Bounded Perturbations of the Resolvent Operators Associated to the $\mathcal{F}$-Spectrum

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Abstract. Recently, we have introduced the $\mathcal{F}$-functional calculus and the $\mathcal{SC}$-functional calculus. Our theory can be developed for operators of the form $T = T_0 + e_1T_1 + \ldots + e_nT_n$ where $(T_0, T_1, \ldots, T_n)$ is an $(n + 1)$-tuple of linear commuting operators. The $\mathcal{SC}$-functional calculus, which is defined for bounded but also for unbounded operators, associates to a suitable slice monogenic function f with values in the Clifford algebra $\mathbb{R}_n$ the operator $f(T)$. The $\mathcal{F}$-functional calculus has been defined, for bounded operators T, by an integral transform. Such an integral transform comes from the Fueter's mapping theorem and it associates to a suitable slice monogenic function f the operator $\breve{f}(T)$, where $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$ and Δ is the Laplace operator. Both functional calculi are based on the notion of $\mathcal{F}$-spectrum that plays the role that the classical spectrum plays for the Riesz-Dunford functional calculus. The aim of this paper is to study the bounded perturbations of the $\mathcal{SC}$-resolvent operator and of the $\mathcal{F}$-resolvent operator. Moreover we will show some examples of equations that lead to the $\mathcal{F}$-spectrum.

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1. Introduction

The recent theory of slice monogenic functions, mainly developed in the papers [2], [3], [8], [9], [10], [12], turned out to be very important because of its applications to the so-called $\mathcal{S}$-functional calculus for n-tuples of non-necessarily commuting operators (bounded or unbounded), see [3] and [11]. The theory admits a quaternionic version of the $\mathcal{S}$-functional calculus for quaternionic linear operators which can be found in [4]. It is crucial to note that slice monogenic functions have a Cauchy formula with slice monogenic kernel that admits two expressions. These

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two expressions of the Cauchy kernel are not equivalent when we want to define a functional calculus for non-necessarily commuting operators. In this paper, we will consider the case of commuting operators and the expression of the Cauchy kernel which gives rise to the definition of $\mathcal{F}$-spectrum which is the natural tool to treat the case of commuting operators.

Let $x = x_0 + e_1x_1 + \ldots + e_nx_n$ and $s = s_0 + e_1s_1 + \ldots + e_ns_n$ be paravectors in $\mathbb{R}^{n+1}$. We consider the Cauchy kernel written in the form

$$\mathcal{S_C}^{-1}(s, x) = (s - \bar{x})(s^2 - 2\mathrm{Re}[x]s + |x|^2)^{-1}$$

which is defined for $x^2 - 2xs_0 + |s|^2 \neq 0$. Let $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}_n$, where $\mathbb{R}_n$ is the real Clifford algebra with n imaginary units, U is a suitable open set that contains the singularities of $\mathcal{S_C}^{-1}(s, x)$. Let I be a 1-vector such that $I^2 = -1$ and let $\mathbb{C}_I$ be the complex plane that contains 1 and I. Then we have the Cauchy formula

$$f(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} \mathcal{S_C}^{-1}(s, x)ds_I f(s), \qquad ds_I = -Ids, \tag{1.1}$$

where the integral does not depend on the open set U and on the imaginary unit I. In the paper [5] we have introduced the so-called $\mathcal{SC}$-functional calculus ($\mathcal{SC}$ stands for *slice-commuting*), which is defined for bounded but also for unbounded commuting operators, starting from the above Cauchy formula

$$f(T) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} \mathcal{S_C}^{-1}(s, T)ds_I f(s), \qquad ds_I = -Ids. \tag{1.2}$$

The definition is well posed because the the integral in (1.2) does not depend on the open set U and on the imaginary unit I. The $\mathcal{SC}$-resolvent operator is defined by

$$\mathcal{S_C}^{-1}(s, T) := (s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}$$

whose associated spectrum is the $\mathcal{F}$-spectrum of T defined as:

$$\sigma_{\mathcal{F}}(T) = \{s \in \mathbb{R}^{n+1} \ : \ s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T} \quad \text{is not invertible}\}.$$

It is important to point out the meaning of the symbols. We have that, by definition, $T = T_0 + e_1T_1 + \ldots + e_nT_n$, $\overline{T} = T_0 - e_1T_1 - \ldots - e_nT_n$, so that $T + \overline{T} = 2T_0$, and since the components of T commute, we have $T\overline{T} = T_0^2 + T_1^2 + \ldots + T_n^2$.

In the paper [6] we have proved the Fueter mapping theorem in integral form using the Cauchy formula (1.2). Precisely we have proved that, when n is an odd number, given the slice monogenic function f, we can associate to it the monogenic function $\breve{f}(x)$ by the integral transform

$$\breve{f}(x) = \frac{1}{2\pi} \int_{\partial(U\cap\mathbb{C}_I)} \gamma_n(s-\overline{x})(s^2-s(x+\overline{x})+|x|^2)^{-\frac{n+1}{2}} ds_I f(s), \quad ds_I = -Ids, \tag{1.3}$$

where γ_n ia a given constant. We recall that the Fueter mapping theorem in differential form is given in [19] for n odd and in Qian's paper [16] in the general case. Later on, Fueter's theorem has been generalized to the case in which a function f as above is multiplied by a monogenic homogeneous polynomial of degree k, see

[15], [20] and to the case in which the function f is defined on an open set U not necessarily chosen in the upper complex plane, see [17].

We point out that formula (1.3) allows us to define the $\mathcal{F}$-functional calculus by replacing $x = x_0 + x_1e_1 + \ldots + x_ne_n$ by $T = T_0 + T_1e_1 + \ldots + T_ne_n$. More precisely, in [6] we have defined the following version of the monogenic functional calculus by setting

$$\breve{f}(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \mathcal{F}_n^{-1}(s,T)ds_I f(s), \tag{1.4}$$

where

$$\mathcal{F}_n^{-1}(s,T) := \gamma_n(s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T+\overline{T}) + T\overline{T})^{-\frac{n+1}{2}}. \tag{1.5}$$

The functional calculus in (1.4) is well defined since the integral does not depend on the open set U and on $I \in \mathbb{S}$. The natural notion of spectrum in this case is again the notion of $\mathcal{F}$-spectrum of T as it is suggested by the definition of the $\mathcal{F}$-resolvent operator defined in (1.5).

The goal of this paper is to prove that bounded perturbations of the $\mathcal{SC}$-resolvent operator and of the $\mathcal{F}$-resolvent operator produce bounded variations of the respective functional calculi.

We conclude by recalling that the well known theory of monogenic functions, see [1], [7], is the natural tool to define the monogenic functional calculus which has been well studied and developed by several authors, see the book of B. Jefferies [14] and the literature therein. For the analogies with the Riesz-Dunford functional calculus see for example the classical books [13] and [18].

2. Preliminary material

The setting in which we will work is the real Clifford algebra $\mathbb{R}_n$ over n imaginary units $e_1,\ldots,e_n$ satisfying the relations $e_ie_j + e_je_i = -2\delta_{ij}$. An element in the Clifford algebra will be denoted by $\sum_A e_Ax_A$ where $A = i_1\ldots i_r$, $i_\ell \in \{1,2,\ldots,n\}$, $i_1 < \ldots < i_r$ is a multi-index, $e_A = e_{i_1}e_{i_2}\ldots e_{i_r}$ and $e_\emptyset = 1$. In the Clifford algebra $\mathbb{R}_n$, we can identify some specific elements with the vectors in the Euclidean space $\mathbb{R}^n$: an element $(x_1,x_2,\ldots,x_n) \in \mathbb{R}^n$ can be identified with a so-called 1-vector in the Clifford algebra through the map $(x_1,x_2,\ldots,x_n) \mapsto \underline{x} = x_1e_1 + \ldots + x_ne_n$. An element $(x_0,x_1,\ldots,x_n) \in \mathbb{R}^{n+1}$ will be identified with the element $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^n x_je_j$ called a paravector. The norm of $x \in \mathbb{R}^{n+1}$ is defined as $|x|^2 = x_0^2 + x_1^2 + \ldots + x_n^2$. The real part x_0 of x will be also denoted by $\mathrm{Re}[x]$. A function $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ is seen as a function $f(x)$ of x (and similarly for a function $f(\underline{x})$ of $\underline{x} \in U \subset \mathbb{R}^{n+1}$).

Definition 2.1. We will denote by $\mathbb{S}$ the sphere of unit 1-vectors in $\mathbb{R}^n$, i.e.,

$$\mathbb{S} = \{\underline{x} = e_1x_1 + \ldots + e_nx_n \ : \ x_1^2 + \ldots + x_n^2 = 1\}.$$

Note that $\mathbb{S}$ is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. The vector space $\mathbb{R}+I\mathbb{R}$ passing through 1 and $I \in \mathbb{S}$ will be denoted by $\mathbb{C}_I$, while an element belonging to $\mathbb{C}_I$ will be denoted by $u + Iv$, for u, $v \in \mathbb{R}$. Observe that $\mathbb{C}_I$, for every $I \in \mathbb{S}$, is a 2-dimensional real subspace of $\mathbb{R}^{n+1}$ isomorphic to the complex plane. The isomorphism turns out to be an algebra isomorphism.

Given a paravector $x = x_0 + \underline{x} \in \mathbb{R}^{n+1}$ let us set

$$I_x = \begin{cases} \dfrac{\underline{x}}{|\underline{x}|} & \text{if } \underline{x} \neq 0, \\ \text{any element of } \mathbb{S} \text{ otherwise.} \end{cases}$$

By definition we have that a paravector x, with $\underline{x} \neq 0$, belongs to $\mathbb{C}_{I_x}$.

Definition 2.2. Given an element $x \in \mathbb{R}^{n+1}$, we define $[x] = \{y \in \mathbb{R}^{n+1} \ : \ y = \mathrm{Re}[x] + I|\underline{x}|\}$, where $I \in \mathbb{S}$.

Remark 2.3. The set $[x]$ is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. When $x \in \mathbb{R}$, then $[x]$ contains x only. In this case, the $(n-1)$-dimensional sphere has radius equal to zero.

Definition 2.4 (Slice monogenic functions). Let $U \subseteq \mathbb{R}^{n+1}$ be an open set and let $f : U \to \mathbb{R}_n$ be a real differentiable function. Let $I \in \mathbb{S}$ and let f_I be the restriction of f to the complex plane $\mathbb{C}_I$. We say that f is a (left) slice monogenic function, or s-monogenic function, if for every $I \in \mathbb{S}$, we have

$$\frac{1}{2}\left(\frac{\partial}{\partial u} + I\frac{\partial}{\partial v}\right) f_I(u + Iv) = 0.$$

We denote by $\mathcal{SM}(U)$ the set of s-monogenic functions on U.

The natural class of domains in which we can develop the theory of s-monogenic functions are the so-called slice domains and axially symmetric domains.

Definition 2.5 (Slice domains). Let $U \subseteq \mathbb{R}^{n+1}$ be a domain. We say that U is a slice domain (s-domain for short) if $U \cap \mathbb{R}$ is non-empty and if $\mathbb{C}_I \cap U$ is a domain in $\mathbb{C}_I$ for all $I \in \mathbb{S}$.

Definition 2.6 (Axially symmetric domains). Let $U \subseteq \mathbb{R}^{n+1}$. We say that U is axially symmetric if, for every $u + Iv \in U$, the whole $(n-1)$-sphere $[u + Iv]$ is contained in U.

Let us now introduce the notations necessary to deal with linear operators. By V and by V_n we denote a Banach space over $\mathbb{R}$ with norm $\|\cdot\|$ and $V \otimes \mathbb{R}_n$, respectively. We recall that V_n is a two-sided Banach module over $\mathbb{R}_n$ and its elements are of the type $\sum_A v_A \otimes e_A$ (where $A = i_1 \dots i_r$, $i_\ell \in \{1, 2, \dots, n\}$, $i_1 < \dots < i_r$ is a multi-index). The multiplications (right and left) of an element $v \in V_n$ with a scalar $a \in \mathbb{R}_n$ are defined as $va = \sum_A v_A \otimes (e_A a)$ and $av = \sum_A v_A \otimes (a e_A)$. For short, in the sequel we will write $\sum_A v_A e_A$ instead of $\sum_A v_A \otimes e_A$. Moreover, we define $\|v\|^2_{V_n} = \sum_A \|v_A\|^2_V$.

Let $\mathcal{B}(V)$ be the space of bounded $\mathbb{R}$-homomorphisms of the Banach space V into itself endowed with the natural norm denoted by $\|\cdot\|_{\mathcal{B}(V)}$. If $T_A \in \mathcal{B}(V)$, we can define the operator $T=\sum_A T_Ae_A$ and its action on $v=\sum_B v_Be_B$ as $T(v)=\sum_{A,B}T_A(v_B)e_Ae_B$. The set of all such bounded operators is denoted by $\mathcal{B}_n(V_n)$ and the norm is defined by $\|T\|_{\mathcal{B}_n(V_n)}=\sum_A\|T_A\|_{\mathcal{B}(V)}$. Note that, in the sequel, we will omit the subscript $\mathcal{B}_n(V_n)$ in the norm of an operator and note also that $\|TS\|\leq\|T\|\|S\|$. A bounded operator $T=T_0+\sum_{j=1}^n e_jT_j$, where $T_\mu\in\mathcal{B}(V)$ for $\mu=0,1,\dots,n$, will be called, with an abuse of notation, an operator in paravector form. The set of such operators will be denoted by $\mathcal{B}_n^{0,1}(V_n)$. The set of bounded operators of the type $T=\sum_{j=1}^n e_jT_j$, where $T_\mu\in\mathcal{B}(V)$ for $\mu=1,\dots,n$, will be denoted by $\mathcal{B}_n^1(V_n)$ and T will be said operator in vector form. We will consider operators of the form $T=T_0+\sum_{j=1}^n e_jT_j$ where $T_\mu\in\mathcal{B}(V)$ for $\mu=0,1,\dots,n$ for the sake of generality, but when dealing with n-tuples of operators, we will embed them into $\mathcal{B}_n(V_n)$ as operators in vector form, by setting $T_0=0$. The subset of those operators in $\mathcal{B}_n(V_n)$ whose components commute among themselves will be denoted by $\mathcal{BC}_n(V_n)$. In the same spirit we denote by $\mathcal{BC}_n^{0,1}(V_n)$ the set of paravector operators with commuting components.

We now recall some definitions and results from [5], [6].

Definition 2.7 (The $\mathcal{F}$-spectrum and the $\mathcal{F}$-resolvent sets). Let $T\in\mathcal{BC}_n^{0,1}(V_n)$. We define the $\mathcal{F}$-spectrum of T as:

$$\sigma_{\mathcal{F}}(T)=\{s\in\mathbb{R}^{n+1}\ :\ \ s^2\mathcal{I}-s(T+\overline{T})+T\overline{T}\text{ is not invertible}\}.$$

The $\mathcal{F}$-resolvent set of T is defined by

$$\rho_{\mathcal{F}}(T)=\mathbb{R}^{n+1}\setminus\sigma_{\mathcal{F}}(T).$$

Theorem 2.8 (Structure of the $\mathcal{F}$-spectrum). *Let $T\in\mathcal{BC}_n^{0,1}(V_n)$ and let $p=p_0+p_1I\in[p_0+p_1I]\subset\mathbb{R}^{n+1}\setminus\mathbb{R}$, such that $p\in\sigma_{\mathcal{F}}(T)$. Then all the elements of the $(n-1)$-sphere $[p_0+p_1I]$ belong to $\sigma_{\mathcal{F}}(T)$. Thus the $\mathcal{F}$-spectrum consists of real points and/or $(n-1)$-spheres.*

Theorem 2.9 (Compactness of $\mathcal{F}$-spectrum). *Let $T\in\mathcal{BC}_n^{0,1}(V_n)$. Then the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$ is a compact non-empty set. Moreover $\sigma_{\mathcal{F}}(T)$ is contained in $\{s\in\mathbb{R}^{n+1}\ :\ |s|\leq\|T\|\ \}$.*

Definition 2.10. Let $T\in\mathcal{BC}_n^{0,1}(V_n)$ and let $U\subset\mathbb{R}^{n+1}$ be an axially symmetric s-domain containing the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$, and such that $\partial(U\cap\mathbb{C}_I)$ is union of a finite number of continuously differential Jordan curves for every $I\in\mathbb{S}$. Let W be an open set in $\mathbb{R}^{n+1}$. A function $f\in\mathcal{SM}(W)$ is said to be locally s-monogenic on $\sigma_{\mathcal{F}}(T)$ if there exists a domain $U\subset\mathbb{R}^{n+1}$ as above such that $\overline{U}\subset W$. We will denote by $\mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ the set of locally s-monogenic functions on $\sigma_{\mathcal{F}}(T)$.

Definition 2.11 (The $\mathcal{SC}$-resolvent operator). Let $T\in\mathcal{BC}_n^{0,1}(V_n)$ and $s\in\rho_{\mathcal{F}}(T)$. We define the $\mathcal{SC}$-resolvent operator as

$$\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T):=(s\mathcal{I}-\overline{T})(s^2\mathcal{I}-s(T+\overline{T})+T\overline{T})^{-1}. \tag{2.1}$$

Definition 2.12 (The $\mathcal{SC}$-functional calculus). Let $T \in \mathcal{BC}_n^{0,1}(V_n)$ and $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$. Let $U \subset \mathbb{R}^{n+1}$ be a domain as in Definition 2.10 and set $ds_I = ds/I$ for $I \in \mathbb{S}$. We define the $\mathcal{SC}$-functional calculus as

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{S_C}^{-1}(s, T)\, ds_I\, f(s). \tag{2.2}$$

Definition 2.13 ($\mathcal{F}$-resolvent operator). Let n be an odd number and let $T \in \mathcal{BC}_n^{0,1}(V_n)$. For $s \in \rho_{\mathcal{F}}(T)$ we define the $\mathcal{F}$-resolvent operator as

$$\mathcal{F}_n^{-1}(s, T) := \gamma_n (s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-\frac{n+1}{2}},$$

where

$$\gamma_n := (-1)^{(n-1)/2} 2^{(n-1)/2} (n-1)! \Big(\frac{n-1}{2}\Big)!.$$

Next we define $\breve{f}(T)$ when $\breve{f}$ is a monogenic function which comes from an s-monogenic function f via Fueter's theorem. The $\mathcal{F}$-functional calculus will be defined for those monogenic functions that are of the form $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$, where f is an s-monogenic function. For the functional calculus associated to standard monogenic functions we mention the book [14].

Definition 2.14 (The $\mathcal{F}$-functional calculus). Let n be an odd number and let $T \in \mathcal{BC}_n^{0,1}(V_n)$. Let U be an open set as in Definition 2.10. Suppose that $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ and let $\breve{f}(x) = \Delta^{\frac{n-1}{2}} f(x)$. We define the $\mathcal{F}$-functional calculus as

$$\breve{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{F}_n^{-1}(s, T)\, ds_I\, f(s). \tag{2.3}$$

Remark 2.15. The definitions of the $\mathcal{SC}$-functional calculus and of the $\mathcal{F}$-functional calculus are well posed since the integrals in (2.2) and in (2.3) are independent of $I \in \mathbb{S}$ and of the open set U.

3. Examples of equations for the $\mathcal{F}$-spectrum

Example (The case of Dirac operator). Let us consider the n-tuple of operators $(\partial_{x_1}, \dots, \partial_{x_n})$, each of them acting on the vector space of functions of class $\mathcal{C}^2$ over an open set $U \subseteq \mathbb{R}^{n+1}$. The vector operator associated to them is the Dirac operator

$$T = \partial_{x_1} e_1 + \ldots + \partial_{x_n} e_n.$$

Let us determine the equation which gives its $\mathcal{F}$-spectrum. We have $\overline{T} = -\partial_{x_1} e_1 - \ldots - \partial_{x_n} e_n$, and, since $\partial_{x_i}\partial_{x_j} = \partial_{x_j}\partial_{x_i}$, for all $i, j = 1, \dots, n$ we also have $T + \overline{T} = 0$ and $T\overline{T} = \partial_{x_1}^2 + \ldots + \partial_{x_n}^2 = \Delta$. The $\mathcal{F}$-spectrum is associated to the equation

$$(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})v = 0 \quad \text{for } \ v \neq 0$$

which, in this case, becomes

$$(s^2\mathcal{I} + \Delta)v = 0 \quad \text{for } \ v \neq 0. \tag{3.1}$$

The paravector s can be considered as an element belonging to a complex plane $s \in \mathbb{C}_{I_0}$, so we can assume that $s = s_0 + s_1 I_0$ is a solution of (3.1) for some I_0. Then the $\mathcal{F}$-spectrum of T is given by

$$\sigma_{\mathcal{F}}(T) = \bigcup_{s\in\mathbb{C}_{I_0} \text{ solution of (3.1)}} \{s = s_0 + s_1 I, \text{ for all } I \in \mathbb{S}\}.$$

Example (The case of second derivatives). Let us consider the second-order operators $(\partial^2_{x_1},\ldots,\partial^2_{x_n})$ each of them acting on the vector space of functions of class $\mathcal{C}^4$ over an open set $U \subseteq \mathbb{R}^{n+1}$, and let us write

$$T = \partial^2_{x_1} e_1 + \ldots + \partial^2_{x_n} e_n.$$

Determine the equation which gives its $\mathcal{F}$-spectrum. We have $\overline{T} = -\partial^2_{x_1} e_1 - \ldots - \partial^2_{x_n} e_n$, and, since $\partial^2_{x_i}\partial^2_{x_j} = \partial^2_{x_j}\partial^2_{x_i}$, for all $i, j = 1, \ldots, n$ we also have $T + \overline{T} = 0$ and $T\overline{T} = \partial^4_{x_1} + \ldots + \partial^4_{x_n}$. The $\mathcal{F}$-spectrum is associated to the equation

$$(s^2\mathcal{I} + \partial^4_{x_1} + \ldots + \partial^4_{x_n})v = 0 \quad \text{for} \quad v \neq 0. \tag{3.2}$$

We solve the equation (3.2) on the complex plane $s \in \mathbb{C}_{I_0}$ for some I_0. Then the $\mathcal{F}$-spectrum of T is given by

$$\sigma_{\mathcal{F}}(T) = \bigcup_{s\in\mathbb{C}_{I_0} \text{ solution of (3.2)}} \{s = s_0 + s_1 I, \text{ for all } I \in \mathbb{S}\}.$$

Example (The case of powers of a real matrix). Let A be a matrix $n \times n$ with real entries and consider the operators $T_j := A^j$ for $j = 1, \ldots, n$. Determine the equation associated to the $\mathcal{F}$-spectrum. It is well known that $T_j T_k = T_k T_j$, for $j, k = 1, \ldots, n$. So we consider the operator

$$T = Ae_1 + \ldots + A^n e_n.$$

We have $\overline{T} = -Ae_1 - \ldots - A^n e_n$, $T + \overline{T} = 0$ and also $T\overline{T} = \sum_{j=1}^n A^{2j}$. The $\mathcal{F}$-spectrum is associated to the equation

$$(s^2\mathcal{I} + \sum_{j=1}^n A^{2j})v = 0 \quad \text{for} \quad v \neq 0. \tag{3.3}$$

Let us conclude this short list of examples with an explicit computation of the $\mathcal{F}$-spectrum.

Example (The case of two triangular commuting matrices). Let us consider a, b, α, $\beta \in \mathbb{R}$ and the two matrices:

$$T_1 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \qquad T_2 = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}.$$

It is easy to verify that $T_1T_2 = T_2T_1$. So we associate to T_1 and T_2 the operator

$$T = T_1e_1 + T_2e_2 = \begin{bmatrix} ae_1 + \alpha e_2 & be_1 + \beta e_2 \\ 0 & ae_1 + \alpha e_2 \end{bmatrix}.$$

We have

$$\overline{T} = -\begin{bmatrix} ae_1 + \alpha e_2 & be_1 + \beta e_2 \\ 0 & ae_1 + \alpha e_2 \end{bmatrix},$$

$T + \overline{T} = 0$ and

$$T\overline{T} = \begin{bmatrix} a^2 + \alpha^2 & 2ab + 2\alpha\beta \\ 0 & a^2 + \alpha^2 \end{bmatrix}.$$

The $\mathcal{F}$-spectrum is associated to the equation

$$\left(s^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a^2 + \alpha^2 & 2ab + 2\alpha\beta \\ 0 & a^2 + \alpha^2 \end{bmatrix}\right) v = 0 \quad \text{for} \quad v \neq 0$$

which becomes

$$\begin{bmatrix} s^2 + a^2 + \alpha^2 & 2ab + 2\alpha\beta \\ 0 & s^2 + a^2 + \alpha^2 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = 0 \quad \text{for} \quad \begin{bmatrix} u \\ w \end{bmatrix} \neq 0.$$

Consider a paravector s on the complex plane $\mathbb{C}_{I_0}$: with some calculation we obtain $s = \pm I_0\sqrt{a^2 + \alpha^2}$. Thus the $\mathcal{F}$-spectrum of T is given by

$$\sigma_{\mathcal{F}}(T) = \{\pm I\sqrt{a^2 + \alpha^2} \text{ for all } I \in \mathbb{S}\}.$$

4. Bounded perturbations of the $\mathcal{SC}$-resolvent

Lemma 4.1. *The set $\mathcal{U}(V_n)$ of elements in $\mathcal{B}_n(V_n)$ which have inverse in $\mathcal{B}_n(V_n)$ is an open set in the uniform topology of $\mathcal{B}_n(V_n)$. If $\mathcal{U}(V_n)$ contains an element A, then it contains the ball*

$$\Sigma = \{B \in \mathcal{B}_n(V_n) : \ \|A - B\| < \|A^{-1}\|^{-1}\}.$$

If $B \in \Sigma$, its inverse is given by the series

$$B^{-1} = A^{-1} \sum_{m \geq 0} [(A - B)A^{-1}]^m. \tag{4.1}$$

Furthermore, the map $A \mapsto A^{-1}$ from $\mathcal{U}(V_n)$ onto $\mathcal{U}(V_n)$ is a homeomorphism in the uniform operator topology.

Proof. See Lemma 7.1 in [3]. □

In order to state our results, we need the following definitions:

Definition 4.2. Let $T \in \mathcal{BC}^{0,1}_n(V_n)$. We denote by $\sigma_L(T)$ the so-called left spectrum of T related to the resolvent operator $(s\mathcal{I} - T)^{-1}$ that is defined as

$$\sigma_L(T) = \{s \in \mathbb{R}^{n+1} : \quad s\mathcal{I} - T \quad \text{is not invertible in } \mathcal{BC}^{0,1}_n(V_n)\},$$

where the notation $s\mathcal{I}$ in $\mathcal{B}^R(V)$ means that $(s\mathcal{I})(v) = sv$.

Definition 4.3. Let $\mathcal{W}$ be a subset of $\mathbb{R}^{n+1}$. We denote by $B(\mathcal{W}, \varepsilon)$, for $\varepsilon > 0$, the ε-neighborhood of $\mathcal{W}$ defined as

$$B(\mathcal{W}, \varepsilon) := \{x \in \mathbb{R}^{n+1} : \quad \inf_{s \in \mathcal{W}} |s - x| < \varepsilon\}.$$

Lemma 4.4. *Let T, $Z \in \mathcal{BC}_n^{0,1}(V_n)$, $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$ and consider*

$$\mathcal{S}_{\mathcal{C}}(s,T) = s\mathcal{I} - (s\mathcal{I} - \overline{T})\, T\, (s\mathcal{I} - \overline{T})^{-1}, \tag{4.2}$$

$$\mathcal{S}_{\mathcal{C}}(s,Z) = s\mathcal{I} - (s\mathcal{I} - \overline{Z})\, Z\, (s\mathcal{I} - \overline{Z})^{-1}. \tag{4.3}$$

Then there exists a strictly positive constant $K(s)$, depending on s and also on the operators T and Z, such that

$$\|\mathcal{S}_{\mathcal{C}}(s,T) - \mathcal{S}_{\mathcal{C}}(s,Z)\| \leq K(s)\|T - Z\|. \tag{4.4}$$

Proof. Consider the chain of equalities

$$\begin{aligned}
&\mathcal{S}_{\mathcal{C}}(s,T) - \mathcal{S}_{\mathcal{C}}(s,Z)\\
&= (s\mathcal{I} - \overline{Z})\, Z\, (s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})\, T\, (s\mathcal{I} - \overline{T})^{-1}\\
&= (s\mathcal{I} - \overline{Z})\, Z\, (s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})\, Z\, (s\mathcal{I} - \overline{Z})^{-1}\\
&\quad + (s\mathcal{I} - \overline{T})\, Z\, (s\mathcal{I} - \overline{Z})^{-1} - (s\mathcal{I} - \overline{T})\, T\, (s\mathcal{I} - \overline{T})^{-1}\\
&= (\overline{T} - \overline{Z})\, Z\, (s\mathcal{I} - \overline{Z})^{-1} + (s\mathcal{I} - \overline{T})[\, Z\, (s\mathcal{I} - \overline{Z})^{-1} - \, T\, (s\mathcal{I} - \overline{T})^{-1}]\\
&= (\overline{T} - \overline{Z})\, Z\, (s\mathcal{I} - \overline{Z})^{-1}\\
&\quad + (s\mathcal{I} - \overline{T})[(Z - T)\, (s\mathcal{I} - \overline{Z})^{-1} + T(\, (s\mathcal{I} - \overline{Z})^{-1} - \, (s\mathcal{I} - \overline{T})^{-1})]\\
&= (\overline{T} - \overline{Z})\, Z\, (s\mathcal{I} - \overline{Z})^{-1}\\
&\quad + (s\mathcal{I} - \overline{T})\Big[(Z - T)\, (s\mathcal{I} - \overline{Z})^{-1} + T\, (s\mathcal{I} - \overline{Z})^{-1}(\overline{Z} - \overline{T})\, (s\mathcal{I} - \overline{T})^{-1}\Big].
\end{aligned}$$

By taking the norm and observing that $\|T - Z\| = \|\overline{T} - \overline{Z}\|$, we have

$$\begin{aligned}
\|\mathcal{S}_{\mathcal{C}}(s,T) - \mathcal{S}_{\mathcal{C}}(s,Z)\| \leq \|T - Z\| \Big(&\|Z\|\, \|(s\mathcal{I} - \overline{Z})^{-1}\|\\
&+ \|s\mathcal{I} - \overline{T}\| \Big[\|(s\mathcal{I} - \overline{Z})^{-1}\| + \|T\|\, \|(s\mathcal{I} - \overline{Z})^{-1}\| \|(s\mathcal{I} - \overline{T})^{-1}\|\Big]\Big).
\end{aligned}$$

If we now set

$$K(s) := \|(s\mathcal{I} - \overline{Z})^{-1}\| \Big(\|Z\| + \|s\mathcal{I} - \overline{T}\| \Big[1 + \|T\|\, \|(s\mathcal{I} - \overline{T})^{-1}\|\Big]\Big), \tag{4.5}$$

we have that $K(s) > 0$ and we get the statement. □

Lemma 4.5. *Let T, $Z \in \mathcal{BC}_n^{0,1}(V_n)$, $s \in \rho_{\mathcal{F}}(T)$, $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$ and suppose that*

$$\|T - Z\| < \frac{1}{K(s)} \|\mathcal{S}_{\mathcal{C}}^{-1}(s,T)\|^{-1},$$

where $K(s)$ is defined in (4.5)*. Then $s \in \rho_{\mathcal{F}}(Z)$ and*

$$\mathcal{S}_{\mathcal{C}}^{-1}(s,Z) - \mathcal{S}_{\mathcal{C}}^{-1}(s,T) = \mathcal{S}_{\mathcal{C}}^{-1}(s,T) \sum_{m \geq 1} [(\mathcal{S}_{\mathcal{C}}(s,T) - \mathcal{S}_{\mathcal{C}}(s,Z))\mathcal{S}_{\mathcal{C}}^{-1}(s,T)]^m. \tag{4.6}$$

Proof. Let us recall (4.2) and (4.3) and set

$$A := \mathcal{S}_{\mathcal{C}}(s,T), \qquad B := \mathcal{S}_{\mathcal{C}}(s,Z), \qquad A^{-1} = \mathcal{S}_{\mathcal{C}}{}^{-1}(s,T). \tag{4.7}$$

By formula (4.1) in Lemma 4.1 with $B^{-1} := \mathcal{S}_{\mathcal{C}}{}^{-1}(s,Z)$, we get

$$\mathcal{S}_{\mathcal{C}}{}^{-1}(s,Z) = \mathcal{S}_{\mathcal{C}}{}^{-1}(s,T) \sum_{m\geq 0} [(\mathcal{S}_{\mathcal{C}}(s,T) - \mathcal{S}_{\mathcal{C}}(s,Z))\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)]^m. \tag{4.8}$$

The series in (4.8) converges since

$$\|(\mathcal{S}_{\mathcal{C}}(s,T) - \mathcal{S}_{\mathcal{C}}(s,Z))\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)\| \leq K(s)\|T-Z\|\|\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)\| < 1,$$

so we get the statement. □

Theorem 4.6. *Let T, $Z \in \mathcal{BC}^{0,1}_n(V_n)$, $s \in \rho_{\mathcal{F}}(T)$, $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$. Let $\varepsilon > 0$ and consider the ε-neighborhood $B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$ of $\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T})$. Then there exists $\delta > 0$ such that, for $\|T - Z\| < \delta$, we have*

$$\sigma_{\mathcal{F}}(Z) \subseteq B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$$

and

$$\|\mathcal{S}_{\mathcal{C}}{}^{-1}(s,Z) - \mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)\| < \varepsilon, \quad for \quad s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon).$$

Proof. Let $\overline{T}, \overline{Z} \in \mathcal{BC}^{0,1}_n(V_n)$ and let $\varepsilon > 0$. Thanks to Lemma 4.1 there exists a $\eta > 0$ such that if

$$\|T - Z\| < \eta$$

then $\sigma_L(\overline{Z}) \subset B(\sigma_L(\overline{T}), \varepsilon)$, where $B(\sigma_L(\overline{T}), \varepsilon)$ is the ε-neighborhood of $\sigma_L(\overline{T})$. So we can always choose η such that $\sigma_L(\overline{Z}) \subset B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$. Consider the function $K(s)$ defined in (4.5) and observe that the constant K_ε defined by

$$K_\varepsilon = \sup_{s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)} K(s)$$

is finite since $s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$, for the above observation $\sigma_L(\overline{Z}) \subset B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$ and because

$$\lim_{s\to\infty} \|(s\mathcal{I} - \overline{Z})^{-1}\| = \lim_{s\to\infty} \|(s\mathcal{I} - \overline{T})^{-1}\| = 0.$$

Observe that since $s \in \rho_{\mathcal{F}}(T)$ the map $s \mapsto \|\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)\|$ is continuous and

$$\lim_{s\to\infty} \|\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)\| = 0,$$

for s in the complement set of $B(\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T}), \varepsilon)$ we have that there exists a positive constant N_ε such that

$$\|\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)\| \leq N_\varepsilon.$$

From Lemma 4.5, if $\delta_1 > 0$ is such that $\|Z - T\| < \frac{1}{K_\varepsilon N_\varepsilon} := \delta_1$, then $s \in \rho_{\mathcal{F}}(Z)$ and

$$\|\mathcal{S}_{\mathcal{C}}{}^{-1}(s,Z)-\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)\| \leq \frac{\|\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)\|^2\ \|\mathcal{S}_{\mathcal{C}}(s,T)-\mathcal{S}_{\mathcal{C}}(s,Z)\|}{1-\|\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)\|\ \|\mathcal{S}_{\mathcal{C}}(s,T)-\mathcal{S}_{\mathcal{C}}(s,Z)\|}$$
$$\leq \frac{N_\varepsilon^2 K_\varepsilon \|Z-T\|}{1-N_\varepsilon K_\varepsilon\|Z-T\|} < \varepsilon$$

if we take

$$\|Z-T\| < \delta_2 := \frac{\varepsilon}{K_\varepsilon(N_\varepsilon^2+\varepsilon N_\varepsilon)}.$$

To get the statement it suffices to set $\delta = \min\{\eta, \delta_1, \delta_2\}$. □

Theorem 4.7. *Let* $T, Z \in \mathcal{BC}_n^{0,1}(V_n)$, $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ *and let* $\varepsilon > 0$. *Then there exists* $\delta > 0$ *such that, for* $\|Z-T\| < \delta$, *we have* $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(Z)}$ *and*

$$\|f(Z)-f(T)\| < \varepsilon.$$

Proof. We recall that operator $f(T)$ is defined by

$$f(T) = \frac{1}{2\pi}\int_{\partial(U\cap\mathbb{C}_I)} \mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)\ ds_I\ f(s)$$

where $U \subset \mathbb{R}^{n+1}$ is a domain as in Definition 2.10, $ds_I = ds/I$ for $I \in \mathbb{S}$. Suppose that U is an ε-neighborhood of $\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T})$ and it is contained in the domain in which f is s-monogenic. By Lemma 4.6 there is a $\delta_1 > 0$ such that $\sigma_{\mathcal{F}}(Z) \subset U$ for $\|Z-T\| < \delta_1$. Consequently $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(Z)}$ for $\|Z-T\| < \delta_1$. By Lemma 4.6 $\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)$ is uniformly near to $\mathcal{S}_{\mathcal{C}}{}^{-1}(s,Z)$ with respect to $s \in \partial(U\cap\mathbb{C}_I)$ for $I \in \mathbb{S}$ if $\|Z-T\|$ is small enough, so for some positive $\delta \leq \delta_1$ we get

$$\|f(T)-f(Z)\| = \frac{1}{2\pi}\|\int_{\partial(U\cap\mathbb{C}_I)} [\mathcal{S}_{\mathcal{C}}{}^{-1}(s,T)-\mathcal{S}_{\mathcal{C}}{}^{-1}(s,Z)]\ ds_I\ f(s)\| < \varepsilon. \qquad \square$$

5. Bounded perturbations of the $\mathcal{F}$-resolvent

Let n be an odd number. For $s \in \rho_{\mathcal{F}}(T)$ the $\mathcal{F}$-resolvent operator associated to T is

$$\mathcal{F}_n^{-1}(s,T) := \gamma_n(s\mathcal{I}-\overline{T})(s^2\mathcal{I}-s(T+\overline{T})+T\overline{T})^{-\frac{n+1}{2}}, \tag{5.1}$$

while its inverse is

$$\mathcal{F}_n(s,T) := \frac{1}{\gamma_n}(s^2\mathcal{I}-s(T+\overline{T})+T\overline{T})^{\frac{n+1}{2}}(s\mathcal{I}-\overline{T})^{-1}, \tag{5.2}$$

for $s \notin \sigma_L(\overline{T})$. Analogously for $s \in \rho_{\mathcal{F}}(Z)$ the $\mathcal{F}$-resolvent operator associated to Z is

$$\mathcal{F}_n^{-1}(s,Z) := \gamma_n(s\mathcal{I}-\overline{Z})(s^2\mathcal{I}-s(Z+\overline{Z})+Z\overline{Z})^{-\frac{n+1}{2}}, \tag{5.3}$$

and it has the inverse

$$\mathcal{F}_n(s,Z) := \frac{1}{\gamma_n}(s^2\mathcal{I}-s(Z+\overline{Z})+Z\overline{Z})^{\frac{n+1}{2}}(s\mathcal{I}-\overline{Z})^{-1}, \tag{5.4}$$

for $s \notin \sigma_L(\overline{Z})$.

Lemma 5.1. *Let n be an odd number, T, $Z \in \mathcal{BC}_n^{0,1}(V_n)$ and let $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$. Then there exists a positive constant $C_n(s)$ depending on s and also on the operators T and Z such that*

$$\|\mathcal{F}_n(s,T) - \mathcal{F}_n(s,Z)\| \leq C_n(s)(|s| + \vartheta)^{n-1}\|T - Z\|, \tag{5.5}$$

where $\vartheta := \max\{\|T\|, \|Z\|\}$.

Proof. For simplicity let us set the positions $\frac{n+1}{2} := k+1$, for $k \in \mathbb{N}$, so that $k = \frac{n-1}{2}$, for $n = 1, 3, 5, \ldots$. The case $k = 0$ has been studied in the previous section. Here we consider $k \geq 1$. We set $\beta_k := \gamma_{2k+1}$ and we define, for $s \in \rho_{\mathcal{F}}(T)$,

$$\widetilde{\mathcal{F}}_k^{-1}(s,T) := \beta_k \ {\mathcal{S}_{\mathcal{C}}}^{-1}(s,T) \ (s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-k}. \tag{5.6}$$

The inverse of operator $\widetilde{\mathcal{F}}_k^{-1}(s,T)$ exists for $s \notin \sigma_L(\overline{T})$ and is given by

$$\widetilde{\mathcal{F}}_k(s,T) = \frac{1}{\beta_k} \ (s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^k \ \mathcal{S}_{\mathcal{C}}(s,T), \tag{5.7}$$

while the inverse of operator $\widetilde{\mathcal{F}}_k^{-1}(s,Z)$ exists for $s \notin \sigma_L(\overline{Z})$ and is given by

$$\widetilde{\mathcal{F}}_k(s,Z) = \frac{1}{\beta_k} \ (s^2\mathcal{I} - s(Z + \overline{Z}) + Z\overline{Z})^k \ \mathcal{S}_{\mathcal{C}}(s,Z). \tag{5.8}$$

Consider (5.7) and (5.8) for $k = 1$; we have

$$\begin{aligned}
&\beta_1[\widetilde{\mathcal{F}}_1(s,T) - \widetilde{\mathcal{F}}_1(s,Z)]\\
&= (s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}) \ \mathcal{S}_{\mathcal{C}}(s,T) - (s^2\mathcal{I} - s(Z + \overline{Z}) + Z\overline{Z}) \ \mathcal{S}_{\mathcal{C}}(s,Z)\\
&= (s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}) \ \mathcal{S}_{\mathcal{C}}(s,T) - (s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}) \ \mathcal{S}_{\mathcal{C}}(s,Z)\\
&\quad + (s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}) \ \mathcal{S}_{\mathcal{C}}(s,Z) - (s^2\mathcal{I} - s(Z + \overline{Z}) + Z\overline{Z}) \ \mathcal{S}_{\mathcal{C}}(s,Z)\\
&= (s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}) \ [\mathcal{S}_{\mathcal{C}}(s,T) - \mathcal{S}_{\mathcal{C}}(s,Z)]\\
&\quad + [-s(T + \overline{T}) + T\overline{T} + s(Z + \overline{Z}) - Z\overline{Z})] \ \mathcal{S}_{\mathcal{C}}(s,Z)\\
&= (s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T}) \ [\mathcal{S}_{\mathcal{C}}(s,T) - \mathcal{S}_{\mathcal{C}}(s,Z)]\\
&\quad + [s(Z - T + \overline{Z} - \overline{T}) + (T - Z)\overline{T} + Z(\overline{T} - \overline{Z})] \ \mathcal{S}_{\mathcal{C}}(s,Z)
\end{aligned}$$

and taking the norm we get

$$\begin{aligned}
&|\beta_1|\|\widetilde{\mathcal{F}}_1(s,T) - \widetilde{\mathcal{F}}_1(s,Z)\|\\
&\leq (|s|^2 + 2|s| \, \|T\| + \|T\overline{T}\|)\|\mathcal{S}_{\mathcal{C}}(s,T) - \mathcal{S}_{\mathcal{C}}(s,Z)\|\\
&\quad + \big[2|s| \, \|Z - T\| + \|T - Z\|(\|\overline{T}\| + \|Z\|)\big] \ \|\mathcal{S}_{\mathcal{C}}(s,Z)\|.\\
&\leq (|s| + \vartheta)^2\|\mathcal{S}_{\mathcal{C}}(s,T) - \mathcal{S}_{\mathcal{C}}(s,Z)\| + \big[2(|s| + \vartheta)\|Z - T\|)\big] \ \|\mathcal{S}_{\mathcal{C}}(s,Z)\|.
\end{aligned}$$

Now observe that

$$(|s| + \vartheta)^{-1}\|\mathcal{S}_{\mathcal{C}}(s,Z)\| \leq (|s| + \vartheta)^{-1}\big[|s| + \|(s\mathcal{I} - \overline{Z})\| \, \|Z\| \, \|(s\mathcal{I} - \overline{Z})^{-1}\|\big] =: M(s) \tag{5.9}$$

where $M(s)$ is a continuous function since $s \notin \sigma_L(\overline{Z})$. Using Lemma 4.4 we get

$$\|\widetilde{\mathcal{F}}_1(s,T) - \widetilde{\mathcal{F}}_1(s,Z)\| \leq \frac{1}{|\beta_1|}[K(s) + 2M(s)](|s| + \vartheta)^2\|Z - T\|. \tag{5.10}$$

We now use the induction principle. We assume that the estimate

$$\|\widetilde{\mathcal{F}}_k(s,T) - \widetilde{\mathcal{F}}_k(s,Z)\| \leq \frac{1}{|\beta_k|}(|s| + \vartheta)^{2k}\Big[K(s) + 2kM(s)\Big]\|Z - T\| \tag{5.11}$$

holds for $k \geq 1$. Observe that (5.8) implies that the estimate

$$\|\mathcal{F}_k(s,Z)\| \leq \frac{1}{|\beta_k|}(|s| + \vartheta)^{2k}\ \|\mathcal{S}_{\mathcal{C}}(s,Z)\| \tag{5.12}$$

holds. We prove that

$$\|\widetilde{\mathcal{F}}_{k+1}(s,T) - \widetilde{\mathcal{F}}_{k+1}(s,Z)\| \leq \frac{1}{|\beta_{k+1}|}(|s| + \vartheta)^{2(k+1)}\Big[K(s) + 2(k+1)M(s)\Big]\|Z - T\|.$$

In fact, we have that

$$\begin{aligned}
&\beta_{k+1}(\widetilde{\mathcal{F}}_{k+1}(s,T) - \widetilde{\mathcal{F}}_{k+1}(s,Z))\\
&= \beta_k(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})\ \widetilde{\mathcal{F}}_k(s,T) - \beta_k(s^2\mathcal{I} - s(Z + \overline{Z}) + Z\overline{Z})\ \widetilde{\mathcal{F}}_k(s,Z)\\
&= \beta_k(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})\ [\widetilde{\mathcal{F}}_k(s,T) - \widetilde{\mathcal{F}}_k(s,Z)]\\
&\quad - \beta_k[s(Z - T + \overline{Z} - \overline{T}) + (T - Z)\overline{T} + Z(\overline{T} - \overline{Z})]\ \widetilde{\mathcal{F}}_k(s,Z)
\end{aligned}$$

and taking the norms we have

$$\begin{aligned}
&|\beta_{k+1}|\|\widetilde{\mathcal{F}}_{k+1}(s,T) - \widetilde{\mathcal{F}}_{k+1}s,Z)\|\\
&\leq |\beta_k|(|s| + \vartheta)^2\|\widetilde{\mathcal{F}}_k(s,T) - \widetilde{\mathcal{F}}_k(s,Z)\| + 2\,\beta_k(|s| + \vartheta)\ \|\mathcal{F}_k(s,Z)\|\|Z - T\|.
\end{aligned}$$

Using (5.11) and (5.12) we obtain

$$\begin{aligned}
&|\beta_{k+1}|\|\widetilde{\mathcal{F}}_{k+1}(s,T) - \widetilde{\mathcal{F}}_{k+1}s,Z)\|\\
&\leq (|s| + \vartheta)^{2k+2}\Big[K(s) + 2kM(s)\Big]\|Z - T\| + 2\,(|s| + \vartheta)^{2k+1}\ \|\mathcal{S}_{\mathcal{C}}(s,Z)\|\|Z - T\|\\
&\leq (|s| + \vartheta)^{2k+2}\Big[K(s) + 2kM(s) + 2\,(|s| + \vartheta)^{-1}\ \|\mathcal{S}_{\mathcal{C}}(s,Z)\|\Big]\|Z - T\|\\
&\leq (|s| + \vartheta)^{2k+2}\Big[K(s) + 2(k+1)M(s)\|\Big]\|Z - T\|.
\end{aligned}$$

Setting $\widetilde{C}_k(s) := \frac{1}{|\beta_k|}[K(s) + 2kM(s)]$ the constant $C_n(s)$ in estimate (5.5) is given by

$$C_n(s) := \frac{1}{|\gamma_n|}[K(s) + (n-1)M(s)]. \tag{5.13}$$

This concludes the proof. □

Lemma 5.2. *Let n be an odd number, T, $Z \in \mathcal{BC}^{0,1}_n(V_n)$, let $s \in \rho_{\mathcal{F}}(T)$, $s \notin \sigma_L(\overline{T}) \cup \sigma_L(\overline{Z})$ and suppose that*

$$\|T - Z\| < \frac{1}{C_n(s)}(|s| + \vartheta)^{-(n-1)}\|\mathcal{F}_n^{-1}(s,T)\|^{-1},$$

where $C_n(s)$ *is defined in* (5.13)*. Then* $s \in \rho_{\mathcal{F}}(Z)$ *and*

$$\mathcal{F}_n^{-1}(s,Z)-\mathcal{F}_n^{-1}(s,T)=\mathcal{F}_n^{-1}(s,T)\sum_{m\geq 1}[(\mathcal{F}_n(s,T)-\mathcal{F}_n(s,Z))\mathcal{F}_n^{-1}(s,T)]^m. \quad (5.14)$$

Proof. Let us recall (5.2), (5.4) and set

$$A:=\mathcal{F}_n(s,T), \qquad B:=\mathcal{F}_n(s,Z), \qquad A^{-1}=\mathcal{F}_n^{-1}(s,T)(s,T). \quad (5.15)$$

By Lemma 4.1, formula (4.1), for $B^{-1}:=\mathcal{F}_n^{-1}(s,Z)$ we get

$$\mathcal{F}_n^{-1}(s,Z)=\mathcal{F}_n^{-1}(s,T)\sum_{m\geq 0}[(\mathcal{F}_n(s,T)-\mathcal{F}_n(s,Z))\mathcal{F}_n^{-1}(s,T)]^m. \quad (5.16)$$

Using the hypothesis, we have that the series converges since

$$\begin{aligned}&\|(\mathcal{F}_n(s,T)-\mathcal{F}_n(s,Z))\mathcal{F}_n^{-1}(s,T)\|\\ &\leq \|(\mathcal{F}_n(s,T)-\mathcal{F}_n(s,Z))\|\,\|\mathcal{F}_n^{-1}(s,T)\|\\ &\leq C_n(s)(|s|+\vartheta)^{n-1}\|Z-T\|\,\|\mathcal{F}_n^{-1}(s,T)\|<1.\end{aligned}$$

□

Theorem 5.3. *Let* n *be an odd number,* T*,* $Z\in\mathcal{BC}_n^{0,1}(V_n)$*,* $s\in\rho_{\mathcal{F}}(T)$*,* $s\notin\sigma_L(\overline{T})\cup\sigma_L(\overline{Z})$*. Let* $\varepsilon>0$ *and consider the* ε*-neighborhood* $B(\sigma_{\mathcal{F}}(T)\cup\sigma_L(\overline{T}),\varepsilon)$ *of* $\sigma_{\mathcal{F}}(T)\cup\sigma_L(\overline{T})$*. Then there exists* $\delta>0$ *such that, for* $\|T-Z\|<\delta$*, we have*

$$\sigma_{\mathcal{F}}(Z)\subseteq B(\sigma_{\mathcal{F}}(T)\cup\sigma_L(\overline{T}),\varepsilon)$$

and

$$\|\mathcal{F}_n^{-1}(s,Z)-\mathcal{F}_n^{-1}(s,T)\|<\varepsilon, \quad for \quad s\notin B(\sigma_{\mathcal{F}}(T)\cup\sigma_L(\overline{T},\varepsilon).$$

Proof. Let $\overline{T},\overline{Z}\in\mathcal{BC}_n^{0,1}(V_n)$ and let $\varepsilon>0$. Thanks to Lemma 4.1 there exists a $\eta>0$ such that if

$$\|T-Z\|<\eta,$$

then $\sigma_L(\overline{Z})\subset B(\sigma_L(\overline{T}),\varepsilon)$, where $B(\sigma_L(\overline{T}),\varepsilon)$ is the ε-neighborhood of $\sigma_L(\overline{T})$. So we can always choose η such that $\sigma_L(\overline{Z})\subset B(\sigma_{\mathcal{F}}(T)\cup\sigma_L(\overline{T}),\varepsilon)$. Consider the function $C_n(s)$ defined in (5.13). The constant $C_{n,\varepsilon}$ defined as

$$C_{n,\varepsilon}=\sup_{s\notin B(\sigma_{\mathcal{F}}(T)\cup\sigma_L(\overline{T},\varepsilon)} C_n(s)$$

is finite because $s\notin B(\sigma_{\mathcal{F}}(T)\cup\sigma_L(\overline{T},\varepsilon)$ and

$$\lim_{s\to\infty}\|(s\mathcal{I}-\overline{Z})^{-1}\|=\lim_{s\to\infty}\|(s\mathcal{I}-\overline{T})^{-1}\|=0.$$

Observe that the since $s\in\rho_{\mathcal{F}}(T)$ map $s\mapsto\|\mathcal{F}_n^{-1}(s,T)\|$ is continuous and

$$\lim_{s\to\infty}\|\mathcal{F}_n^{-1}(s,T)\|=0,$$

and so for s in the complement set of $B(\sigma_{\mathcal{F}}(T)\cup\sigma_L(\overline{T},\varepsilon)$ we have that there exists a positive constant M_ε such that

$$\|\mathcal{F}_n^{-1}(s,T)\|\leq M_\varepsilon.$$

From Lemma 5.2 if $\delta_1 > 0$ is such that

$$\|Z - T\| < \frac{1}{C_{n,\varepsilon} M_\varepsilon} := \delta_3,$$

then $s \in \rho_{\mathcal{F}}(Z)$ and

$$\begin{aligned}
&\|\mathcal{F}_n^{-1}(s,Z) - \mathcal{F}_n^{-1}(s,T)\| \\
&\leq \frac{\|\mathcal{F}_n^{-1}(s,T)\|^2 \ \|\mathcal{F}_n(s,T) - \mathcal{F}_n(s,Z)\|}{1 - \|\mathcal{F}_n^{-1}(s,T)\| \ \|\mathcal{F}_n(s,T) - \mathcal{F}(s,Z)\|} \\
&\leq \frac{M_\varepsilon^2 C_{n,\varepsilon}\|Z - T\|}{1 - M_\varepsilon C_{n,\varepsilon}\|Z - T\|} < \varepsilon
\end{aligned}$$

if we take

$$\|Z - T\| < \delta_4 := \frac{\varepsilon}{C_{n,\varepsilon}(M_\varepsilon^2 + \varepsilon M_\varepsilon)}.$$

To get the statement it suffices to set $\delta = \min\{\eta, \delta_3, \delta_4\}$. □

Theorem 5.4. *Let n be an odd number, $T, Z \in \mathcal{BC}_n^{0,1}(V_n)$, $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(T)}$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that, for $\|Z - T\| < \delta$, we have $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(Z)}$ and*

$$\|\breve{f}(Z) - \breve{f}(T)\| < \varepsilon.$$

Proof. We recall that

$$\breve{f}(T) = \frac{1}{2\pi}\int_{\partial(U \cap \mathbb{C}_I)} \mathcal{F}_n^{-1}(s,T)\ ds_I\ f(s)$$

and $U \subset \mathbb{R}^{n+1}$ is a domain as in Definition 2.10, $ds_I = ds/I$ for $I \in \mathbb{S}$. Let U be an ε-neighborhood of $\sigma_{\mathcal{F}}(T) \cup \sigma_L(\overline{T})$ contained in the domain in which f is s-monogenic. By Lemma 5.3 there is a $\delta_1 > 0$ such that $\sigma_{\mathcal{F}}(Z) \subset U$ for $\|Z - T\| < \delta_1$. Consequently, $f \in \mathcal{SM}_{\sigma_{\mathcal{F}}(Z)}$ for $\|Z - T\| < \delta_1$. By Lemma 5.3, $\mathcal{F}_n^{-1}(s,T)$ is uniformly near to $\mathcal{F}_n^{-1}(s,Z)$ with respect to $s \in \partial(U \cap \mathbb{C}_I)$ for $I \in \mathbb{S}$ if $\|Z - T\|$ is small enough, so for some positive $\delta \leq \delta_1$ we get

$$\|\breve{f}(T) - \breve{f}(Z)\| = \frac{1}{2\pi}\|\int_{\partial(U \cap \mathbb{C}_I)} [\mathcal{F}_n^{-1}(s,T) - \mathcal{F}_n^{-1}(s,Z)]\ ds_I\ f(s)\| < \varepsilon. \qquad \square$$

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