Preface

Since the second half of the 20th century, the Riemannian and semi - Riemannian geometries have been active areas of research in differential geometry and its applications to a variety of subjects in mathematics and physics. A recent survey in Marcel Berger's book [60] includes the major developments of Riemannian geometry since 1950, citing the works of differential geometers of that time. During the mid 1970s, the interest shifted towards Lorentzian geometry, the mathematical theory used in general relativity. Since then there has been an amazing leap in the depth of the connection between modern differential geometry and mathematical relativity, both from the local and the global point of view. Most of the work on global Lorentzian geometry has been described in a standard book by Beem and Ehrlich [34] and in their second edition in 1996, with Easley.

As for any semi-Riemannian manifold there is a natural existence of null (lightlike) subspaces, in 1996, Duggal-Bejancu published a book [149] on the lightlike (degenerate) geometry of submanifolds needed to fill an important missing part in the general theory of submanifolds. Since then the large number of papers published on lightlike hypersurfaces and general theory of submanifolds of semi-Riemannian manifolds has created a demand for publication of this volume as an update on the study of lightlike geometry.

The objective is to focus on all new geometric results (in particular, those available only after publication of the Duggal–Bejancu book) on lightlike geometry with proofs and their physical applications in mathematical physics.

Chapter 1 covers preliminaries, followed by up-to-date mathematical results in Chapters 2, 4 and 5 on lightlike hypersurfaces, half-lightlike, coistropic and r-lightlike submanifolds of semi-Riemannian manifolds, respectively. Due to degenerate induced metric of a lightlike submanifold M, we use a non-degenerate screen distribution $S(TM)$ to project induced objects on M. Unfortunately, $S(TM)$ is not in general unique. Since 1996 considerable work has been done in the search for canonical or unique screens. We highlight that each of these three chapters contain theorems on the existence of unique screen distributions subject to some reasonable geometric conditions. Chapter 3 is focused on applications of lightlike hypersurfaces in two active ongoing research areas in mathematical physics. First, we deal with *black hole horizons.* We prove a *Global Null Splitting Theorem* and relate it with physically significant works of Galloway [197], Ashtekar and

Krishnan's works [16] on *dynamical horizons* and Sultana-Dyer's work [378, 379] on *conformal Killing horizons*, with references to a host of related researchers. Secondly, we present the latest work on *Osserman lightlike hypersurfaces* [20].

Motivation of Chapters 6–9 comes from the historical development of the general theory of Cauchy-Riemann (CR) submanifolds [45] and their use in mathematical physics, as follows:

In the early 1930s, Riemannian geometry and the theory of complex variables were synthesized by Kähler [250] whose work developed (during 1950) into complex manifold theory [169, 302]. A Riemann surface, C^n and its projective space CP^{n-1} are simple examples of complex manifolds. This interrelation between the above two main branches of mathematics developed into what is now known as Kählerian and Sasakian [367] geometry. Almost complex [407], almost contact [66, 68] and quaternion Kähler manifolds [239, 366] and their complex, totally real, CR and slant submanifolds [45, 99, 102, 314, 410] are some of the most interesting topics of Riemannian geometry. By a CR submanifold we mean a real submanifold M of an almost Hermitian manifold $(\bar{M}, J, \bar{g})$, carrying a J-invariant distribution D (i.e., $JD = D$) and whose $\bar{g}$-orthogonal complement is J-anti-invariant (i.e., $JD^\perp \subseteq T(M)^\perp$), where $T(M)^\perp \to M$ is the normal bundle of M in $\bar{M}$. The CR submanifolds were introduced as an umbrella of a variety (such as invariant, anti-invariant, semi-invariant, generic) of submanifolds. Details on these may be seen in [45, 102, 412, 413]. On the other hand, a CR manifold (independent of its landing space) is a C^∞ differentiable manifold M with a holomorphic subbundle H of its complexified tangent bundle $T(M) \otimes C$, such that $H \cap \bar{H} = \{0\}$ and H is involutive (i.e., $[X, Y] \in H$ for every $X, Y \in H$). For an update on CR manifolds (which is out of the scope of this book) we refer two recent books [132] by Dragomir and Tomassini and [26] by Barletta, Dragomir and Duggal. Here we highlight that Blair and Chen [69] were the first to interrelate these two concepts by proving that *proper CR submanifolds, of a Hermitian manifold, are CR manifolds.* Since then there has been active interrelation between the geometries of real and complex manifolds, with physical applications. Complex manifolds have two interesting classes of Kähler manifolds, namely, (i) Calabi-Yau manifolds with application to super string theory (see Candelas et al. [90]) and (ii) Teichmuller spaces applicable to relativity (see Tromba [396]).

The study of the above mentioned variety of geometric structures was primarily confined to Riemannian manifolds and their submanifolds, which carry a positive definite metric tensor, until in early 1980, when Beem-Ehrlich [34] published a book on *Global Lorentzian Geometry*, a paper by Barros and Romero [28] on *Indefinite Kähler Manifolds* and a book by O'Neill [317] on *Semi-Riemannian Geometry with Applications to Relativity.* Since then a considerable amount of work has been done on the study of semi-Riemannian geometry and its submanifolds, in particular, see a recent book by García-Río et al. [201], Sharma [373] and Duggal [133, 135, 136], including the use of indefinite Kählerian, Sasakian and quaternion structures. As a result, now we know that there are similarities and differences between the Riemannian and the semi-Riemannian geometries, in particular, reference to the Lorentzian case used in relativity. In the case of lightlike submanifolds, its geometry is quite different than the counter part of non-degenerate submanifolds. To highlight this, in Duggal-Bejancu's book [149] there is a discussion on CR lightlike submanifolds of an indefinite Kähler manifold. Unfortunately, contrary to the non-degenerate case, CR lightlike submanifolds are non-trivial (i.e., they do not include invariant (complex) and real parts). To fill in this needed information, in Chapter 6 we present the latest work of the authors [159, 160] leading to a new class called *Generalized CR lightlike submanifolds* which represents an umbrella of invariant, screen real and CR lightlike submanifolds. We also present lightlike versions of slant submanifolds and totally real submanifolds.

Motivated by a significant use of contact geometry in differential equations, optics and phase space of a dynamical system (see details in Arnold [9], Maclane [292] and Nazaikinskii [304] and many more references therein), in Chapter 7, we present the first-ever collection of the authors' recent work [161, 162] on *lightlike submanifolds of indefinite Sasakian manifolds*, leading up to another umbrella of invariant, screen real and contact CR-lightlike submanifolds. We highlight that Chapters 6 and 7 fulfill the purpose (see Bejancu [45]) of having an umbrella of all possible lightlike submanifolds of indefinite Kählerian and Sasakian manifolds.

In Chapter 8, we study lightlike submanifolds of indefinite quaternion Kähler manifolds, using the concept of QR-lightlike, screen QR-lightlike and quaternion CR-lightlike submanifolds and give many examples.

In Chapter 9, we present applications of lightlike geometry to null 2-surfaces in spacetimes, lightlike version of harmonic maps and morphisms, CR-structures in general relativity and lightlike contact geometry in physics. Results included in this volume should stimulate future research on lightlike geometry and its applications. To the best of our knowledge, there does not exist any other book covering the material in this volume, other than a fresh and improved version of a small part from [149]. Equations are numbered within each chapter and its section. To illustrate this, a triplet (a, b, c) stands for each equation such that a, b and c are labeled for the chapter, the section and the number of equation in that section accordingly. There is an extensive list of bibliography and subject index. Our approach, in this book, has the following special features:

- Extensive list of cited references on semi-Riemannian geometry and its non-degenerate submanifolds is provided for the readers to easily understand the main focus on lightlike geometry.

- Each chapter starts with basic material on the no-degenerate submanifolds needed for that chapter. We expect that this approach will help readers to understand each chapter independently without knowing all the prerequisites in the beginning.

- The sequence of chapters is arranged so that the understanding of a chapter stimulates interest in reading the next one and so on.

- Physical applications are discussed separately (see Chapters 3 and 9) from the mathematical theory.
- Overall the presentation is self contained, fairly accessible and in some special cases supported by references.

This book is intended for graduate students and researchers who have good knowledge of semi-Riemannian geometry and its submanifolds and interest in the theory and applications of lightlike submanifolds. The book is also suitable for a senior level graduate course in differential geometry.

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