

Variants of the Effective Nullstellensatz and Residue Calculus

Mats Andersson and Elizabeth Wulcan

Abstract. We describe how one can obtain effective versions of the Nullstellensatz and variations by a combination of residue calculus and a geometric estimate for so-called distinguished varieties.

Mathematics Subject Classification (2000). 14Q20, 32A27, 32B99.

Keywords. Effective Nullstellensatz, polynomial division, residue calculus, Briançon-Skoda theorem.

1. Introduction

In [1] the first author introduced a framework to obtain effective membership results for polynomial ideals in $\mathbb{C}^n$ by means of an interplay between geometry and residue calculus. In this note we describe this framework and how various classical and more recent such results fit into it, including an almost optimal version of the effective Nullstellensatz. Most of the results, or closely related ones, can be found (at least implicitly) in [13], [12], [1], [4], [6], [9], or [11]. The aim is to present the ideas rather than elaborate all technical details, for which we instead give suitable references. We will use basic facts about line bundles on complex projective space $\mathbb{P}^n$ and a geometric estimate used in [9] and [11]; however no prior knowledge of multivariable residue calculus will be assumed. Our hope is that this note will serve as an invitation to residue calculus techniques. We conclude with a discussion about the worst case scenario regarding the effective Nullstellensatz.

Let $F_1, \dots, F_m$ be polynomials in $\mathbb{C}^n$ of degree at most d with no common zeros. By the Nullstellensatz there are polynomials Q_j such that

$$F_1 Q_1 + \cdots + F_m Q_m = 1. \tag{1.1}$$

The first author was partially supported by the VR (the Swedish Research Council). The second author was partially supported by the VR and by NSF.

It is proved by Kollár, [13], and Jelonek, [12], that one can find Q_j such that[1]

$$\deg(F_j Q_j) \le d^\mu,$$

where, throughout this paper,

$$\mu := \min(m, n).$$

This degree bound is optimal[2]. We will see however that one can get sharper degree estimates with extra hypotheses on the common zero set of the polynomials at infinity.

We first describe the framework. Let $F_1, \ldots, F_m$ be polynomials of degree $\le d$ and let Φ be any polynomial. Let $z = (z_0, \ldots, z_n)$, $z' = (z_1, \ldots, z_n)$, and let $f_j(z) := z_0^d F_j(z'/z_0)$ be the d-homogenizations of F_j. It is natural to consider the f_j as holomorphic sections of the line bundle $\mathcal{O}(d)$ over $\mathbb{P}^n$. Let $\psi(z) = z_0^\rho \Phi(z'/z_0)$ be the ρ-homogenization of Φ, $\rho \ge \deg \Phi$. Then there is a representation

$$F_1 Q_1 + \cdots + F_m Q_m = \Phi \tag{1.2}$$

in $\mathbb{C}^n$ with $\deg(F_j Q_j) \le \rho$ if and only if there are $(\rho - d)$-homogeneous forms q_j such that

$$f_1 q_1 + \cdots + f_m q_m = \psi. \tag{1.3}$$

Thus the question is reduced to an equation for sections of holomorphic line bundles over $\mathbb{P}^n$. We associate to the f_j a (residue) current R^f of Bochner-Martinelli type on $\mathbb{P}^n$ with support on the common zero set $Z \subset \mathbb{P}^n$ of the f_j. The basic result is

Proposition 1.1. *With the notation above, if $\psi R^f = 0$, and $\rho \ge d(n+1) - n$ or $m \le n$, then there is a representation* (1.2) *with* $\deg(F_j Q_j) \le \rho$.

The sections f_j generate a coherent ideal sheaf $\mathcal{J}$ over $\mathbb{P}^n$, and to such a sheaf there are associated so-called *distinguished varieties* Z_j in the sense of Fulton-MacPherson, whose union is equal to Z, see [10]. Let c_∞ be the maximal codimension of those Z_j that are contained in the hyperplane at infinity $H_\infty = \mathbb{P}^n \setminus \mathbb{C}^n$. The codimension of Z_j cannot exceed the number of generators m, see, e.g., [9]. Thus

$$c_\infty \le \mu.$$

If Φ is a polynomial and locally in $\mathbb{C}^n$ there is a constant $C > 0$ such that

$$|\Phi| \le C|F|^\mu, \tag{1.4}$$

then actually Φ belongs to the polynomial ideal (F_j) generated by F_j in $\mathbb{C}^n$; this follows from the Briançon-Skoda theorem, [7].

[1] For $d = 2$ and $m > n$, d^μ should be replaced by $2d^\mu$, see [17].
[2] There is a more precise result where the various degrees of the F_j are taken into account; however throughout this note, for simplicity, we keep the same upper bound d for all the polynomials.

Theorem 1.2. *Let F_j be polynomials of degree at most d in $\mathbb{C}^n$.*

(i) *If Φ is a polynomial such that* (1.4) *holds locally in $\mathbb{C}^n$, then there is a representation* (1.2) *with*

$$\deg(F_jQ_j) \leq \max(\deg \Phi + \mu d^{c_\infty}, \gamma), \tag{1.5}$$

where $\gamma = d(n+1) - n$ if $m > n$ and $\gamma = 0$ if $m \leq n$.

(ii) *If* $\operatorname{codim}\{F_1 = \cdots = F_m = 0\} \geq m$ *in $\mathbb{C}^n$ and $\Phi \in (F_j)$, then there is a representation* (1.2) *such that* (1.5) *holds.*

If there are no distinguished varieties contained in H_∞ then we interpret d^{c_∞} as 0. Part (i) can be seen as an effective Briançon-Skoda theorem. It was proved by Hickel, [11], but with the bound $\min(m, n+1)d^\mu$ rather than μd^{c_∞}. The ideas in [11] are very close to the ones used in [9]. The factor $(n+1)$ comes from an application of a global Briançon-Skoda type theorem. In our approach ψ just has to annihilate R^f, i.e., $\psi R^f = 0$; this is a purely local matter, and therefore it is enough with the local Briançon-Skoda power μ. This local nature is even more important in the proof of part (ii), where the residue R^f is annihilated for "different" reasons in $\mathbb{C}^n$ and at H_∞. The statement (ii) appeared in [4] but with d^μ instead of d^{c_∞}. From (i) we deduce the following version of the Nullstellensatz.

Corollary 1.3. *If F_j have no common zeros in $\mathbb{C}^n$, then there are polynomials Q_j such that* (1.1) *holds and*

$$\deg(F_jQ_j) \leq \mu d^{c_\infty}. \tag{1.6}$$

This result appeared (in Example 1) in [9], but with the factor $(n+1)$ instead of μ. It is weaker than the optimal result of Kollár and Jelonek because of the presence of the factor μ in front of d^{c_∞}. On the other hand, if $c_\infty < \mu$, i.e., there are no distinguished points, and $d > \mu$, then (1.6) is sharper. Actually, as soon as there are "many" distinguished varieties one gets a sharper estimate; this is discussed in Section 7 in connection with Kollár's example in [13].

The second part of Theorem 1.2 implies a Max Noether type result.

Corollary 1.4. *If $m \leq n$,* $\operatorname{codim} Z = m$, *no irreducible component of Z is contained in H_∞, and Φ is a polynomial in the ideal (F_j), then there is a representation* (1.2) *with* $\deg(F_jQ_j) \leq \deg \Phi$.

Proof. Since the union of all the distinguished varieties is equal to Z, and no distinguished variety has codimension larger than m, it follows that no distinguished variety is contained in H_∞, and hence $d^{c_\infty} = 0$. Thus the corollary follows from part (ii) of the theorem. □

This statement appeared already in [2]. If $m = n$, and thus Z is discrete, this is the classical $AF + BG$ theorem due to Max Noether, [16].

In case Z is empty, Theorem 1.2 (ii) implies the classical Macaulay theorem, [15]:

Corollary 1.5. *If f_j have no common zeros on $\mathbb{P}^n$ and Φ is any polynomial, then there is a representation* (1.2) *with* $\deg(F_jQ_j) \le \max(\deg\Phi, d(n+1)-n)$.

We can just as well consider sections f_j of an ample line bundle $L \to X$ over a smooth projective manifold X. With the same arguments we then get, e.g., the following variant of the main result in [9]. To each distinguished variety Z_j there is an associated positive order r_j, see Section 5.

Theorem 1.6. *Let $f_1, \dots, f_m$ be global holomorphic sections of an ample line bundle L over a smooth projective variety X, and let ψ be a holomorphic section of $L^s \otimes A \otimes K_X$, where A is ample, or big and nef, and assume that $s \ge \min(m, n+1)$.*

(i) *If*
$$|\psi| \le C|f|^\mu, \tag{1.7}$$
then there are sections q_j of $L^{s-1} \otimes A \otimes K_X$ such that
$$f_1q_1 + \cdots + f_mq_m = \psi. \tag{1.8}$$

(ii) *If ψ vanishes to order μr_j at a generic point on Z_j for each j, then there are sections q_j of $L^{s-1} \otimes A \otimes K_X$ such that* (1.8) *holds.*

We will see below that (i) implies (ii), which is (a slightly improved version of) the main result in [9]; in [9] the hypothesis is that ψ vanishes to order $\min(m, n+1)r_j$.

2. Product ideals

By a small variation of the set-up we can obtain similar results for products of polynomial ideals. For $j = 1, \dots, r$, let $F^j = (F^j_1, \dots, F^j_{m_j})$ be a tuple of of polynomials of degree (at most) d^j. For each j we then have a number c^j_∞ defined as before.

Theorem 2.1. *Assume that Φ is a polynomial such that*
$$|\Phi| \le C|F^1|^{s_1} \cdots |F^r|^{s_r}$$
locally in $\mathbb{C}^n$ for $s_1 + \cdots + s_r \le n + r - 1, \quad 1 \le s_j \le m_j$. Then Φ belongs to the product ideal $(F^1_j) \cdots (F^r_j)$ and there is a representation
$$\Phi = \sum_{1 \le \ell_j \le m_j} F^1_{\ell_1} \cdots F^r_{\ell_r} Q_{\ell_1 \cdots \ell_r}$$
with $\deg(F^1_{\ell_1} \cdots F^r_{\ell_r} Q_{\ell_1\cdots\ell_r}) \le \max(\deg\Phi + \mu d^{c_\infty}, d\hat{\mu} - n)$, *where*
$$\mu d^{c_\infty} := \max\Big\{\sum_1^r s_j d_j^{c^j_\infty};\ s_1 + \cdots + s_r \le n + r - 1,\ 1 \le s_j \le m_j\Big\}$$
and
$$d\hat{\mu} := \max\Big\{\sum_1^r s_j d_j;\ s_1 + \cdots + s_r \le n + r,\ 1 \le s_j \le m_j\Big\}.$$

In particular, one can take all F^j equal to one single tuple F and get a result for membership in the ideal $(F)^r$. However, in this case the proof gives a somewhat sharper estimate, cf. [6] and [9].

Theorem 2.2. *If $F_1, \ldots, F_m$ are polynomials of degree at most d and Φ is a polynomial such that $|\Phi| \le C|F|^{\mu+r-1}$ locally on $\mathbb{C}^n$, then Φ belongs to the ideal $(F_j)^r$ and there is a representation*

$$\Phi = \sum_{I_1+\cdots+I_m=r} F_1^{I_1}\cdots F_m^{I_m} Q_I$$

with

$$\deg(F_1^{I_1}\cdots F_m^{I_m} Q_I) \le \max\big(\deg\Phi + (\mu+r-1)d^{c_\infty}, d(\min(m,n+1)+r-1)-n\big).$$

3. Division problems and residues, the basic set-up

Let X be a smooth projective variety of dimension n, let f_j be holomorphic global sections of a Hermitian line bundle $L \to X$, and let $\mathcal{J}$ be the associated ideal sheaf with zero set Z. The reader who only cares about polynomials in $\mathbb{C}^n$ should take $X = \mathbb{P}^n$ and $L = \mathcal{O}(d)$. Let E^j be disjoint trivial line bundles with basis elements e_j, and define the rank m bundle

$$E = L^{-1}\otimes E^1 \oplus \cdots \oplus L^{-1}\otimes E^m$$

over X. Then $f = \sum f_j e_j^*$, where e_j^* is the dual basis, is a section of the dual bundle E^*; it induces the Koszul complex

$$0 \to \Lambda^m E \xrightarrow{\delta_f} \cdots \xrightarrow{\delta_f} \Lambda^2 E \xrightarrow{\delta_f} E \to \mathbb{C} \to 0,$$

where δ_f is interior multiplication with f. Notice that

$$\Lambda^k E = L^{-k}\otimes \Lambda^k(E^1\oplus\cdots\oplus E^m). \tag{3.1}$$

We will consider $(0,q)$-forms (or currents) with values in these vector bundles. We therefore form the exterior algebra over $E \oplus T^*_{0,1}(X)$. In this way, e.g., $d\bar{z}_j \wedge e_\ell = -e_\ell\wedge d\bar{z}_j$. A $(0,q)$-form ξ with values in $\Lambda^k E$ can be written

$$\xi = \sum_{|I|=k}{}' \xi_I\wedge e_{I_1}\wedge\ldots\wedge e_{I_k},$$

where ξ_I are $(0,q)$-forms with values in L^{-k} and the prime means that the summation is performed over increasing multi-indices. One can apply both δ_f and $\bar\partial$ to such forms, and it is easy to check that δ_f and $\bar\partial$ anti-commute, i.e.,

$$\delta_f\circ\bar\partial = -\bar\partial\circ\delta_f. \tag{3.2}$$

We have the associated sheaf complex

$$0\to\mathcal{O}(\Lambda^m E)\xrightarrow{\delta_f}\cdots\xrightarrow{\delta_f}\mathcal{O}(\Lambda^2 E)\xrightarrow{\delta_f}\mathcal{O}(E)\to\mathcal{O}. \tag{3.3}$$

Given a global holomorphic section ψ of the Hermitian line bundle $S \to X$, we want to find sections q_j of $S \otimes L^{-1}$ such that

$$f_1 q_1 + \cdots + f_m q_m = \psi. \tag{3.4}$$

This precisely means that we look for a holomorphic section $q = \sum_1^m q_j e_j$ such that $\delta_f q = \psi$. A necessary condition of course is that this equation is solvable locally; i.e., that ψ belongs to the sheaf $\mathcal{J}$. If this holds, then it is easy to produce, by means of a partition of unity, a smooth global section v_1 such that $\delta_f v_1 = \psi$. Assume that we have a form (or current)

$$v = v_1 + \cdots + v_n,$$

where v_k has bidegree $(0, k-1)$ and takes values in $S \otimes \Lambda^k E$, such that

$$\delta_f v_1 = \psi, \quad \delta_f v_{k+1} = \bar\partial v_k, \ k \geq 1. \tag{3.5}$$

Introducing the symbol $\nabla_f = \delta_f - \bar\partial$ we can write (3.5) compactly as

$$\nabla_f v = \psi. \tag{3.6}$$

One readily checks that

$$\nabla_f (\xi \wedge \eta) = \nabla_f \xi \wedge \eta + (-1)^{\deg \xi} \xi \wedge \nabla_f \eta, \qquad \nabla_f^2 = 0. \tag{3.7}$$

Notice that ∇_f acts on currents as well.

Proposition 3.1. *If there is a global current solution to* (3.6) *and*

$$H^{k-1}(X, S \otimes L^{-k}) = 0, \quad 2 \leq k \leq \min(m, n+1), \tag{3.8}$$

then there is a global holomorphic solution to $\delta_f q = \psi$.

Proof. It follows from (3.5) that $\bar\partial v_{\min(m,n+1)}$ vanishes. In fact, it is equal to $\delta_f v_{\min(m,n+1)+1}$, but $v_{\min(m,n+1)+1} = 0$ for degree reasons: If $m \leq n$ then it vanishes since the Koszul complex terminates at level m; if $m > n$ it vanishes since it is then a $(0, n+1)$-form.

In view of (3.1), $S \otimes \Lambda^k E$ is a direct sum of line bundles $S \otimes L^{-k}$. If $H^{k-1}(X, S \otimes L^{-k}) = 0$ for $k = \min(m, n+1)$ we can thus find a global solution to $\bar\partial w_{\min(m,n+1)} = v_{\min(m,n+1)}$. Then $v_{\min(m,n+1)-1} + \delta_f w_{\min(m,n+1)}$ is $\bar\partial$-closed and again we can solve a global $\bar\partial$-equation provided that $H^{k-1}(X, S \otimes L^{-k}) = 0$ for $k = \min(m, n+1) - 1$. Proceeding in this way we can successively solve

$$\bar\partial w_k = v_k + \delta_f w_{k+1}, \quad k \geq 2,$$

if (3.8) holds. Finally $q := v_1 + \delta_f w_2$ is $\bar\partial$-closed, thus a holomorphic section of $S \otimes E$, and $\delta_f q = \psi$ as desired. □

If f_j locally defines a complete intersection, i.e., $\operatorname{codim} Z = m$, then the sheaf complex (3.3) is exact, and if ϕ is in $\mathcal{J}$, then one can easily find a global smooth solution to (3.6); in general however there is no global, not even current, solution[3].

[3]If there is a solution to (3.6), then ψ is in $\mathcal{J}$; this follows as above since all the $\bar\partial$-equations are solvable locally.

In order to find a global solution to (3.6) we will use residue calculus. Notice that we have the natural norm

$$|f|^2 = \sum_j |f_j|^2.$$

Let σ be the section over $X \setminus Z$ with pointwise minimal norm such that $f \cdot \sigma = \delta_f \sigma = 1$. This means that

$$\sigma = \sum_j \frac{f_j^* e_j}{|f|^2},$$

where f_j^* is the section of L^{-1} of minimal norm such that $f_j f_j^* = |f_j|^2$. We now consider the smooth forms $u_k = \sigma \wedge (\bar\partial \sigma)^{k-1}$ in $X \setminus Z$, and put

$$u = u_1 + u_2 + \cdots + u_n.$$

It is readily checked that $\nabla_f u = 1$ in $X \setminus Z$. An elegant way to see this is to observe that $u = \sigma / \nabla_f \sigma$ so that, cf. (3.7), $\nabla_f u = \nabla_f \sigma / \nabla_f \sigma = 1$, cf. [1]. If ψ is holomorphic, then $\nabla_f(u\psi) = \psi$ in $X \setminus Z$. We want to extend this equality across Z. There is indeed a natural current extension U of u across Z:

Proposition 3.2. *The form-valued function*

$$\lambda \mapsto |f|^{2\lambda} u,$$

a priori defined for $\operatorname{Re}\lambda \gg 0$*, has a current-valued analytic continuation to* $\operatorname{Re}\lambda > -\epsilon$ *and the value at* $\lambda = 0$ *is a current extension* U *of* u *across* Z*.*

The first statement means that for each test form ξ, the function

$$\lambda \mapsto \int |f|^{2\lambda} u \wedge \xi$$

admits the analytic continuation. We provide a proof in the next section.

Since $\nabla_f u = 1$ in $X \setminus Z$ it follows that

$$\nabla_f(|f|^{2\lambda} u) = 1 - (1 - |f|^{2\lambda}) - \bar\partial |f|^{2\lambda} \wedge u$$

if $\operatorname{Re}\lambda \gg 0$, and hence by uniqueness of analytic continuation we get that

$$\nabla_f U = 1 - R^f, \tag{3.9}$$

where R^f is the value at $\lambda = 0$ of $R^\lambda := 1 - |f|^{2\lambda} + \bar\partial |f|^{2\lambda} \wedge u$. It follows that R^f is a current[4] with support on Z. In view of (3.7) and (3.9) we have that

$$\nabla_f R^f = 0. \tag{3.10}$$

If $R^f \psi = 0$, which clearly holds if ψ vanishes enough on Z, then, by (3.9),

$$\nabla_f(U\psi) = (1 - R^f)\psi = \psi,$$

since $\nabla_f \psi = -\bar\partial \psi = 0$. Combining with Proposition 3.1 we thus have

[4]The component $R_0^f := (1 - |f|^{2\lambda})|_{\lambda=0}$ is zero unless $f \equiv 0$ in which case it is 1.

Proposition 3.3. *Let $f_1, \dots, f_m$ be holomorphic sections of $L \to X$ and let R^f be the associated residue current. If ψ is a global holomorphic section of S such that $R^f\psi = 0$ and if in addition* (3.8) *holds, then there are holomorphic sections q_j of $L^{-1} \otimes S$ such that $\sum f_j q_j = \psi$.*

Proof of Proposition 1.1. It is well known that $H^k(\mathbb{P}^n, \mathcal{O}(\ell)) = 0$ if either $1 \le k \le n-1$, or $q = n$ and $\ell \ge -n$, see, e.g., [8]. Thus Proposition 1.1 follows from Proposition 3.3 with $X = \mathbb{P}^n$, $L = \mathcal{O}(d)$, and $S = \mathcal{O}(\rho)$. $\square$

The global residue current R^f was introduced in [1], very much inspired by a local analogue that was defined in [18].

4. Residue calculus

If s is a complex variable, then $1/s$ is locally integrable and thus a distribution. By Cauchy's formula we have that

$$\int_s \bar\partial(1/s)\wedge\xi(s)ds = 2\pi i\xi(0) \tag{4.1}$$

for test forms ξds. One can define the distributions $1/s^m$ inductively for positive integers m by the formula

$$-m/s^{m+1} = (\partial/\partial s)(1/s^m). \tag{4.2}$$

Assume now that s_j are coordinates in $\mathbb{C}^n$, and let $s^\alpha = s_1^{\alpha_1}\cdots s_r^{\alpha_r}$ be a monomial, $r \le n$, and α_k positive integers. If a is a non-vanishing smooth function and ξ is a test form, then

$$\int \frac{|s^\alpha a|^{2\lambda}}{s^\alpha}\wedge\xi,$$

a priori defined for $\mathrm{Re}\,\lambda \gg 0$, has an analytic continuation to $\mathrm{Re}\,\lambda > -\epsilon$ and the value at $\lambda = 0$ is equal to the action of $1/s^\alpha$ on ξ, where $1/s^\alpha$ is the tensor product of the one-variable distributions $1/s_j^{\alpha_j}$. In particular the value at $\lambda = 0$ is independent of a. It is elementary to prove this when $n = 1$, and the general case then follows. Observe that the action of $\bar\partial(1/s^\alpha)$ on a test form ξ is the value at $\lambda = 0$ of

$$\int \frac{\bar\partial|s^\alpha a|^{2\lambda}}{s^\alpha}\wedge\xi. \tag{4.3}$$

Clearly $\bar\partial(1/s^\alpha)$ has support where $s^\alpha = 0$ and moreover

$$s^\alpha\bar\partial\frac{1}{s^\alpha} = 0. \tag{4.4}$$

In fact,

$$\int \bar\partial|s^\alpha a|^{2\lambda}\wedge\xi = -\int |s^\alpha a|^{2\lambda}\bar\partial\xi$$

so the value at $\lambda = 0$ is

$$\int \bar\partial\xi = \int d\xi = 0$$

since ξ has compact support. Notice also that

$$\bar{s}_1 \dots \bar{s}_r \bar{\partial} \frac{1}{s^\alpha} = d(\bar{s}_1 \dots \bar{s}_r) \wedge \bar{\partial} \frac{1}{s^\alpha} = 0; \tag{4.5}$$

this follows from the corresponding one-variable statement, which in turn is quite immediate in view of (4.2) noting that $\bar{s}\bar{\partial}(1/s) = 0$, cf. (4.1).

Proof of Proposition 3.2. It follows from Hironaka's theorem that one can find a smooth modification (a proper mapping that is a biholomorphism outside a hypersurface) $\pi \colon \tilde{X} \to X$ such that the pullback of the sheaf $\mathcal{J}$ to $\tilde{X}$ is principal, i.e., generated by a section f^0 of a line bundle $\mathcal{O}(-Y)$, where $Y = \sum \alpha_j Y_j$ is the divisor of f^0, Y_j are smooth and have normal crossings. Then $\pi^* f_j = f^0 f'_j$, where f'_j are sections of $\pi^* L \otimes \mathcal{O}(Y)$, where $\mathcal{O}(Y) = \mathcal{O}(-Y)^{-1}$, the tuple $f' = (f'_1, \dots, f'_m)$ is non-vanishing, and locally f^0 is a monomial in appropriate local coordinates. More precisely,

$$f^0 = \prod s_j^{\alpha_j}, \tag{4.6}$$

where s_j are sections that define Y_j, and locally s_j (or rather their representations in local frames) are part of a coordinate system.

Now, $\pi^* \sigma = (1/f^0)\sigma'$, where σ' is a smooth section of $\pi^* L^{-1} \otimes \mathcal{O}(-Y)$. In fact, if we choose any metric on $\mathcal{O}(-Y)$ and take the induced metric on $\mathcal{O}(Y) \otimes \pi^* L$, then $\pi^* f_j \pi^* f_j^* = \pi^* |f_j|^2 = |f^0 f'|^2 = f^0 (f^0)^* f'_j (f'_j)^*$ so that $\pi^* f_j^* = (f^0)^* (f'_j)^*$. It follows that

$$\pi^* \sigma = \pi^* \frac{\sum f_j^* e_j}{|f|^2} = \frac{1}{f^0} \frac{\sum_j (f'_j)^* \pi^* e_j}{|f'|^2}.$$

Thus

$$\pi^* u_k = \frac{1}{(f^0)^k} \sigma' \wedge (\bar{\partial} \sigma')^{k-1} = \frac{1}{(f^0)^k} u'_k,$$

where u'_k is smooth, and if ξ is a test form we have that

$$\int_X |f|^{2\lambda} u_k \wedge \xi = \int_{\tilde{X}} |f^0|^{2\lambda} |f'|^{2\lambda} \frac{1}{(f^0)^k} u'_k \wedge \pi^* \xi \tag{4.7}$$

if $\operatorname{Re} \lambda \gg 0$. In view of (4.6) and the discussion above we see that the right-hand side of (4.7) admits an analytic continuation to $\operatorname{Re} \lambda > -\epsilon$, and so the left-hand side does. □

It follows from the proof that $U = \pi_* ((1/(f^0)^k) \wedge u'_k)$. Notice also that

$$R_k^f = \pi_* \tilde{R}_k, \tag{4.8}$$

cf. (4.3), where

$$\tilde{R}_k = \bar{\partial} \frac{1}{(f^0)^k} \wedge u'_k. \tag{4.9}$$

For degree reasons, $R_k^f = 0$ when $k > \mu$.

Theorem 4.1 (Duality theorem). *If* $\operatorname{codim} Z = m$ *and* ψ *is holomorphic, then locally* $\psi \in \mathcal{J}$ *if and only if* $\psi R^f = 0$.

This statement can be deduced from the analogous classical theorem due to Passare and Dickenstein-Sessa for the so-called Coleff-Herrera product defined by the f_j, but it is easier to give a direct proof.

Proof. The "if"-part is already proved in the previous section, and does not depend on $\operatorname{codim} Z$. For the converse, we first claim that $R^f_k = 0$ if $k < m$. In fact, if h is a holomorphic function that (locally) vanishes on Z, then $\pi^* h$ vanishes on Y, and therefore, locally, it must contain each coordinate factor in f^0. In view of (4.5), therefore, $\pi^* d\bar{h} \wedge \tilde{R}_k = 0$, and by (4.8) it follows that $d\bar{h} \wedge R^f_k = \pi_*(\pi^* d\bar{h} \wedge \tilde{R}_k) = 0$. Consider now a neighborhood of a point on Z_{reg} and choose a coordinate system w such that $w_1, \ldots, w_m$ vanish on Z. Then $d\bar{w}_j \wedge R^f_k = 0$ for $j = 1, \ldots, m$ and hence R^f_k must be of the form $\alpha d\bar{w}_1 \wedge \cdots \wedge d\bar{w}_m$, and so it vanishes unless $k = m$. It follows that R^f_k must have support on Z_{sing}.

Assume now that h is holomorphic and vanishes (locally) on the regular part of Z_{sing}. Then $|h|^{2\lambda} R^f_k$ vanishes when $\lambda = 0$. From this one can deduce, cf. [5], that R^f_k is unaffected if we redefine it as the direct image of only those terms in the development of $\bar{\partial}(1/(f^0)^k) \wedge u'_k$, where $\bar{\partial}$ falls on a factor $1/s_j^{\alpha_j}$ such that the zero set of s_j is contained in $\pi^{-1} Z_{\text{sing}}$. As before thus $d\bar{h} \wedge R^f_k = 0$. Arguing as above we find that R^f_k has support on the singular part of Z_{sing}. By finite induction we conclude that $R^f_k = 0$.

Thus $R^f = R^f_m$. If now $\psi = \delta_f \xi$ (locally somewhere on $\mathbb{P}^n$) for a holomorphic ξ, then by (3.7) and (3.10),

$$\psi R^f = \nabla_f \xi R^f = \nabla_f (\xi \wedge R^f).$$

However, for degree (with respect to $\Lambda^{\bullet} E$) reasons $\xi \wedge R^f = \xi \wedge (R^f_{m-1} + R^f_{m-2} + \cdots) = 0$, and thus $\psi R^f = 0$. □

Remark 1. The claim in the proof above is an instance of a general dimension principle (proved basically in the same way) that a *pseudomeromorphic*, a notion introduced in [5], current of bidegree $(*, q)$ that has support on a subvariety of codimension strictly larger than q must vanish, see [5]. □

5. Integral closure and distinguished varieties

Let $f_1, \ldots, f_m$ be global holomorphic sections of a Hermitian line bundle $L \to X$, and let $\mathcal{J}$ be the coherent ideal sheaf they generate. Let

$$\pi_+ \colon X_+ \to X$$

be the normalization of the blow-up of X along $\mathcal{J}$, and let $Y^+ = \sum r_j Y^+_j$ be the exceptional divisor; here Y^+_j are irreducible Cartier divisors. The images $Z_j = \pi_+ Y^+_j$ are called the Fulton-MacPherson *distinguished varieties* associated with $\mathcal{J}$, cf. [10]. As in the case with the smooth modification in the proof of Proposition 3.2, we have a factorization $\pi^*_+ f = f^0_+ f'_+$, where f^0_+ is a section that defines the divisor

Y^+. However, X_+ is not necessarily smooth, and in any case we may not assume that f^0_+ is locally like a monimial, i.e., we do not have normal crossings.

Recall that a (germ of a function) ψ belongs to the *integral closure* $\overline{\mathcal{J}_x}$ of the ideal $\mathcal{J}_x$ in the local ring $\mathcal{O}_x$ at x if $\pi_+^*\psi$ vanishes to order (at least) r_j on Y_j^+ for all j such that $x \in Z_j$. This holds if and only if $|\pi_+^*\psi| \le C|f^0|$ (in a neighborhood of $\pi^{-1}(x)$), which in turn holds if and only if $|\psi| \le C|f|$ in some neighborhood of x. It follows that

$$|\psi| \le C|f|^\ell \quad \text{if and only if} \quad \psi \in \overline{\mathcal{J}_x^\ell}. \tag{5.1}$$

We will use the geometric estimate

$$\sum r_j \deg_L Z_j \le \deg_L X, \tag{5.2}$$

from [9] (Proposition 3.1); see also [14], formula (5.20). Here

$$\deg_L Z_j = \int_{Z_j} \omega_L^{\dim Z_j},$$

where ω_L is the first Chern form for L.

If $X = \mathbb{P}^n$ and $L = \mathcal{O}(d)$ with the natural metric, the first Chern form is $d\Omega$, where $\Omega = dd^c \log |z|^2$. By (5.2) we therefore have that

$$\sum_j r_j \int_{Z_j} (d\Omega)^{\dim Z_j} \le \int_X (d\Omega)^n$$

which implies, cf. (*) p. 432 in [9], that

$$\sum_j r_j d^{\dim Z_j} \deg Z_j \le d^n. \tag{5.3}$$

6. Proofs of the theorems

Proof of Theorem 1.2. Take $\rho \ge \deg \Phi + \mu d^{c_\infty}$, and as before let $\psi = z_0^{\rho - \deg \Phi} \phi$, where ϕ is the $\deg \Phi$-homogenization of Φ and thus a holomorphic section of $\mathcal{O}(\deg \Phi)$.

Consider the normalization of the blow-up $\pi_+ \colon X^+ \to \mathbb{P}^n$ along $\mathcal{J}$ and let $Y^+ = \sum_j r_j Y_j^+$ be the exceptional divisor as before. If $\pi_+ Y_j^+$ is not fully contained in the hyperplane at infinity H_∞, then the hypothesis (1.4) implies that $\pi_+^*\phi$ and hence $\pi_+^*\psi$ vanish to order μr_j on Y_j^+, cf. the discussion in the previous section. On the other hand, if $\pi_+ Y_j^+$ is contained in H_∞, then $\pi_+^* z_0$ vanishes on Y_j^+ and hence $\pi_+^*\psi$ must vanish to order $\rho - \deg \Phi$ on Y_j^+. However, $\rho - \deg \Phi \ge \mu d^{c_\infty}$ and by (5.3) and the definition of c_∞ it follows that $d^{c_\infty} \ge r_j$. In view of (5.1) we conclude that

$$|\psi| \le C|f|^\mu \tag{6.1}$$

on $\mathbb{P}^n$.

We will now use the same notation as in the proof of Proposition 3.2. The hypothesis (6.1) implies that $|\pi^*\psi| \leq C|f^0|^\mu$ in $\tilde{X}$; since locally f^0 is (a non-vanishing) holomorphic function times the monomial s^α it follows that $\pi^*\psi$ must contain the factor $s^{\mu\alpha} \sim (f^0)^\mu$. From (4.9) and (4.4) we have that $(\pi^*\psi)\tilde{R}_k = 0$ since $k \leq \mu$, and we can conclude that $\psi R^f = \pi_*\big(\pi^*\psi\tilde{R}\big) = 0$ as wanted. Now part (i) of Theorem 1.2 follows from Proposition 1.1.

The proof of Theorem 1.2 (ii) requires a more delicate argument. Again we have to prove that $\psi R^f = 0$ under the stated assumptions. Following [5] we can decompose R^f as

$$R^f = \mathbf{1}_{\mathbb{C}^n} R^f + \mathbf{1}_{H_\infty} R^f$$

where the first term is an extension to $\mathbb{P}^n$ of the natural restriction of R^f to $\mathbb{C}^n$, and the second term has support on H_∞. To see this, notice that

$$R^f_k = \sum_j \pi_*\Big[\bar\partial \frac{1}{s_j^{k\alpha_j}} \frac{1}{\Pi_{i\neq j} s_i^{k\alpha_i}} \wedge u'_k\Big] =: \pi_* \sum_j \tilde{R}_{kj} =: \sum_j R^f_{kj}.$$

Let h be the section z_0 of $\mathcal{O}(1)$ and define

$$\mathbf{1}_{\mathbb{C}^n} R^f = |h|^{2\lambda} R^f|_{\lambda=0}. \tag{6.2}$$

The existence of the analytic continuation follows as in the proof of Proposition 3.2, and if we define $\mathbf{1}_{H_\infty} R^f = R^f - \mathbf{1}_{\mathbb{C}^n} R^f$ it is readily checked that

$$\mathbf{1}_{H_\infty} R^f = \sum_k \sum_{\pi Y_j \subset H_\infty} R^f_{kj}. \tag{6.3}$$

Clearly this current has support on H_∞. By the duality theorem (Theorem 4.1), $\psi R^f = 0$ in $\mathbb{C}^n$, and thus $\psi |h|^{2\lambda} R^f$ vanishes for $\operatorname{Re}\lambda \gg 0$. From (6.2) we conclude that $\psi \mathbf{1}_{\mathbb{C}^n} R^f = 0$.

It is well known that π factorizes over π_+, i.e., we have

$$\tilde{X} \xrightarrow{\tau} X_+ \xrightarrow{\pi_+} X.$$

Now consider a fixed $Y_j \subset \tilde{X}$ such that $\pi Y_j \subset H_\infty$. First assume that τ maps Y_j onto one of the Y_i^+. We know that $\pi_+^*\psi$ vanishes at least to the same order as $(f_+^0)^\mu$ (i.e., μr_i) on Y_i^+ and hence $\pi^*\psi = \tau^*\pi_+^*\psi$ must vanish to the same order as $(f^0)^\mu = \tau^*(f_+^0)^\mu$ on Y_j. It follows that $\pi^*\psi\tilde{R}_{kj} = 0$ and thus $\psi R^f_{kj} = 0$. Now assume that τY_j has codimension ≥ 2 in X_+. There is a smooth form $u'_{+,k}$ in X_+, defined precisely as u'_k is defined in $\tilde{X}$, see the proof of Proposition 3.2, such that $\tau^* u'_{+,k} = u'_k$. Thus

$$\tau_*\tilde{R}_{kj} = \tau_*\Big[\bar\partial \frac{1}{s_j^{k\alpha_j}} \frac{1}{\Pi_{i\neq j} s_i^{k\alpha_i}}\Big] \wedge u'_{+,k}. \tag{6.4}$$

By the dimension principle[5], cf. Remark 1, the first factor on the right-hand side of (6.4) must vanish, since it has bidegree $(0,1)$ and support on a variety of codimension at least 2. Thus $\tau_*\tilde{R}_{kj}=0$ and hence $\pi_*\tilde{R}_{kj}=(\pi_+)_*\tau_*\tilde{R}_{kj}=0$. In view of (6.3) it follows that $\psi\mathbf{1}_{H_\infty}R^f=0$. Summing up we conclude that $\psi R^f=0$. □

For a slightly different proof of part (ii), see [6].

Proof of Theorem 1.6. The hypothesis (1.7) implies that $|\pi^*\psi|\le C|f^0|^\mu$ in $\tilde{X}$. As in the previous proof we conclude that $\psi R^f=0$. Now let $S=L^s\otimes A\otimes K_X$ with $s\ge\min(m,n+1)$. Then $L^{-k}\otimes S\otimes K_X^{-1}=L^{s-k}\otimes A$ is ample or at least big and nef when $k\le\min(m,n+1)$. It follows from the Kodaira and/or Kawamata-Viehweg vanishing theorems that the cohomology groups in (3.8) vanish, and so Theorem 1.6 (i) follows from Proposition 3.3.

If ψ vanishes to order μr_j at a generic point on Z_j, then it is not hard to see that $\pi_+^*\psi$ vanishes to order μr_j on Y_j; see [14] Section 10.5 for details (e.g., the proof of Lemma 10.5.2). If this holds for each j we thus have, cf. (5.1), that $|\psi|\le C|f|^\mu$. Thus part (ii) follows. □

Theorems 2.1 and 2.2 are proved completely analogously, but instead of the Koszul complex we use a certain product of Koszul complexes, cf. [3], page 368. We omit the details.

7. The worst possible situation for the Nullstellensatz

Let us now sum up our proof of the Nullstellensatz, Corollary 1.3, so assume that $\Phi=1$, $Z\subset H_\infty=\{[z]\in\mathbb{P}^n;\ z_0=0\}$, and let $\psi=z_0^\rho$. If $\pi_+^*\psi$ vanishes to order μr_j on Y_j^+ for each j, then we have (1.1) with $\deg F_jQ_j\le\rho$ (provided that $\rho\ge\gamma$).

In view of (5.3), in most cases each r_j and also μr_j will be much smaller than d^n so one gets a degree bound that is much smaller than d^n. The worst case scenario should be when one has just one distinguished point $\{p\}=\pi_+Y_1^+$ where Y_1^+ has multiplicity $r_1=d^n$. In addition, $\pi_+^*z_0$ must vanish just to order 1 on Y_1^+. As we will see now this is precisely the situation in the following example that appeared in Kollár's paper [13].

Example 1. Let

$$F_1(z)=1-z_1z_n^{d-1},\quad F_2(z)=z_1^d-z_2z_n^{d-1},\ldots$$
$$\ldots,F_{n-1}=z_{n-2}^d-z_{n-1}z_n^{d-1},\quad F_n(z)=z_{n-1}^d.$$

It is readily seen that F_j have no common zeros in $\mathbb{C}^n$, and hence by Kollár's theorem there are Q_j such that

$$F_1Q_1+\cdots+F_nQ_n=1,\qquad \deg F_jQ_j\le d^n. \tag{7.1}$$

[5]Here we use residue calculus on a possibly non-smoooth variety; this does not offer any substantial new difficulties, see, e.g., [6].

On the curve

$$t \mapsto \gamma(t) = (t^{d-1}, t^{d^2-1}, \dots, t^{d^{n-1}-1}, 1/t)$$

we get the equality

$$1 = F_n(\gamma(t))Q_n(\gamma(t)) = t^{d^n-d}Q_n(t^{d-1}, t^{d^2-1}, \dots, t^{d^{n-1}-1}, 1/t),$$

which implies that Q_n must have degree at least $d^n - d$, and hence $\deg F_n Q_n \geq d^n$. Thus the bound in Kollar's theorem is optimal. $\square$

It is not hard to find explicit Q_k such that (7.1) holds: Let f_k be the d-homogenizations of F_k and take the homogeneous polynomials p_k such that

$$f_k p_k = z_{k-1}^{d^{n-k+1}} - z_k^{d^{n-k}} z_n^{d^{n-k+1}-d^{n-k}}, \ k = 1, \dots, n-1, \quad p_n f_n = z_{n-1}^d.$$

It is then easy to produce forms q_k such that $\sum f_k q_k = z_0^{d^n}$.

Let us consider this example in some more detail. In the affinization of $\mathbb{P}^n$ where $z_n = 1$ we have affine variables $z_0, \dots, z_{n-1}$. The resulting polynomial ideal is

$$J = (z_1 - z_0^d, z_2 - z_1^d, \dots, z_{n-1} - z_{n-2}^d, z_{n-1}^d),$$

which has a single zero at the point $p = (0, \dots, 0)$, i.e., $[0, \dots, 0, 1]$ in homogeneous coordinates. It is readily checked that there are no other zeros on H_∞, as expected in view of the discussion above.

We have another proof that if q_j are homogeneous forms such that $\sum f_j q_j = z_0^\ell$ then $\ell \geq d^n$. In fact, if this holds, then in particular z_0^ℓ must belong to the local ideal J_p. Notice that

$$\begin{aligned} J_p &= (z_1 - z_0^d, z_2 - z_1^d, \dots, z_{n-1} - z_{n-2}^d, z_{n-2}^{d^2}) \\ &= (z_1 - z_0^d, z_2 - z_1^d, \dots, z_{n-1} - z_{n-2}^d, z_{n-3}^{d^3}) \\ &= \cdots \\ &= (z_1 - z_0^d, z_2 - z_1^d, \dots, z_{n-1} - z_{n-2}^d, z_0^{d^n}). \end{aligned}$$

By a holomorphic change of variables, we have $J_p = (w_1, \dots, w_{n-1}, z_0^{d^n})$ and it is now obvious that $\ell \geq d^n$ if z_0^ℓ is in J_p.

A final remark. In this example the forms f_j actually define a complete intersection so $z_0^{d^n} R^f = 0$ by the duality theorem; thus our framework, i.e., Proposition 1.1, actually produces an optimal solution, i.e., such that (7.1) holds. In the same way, as long as we have n generators $f_1, \dots, f_n$ and only isolated distinguished points p_i, f_j is a complete intersection there and therefore $z_0^{d^n} R^f = 0$, since by the local Bezout theorem $z_0^{d^n}$ belongs to each local ideal $\mathcal{J}_{p_i}$. In this case we thus get the optimal Nullstellensatz, without the annoying factor μ in front of d^n. Unfortunately, we do not know how to get rid of this factor in general.

References

[1] M. Andersson: *Residue currents and ideals of holomorphic functions*, Bull. Sci. Math. **128**, (2004), 481–512.

[2] M. Andersson: *The membership problem for polynomial ideals in terms of residue currents*, Ann. Inst. Fourier **56** (2006), 101–119.

[3] M. Andersson: *Explicit versions of the Briancon-Skoda theorem with variations*, Michigan Math. J. **54(2)** (2006), 361–373.

[4] M. Andersson & E. Götmark: *Explicit representation of membership in polynomial ideals*, Math. Ann. **349** (2011), 345–365.

[5] M. Andersson & E. Wulcan: *Decomposition of residue currents*, J. Reine Angew. Math. **638** (2010), 103–118.

[6] M. Andersson & E. Wulcan: *On the effective membership problem on singular varieties*, Preprint, Göteborg 2011.

[7] J. Briançon & H. Skoda: *Sur la clôture intégrale d'un idéal de germes de fonctions holomorphes en un point de* $\mathbb{C}^n$, C. R. Acad. Sci. Paris Sér. A 278 (1974), 949–951.

[8] J-P Demailly: *Complex Analytic and Differential Geometry*, Monograph Grenoble (1997).

[9] L. Ein & R. Lazarsfeld: *A geometric effective Nullstellensatz*, Invent. math. **135** (1999), 427–448.

[10] W. Fulton: *Intersection theory. Ergebnisse der Mathematik und ihrer Grenzgebiete*, Springer-Verlag, Berlin, 1984. xi+470 pp.

[11] M. Hickel: *Solution d'une conjecture de C. Berenstein-A. Yger et invariants de contact à l'infini*, Ann. Inst. Fourier **51** (2001), 707–744.

[12] Z. Jelonek: *On the effective Nullstellensatz*, Invent. math. **162** 1–17 (2005).

[13] J. Kollár: *Sharp effective Nullstellensatz*, J. American Math. Soc. **1** (1988), 963–975.

[14] R. Lazarsfeld: *Positivity in algebraic geometry I and II*, Springer-Verlag 2004.

[15] F.S. Macaulay: *The algebraic theory of modular systems*, Cambridge Univ. Press, Cambridge 1916.

[16] M. Nöther: *Über einen Satz aus der Theorie der algebraischen Functionen*, Math. Ann. (1873), 351–359.

[17] M. Sombra: *A sparse effective Nullstellensatz*, Adv. in Appl. Math. **22** (1999) 271–295.

[18] M. Passare & A. Tsikh & A. Yger: *Residue currents of the Bochner-Martinelli type*, Publ. Mat. **44** (2000), 85–117.

Mats Andersson and Elizabeth Wulcan
Department of Mathematics
Chalmers University of Technology
and the University of Gothenburg
S-412 96 Gothenburg, Sweden
e-mail: matsa@chalmers.se
wulcan@chalmers.se

http://www.springer.com/978-3-0348-0141-6

Notions of Positivity and the Geometry of Polynomials
Brändén, P.; Passare, M.; Putinar, M. (Eds.)
2011, XX, 404 p., Hardcover
ISBN: 978-3-0348-0141-6
A product of Birkhäuser Basel