Chapter II
\(\ell_p\)-spaces

6 Entropy numbers and eigenvalues

6.1 Preliminaries and notation

The main aim of Chapter II is to study entropy numbers in (weighted) \(\ell_p\)-spaces. This will be done in the Sections 7–9. In the present section we describe briefly the necessary abstract background without proofs. We follow closely [ET96] where proofs, further details, explanations, and more references are given.

A quasi-norm on a complex linear space \(B\) is a map \(\| \cdot \|_B\) from \(B\) to the non-negative reals \(\mathbb{R}_+\) such that
\[
\|\lambda x\|_B = |\lambda| \|x\|_B \quad \text{for all scalars } \lambda \in \mathbb{C} \text{ and all } x \in B,
\]
there is a constant \(C\) such that for all \(x \in B\) and \(y \in B\)
\[
\|x + y\|_B \leq C(\|x\|_B + \|y\|_B).
\]
(6.3)

Plainly \(C \geq 1\); if \(C = 1\) is allowed then \(\| \cdot \|_B\) is a norm in \(B\). As usual, \(B\) is called a quasi-Banach space if every Cauchy sequence with respect to \(\| \cdot \|_B\) is a convergent sequence.

Given any \(p \in (0, 1]\), a \(p\)-norm on a complex linear space \(B\) is a map \(\| \cdot \|_B\) \(\rightarrow \mathbb{R}_+\) which satisfies (6.1), (6.2), and instead of (6.3),
\[
\|x + y\|_B^p \leq \|x\|_B^p + \|y\|_B^p \quad \text{for } x, y \in B.
\]
(6.4)

Two quasi-norms or \(p\)-norms \(\| \cdot \|_1\) and \(\| \cdot \|_2\) are said to be equivalent if there is a constant \(c \geq 1\) such that for all \(x \in B\),
\[
c^{-1}\|x\|_1 \leq \|x\|_2 \leq c \|x\|_1.
\]
(6.5)

It can be shown (see [Kön86], p. 47 or [DeVL93], p. 20) that if \(\| \cdot \|_1\) is a quasi-norm on \(B\) then there exists \(p \in (0, 1]\) and a \(p\)-norm \(\| \cdot \|_2\) on \(B\) which is equivalent to \(\| \cdot \|_1\).

Let \(A, B\) be quasi-Banach spaces and let \(T : A \rightarrow B\) linear. Just as for the Banach space case, \(T\) will be called bounded or continuous if
\[
\|T\| = \sup\{\|Ta\| : a \in A, \|a\|_A \leq 1\} < \infty.
\]
(6.6)
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The family of all such $T$ will be denoted by $L(A, B)$ or $L(A)$ if $A = B$. Otherwise terminology which is standard in the context of Banach spaces will be taken without further comment to quasi-Banach spaces. In particular if $T \in L(B)$ then $\sigma(T)$ stands for its spectrum.

In [ET96], pp. 3–7, we developed a Riesz theory for compact operators $T \in L(B)$ in quasi-Banach spaces $B$ parallel to the well-known assertions in the Banach spaces case. Especially, if $T \in L(B)$ is compact, then $\sigma(T) \setminus \{0\}$ consists of an at most countably infinite number of eigenvalues of finite algebraic multiplicity which may accumulate only at the origin.

If $B$ is a quasi-Banach space then $U_B = \{b \in B : \|b\| \leq 1\}$ stands for the unit ball in $B$.

6.2 Definition Let $A, B$ be quasi-Banach spaces and let $T \in L(A, B)$. Then for all $k \in \mathbb{N}$, the $k$th entropy number $e_k(T)$ of $T$ is defined by

$$e_k(T) = \inf \left\{ \varepsilon > 0 : T(U_A) \subset \bigcup_{j=1}^{2^{k-1}} (b_j + \varepsilon U_B) \text{ for some } b_1, \ldots, b_{2^{k-1}} \in B \right\}.$$  (6.7)

6.3 Remark This formulation coincides with the definition given in [ET96], p. 7, which simply generalizes to quasi-Banach spaces what has been done before for Banach spaces. Further comments and some discussions may be found in [ET96], pp. 7–9, and, in greater detail, in [CaS90] and [EEv87].

6.4 Proposition Let $A, B, C$ be quasi-Banach spaces, let $S, T \in L(A, B)$ and suppose that $R \in L(B, C)$.

(i) $\|T\| \geq e_1(T) \geq e_2(T) \geq \ldots ; e_1(T) = \|T\|$ if $B$ is a Banach space.

(ii) For all $k, l \in \mathbb{N}$

$$e_{k+l-1}(R \circ S) \leq e_k(R) e_l(S).$$  (6.8)

(iii) If $B$ is a $p$-Banach space, where $0 < p \leq 1$, then for all $k, l \in \mathbb{N}$

$$e_{k+l-1}^p(S + T) \leq e_k^p(S) + e_l^p(T).$$  (6.9)

6.5 Remark This formulation coincides with Lemma 1 in [ET96], pp. 7,8, where also a simple proof may be found. In case of quasi-Banach spaces it may happen that $\|T\| > e_1(T)$, see [ET96], Remark 4 on p. 9.
6.6 Compact operators
Recall that \( T \in L(B) \) is compact if, and only if, for every \( \varepsilon > 0 \) there is a finite \( \varepsilon \)-net covering \( T(U_B) \). By (6.7) this is the same as

\[
T \in L(B) \text{ is compact if, and only if, } e_k(T) \to 0 \text{ for } k \to \infty. \tag{6.10}
\]

6.7 Interpolation properties
The entropy numbers behave very well with respect to real interpolation of quasi-Banach spaces. We gave in [ET96], pp. 13–15, a rather careful treatment of this subject, which in turn was based on [HaT94a]. Further properties in the context of Banach spaces and historical comments may be found in [Tri78], 1.16.2, and [Pie80], 12.1.

6.8 Eigenvalues
Again let \( B \) be a (complex) quasi-Banach space and let \( T \in L(B) \) be compact. As we mentioned at the end of 6.1 the spectrum of \( T \), apart from the point 0, consists solely of eigenvalues of finite algebraic multiplicity: let \( \{\mu_k(T)\}_{k \in \mathbb{N}} \) be the sequence of all non-zero eigenvalues of \( T \), repeated according to algebraic multiplicity and ordered so that

\[
|\mu_1(T)| \geq |\mu_2(T)| \geq \ldots \to 0. \tag{6.11}
\]

If \( T \) has only \( m(< \infty) \) distinct eigenvalues and \( M \) is the sum of their algebraic multiplicities we put \( \mu_n(T) = 0 \) for all \( n > M \).

6.9 Theorem Let \( T \) and \( \{\mu_k(T)\}_{k \in \mathbb{N}} \) be as in 6.8. Then

\[
\left( \prod_{m=1}^{k} |\mu_m(T)| \right)^{1/k} \leq \inf_{n \in \mathbb{N}} 2^{\frac{n}{n}} e_n(T), \quad k \in \mathbb{N}. \tag{6.12}
\]

6.10 Corollary For all \( k \in \mathbb{N} \)

\[
|\mu_k(T)| \leq \sqrt{2} e_k(T). \tag{6.13}
\]

6.11 Remark This is Carl’s famous inequality which connects spectral properties of compact operators with the geometry of the map \( T \) described in terms of entropy numbers. (6.13) in the context of Banach spaces was proved by Carl in [Carl81]. In [ET96], pp. 18–20, we gave a proof of (6.12) which generalizes the proof given in [CaT80] from Banach spaces to quasi-Banach spaces. Plainly, (6.13) follows from (6.11) and (6.12) with \( n = k \).

6.12 Remark Further results, comments, references and, in particular comparisons of entropy numbers with other geometric quantities, especially approximation numbers, may be found in [ET96], [CaS90], [EEv87], [Kön86], [Pie87], and [LGM96].
7 The spaces $\ell^M_p$

7.1 Preliminaries and notation

We follow again [ET96], p. 97. Let $M \in \mathbb{N}$ and let $0 < p \leq \infty$. By $\ell^M_p$ we shall mean the linear space of all complex $M$-tuples $y = (y_j)$, endowed with the quasi-norm

$$
\|y\|_{\ell^M_p} = \left( \sum_{j=1}^{M} |y_j|^p \right)^{\frac{1}{p}}, \quad \text{if} \quad 0 < p < \infty,
$$

(7.1)

and

$$
\|y\|_{\ell^M_\infty} = \sup_j |y_j|, \quad \text{if} \quad p = \infty.
$$

(7.2)

Let

$$
U^M_p = \{ y \in \ell^M_p : \|y\|_{\ell^M_p} \leq 1 \}
$$

(7.3)

be the closed unit ball in $\ell^M_p$. Since $\mathbb{C}^M$ may be identified with $\mathbb{R}^{2M}$, we shall understand by the volume of $U^M_p$ the Lebesgue measure of

$$
\left\{ (x_1, \ldots, x_{2M}) \in \mathbb{R}^{2M} : \sum_{j=1}^{M} (x_{2j-1}^2 + x_{2j}^2)^{\frac{p}{2}} \leq 1 \right\}.
$$

(7.4)

Let $p \in (0, \infty]$ be given. There are two positive constants $c_1$ and $c_2$ (which may depend on $p$) such that for all $M \in \mathbb{N}$

$$
c_1 M^{-\frac{1}{p}} \leq \left( \text{vol } U^M_p \right)^{\frac{1}{2M}} \leq c_2 M^{-\frac{1}{p}}.
$$

(7.5)

This follows from the Proposition in [ET96], p. 97, and the end of the proof on p. 98.

Plainly the identity from $\ell^M_{p_1}$ in $\ell^M_{p_2}$ is a compact operator. Our aim is to estimate the corresponding entropy numbers according to Definition 6.2. In what follows we assume that $\log = \log_2$ is taken with respect to the base 2. First we complement the results in [ET96], p. 98.

7.2 Proposition Let $0 < p_1 \leq \infty, 0 < p_2 \leq \infty$ and for each $k \in \mathbb{N}$ let $e_k$ be the entropy numbers of the embedding

$$
id : \ell^M_{p_1} \rightarrow \ell^M_{p_2}.
$$

Then

$$
e_k \geq c \quad \text{if} \quad 1 \leq k \leq \log(2M),
$$

(7.6)

and

$$
e_k \geq c \frac{1}{(2M)^{\frac{1}{p_2}}} \left( \frac{p_2}{p_1} \right)^{\frac{1}{p_1}} \quad \text{if} \quad k \in \mathbb{N},
$$

(7.7)

where $c$ is a positive constant which is independent of $M$ (and $k$) but may depend upon $p_1$ and $p_2$. 


Proof.
Step 1. We prove (7.6). Let $y = (y_j) \in \ell_p^M$ for some $p$ where all components $y_j$ are zero with exception of one component which is either 1 or $-1$. There are $2M$ such elements belonging to $U_{p_1}^M$ and $U_{p_2}^M$. Let $y^1$ and $y^2$ be two such points and assume that they belong to the same $\ell_{p_2}^M$-ball of radius $\varepsilon$, hence

$$y^1 \in x + \varepsilon U_{p_2}^M \quad \text{and} \quad y^2 \in x + \varepsilon U_{p_2}^M \quad \text{for some} \quad x \in \ell_{p_2}^M.$$  (7.8)

For some $c$ which is independent of $M$ and $\overline{p_2} = \min(p_2, 1)$ we have

$$c \leq \| y^1 - y^2 \|_{\ell_{p_2}^M} \leq \| y^1 - x \|_{\ell_{p_2}^M} \overline{p_2} + \| y^2 - x \|_{\ell_{p_2}^M} \overline{p_2} \leq 2 \varepsilon \overline{p_2}. \quad \text{(7.9)}$$

Now (7.6) follows from (7.9), $2^{k-1} \leq M < 2M$ and (6.7).

Step 2. We prove (7.7). We cover $U_{p_1}^M$ with $2^{k-1}$ balls in $\ell_{p_2}^M$ of radius $\varepsilon$ chosen in an appropriate way. Then we have by the interpretation (7.4)

$$\text{vol } U_{p_1}^M \leq 2^{k-1} \varepsilon^{2M} \text{vol } U_{p_2}^M \leq 2^{k} e_k^{2M} \text{vol } U_{p_2}^M. \quad \text{(7.10)}$$

Now (7.7) follows from (7.10) and (7.5).

7.3 Theorem Let $0 < p_1 \leq p_2 \leq \infty$ and for each $k \in \mathbb{N}$ let $e_k$ be the $k$th entropy number of the embedding

$$\text{id} : \ell_{p_1}^M \to \ell_{p_2}^M.$$  

Then

$$c_1 \leq e_k \leq c_2 \quad \text{if} \quad 1 \leq k \leq \log(2M), \quad \text{(7.11)}$$

$$e_k \leq c_2 \left( k^{-1} \log(1 + \frac{2M}{k}) \right)^{\frac{1}{p_1} - \frac{1}{p_2}} \quad \text{if} \quad \log(2M) \leq k \leq 2M, \quad \text{(7.12)}$$

$$c_1 2^{- \frac{k}{M} (2M) \frac{1}{p_2} - \frac{1}{p_1}} \leq e_k \leq c_2 2^{- \frac{k}{M} (2M) \frac{1}{p_2} - \frac{1}{p_1}} \quad \text{if} \quad k \geq 2M, \quad \text{(7.13)}$$

where $c_1$ and $c_2$ are positive constants which are independent of $M$ (and $k$) but may depend upon $p_1$ and $p_2$.

7.4 Remark The estimate from below is covered by Proposition 7.2. A proof of the estimate from above may be found in [ET96], pp. 98–101.

7.5 Remark If $1 \leq p_1 < p_2 \leq \infty$, then also the estimate (7.12) is an equivalence as in the two other cases: see [Schü84] and [Kön86], 3.c.8, pp. 190–191.
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8 Weighted $\ell_p$-spaces

8.1 Preliminaries and notation

Let $d > 0$, $\delta \geq 0$ and $(M_j)_{j \in \mathbb{N}_0}$ be a sequence of natural numbers. We always assume that there are two positive numbers $c_1$ and $c_2$ with
\[ c_1 \leq M_j 2^{-jd} \leq c_2 \quad \text{for every} \quad j \in \mathbb{N}_0. \]  \hspace{1cm} (8.1)

Let $0 < p \leq \infty$ and $0 < q \leq \infty$. Then by $\ell_q(2^{j\delta} \ell_p^{M_j})$ we shall mean the linear space of all complex sequences $x = (x_{j,l} : j \in \mathbb{N}_0; \ l = 1, \ldots, M_j)$ endowed with the quasi-norm
\[ \|x|\ell_q(2^{j\delta} \ell_p^{M_j})\| = \left( \sum_{j=0}^{\infty} \left( \sum_{l=1}^{M_j} 2^{j\delta p} |x_{j,l}|^p \right)^{\frac{1}{q}} \right)^{\frac{1}{q}} \]  \hspace{1cm} (8.2)

with the obvious modifications according to (7.2) if $p = \infty$ and/or $q = \infty$. In case of $\delta = 0$ we write $\ell_q(\ell_p^{M_j})$ and if, in addition $p = q$, then we have the $\ell_p$-spaces with the components ordered in the given way. Plainly, $\ell_q(2^{j\delta} \ell_p^{M_j})$ consists of dyadic blocks of spaces $\ell_p^{M_j}$ as introduced in 7.1 clipped together via the weights $2^{j\delta}$. We are interested in the counterpart of Theorem 7.3. Let $d > 0$, $\delta > 0$ and
\[ 0 < p_1 \leq p_2 \leq \infty, \quad 0 < q_1 \leq \infty, \quad 0 < q_2 \leq \infty. \]  \hspace{1cm} (8.3)

Then the identity map
\[ id : \ell_{q_1}(2^{j\delta} \ell_{p_1}^{M_j}) \to \ell_{q_2}(\ell_{p_2}^{M_j}) \]  \hspace{1cm} (8.4)
is compact, where $M_j$ is restricted by (8.1). To prove this claim we use the decomposition
\[ id = \sum_{j=0}^{\infty} id_j \]  \hspace{1cm} (8.5)

where
\[ id_j x = (\delta_{jk} x_{k,l} : k \in \mathbb{N}_0; \ l = 1, \ldots, M_k) = (0, \ldots, 0, x_{j,1}, \ldots, x_{j,M_j}, 0, 0, \ldots) \]  \hspace{1cm} (8.6)
selects the $j$th block. We have
\[ \|id_j x|\ell_{q_2}(\ell_{p_2}^{M_k})\| = \|(x_{j,l})|\ell_{p_2}^{M_j}\| \]
\[ \leq \|(x_{j,l})|\ell_{p_1}^{M_j}\| \]
\[ \leq 2^{-j\delta} \|x|\ell_{q_1}(2^{k\delta} \ell_{p_1}^{M_k})\|. \]  \hspace{1cm} (8.7)

Now by (8.5) and (8.7) it follows that $id$ is compact.
8.2 Theorem Let $d > 0$, $\delta > 0$, and $M_j \in \mathbb{N}$ with (8.1). Let $p_1$, $p_2$, $q_1$, $q_2$ be given by (8.3). Let $e_k$ be the entropy numbers of the compact operator $id$ in (8.4) according to Definition 6.2. There are two positive numbers $c$ and $C$ such that

$$c \ k^{-\frac{d}{\delta} + \frac{1}{p_2} - \frac{1}{p_1}} \leq e_k \leq C \ k^{-\frac{d}{\delta} + \frac{1}{p_2} - \frac{1}{p_1}}, \ k \in \mathbb{N}. \quad (8.8)$$

Proof.

Step 1. First we prove the left-hand side of (8.8). In the commutative diagram

the operator $id_j$ is given as in (8.6), now acting in the indicated slightly modified way, whereas $id^j$ maps $\ell_{p_1}^{M_j}$ identically onto $2^{j \delta} \ell_{p_1}^{M_j}$ interpreted as a dyadic block of $\ell_{q_1}(2^{j \delta} \ell_{p_1}^{M_j})$. In what follows we reserve $id$ for the identity given by (8.4), otherwise we indicate the spaces involved. Hence

$$id \left( \ell_{p_1}^{M_j} \rightarrow \ell_{p_2}^{M_j} \right) = id_j \circ id \circ id^j, \ j \in \mathbb{N}. \quad (8.9)$$

Plainly,

$$\|id^j\| = 2^{j \delta} \quad \text{and} \quad \|id_j\| = 1, \quad (8.10)$$

and consequently by (6.8)

$$e_k \left( id : \ell_{p_1}^{M_j} \rightarrow \ell_{p_2}^{M_j} \right) \leq 2^{j \delta} e_k, \ k \in \mathbb{N}, \ j \in \mathbb{N}. \quad (8.11)$$

By (7.13) with $k = 2M_j$ we obtain

$$e_{2M_j} \geq c \ 2^{-j \delta} 2^{j d \left( \frac{1}{p_2} - \frac{1}{p_1} \right)}, \ j \in \mathbb{N}. \quad (8.12)$$

By (8.1) and the monotonicity properties of the entropy numbers described in Proposition 6.4(i) it follows that

$$e_k \geq c \ k^{-\frac{d}{\delta} + \frac{1}{p_2} - \frac{1}{p_1}}, \ k \in \mathbb{N}, \quad (8.13)$$

for some $c > 0$. 
Step 2. The estimate from above is more complicated. Let $J \in \mathbb{N}$ and

$$L \delta = J \delta + Jd\left(\frac{1}{p_1} - \frac{1}{p_2}\right); \quad (8.14)$$

in particular $L \geq J$. We put $e_t = e_{[t]}$ if $t \geq 1$ and assume $L \in \mathbb{N}$. We wish to prove

$$e_{2^Jd} \leq c \ 2^{-j \delta + Jd\left(\frac{1}{p_2} - \frac{1}{p_1}\right)}, \quad J \in \mathbb{N}. \quad (8.15)$$

This is equivalent to the estimate from above in (8.8). We split the sum in (8.5) in three parts,

$$\text{id} = \sum_{j=0}^{L} \text{id}_j + \sum_{j=L+1}^{L} \text{id}_j + \sum_{j=L+1}^{\infty} \text{id}_j. \quad (8.16)$$

Of course here $\text{id}_j$ is considered as a map between the two spaces in (8.4) according to (8.6) in contrast to $\text{id}_j$ in the above diagram in Step 1. There is no danger of confusion. In particular by (8.7) we have

$$\left\| \sum_{j=L+1}^{\infty} \text{id}_j \right\| \leq c \ 2^{-L \delta}, \quad (8.17)$$

which by (8.14) coincides with the right-hand side of (8.15). Let $\varrho = \min(1, p_2, q_2)$. It is easy to see that $\ell_\varrho(\ell_{p_2}^{M_j})$ is a $\varrho$-Banach space. Then we obtain by (6.9), (8.16), and (8.17)

$$e_k^\varrho \leq c \ 2^{-L \delta} + \sum_{j=0}^{L} e_{k_j}^\varrho(\text{id}_j) + \sum_{j=L+1}^{L} e_{k_j}^\varrho(\text{id}_j) \quad (8.18)$$

where $k = \sum_{j=0}^{L} k_j$. By (8.7) we have

$$e_{k_j}(\text{id}_j) = 2^{-j \delta} e_{k_j} \left( \text{id} : \ell_{p_1}^{M_j} \to \ell_{p_2}^{M_j} \right), \quad (8.19)$$

and hence by Theorem 7.3

$$e_{k_j}(\text{id}_j) \leq c \ 2^{-j \delta} \left[ k_j^{-1} \log \left( c \ 2^{id_k_j} k_j^{-1} \right) \right]^\frac{1}{p_1} - \frac{1}{p_2} \quad \text{if} \quad k_j \leq 2M_j, \quad (8.20)$$

which covers also (7.11) and

$$e_{k_j}(\text{id}_j) \leq c \ 2^{-j \delta} 2^{-k_j M_j} \left( 2M_j \right)^{\frac{1}{p_2} - \frac{1}{p_1}} \quad \text{if} \quad k_j > 2M_j. \quad (8.21)$$
Now we choose
\[ k_j = 2^{jd} 2^{-(j-j)\varepsilon} \quad \text{if} \quad j = 0, \ldots, J \]  
(8.22)
and
\[ k_j = 2^{jd} 2^{-(j-j)\varkappa} \quad \text{if} \quad j = J + 1, \ldots, L, \]  
(8.23)
where \( \varepsilon \) and \( \varkappa \) are positive numbers which will be chosen later on. We obtain
\[ k \sim 2^{jd}, \]  
(8.24)
where \( \sim \) indicates equivalences (two-sided estimates up to unimportant positive constants which are independent of \( J \)). We deal with the two sums in (8.18) separately.

**Step 3.** Let \( j = 0, \ldots, J \). By (8.22) we have
\[ k_j = 2^{jd} 2^{(j-j)(d-\varepsilon)} \geq 2^{jd}, \]  
(8.25)
where we choose \( 0 < \varepsilon < d \). By (8.1) and (8.20), (8.21) it follows that we can always apply (8.21) in that case. Then we obtain
\[ e_{k_j} (id_j) \leq c 2^{\lambda_j} \]  
(8.26)
with
\[ \lambda_j = -J \delta + jd \left( \frac{1}{p_2} - \frac{1}{p_1} \right) + (J - j) \left( \delta - d \left( \frac{1}{p_2} - \frac{1}{p_1} \right) \right) - c 2^{(j-j)(d-\varepsilon)} \]  
(8.27)
and consequently
\[ \sum_{j=0}^{J} e_{k_j}^0 (id_j) \leq c 2^{-J \delta + jd \left( \frac{1}{p_2} - \frac{1}{p_1} \right)} \]  
(8.28)

**Step 4.** Let \( j = J + 1, \ldots, L \). By (8.23) we have
\[ k_j \leq c 2^{jd}. \]  
(8.29)
Hence we can always apply (8.20) and obtain
\[ e_{k_j} (id_j) \leq c 2^{-j \delta} \left[ 2^{-jd+(j-j)\varkappa} \log \left( c 2^{(j-j)d} 2^{(j-j)\varkappa} \right) \right] \frac{1}{p_1} - \frac{1}{p_2} \]
\[ \leq c 2^{-J \delta + jd \left( \frac{1}{p_2} - \frac{1}{p_1} \right)} 2^{(j-j)\delta + \varkappa \left( \frac{1}{p_2} - \frac{1}{p_1} \right)} [(d + \varkappa)(j - J)] \frac{1}{p_1} - \frac{1}{p_2}. \]  
(8.30)
We choose \( \varkappa > 0 \) such that
\[ \varkappa \left( \frac{1}{p_2} - \frac{1}{p_1} \right) < \delta \]  
(8.31)
and obtain
\[ \sum_{j=J+1}^{L} e_{k_j}^0 (id_j) \leq c 2^{-J \delta + jd \left( \frac{1}{p_2} - \frac{1}{p_1} \right)}. \]  
(8.32)

**Step 5.** By (8.24), (8.14), (8.18), (8.28), and (8.32) we have
\[ e_{c 2^{jd}} \leq c' 2^{-J \delta + jd \left( \frac{1}{p_2} - \frac{1}{p_1} \right)}, \quad J \in \mathbb{N}, \]  
(8.33)
where \( c \) and \( c' \) are appropriate positive constants. This coincides essentially with (8.15) and completes the proof of the right-hand side of (8.8).
8.3 Remark In the case of Banach spaces, which means that the numbers \( p_1, p_2, q_1 \) and \( q_2 \) in (8.3) are larger than or equal to 1, the above theorem is more or less known, see [Kühn84]. In that paper the proof is based on interpolation properties of entropy numbers and entropy numbers of diagonal operators in \( \ell_p \)-spaces due to Carl, see [CaS90].

8.4 Estimates of constants
Theorem 8.2 is the basis for the study of entropy numbers of embedding operators in function spaces. Usually, in non-limiting situations, all parameters \( p_1, p_2, q_1, q_2, d \) and \( \delta \) are fixed and there is no need to have additional information on the dependence of \( c \) and \( C \) in (8.8) on these parameters. However in some limiting cases we deal with a sequence of target spaces and we have to know how \( C \) in (8.8) depends on \( p_2, q_2 \) and \( \delta \), whereas the dependence of \( C \) on \( p_1, q_1 \) and \( d \) is not so interesting for our later purposes. To facilitate the estimates we assume in addition

\[
1 \leq p_2 \leq \infty, \quad 1 \leq q_2 \leq \infty, \quad \text{and} \quad 0 < \delta \leq 1.
\]

This is not really necessary, but sufficient for our later purposes.

8.5 Corollary Under the hypotheses of Theorem 8.2, complemented by (8.34), we have

\[
e_k \leq c \delta^{-1-2(\frac{1}{p_1} - \frac{1}{p_2})} k^{-\frac{\delta}{\delta} + \frac{1}{p_2} - \frac{1}{p_1}}, \quad k \in \mathbb{N},
\]

for some positive constant \( c \) which is independent of \( p_2, q_2 \), and \( \delta \) (but may depend on \( p_1, q_1 \), and \( d \)).

Proof. We follow the arguments in the Steps 2–5 of the proof of Theorem 8.2. By (8.34) we have \( \varrho = 1 \) in (8.18). We estimate the constant \( c \) in the first term on the right-hand side of (8.18). By (8.7) and (8.17) we have

\[
\left\| \sum_{j=L+1}^{\infty} id_j \right\| \leq \sum_{j=L+1}^{\infty} \|id_j\| \leq \sum_{j=L+1}^{\infty} 2^{-j\delta} \leq c \delta^{-1} 2^{-L\delta},
\]

where \( c \) is independent of \( \delta \). Next we remark that we may assume that the constant \( c_2 \) in (7.11)–(7.13) is independent of \( p_2 \). We refer to [ET96], Remark 2 on p. 101. But this is not a deep result. It follows immediately from Theorem 7.3 with \( p_2 = \infty \) and the interpolation properties of the entropy numbers mentioned in 6.7. Having this in mind it follows that the constants \( c \) in (8.20) and (8.21) are independent of \( p_2, q_2 \) and \( \delta \). We may choose \( \varepsilon = \frac{\delta}{2} \) in (8.22) and (8.25). Hence \( \varepsilon \) is not of interest for us. As for \( \kappa \) in (8.23) and (8.31) we may choose \( \kappa = \frac{\delta p_1}{2} \). Then (8.24) must be substituted now by

\[
2^{jd} \leq k \leq \frac{c}{\delta} 2^{jd},
\]

where \( c \) is independent of \( \delta \) (and \( p_2 \) and \( q_2 \)). Now by the above remarks about the constants in Theorem 7.3 and \( \delta \leq 1 \) it follows from (8.26) and (8.27) that the
constant $c$ in (8.28) (now with $\varrho = 1$) is independent of $p_2$, $q_2$ and $\delta$. We estimate the constant $c$ in (8.32) (again with $\varrho = 1$). By the above choice of $\varkappa$ it follows from (8.30)

$$
\sum_{j=1}^L e_{kj}(id_j) \leq c_1 2^{-J\delta + Jd\left(\frac{1}{p_2} - \frac{1}{p_1}\right)} \frac{1}{\gamma_2 - \frac{1}{p_1}} \sum_{l=1}^{L-1} 2^{-c_2 \delta l} \frac{1}{\gamma_1 - \frac{1}{p_2}}
$$

(8.38)

where $c_1$ and $c_2$ are independent of $\delta$, $p_2$ and $q_2$. The last factor can be estimated from above by

$$
\int_0^\infty e^{-c_3 \delta t} \left(\frac{1}{\gamma_1 - \frac{1}{p_2}}\right) dt \leq c_4 \delta^{-1}.
$$

(8.39)

Now by (8.18), (8.36), (8.28) and (8.38) with (8.39) we obtain

$$
e_k \leq c \delta^{-1 - \frac{1}{p_2} + \frac{1}{p_1}} 2^{-J\delta + Jd\left(\frac{1}{p_2} - \frac{1}{p_1}\right)},
$$

(8.40)

where $c$ is independent of $\delta$, $p_2$, and $q_2$, and $k$ is given by (8.37). With $2^d \sim k\delta$ in (8.40) we have

$$
e_k \leq c \delta^{-1 - 2\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} k^{-\frac{\delta}{p_2} + \frac{1}{p_2} - \frac{1}{p_1}}, \quad k \in \mathbb{N}.
$$

(8.41)

The proof of (8.35) is complete.

8.6 Remark The restrictions $p_2 \geq 1$ and $q_2 \geq 1$ are unimportant. Otherwise one has $\varrho < 1$ in (8.18). There is no problem to follow the above reasoning in this more general case.

8.7 Comparison The estimates for entropy numbers of compact embeddings between function spaces will be based in non-limiting cases on (8.8), whereas in some limiting cases we need the additional information given in the above corollary. Although the context is slightly different (so far) one can compare the exponent $1 + 2\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ of $\delta$ in (8.35) with the exponents $1 + \frac{2}{p}$ in [ET96], p. 130, formula (8), and $-\frac{2\varepsilon}{p} - \varepsilon$ in [ET96], p. 139, formula (3), where any $\varepsilon > 0$ is admitted. It comes out that $1 + \frac{2}{p}$ originates precisely from $1 + 2\left(\frac{1}{p_1} - \frac{1}{p_2}\right)$ (restricted to the treated case), whereas $-\frac{2\varepsilon}{p} - \varepsilon$ is somewhat better ($\varepsilon = 1$ would be the direct counterpart). On the other hand the situation considered in [ET96], p. 139, is more special. We return in 23.5 to these comparisons in greater detail and shed more light upon these admittedly somewhat cryptical remarks.
8.8 A digression: Matrix operators

It is not our aim to discuss the spectral theory of compact operators acting in $\ell_p$-spaces. This has been done in great detail by A. Pietsch, B. Carl and other mathematicians. We refer to [Pie87], esp. pp. 230–231, [Kön86], esp. pp. 150–151, and [CaS90]. Our intention here is simply to demonstrate the power of Theorem 8.2. For that purpose we estimate the distribution of eigenvalues of some matrix operators in $\ell_p$-spaces. We avoid any technical complications and we are far from the most general case which can be treated in that way. In this sense we leave it to the interested reader to compare the results obtained here with the more systematic treatments in the above-mentioned books. Let $0 < p \leq \infty$; recall that $\ell_p$ is the linear space of all complex sequences $x = (x_k : k \in \mathbb{N})$ endowed with the quasi-norm

$$\|x\|_{\ell_p} = \left( \sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}},$$

with the obvious modification if $p = \infty$. Let

$$A = (a_{lk} : l \in \mathbb{N}, k \in \mathbb{N}), \quad a_{lk} \in \mathbb{C},$$

and as usual let

$$Ax = \left( \sum_{k=1}^{\infty} a_{lk} x_k : l \in \mathbb{N} \right) \text{ for } x = (x_k : k \in \mathbb{N}).$$

Let $d > 0$ and $\delta > 0$, and $M_j \sim 2^{jd}$ according to (8.1) with $j \in \mathbb{N}_0$. We put

$$M_j^0 = 0, \quad M_j = \sum_{m=0}^{j-1} M_m \sim 2^{jd}, \quad j \in \mathbb{N}; \quad \text{and} \quad M_0 = 0,$$

and assume that the entries $a_{lk}$ can be represented as

$$a_{lk} = 2^{-jd} b_{lm}^j, \quad l \in \mathbb{N},$$

$$k = M_j + m \text{ for } j \in \mathbb{N}_0 \text{ and } m = 1, \ldots, M_j,$$

and

$$\sum_{l=0}^{\infty} \left( \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} |b_{lm}^j| \right)^p < \infty, \quad 0 < p \leq \infty,$$

with the obvious modification if $p = \infty$. In other words, for fixed $l$ we sum first the absolute values of the entries in the $l$th row. Hence,

$$B = (b_{lk} = b_{lm}^j : l \in \mathbb{N}, k \text{ given by (8.47)})$$
is a so-called *Hille-Tamarkin matrix*, see the above references. In particular, \( A \) can be decomposed by

\[
A = B \circ D,
\]

where \( D \) is a diagonal matrix with the entries \( d_k = 2^{-j\delta} \), where \( k \) is given by (8.47). As we shall see, \( A \) generates a compact operator in \( \ell_p \). Then we can apply the Riesz theory mentioned in 6.1, 6.6 and 6.8. In particular, the non-zero eigenvalues \( \mu_k(A) \) of \( A \), repeated according to algebraic multiplicity, can be ordered as in (6.11).

**8.9 Proposition** Let \( 0 < p \leq \infty \) and let \( A \) be the above operator. Then there is a positive constant \( c \) such that

\[
|\mu_k(A)| \leq c k^{-\frac{j}{p}} - \frac{1}{p}, \quad k \in \mathbb{N}.
\]

*Proof.* We decompose \( A \) as

\[
A = B \circ id \circ D,
\]

where \( D \) and \( B \) have the above meaning. We claim

\[
\begin{align*}
D &: \ell_p \rightarrow \ell_p (2j^\delta \ell_p^M), \\
\text{id} &: \ell_p (2j^\delta \ell_p^M) \rightarrow \ell_\infty, \\
B &: \ell_\infty \rightarrow \ell_p.
\end{align*}
\]

The first line is obvious where we used the notation introduced in (8.2). By Theorem 8.2 the operator \( \text{id} \) is compact and

\[
e_k(id) \leq C k^{-\frac{j}{p}} - \frac{1}{p}, \quad k \in \mathbb{N}.
\]

Let \( x = (x_k : k \text{ given by } (8.47)) \in \ell_\infty \). Then by (8.49)

\[
|(Bx)_l| = \left| \sum_{j=0}^{\infty} \sum_{m=1}^{M_l} b_{lm}^j x_{M_j+m} \right| \leq \|x\|_{\ell_\infty} \sum_{j=0}^{\infty} \sum_{m=1}^{M_l} |b_{lm}^j|.
\]

Now the last line in (8.53) is a consequence of (8.48) and (8.55). Since \( D \) and \( B \) are bounded, (8.52), (8.54), and (6.8) prove

\[
e_k(A) \leq c k^{-\frac{j}{p}} - \frac{1}{p}, \quad k \in \mathbb{N}.
\]

Finally, (8.51) is a consequence of Corollary 6.10.
8.10 Remark It can be easily seen that the exponent in (8.51) is sharp: Let \( A \) be a diagonal operator, \( a_{kl} = 0 \) if \( k \neq l \) and

\[
a_{kk} = k^{-\frac{k}{d} - \frac{1}{p}} (\log k)^{\alpha}.
\]

(8.57)

By (8.47) and (8.48) with (8.45) we have

\[
\sum_{l=0}^{\infty} \left( \sum_{j=0}^{\infty} \sum_{m=1}^{M_j} |b_{lm}| \right)^p \sim \sum_{j=0}^{\infty} 2^{-jd} j^\alpha 2^{jd} < \infty
\]

(8.58)

if \( \alpha < -\frac{1}{p} \). In that case \( A \) has the required properties. On the other hand, \( a_{kk} \) are the eigenvalues of \( A \). Hence the exponent in (8.51) is the best possible.

9 Weighted \( \ell_p \)-spaces: a generalization

9.1 Preliminaries and notation

Unfortunately Theorem 8.2 and Corollary 8.5 are not completely sufficient for our later purposes. We need something like an \( \ell_u \)-version of these two assertions. Fortunately it comes out that these generalizations are nothing more than a technical appendix to the results just mentioned. We use the same notation as in 8.1. In particular, let \( d > 0, \delta \geq 0 \) and \((M_j)_{j \in \mathbb{N}_0}\) be a sequence of natural numbers with (8.1) for some positive numbers \( c_1 \) and \( c_2 \). Let again \( \ell_q(2^{j\delta} \ell_p^{M_j}) \) with \( 0 < p \leq \infty \) and \( 0 < q \leq \infty \) be the quasi-Banach space introduced in 8.1 and quasi-normed by (8.2). Let, in addition, \( \mu \geq 0 \) and \( 0 < u \leq \infty \). Then by

\[
\ell_u \left[ 2^{\mu m} \ell_q (2^{j\delta} \ell_p^{M_j}) \right]
\]

we shall mean the linear space of all \( \ell_q(2^{j\delta} \ell_p^{M_j}) \)-valued sequences \( x = (x^m : m \in \mathbb{N}_0) \) endowed with the quasi-norm

\[
\left\| x \right\|_{\ell_u \left[ 2^{\mu m} \ell_q (2^{j\delta} \ell_p^{M_j}) \right]} = \left( \sum_{m=0}^{\infty} 2^{\mu mu} \left\| x^m \ell_q (2^{j\delta} \ell_p^{M_j}) \right\|^u \right)^{\frac{1}{u}}
\]

(9.1)

with the obvious modification according to the vector-valued version of (7.2) if \( u = \infty \). In case of \( \mu = \delta = 0 \) we write \( \ell_u \left[ \ell_q(\ell_p^{M_j}) \right] \) in accordance with the notation introduced in 8.1. We are interested in an extension of Theorem 8.2. Let \( d > 0, \delta > 0, \mu > 0, \)

\[
0 < p_1 \leq p_2 \leq \infty
\]

(9.2)

and

\[
0 < q_1 \leq \infty, \ 0 < q_2 \leq \infty, \ 0 < u_1 \leq \infty, \ 0 < u_2 \leq \infty.
\]

(9.3)
9 Weighted $\ell_p$-spaces: a generalization

Then the identity map

$$id : \ell_{u_1} \left[ 2^{\mu m} \ell_{q_1} (2^{i\delta} \ell_{p_1}^M) \right] \rightarrow \ell_{u_2} [\ell_{q_2} (\ell_{p_2}^M)]$$

(9.4)
is compact. This is simply the extension of what had been said in 8.1, see (8.4)–(8.7), from the scalar case to the $\ell_{u_1}$-valued case. However this generalization is an immediate consequence of $\mu > 0$. Now Theorem 8.2 can be rather easily extended to the vector-valued case.

9.2 Theorem

Let $d > 0$, $\delta > 0$, $\mu > 0$, and $M_j \in \mathbb{N}$ with (8.1). Let $p_1$, $p_2$, $q_1$, $q_2$, $u_1$, $u_2$ be given by (9.2) and (9.3). Let $e_k$ be the entropy numbers of the compact operator $id$ according to (9.4). There are two positive numbers $c$ and $C$ such that

$$c k^{-\frac{\delta}{\mu} + \frac{1}{p_2} - \frac{1}{p_1}} \leq e_k \leq C k^{-\frac{\delta}{\mu} + \frac{1}{p_2} - \frac{1}{p_1}}, \quad k \in \mathbb{N}.$$  

(9.5)

Proof.

Step 1. The estimate from below is covered by Step 1 of the proof of Theorem 8.2.

Step 2. We reduce the estimate from above to the corresponding scalar case in Theorem 8.2. Let

$$id_m : x \mapsto x^m, \quad \text{where} \quad x = (x^l)_{l \in \mathbb{N}_0}$$

(9.6)

has the same meaning as in (9.1). Then we have

$$id = \sum_{m=0}^{\infty} id_m.$$  

(9.7)

Let, for brevity, $a = \frac{\delta}{\mu} + \frac{1}{p_1} - \frac{1}{p_2}$, and let $J \in \mathbb{N}$ and

$$L = \left[ \frac{a}{\mu} J \right] \in \mathbb{N}_0.$$  

(9.8)

Of course $a > 0$. Then it follows that

$$\left\| \sum_{l=L}^{\infty} id_l \right\| \leq c 2^{-\mu L} \leq c' 2^{-a J}.$$  

(9.9)

Let

$$k_l = 2^l 2^{-l \varepsilon} \quad \text{where} \quad l = 0, \ldots, L \quad \text{and} \quad \varepsilon a < \mu.$$  

(9.10)

Then we have

$$k = \sum_{l=0}^{L} k_l \sim 2^J$$

(9.11)

and by (8.8)

$$e_{k_l} (id_l) \leq c 2^{-\mu l} 2^{-a J} 2^{l \varepsilon}, \quad l = 0, \ldots, L.$$  

(9.12)

Now by (9.9)–(9.11) and (6.9) it follows that

$$e_{c_1 2^l} (id) \leq c_2 2^{-a J}, \quad J \in \mathbb{N},$$  

(9.13)

for some $c_1 > 0$ and $c_2 > 0$. This proves the right-hand side of (9.5).
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9.3 Remark We are also interested in an extension of Corollary 8.5 to the vector-valued case. In accordance with (8.34) we assume in addition

$$1 \leq p_2 \leq \infty, \ 1 \leq q_2 \leq \infty, \ 1 \leq u_2 \leq \infty, \text{ and } 0 < \delta \leq 1. \quad (9.14)$$

Again, these conditions are not really necessary, but sufficient for our later purposes.

9.4 Corollary Under the hypotheses of Theorem 9.2 complemented by (9.14) we have

$$e_k \leq c \delta^{1-2\left(\frac{1}{p_1} - \frac{1}{p_2}\right)} k^{-\frac{\delta}{d} + \frac{1}{p_2} - \frac{1}{p_1}}, \quad k \in \mathbb{N}, \quad (9.15)$$

for some positive constant $c$ which is independent of $p_2, q_2, u_2$, and $\delta$ (but may depend on $p_1, q_1, u_1, d$ and $\mu$).

Proof. By slight modification we may assume that $L$ in (9.8) and $\varepsilon$ in (9.10) are chosen independently of the indicated numbers. Then we have the same situation in (9.9) and (9.11) by a similar argument as in (8.36). Replacing the constant $c$ in (9.12) by the corresponding constant on the right-hand side of (8.35) we obtain (9.13) with the desired constant. This proves (9.15).

9.5 Remark As for comments we refer to 8.6 and 8.7.
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