Bounded Perturbations of the Resolvent Operators Associated to the $\mathcal{F}$-Spectrum

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Abstract. Recently, we have introduced the $\mathcal{F}$-functional calculus and the SC-functional calculus. Our theory can be developed for operators of the form $T = T_0 + \epsilon_1 T_1 + \ldots + \epsilon_n T_n$ where $(T_0, T_1, \ldots, T_n)$ is an $(n + 1)$-tuple of linear commuting operators. The SC-functional calculus, which is defined for bounded but also for unbounded operators, associates to a suitable slice monogenic function $f$ with values in the Clifford algebra $\mathbb{R}_n$ the operator $f(T)$. The $\mathcal{F}$-functional calculus has been defined, for bounded operators $T$, by an integral transform. Such an integral transform comes from the Fueter’s mapping theorem and it associates to a suitable slice monogenic function $f$ the operator $\hat{f}(T)$, where $\hat{f}(x) = \Delta^{n-1} f(x)$ and $\Delta$ is the Laplace operator. Both functional calculi are based on the notion of $\mathcal{F}$-spectrum that plays the role that the classical spectrum plays for the Riesz-Dunford functional calculus. The aim of this paper is to study the bounded perturbations of the SC-resolvent operator and of the $\mathcal{F}$-resolvent operator. Moreover we will show some examples of equations that lead to the $\mathcal{F}$-spectrum.

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1. Introduction

The recent theory of slice monogenic functions, mainly developed in the papers [2], [3], [8], [9], [10], [12], turned out to be very important because of its applications to the so-called $\mathcal{S}$-functional calculus for $n$-tuples of non-necessarily commuting operators (bounded or unbounded), see [3] and [11]. The theory admits a quaternionic version of the $\mathcal{S}$-functional calculus for quaternionic linear operators which can be found in [4]. It is crucial to note that slice monogenic functions have a Cauchy formula with slice monogenic kernel that admits two expressions. These
two expressions of the Cauchy kernel are not equivalent when we want to define a functional calculus for non-necessarily commuting operators. In this paper, we will consider the case of commuting operators and the expression of the Cauchy kernel which gives rise to the definition of $\mathcal{F}$-spectrum which is the natural tool to treat the case of commuting operators.

Let $x = x_0 + e_1 x_1 + \ldots + e_n x_n$ and $s = s_0 + e_1 s_1 + \ldots + e_n s_n$ be paravectors in $\mathbb{R}^{n+1}$. We consider the Cauchy kernel written in the form

$$\mathcal{S}_C^{-1}(s, x) = (s - \bar{x})(s^2 - 2Re[x]s + |x|^2)^{-1}$$

which is defined for $x^2 - 2xs_0 + |s|^2 \neq 0$. Let $f : U \subset \mathbb{R}^{n+1} \to \mathbb{R}_n$, where $\mathbb{R}_n$ is the real Clifford algebra with $n$ imaginary units, $U$ is a suitable open set that contains the singularities of $\mathcal{S}_C^{-1}(s, x)$. Let $I$ be a 1-vector such that $I^2 = -1$ and let $\mathbb{C}_I$ be the complex plane that contains 1 and $I$. Then we have the Cauchy formula

$$f(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{S}_C^{-1}(s, x)ds_I f(s), \quad ds_I = -Ids,$$  \hspace{1cm} (1.1)

where the integral does not depend on the open set $U$ and on the imaginary unit $I$. In the paper [5] we have introduced the so-called $\mathcal{S}\mathcal{C}$-functional calculus ($\mathcal{S}\mathcal{C}$ stands for slice-commuting), which is defined for bounded but also for unbounded commuting operators, starting from the above Cauchy formula

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{S}_C^{-1}(s, T)ds_I f(s), \quad ds_I = -Ids.$$  \hspace{1cm} (1.2)

The definition is well posed because the the integral in (1.2) does not depend on the open set $U$ and on the imaginary unit $I$. The $\mathcal{S}\mathcal{C}$-resolvent operator is defined by

$$\mathcal{S}_C^{-1}(s, T) := (s\mathcal{I} - \overline{T})(s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-1}$$

whose associated spectrum is the $\mathcal{F}$-spectrum of $T$ defined as:

$$\sigma_{\mathcal{F}}(T) = \{s \in \mathbb{R}^{n+1} : s^2\mathcal{I} - s(T + \overline{T}) + T\overline{T} \text{ is not invertible}\}.$$ 

It is important to point out the meaning of the symbols. We have that, by definition, $T = T_0 + e_1 T_1 + \ldots + e_n T_n$, $\overline{T} = T_0 - e_1 T_1 - \ldots - e_n T_n$, so that $T + \overline{T} = 2T_0$, and since the components of $T$ commute, we have $T\overline{T} = T_0^2 + T_1^2 + \ldots + T_n^2$.

In the paper [6] we have proved the Fueter mapping theorem in integral form using the Cauchy formula (1.2). Precisely we have proved that, when $n$ is an odd number, given the slice monogenic function $f$, we can associate to it the monogenic function $\hat{f}(x)$ by the integral transform

$$\hat{f}(x) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_I)} \gamma_n (s - \overline{x})(s^2 - s(x + \overline{x}) + |x|^2)^{-\frac{n+1}{2}}ds_I f(s), \quad ds_I = -Ids,$$  \hspace{1cm} (1.3)

where $\gamma_n$ ia a given constant. We recall that the Fueter mapping theorem in differential form is given in [19] for $n$ odd and in Qian’s paper [16] in the general case. Later on, Fueter’s theorem has been generalized to the case in which a function $f$ as above is multiplied by a monogenic homogeneous polynomial of degree $k$, see
[15], [20] and to the case in which the function $f$ is defined on an open set $U$ not necessarily chosen in the upper complex plane, see [17].

We point out that formula (1.3) allows us to define the $\mathcal{F}$-functional calculus by replacing $x = x_0 + x_1e_1 + \ldots + x_ne_n$ by $T = T_0 + T_1e_1 + \ldots + T_ne_n$. More precisely, in [6] we have defined the following version of the monogenic functional calculus by setting

$$\hat{f}(T) = \frac{1}{2\pi i} \int_{\partial(U \cap \mathbb{C}_I)} \mathcal{F}_n^{-1}(s, T) ds f(s), \quad (1.4)$$

where

$$\mathcal{F}_n^{-1}(s, T) := \gamma_n(s \mathcal{I} - \mathcal{T})(s^2 \mathcal{I} - s(T + \mathcal{T}) + T \mathcal{T})^{-\frac{n+1}{2}}. \quad (1.5)$$

The functional calculus in (1.4) is well defined since the integral does not depend on the open set $U$ and on $I \in \mathcal{S}$. The natural notion of spectrum in this case is again the notion of $\mathcal{F}$-spectrum of $T$ as it is suggested by the definition of the $\mathcal{F}$-resolvent operator defined in (1.5).

The goal of this paper is to prove that bounded perturbations of the $\mathcal{S}\mathcal{C}$-resolvent operator and of the $\mathcal{F}$-resolvent operator produce bounded variations of the respective functional calculi.

We conclude by recalling that the well known theory of monogenic functions, see [1], [7], is the natural tool to define the monogenic functional calculus which has been well studied and developed by several authors, see the book of B. Jefferies [14] and the literature therein. For the analogies with the Riesz-Dunford functional calculus see for example the classical books [13] and [18].

2. Preliminary material

The setting in which we will work is the real Clifford algebra $\mathbb{R}_n$ over $n$ imaginary units $e_1, \ldots, e_n$ satisfying the relations $e_ie_j + e_je_i = -2\delta_{ij}$. An element in the Clifford algebra will be denoted by $\sum_A e_Ax_A$ where $A = i_1 \ldots i_r$, $i_r \in \{1, 2, \ldots, n\}$, $i_1 < \ldots < i_r$ is a multi-index, $e_A = e_{i_1}e_{i_2}\ldots e_{i_r}$ and $e_0 = 1$. In the Clifford algebra $\mathbb{R}_n$, we can identify some specific elements with the vectors in the Euclidean space $\mathbb{R}^n$: an element $(x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ can be identified with a so-called 1-vector in the Clifford algebra through the map $(x_1, x_2, \ldots, x_n) \mapsto \underline{x} = x_1e_1 + \ldots + x_n e_n$.

An element $(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1}$ will be identified with the element $x = x_0 + \underline{x} = x_0 + \sum_{j=1}^n x_j e_j$ called a paravector. The norm of $x \in \mathbb{R}^{n+1}$ is defined as $|x|^2 = x_0^2 + x_1^2 + \ldots + x_n^2$. The real part $x_0$ of $x$ will be also denoted by Re[$x$]. A function $f : U \subseteq \mathbb{R}^{n+1} \to \mathbb{R}_n$ is seen as a function $f(x)$ of $x$ (and similarly for a function $f(\underline{x})$ of $\underline{x} \in U \subseteq \mathbb{R}^{n+1}$).

Definition 2.1. We will denote by $\mathcal{S}$ the sphere of unit 1-vectors in $\mathbb{R}^n$, i.e.,

$$\mathcal{S} = \{\underline{x} = e_1x_1 + \ldots + e_nx_n : x_1^2 + \ldots + x_n^2 = 1\}.$$

Note that $\mathcal{S}$ is an $(n-1)$-dimensional sphere in $\mathbb{R}^{n+1}$. The vector space $\mathbb{R}+I\mathbb{R}$ passing through 1 and $I \in \mathcal{S}$ will be denoted by $\mathbb{C}_I$, while an element belonging
to \( \mathbb{C}_I \) will be denoted by \( u + Iv \), for \( u, v \in \mathbb{R} \). Observe that \( \mathbb{C}_I \), for every \( I \in \mathbb{S} \), is a 2-dimensional real subspace of \( \mathbb{R}^{n+1} \) isomorphic to the complex plane. The isomorphism turns out to be an algebra isomorphism.

Given a paravector \( x = x_0 + x \in \mathbb{R}^{n+1} \) let us set
\[
I_x = \begin{cases} \frac{x}{|x|} & \text{if } x \neq 0, \\ \text{any element of } \mathbb{S} & \text{otherwise.} \end{cases}
\]

By definition we have that a paravector \( x \), with \( x \neq 0 \), belongs to \( \mathbb{C}_{I_x} \).

**Definition 2.2.** Given an element \( x \in \mathbb{R}^{n+1} \), we define \( [x] = \{ y \in \mathbb{R}^{n+1} : y = \text{Re}[x] + I|x| \} \), where \( I \in \mathbb{S} \).

**Remark 2.3.** The set \( [x] \) is an \((n - 1)\)-dimensional sphere in \( \mathbb{R}^{n+1} \). When \( x \in \mathbb{R} \), then \( [x] \) contains \( x \) only. In this case, the \((n - 1)\)-dimensional sphere has radius equal to zero.

**Definition 2.4 (Slice monogenic functions).** Let \( U \subseteq \mathbb{R}^{n+1} \) be an open set and let \( f : U \to \mathbb{R}^n \) be a real differentiable function. Let \( I \in \mathbb{S} \) and let \( f_I \) be the restriction of \( f \) to the complex plane \( \mathbb{C}_I \). We say that \( f \) is a (left) slice monogenic function, or \( s \)-monogenic function, if for every \( I \in \mathbb{S} \), we have
\[
\frac{1}{2} \left( \frac{\partial}{\partial u} + I \frac{\partial}{\partial v} \right) f_I(u + Iv) = 0.
\]

We denote by \( SM(U) \) the set of \( s \)-monogenic functions on \( U \).

The natural class of domains in which we can develop the theory of \( s \)-monogenic functions are the so-called slice domains and axially symmetric domains.

**Definition 2.5 (Slice domains).** Let \( U \subseteq \mathbb{R}^{n+1} \) be a domain. We say that \( U \) is a slice domain (s-domain for short) if \( U \cap \mathbb{R} \) is non-empty and if \( \mathbb{C}_I \cap U \) is a domain in \( \mathbb{C}_I \) for all \( I \in \mathbb{S} \).

**Definition 2.6 (Axially symmetric domains).** Let \( U \subseteq \mathbb{R}^{n+1} \). We say that \( U \) is axially symmetric if, for every \( u + Iv \in U \), the whole \((n - 1)\)-sphere \( |u + Iv| \) is contained in \( U \).

Let us now introduce the notations necessary to deal with linear operators. By \( V \) and by \( V_n \) we denote a Banach space over \( \mathbb{R} \) with norm \( \| \cdot \| \) and \( V \otimes \mathbb{R}_n \), respectively. We recall that \( V_n \) is a two-sided Banach module over \( \mathbb{R}_n \) and its elements are of the type \( \sum_A v_A \otimes e_A \) (where \( A = i_1 \ldots i_r, i_\ell \in \{1, 2, \ldots, n\}, i_1 < \ldots < i_r \) is a multi-index). The multiplications (right and left) of an element \( v \in V_n \) with a scalar \( a \in \mathbb{R}_n \) are defined as \( va = \sum_A v_A \otimes (e_A a) \) and \( av = \sum_A v_A \otimes (ae_A) \). For short, in the sequel we will write \( \sum_A v_A e_A \) instead of \( \sum_A v_A \otimes e_A \). Moreover, we define \( \|v\|_{V_n}^2 = \sum_A \|v_A\|^2_V \).

Let \( B(V) \) be the space of bounded \( \mathbb{R} \)-homomorphisms of the Banach space \( V \) into itself endowed with the natural norm denoted by \( \| \cdot \|_{B(V)} \). If \( T_A \in B(V) \), we
can define the operator $T = \sum_A T_A e_A$ and its action on $v = \sum_B v_B e_B$ as $T(v) = \sum_A B_v e_A T(v_B) e_A B_v$. The set of all such bounded operators is denoted by $\mathcal{B}_n(V_n)$ and the norm is defined by $\|T\|_{\mathcal{B}_n(V_n)} = \sum_A B_v \|T_A\|_{\mathcal{B}(V)}$. Note that, in the sequel, we will omit the subscript $\mathcal{B}_n(V_n)$ in the norm of an operator and note also that $\|TS\| \leq \|T\|_1 S$. A bounded operator $T = T_0 + \sum_{j=1}^n e_j T_j$, where $T_\mu \in \mathcal{B}(V)$ for $\mu = 0, 1, \ldots, n$, will be called, with an abuse of notation, an operator in paravector form. The set of such operators will be denoted by $\mathcal{B}_n^0(V_n)$. The set of bounded operators of the type $T = \sum_{j=1}^n e_j T_j$, where $T_\mu \in \mathcal{B}(V)$ for $\mu = 1, \ldots, n$, will be denoted by $\mathcal{B}_n^1(V_n)$ and $T$ will be said operator in vector form. We will consider operators of the form $T = T_0 + \sum_{j=1}^n e_j T_j$ where $T_\mu \in \mathcal{B}(V)$ for $\mu = 0, 1, \ldots, n$ for the sake of generality, but when dealing with $n$-tuples of operators, we will embed them into $\mathcal{B}_n(V_n)$ as operators in vector form, by setting $T_0 = 0$. The subset of those operators in $\mathcal{B}_n(V_n)$ whose components commute among themselves will be denoted by $\mathcal{B}_n^0(V_n)$. In the same spirit we denote by $\mathcal{B}_n^0(V_n)\mathcal{C}_n(\mathbb{V}_n)$ the set of paravector operators with commuting components.

We now recall some definitions and results from [5], [6].

**Definition 2.7 (The $\mathcal{F}$-spectrum and the $\mathcal{F}$-resolvent sets).** Let $T \in \mathcal{BC}_n^0(V_n)$. We define the $\mathcal{F}$-spectrum of $T$ as:

$$\sigma_{\mathcal{F}}(T) = \{s \in \mathbb{R}^{n+1} : s^2 \mathcal{I} - s(T + \mathcal{T}) + T\mathcal{T} \text{ is not invertible}\}.$$

The $\mathcal{F}$-resolvent set of $T$ is defined by

$$\rho_{\mathcal{F}}(T) = \mathbb{R}^{n+1} \setminus \sigma_{\mathcal{F}}(T).$$

**Theorem 2.8 (Structure of the $\mathcal{F}$-spectrum).** Let $T \in \mathcal{BC}_n^0(V_n)$ and let $p = p_0 + p_1 I \in [p_0 + p_1 I] \subset \mathbb{R}^{n+1} \setminus \mathbb{R}$, such that $p \in \sigma_{\mathcal{F}}(T)$. Then all the elements of the $(n-1)$-sphere $[p_0 + p_1 I]$ belong to $\sigma_{\mathcal{F}}(T)$. Thus the $\mathcal{F}$-spectrum consists of real points and/or $(n-1)$-spheres.

**Theorem 2.9 (Compactness of $\mathcal{F}$-spectrum).** Let $T \in \mathcal{BC}_n^0(V_n)$. Then the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$ is a compact non-empty set. Moreover $\sigma_{\mathcal{F}}(T)$ is contained in $\left\{s \in \mathbb{R}^{n+1} : |s| \leq \||T||\right\}$.

**Definition 2.10.** Let $T \in \mathcal{BC}_n^0(V_n)$ and let $U \subset \mathbb{R}^{n+1}$ be an axially symmetric $s$-domain containing the $\mathcal{F}$-spectrum $\sigma_{\mathcal{F}}(T)$, and such that $\partial(U \cap \mathbb{C}_I)$ is union of a finite number of continuously differential Jordan curves for every $I \in \mathbb{S}$. Let $W$ be an open set in $\mathbb{R}^{n+1}$. A function $f \in SM(W)$ is said to be locally $s$-monogenic on $\sigma_{\mathcal{F}}(T)$ if there exists a domain $U \subset \mathbb{R}^{n+1}$ as above such that $\overline{U} \subset W$. We will denote by $SM_{\sigma_{\mathcal{F}}(T)}$ the set of locally $s$-monogenic functions on $\sigma_{\mathcal{F}}(T)$.

**Definition 2.11 (The $\mathcal{SC}$-resolvent operator).** Let $T \in \mathcal{BC}_n^0(V_n)$ and $s \in \rho_{\mathcal{F}}(T)$. We define the $\mathcal{SC}$-resolvent operator as

$$S_{\mathcal{C}}^{-1}(s, T) := (s\mathcal{I} - T)(s^2\mathcal{I} - s(T + \mathcal{T}) + T\mathcal{T})^{-1}.$$  

(2.1)
Definition 2.12 (The $SC$-functional calculus). Let $T \in BC_{n}^{0,1}(V_{n})$ and $f \in S\mathcal{M}_{\sigma}(T)$. Let $U \subset \mathbb{R}^{n+1}$ be a domain as in Definition 2.10 and set $ds_{I} = ds/I$ for $I \in \mathbb{S}$. We define the $SC$-functional calculus as

$$f(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{I})} S_{c}^{-1}(s, T) \, ds_{I} \, f(s). \quad (2.2)$$

Definition 2.13 ($\mathcal{F}$-resolvent operator). Let $n$ be an odd number and let $T \in BC_{n}^{0,1}(V_{n})$. For $s \in \rho_{\mathcal{F}}(T)$ we define the $\mathcal{F}$-resolvent operator as

$$\mathcal{F}_{n}^{-1}(s, T) := \gamma_{n}(s\mathcal{I} - \overline{T})(s^{2}\mathcal{I} - s(T + \overline{T}) + T\overline{T})^{-\frac{n-1}{2}},$$

where

$$\gamma_{n} := (-1)^{(n-1)/2}2^{(n-1)/2}(n - 1)\left(\frac{n - 1}{2}\right)!. \quad \text{(2.3)}$$

Next we define $\tilde{f}(T)$ when $\tilde{f}$ is a monogenic function which comes from an $s$-monogenic function $f$ via Fueter’s theorem. The $\mathcal{F}$-functional calculus will be defined for those monogenic functions that are of the form $\tilde{f}(x) = \Delta^{\frac{n-1}{2}}f(x)$, where $f$ is an $s$-monogenic function. For the functional calculus associated to standard monogenic functions we mention the book [14].

Definition 2.14 (The $\mathcal{F}$-functional calculus). Let $n$ be an odd number and let $T \in BC_{n}^{0,1}(V_{n})$. Let $U$ be an open set as in Definition 2.10. Suppose that $f \in S\mathcal{M}_{\sigma}(T)$ and let $\tilde{f}(x) = \Delta^{\frac{n-1}{2}}f(x)$. We define the $\mathcal{F}$-functional calculus as

$$\tilde{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap \mathbb{C}_{I})} \mathcal{F}_{n}^{-1}(s, T) \, ds_{I} \, f(s). \quad (2.3)$$

Remark 2.15. The definitions of the $SC$-functional calculus and of the $\mathcal{F}$-functional calculus are well posed since the integrals in (2.2) and in (2.3) are independent of $I \in \mathbb{S}$ and of the open set $U$.

3. Examples of equations for the $\mathcal{F}$-spectrum

Example (The case of Dirac operator). Let us consider the $n$-tuple of operators $(\partial_{x_{1}}, \ldots, \partial_{x_{n}})$, each of them acting on the vector space of functions of class $C^{2}$ over an open set $U \subset \mathbb{R}^{n+1}$. The vector operator associated to them is the Dirac operator

$$T = \partial_{x_{1}}e_{1} + \ldots + \partial_{x_{n}}e_{n}.$$ 

Let us determine the equation which gives its $\mathcal{F}$-spectrum. We have $\overline{T} = -\partial_{x_{1}}e_{1} - \ldots - \partial_{x_{n}}e_{n}$, and, since $\partial_{x_{i}}\partial_{x_{j}} = \partial_{x_{j}}\partial_{x_{i}}$, for all $i, j = 1, \ldots, n$ we also have $T + \overline{T} = 0$ and $T\overline{T} = \partial_{x_{1}}^{2} + \ldots + \partial_{x_{n}}^{2} = \Delta$. The $\mathcal{F}$-spectrum is associated to the equation

$$(s^{2}\mathcal{I} - s(T + \overline{T}) + T\overline{T})v = 0 \quad \text{for} \quad v \neq 0$$

which, in this case, becomes

$$(s^{2}\mathcal{I} + \Delta)v = 0 \quad \text{for} \quad v \neq 0. \quad (3.1)$$
The paravector $s$ can be considered as an element belonging to a complex plane $s \in \mathbb{C}_{I_0}$, so we can assume that $s = s_0 + s_1I_0$ is a solution of (3.1) for some $I_0$. Then the $\mathcal{F}$-spectrum of $T$ is given by

$$\sigma_{\mathcal{F}}(T) = \bigcup_{s \in \mathbb{C}_{I_0} \text{ solution of (3.1)}}\{s = s_0 + s_1I, \text{ for all } I \in S\}.$$ 

**Example** (The case of second derivatives). Let us consider the second-order operators $(\partial^2_{x_1}, \ldots, \partial^2_{x_n})$ each of them acting on the vector space of functions of class $C^4$ over an open set $U \subseteq \mathbb{R}^{n+1}$, and let us write

$$T = \partial^2_{x_1}e_1 + \ldots + \partial^2_{x_n}e_n.$$ 

Determine the equation which gives its $\mathcal{F}$-spectrum. We have $\overline{T} = -\partial^2_{x_1}e_1 - \ldots - \partial^2_{x_n}e_n$, and, since $\partial^2_{x_i}\partial^2_{x_j} = \partial^2_{x_j}\partial^2_{x_i}$, for all $i, j = 1, \ldots, n$ we also have $T + \overline{T} = 0$ and $TT = \partial^4_{x_1} + \ldots + \partial^4_{x_n}$. The $\mathcal{F}$-spectrum is associated to the equation

$$(s^2I + \partial^4_{x_1} + \ldots + \partial^4_{x_n})v = 0 \quad \text{for } v \neq 0. \quad (3.2)$$

We solve the equation (3.2) on the complex plane $s \in \mathbb{C}_{I_0}$ for some $I_0$. Then the $\mathcal{F}$-spectrum of $T$ is given by

$$\sigma_{\mathcal{F}}(T) = \bigcup_{s \in \mathbb{C}_{I_0} \text{ solution of (3.2)}}\{s = s_0 + s_1I, \text{ for all } I \in S\}.$$ 

**Example** (The case of powers of a real matrix). Let $A$ be a matrix $n \times n$ with real entries and consider the operators $T_j := A^j$ for $j = 1, \ldots, n$. Determine the equation associated to the $\mathcal{F}$-spectrum. It is well known that $T_jT_k = T_kT_j$, for $j, k = 1, \ldots, n$. So we consider the operator

$$T = Ae_1 + \ldots + A^ne_n.$$ 

We have $\overline{T} = -Ae_1 - \ldots - A^ne_n$, $T + \overline{T} = 0$ and also $TT = \sum_{j=1}^{n} A^{2j}$. The $\mathcal{F}$-spectrum is associated to the equation

$$(s^2I + \sum_{j=1}^{n} A^{2j})v = 0 \quad \text{for } v \neq 0. \quad (3.3)$$

Let us conclude this short list of examples with an explicit computation of the $\mathcal{F}$-spectrum.

**Example** (The case of two triangular commuting matrices). Let us consider $a, b, \alpha, \beta \in \mathbb{R}$ and the two matrices:

$$T_1 = \begin{bmatrix} a & b \\ 0 & a \end{bmatrix}, \quad T_2 = \begin{bmatrix} \alpha & \beta \\ 0 & \alpha \end{bmatrix}. $$

It is easy to verify that $T_1T_2 = T_2T_1$. So we associate to $T_1$ and $T_2$ the operator

$$T = T_1e_1 + T_2e_2 = \begin{bmatrix} ae_1 + \alpha e_2 & be_1 + \beta e_2 \\ 0 & ae_1 + \alpha e_2 \end{bmatrix}. $$
We have
\[ T = \begin{bmatrix} ae_1 + \alpha e_2 & be_1 + \beta e_2 \\ 0 & ae_1 + \alpha e_2 \end{bmatrix}, \]
and
\[ T + T^* = 0 \]
\[ TT = \begin{bmatrix} a^2 + \alpha^2 & 2ab + 2\alpha \beta \\ 0 & a^2 + \alpha^2 \end{bmatrix}. \]

The \( F \)-spectrum is associated to the equation
\[ \left( s^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a^2 + \alpha^2 & 2ab + 2\alpha \beta \\ 0 & a^2 + \alpha^2 \end{bmatrix} \right) v = 0 \quad \text{for} \quad v \neq 0 \]
which becomes
\[ \begin{bmatrix} s^2 + a^2 + \alpha^2 & 2ab + 2\alpha \beta \\ 0 & s^2 + a^2 + \alpha^2 \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix} = 0 \quad \text{for} \quad \begin{bmatrix} u \\ w \end{bmatrix} \neq 0. \]

Consider a paravector \( s \) on the complex plane \( \mathbb{C}_{I_0} \): with some calculation we obtain
\[ s = \pm I_0 \sqrt{a^2 + \alpha^2}. \]
Thus the \( F \)-spectrum of \( T \) is given by
\[ \sigma_F(T) = \{ \pm I \sqrt{a^2 + \alpha^2} \text{ for all } I \in \mathbb{S} \}. \]

4. Bounded perturbations of the \( SC \)-resolvent

**Lemma 4.1.** The set \( \mathcal{U}(V_n) \) of elements in \( \mathcal{B}_n(V_n) \) which have inverse in \( \mathcal{B}_n(V_n) \) is an open set in the uniform topology of \( \mathcal{B}_n(V_n) \). If \( \mathcal{U}(V_n) \) contains an element \( A \), then it contains the ball
\[ \Sigma = \{ B \in \mathcal{B}_n(V_n) : \| A - B \| < \| A^{-1} \|^{-1} \}. \]
If \( B \in \Sigma \), its inverse is given by the series
\[ B^{-1} = A^{-1} \sum_{m \geq 0} [(A - B)A^{-1}]^m. \quad (4.1) \]
Furthermore, the map \( A \mapsto A^{-1} \) from \( \mathcal{U}(V_n) \) onto \( \mathcal{U}(V_n) \) is a homeomorphism in the uniform operator topology.

**Proof.** See Lemma 7.1 in [3]. \( \square \)

In order to state our results, we need the following definitions:

**Definition 4.2.** Let \( T \in \mathcal{B}_n^{0,1}(V_n) \). We denote by \( \sigma_L(T) \) the so-called left spectrum of \( T \) related to the resolvent operator \((sI - T)^{-1}\) that is defined as
\[ \sigma_L(T) = \{ s \in \mathbb{R}^{n+1} : sI - T \text{ is not invertible in } \mathcal{B}_n^{0,1}(V_n) \}, \]
where the notation \( sI \) in \( \mathcal{B}_n^{0,1}(V_n) \) means that \((sI)(v) = sv\).

**Definition 4.3.** Let \( \mathcal{W} \) be a subset of \( \mathbb{R}^{n+1} \). We denote by \( B(\mathcal{W}, \varepsilon) \), for \( \varepsilon > 0 \), the \( \varepsilon \)-neighborhood of \( \mathcal{W} \) defined as
\[ B(\mathcal{W}, \varepsilon) := \{ x \in \mathbb{R}^{n+1} : \inf_{s \in \mathcal{W}} |s - x| < \varepsilon \}. \]
Lemma 4.4. Let \( T, Z \in \mathcal{BC}_{n}^{0,1}(V_{n}), \ s \notin \sigma_{L}(T) \cup \sigma_{L}(Z) \) and consider
\[
S_{\mathcal{C}}(s, T) = sI - (sI - T)T(sI - T)^{-1},
\]
\[
S_{\mathcal{C}}(s, Z) = sI - (sI - Z)Z(sI - Z)^{-1}.
\]

Then there exists a strictly positive constant \( K(s) \), depending on \( s \) and also on the operators \( T \) and \( Z \), such that
\[
\|S_{\mathcal{C}}(s, T) - S_{\mathcal{C}}(s, Z)\| \leq K(s)\|T - Z\|. \tag{4.4}
\]

Proof. Consider the chain of equalities
\[
S_{\mathcal{C}}(s, T) - S_{\mathcal{C}}(s, Z)
\]
\[
= (sI - Z)Z(sI - Z)^{-1} - (sI - T)T(sI - T)^{-1}
\]
\[
= (sI - Z)Z(sI - Z)^{-1} - (sI - T)Z(sI - Z)^{-1}
\]
\[
+ (sI - T)Z(sI - Z)^{-1} - (sI - T)T(sI - T)^{-1}
\]
\[
= (T - Z)Z(sI - Z)^{-1} + (sI - T)[Z(sI - Z)^{-1} - T(sI - T)^{-1}]
\]
\[
= (T - Z)Z(sI - Z)^{-1}
\]
\[
+ (sI - T)[(Z - T)(sI - Z)^{-1} + T((sI - Z)^{-1} - (sI - T)^{-1})]
\]
\[
= (T - Z)Z(sI - Z)^{-1}
\]
\[
+ (sI - T)[(Z - T)(sI - Z)^{-1} + T(sI - Z)^{-1}(Z - T)(sI - T)^{-1}].
\]

By taking the norm and observing that \( \|T - Z\| = \|T - Z\| \), we have
\[
\|S_{\mathcal{C}}(s, T) - S_{\mathcal{C}}(s, Z)\| \leq \|T - Z\| \( \|Z\| \|sI - Z\|^{-1}\|
\]
\[
+ \|sI - T\| \left[ \|sI - Z\|^{-1}\| + \|T\| \|sI - Z\|^{-1}\(\|sI - T\|^{-1}\| \right]\).
\]

If we now set
\[
K(s) := \|sI - Z\|^{-1}\left( \|Z\| + \|sI - T\| \left[ 1 + \|T\| \|sI - T\|^{-1}\right]\right), \tag{4.5}
\]
we have that \( K(s) > 0 \) and we get the statement. \( \square \)

Lemma 4.5. Let \( T, Z \in \mathcal{BC}_{n}^{0,1}(V_{n}), \ s \in \rho_{\mathcal{F}}(T), \ s \notin \sigma_{L}(T) \cup \sigma_{L}(Z) \) and suppose that
\[
\|T - Z\| < \frac{1}{K(s)}\|S_{\mathcal{C}}^{-1}(s, T)\|^{-1},
\]
where \( K(s) \) is defined in (4.5). Then \( s \in \rho_{\mathcal{F}}(Z) \) and
\[
S_{\mathcal{C}}^{-1}(s, Z) - S_{\mathcal{C}}^{-1}(s, T) = S_{\mathcal{C}}^{-1}(s, T) \sum_{m \geq 1} [(S_{\mathcal{C}}(s, T) - S_{\mathcal{C}}(s, Z))S_{\mathcal{C}}^{-1}(s, T)]^{m}. \tag{4.6}
\]
Proof. Let us recall (4.2) and (4.3) and set
\[
A := \mathcal{S}_c(s, T), \quad B := \mathcal{S}_c(s, Z), \quad A^{-1} = \mathcal{S}_c^{-1}(s, T).
\]
By formula (4.1) in Lemma 4.1 with \(B^{-1} := \mathcal{S}_c^{-1}(s, Z)\), we get
\[
\mathcal{S}_c^{-1}(s, Z) = \mathcal{S}_c^{-1}(s, T) \sum_{m \geq 0} \left( (\mathcal{S}_c(s, T) - \mathcal{S}_c(s, Z)) \mathcal{S}_c^{-1}(s, T) \right)^m.
\]

The series in (4.8) converges since
\[
\| (\mathcal{S}_c(s, T) - \mathcal{S}_c(s, Z)) \mathcal{S}_c^{-1}(s, T) \| \leq K(s) \| T - Z \| \| \mathcal{S}_c^{-1}(s, T) \| < 1,
\]
so we get the statement.

\[\square\]

Theorem 4.6. Let \(T, Z \in BC_n^0(V_n), s \in \rho_F(T), s \not\in \sigma_L(T) \cup \sigma_L(Z)\). Let \(\varepsilon > 0\) and consider the \(\varepsilon\)-neighborhood \(B(\sigma_F(T) \cup \sigma_L(T), \varepsilon)\) of \(\sigma_F(T) \cup \sigma_L(T)\). Then there exists \(\delta > 0\) such that, for \(\| T - Z \| < \delta\), we have
\[
\sigma_F(Z) \subseteq B(\sigma_F(T) \cup \sigma_L(T), \varepsilon)
\]
and
\[
\| \mathcal{S}_c^{-1}(s, Z) - \mathcal{S}_c^{-1}(s, T) \| < \varepsilon, \text{ for } s \not\in B(\sigma_F(T) \cup \sigma_L(T), \varepsilon).
\]

Proof. Let \(\overline{T}, \overline{Z} \in BC_n^0(V_n)\) and let \(\varepsilon > 0\). Thanks to Lemma 4.1 there exists a \(\eta > 0\) such that if
\[
\| T - Z \| < \eta
\]
then \(\sigma_L(\overline{Z}) \subset B(\sigma_L(\overline{T}), \varepsilon)\), where \(B(\sigma_L(\overline{T}), \varepsilon)\) is the \(\varepsilon\)-neighborhood of \(\sigma_L(\overline{T})\). So we can always choose \(\eta\) such that \(\sigma_L(\overline{Z}) \subset B(\sigma_F(T) \cup \sigma_L(T), \varepsilon)\). Consider the function \(K(s)\) defined in (4.5) and observe that the constant \(K_\varepsilon\) defined by
\[
K_\varepsilon = \sup_{s \not\in B(\sigma_F(T) \cup \sigma_L(T), \varepsilon)} K(s)
\]
is finite since \(s \not\in B(\sigma_F(T) \cup \sigma_L(T), \varepsilon)\), for the above observation \(\sigma_L(\overline{Z}) \subset B(\sigma_F(T) \cup \sigma_L(T), \varepsilon)\) and because
\[
\lim_{s \to \infty} \|(sI - \overline{Z})^{-1}\| = \lim_{s \to \infty} \|(sI - \overline{T})^{-1}\| = 0.
\]
Observe that since \(s \in \rho_F(T)\) the map \(s \mapsto \| \mathcal{S}_c^{-1}(s, T) \|\) is continuous and
\[
\lim_{s \to \infty} \| \mathcal{S}_c^{-1}(s, T) \| = 0,
\]
for \(s\) in the complement set of \(B(\sigma_F(T) \cup \sigma_L(T), \varepsilon)\) we have that there exists a positive constant \(N_\varepsilon\) such that
\[
\| \mathcal{S}_c^{-1}(s, T) \| \leq N_\varepsilon.
\]
From Lemma 4.5, if \(\delta_1 > 0\) is such that \(\| Z - T \| < \frac{1}{K_\varepsilon N_\varepsilon} := \delta_1\), then \(s \in \rho_F(Z)\) and
We recall that operator $Z$ is
\[
\|S_c^{-1}(s, Z) - S_c^{-1}(s, T)\| \leq \frac{\|S_c^{-1}(s, T)\|^2 \|S_c(s, T) - S_c(s, Z)\|}{1 - \|S_c^{-1}(s, T)\| \|S_c(s, T) - S_c(s, Z)\|} \\
\leq \frac{N^2_c K\|Z - T\|}{1 - N^2_c K\|Z - T\|} < \varepsilon
\]
if we take
\[
\|Z - T\| < \delta_2 := \frac{\varepsilon}{K\varepsilon(N^2 + \varepsilon N\varepsilon)}.
\]
To get the statement it suffices to set $\delta = \min\{\eta, \delta_1, \delta_2\}$.

**Theorem 4.7.** Let $T, Z \in \mathcal{BC}_n^0(V_n), f \in \mathcal{SM}_{\sigma_f(T)}$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that, for $\|Z - T\| < \delta$, we have $f \in \mathcal{SM}_{\sigma_f(Z)}$ and
\[
\|f(Z) - f(T)\| < \varepsilon.
\]
**Proof.** We recall that operator $f(T)$ is defined by
\[
f(T) = \frac{1}{2\pi} \int_{\partial(U \cap C_t)} S_c^{-1}(s, T) \, ds_I \, f(s)
\]
where $U \subset \mathbb{R}^{n+1}$ is a domain as in Definition 2.10, $ds_I = ds/I$ for $I \in \mathcal{S}$. Suppose that $U$ is an $\varepsilon$-neighborhood of $\sigma_f(T) \cup \sigma_L(T)$ and it is contained in the domain in which $f$ is s-monogenic. By Lemma 4.6 there is a $\delta_1 > 0$ such that $\sigma_f(Z) \subset U$ for $\|Z - T\| < \delta_1$. Consequently $f \in \mathcal{SM}_{\sigma_f(Z)}$ for $\|Z - T\| < \delta_1$. By Lemma 4.6 $S_c^{-1}(s, T)$ is uniformly near to $S_c^{-1}(s, Z)$ with respect to $s \in \partial(U \cap C_I)$ for $I \in \mathcal{S}$ if $\|Z - T\|$ is small enough, so for some positive $\delta \leq \delta_1$ we get
\[
\|f(T) - f(Z)\| = \frac{1}{2\pi} \left\| \int_{\partial(U \cap C_I)} [S_c^{-1}(s, T) - S_c^{-1}(s, Z)] \, ds_I \, f(s) \right\| < \varepsilon. \]

**5. Bounded perturbations of the $F$-resolvent**

Let $n$ be an odd number. For $s \in \rho_f(T)$ the $F$-resolvent operator associated to $T$ is
\[
F_n^{-1}(s, T) := \frac{\gamma_n(sI - T)(s^2I - s(T + T) + TT)}{\gamma_n(s^2I - s(T + T) + TT)} - \frac{n+1}{2}, \quad (5.1)
\]
while its inverse is
\[
F_n(s, T) := \frac{1}{\gamma_n}(s^2I - s(T + T) + TT) - \frac{n+1}{2}(sI - T)^{-1}, \quad (5.2)
\]
for $s \notin \sigma_L(T)$. Analogously for $s \in \rho_f(Z)$ the $F$-resolvent operator associated to $Z$ is
\[
F_n^{-1}(s, Z) := \frac{\gamma_n(sI - Z)(s^2I - s(Z + Z) + ZZ)}{\gamma_n(s^2I - s(Z + Z) + ZZ)} - \frac{n+1}{2}, \quad (5.3)
\]
and it has the inverse
\[
F_n(s, Z) := \frac{1}{\gamma_n}(s^2I - s(Z + Z) + ZZ) - \frac{n+1}{2}(sI - Z)^{-1}, \quad (5.4)
\]
for $s \notin \sigma_L(Z)$. 
Lemma 5.1. Let \( n \) be an odd number, \( T, Z \in \mathcal{B}c_{1}^n(V_n) \) and let \( s \notin \sigma_L(T) \cup \sigma_L(Z) \). Then there exists a positive constant \( C_n(s) \) depending on \( s \) and also on the operators \( T \) and \( Z \) such that
\[
\|F_n(s, T) - F_n(s, Z)\| \leq C_n(s) (|s| + \vartheta)^{n-1} \|T - Z\|,
\]
where \( \vartheta := \max\{\|T\|, \|Z\|\} \).

Proof. For simplicity let us set the positions \( \frac{n+1}{2} := k + 1 \), for \( k \in \mathbb{N} \), so that \( k = \frac{n-1}{2} \), for \( n = 1, 3, 5, \ldots \). The case \( k = 0 \) has been studied in the previous section. Here we consider \( k \geq 1 \). We set \( \beta_k := \gamma_{2k+1} \) and we define, for \( s \in \rho_F(T) \),
\[
\tilde{F}_k^{-1}(s, T) := \beta_k \sigma_C^{-1}(s, T) (s^2I - s(T + \overline{T}) + T) \cdot \overline{T})^{-k}.
\]
The inverse of operator \( \tilde{F}_k^{-1}(s, T) \) exists for \( s \notin \sigma_L(\overline{T}) \) and is given by
\[
\tilde{F}_k(s, T) = \frac{1}{\beta_k} (s^2I - s(T + \overline{T}) + T) \cdot \overline{T})^{k} \cdot \sigma_C(s, T),
\]
while the inverse of operator \( \tilde{F}_k^{-1}(s, Z) \) exists for \( s \notin \sigma_L(\overline{Z}) \) and is given by
\[
\tilde{F}_k(s, Z) = \frac{1}{\beta_k} (s^2I - s(Z + \overline{Z}) + Z) \cdot \overline{T})^{k} \cdot \sigma_C(s, Z).
\]
Consider (5.7) and (5.8) for \( k = 1 \); we have
\[
\beta_1[\tilde{F}_1(s, T) - \tilde{F}_1(s, Z)]
\]
\[
= (s^2I - s(T + \overline{T}) + T) \cdot \overline{T}) \cdot \sigma_C(s, T) - (s^2I - s(Z + \overline{Z}) + Z) \cdot \overline{T}) \cdot \sigma_C(s, Z)
\]
\[
= (s^2I - s(T + \overline{T}) + T) \cdot \overline{T}) \cdot \sigma_C(s, T) - (s^2I - s(T + \overline{T}) + T) \cdot \overline{T}) \cdot \sigma_C(s, Z)
\]
\[
+ (s^2I - s(T + \overline{T}) + T) \cdot \overline{T}) \cdot \sigma_C(s, Z) - (s^2I - s(Z + \overline{Z}) + Z) \cdot \overline{T}) \cdot \sigma_C(s, Z)
\]
\[
= (s^2I - s(T + \overline{T}) + T) \cdot \overline{T}) [\sigma_C(s, T) - \sigma_C(s, Z)]
\]
\[
+ [-s(T + \overline{T}) + T \cdot \overline{T} + s(Z + \overline{Z}) - Z] \cdot \overline{T}) \cdot \sigma_C(s, Z)
\]
\[
= (s^2I - s(T + \overline{T}) + T) \cdot \overline{T}) [\sigma_C(s, T) - \sigma_C(s, Z)]
\]
\[
+ [s(Z - T + \overline{Z} - \overline{T}) + (T - Z) \cdot \overline{T} + Z \cdot \overline{T}] \cdot \overline{T}) \cdot \sigma_C(s, Z)
\]
and taking the norm we get
\[
\|\beta_1[\tilde{F}_1(s, T) - \tilde{F}_1(s, Z)]\|
\]
\[
\leq (|s|^2 + 2|s| \cdot \|T\| + \|T \cdot \overline{T})\|) \cdot \|\sigma_C(s, T) - \sigma_C(s, Z)\|
\]
\[
+ [2|s| \cdot \|Z - T\| + \|T - Z\| \cdot (\|T\| + \|Z\|)] \cdot \|\sigma_C(s, Z)\|
\]
\[
\leq (|s| + \vartheta)^2 \cdot \|\sigma_C(s, T) - \sigma_C(s, Z)\| + [2(|s| + \vartheta) \cdot \|Z - T\|] \cdot \|\sigma_C(s, Z)\|
\]
Now observe that
\[
(|s| + \vartheta)^{-1} \cdot \|\sigma_C(s, Z)\| \leq (|s| + \vartheta)^{-1} [\|s| + \|s(I - \overline{Z})\| \cdot \|Z\| \cdot \|s(I - \overline{Z})^{-1}\|] =: M(s)
\]
(5.9)
where $M(s)$ is a continuous function since $s \not\in \sigma_L(Z)$. Using Lemma 4.4 we get

$$
\|\tilde{F}_1(s, T) - \tilde{F}_1(s, Z)\| \leq \frac{1}{|\beta_1|}[K(s) + 2M(s)](|s| + \vartheta)^2\|Z - T\|. 
$$ (5.10)

We now use the induction principle. We assume that the estimate

$$
\|\tilde{F}_k(s, T) - \tilde{F}_k(s, Z)\| \leq \frac{1}{|\beta_k|}(|s| + \vartheta)^{2k}[K(s) + 2kM(s)]\|Z - T\| 
$$ (5.11)

holds for $k \geq 1$. Observe that (5.8) implies that the estimate

$$
\|\mathcal{F}_k(s, Z)\| \leq \frac{1}{|\beta_k|}(|s| + \vartheta)^{2k}\|\mathcal{S}_C(s, Z)\| 
$$ (5.12)

holds. We prove that

$$
\|\tilde{F}_{k+1}(s, T) - \tilde{F}_{k+1}(s, Z)\| \leq \frac{1}{|\beta_{k+1}|}(|s| + \vartheta)^{2(k+1)}[K(s) + 2(k+1)M(s)]\|Z - T\|. 
$$

In fact, we have that

$$
\begin{align*}
\beta_{k+1}(\tilde{F}_{k+1}(s, T) - \tilde{F}_{k+1}(s, Z)) &= \beta_k(s^2I - s(T + T\bar{T}) + T\bar{T}) \tilde{F}_k(s, T) - \beta_k(s^2I - s(Z + \bar{Z}) + Z\bar{Z}) \tilde{F}_k(s, Z) \\
&= \beta_k(s^2I - s(T + T\bar{T}) + T\bar{T}) [\tilde{F}_k(s, T) - \tilde{F}_k(s, Z)] \\
&= -\beta_k(s(Z - T + \bar{Z} - T) + (T - Z)\bar{T} + Z(T - \bar{Z})] \tilde{F}_k(s, Z)
\end{align*}
$$

and taking the norms we have

$$
|\beta_{k+1}|\|\tilde{F}_{k+1}(s, T) - \tilde{F}_{k+1}(s, Z)\| 
$$

$$
\leq |\beta_k|(|s| + \vartheta)^2\|\tilde{F}_k(s, T) - \tilde{F}_k(s, Z)\| + 2|\beta_k|(|s| + \vartheta) \|\mathcal{F}_k(s, Z)\|\|Z - T\|.
$$

Using (5.11) and (5.12) we obtain

$$
\begin{align*}
&|\beta_{k+1}|\|\tilde{F}_{k+1}(s, T) - \tilde{F}_{k+1}(s, Z)\| \\
&\leq (|s| + \vartheta)^{2k+2}[K(s) + 2kM(s)]\|Z - T\| + 2(|s| + \vartheta)^{2k+1}\|\mathcal{S}_C(s, Z)\|\|Z - T\| \\
&\leq (|s| + \vartheta)^{2k+2}[K(s) + 2kM(s) + 2(|s| + \vartheta)^{-1}\|\mathcal{S}_C(s, Z)\|]\|Z - T\| \\
&\leq (|s| + \vartheta)^{2k+2}[K(s) + 2(k+1)M(s)]\|Z - T\|.
\end{align*}
$$

Setting $\tilde{C}_k(s) := \frac{1}{|\beta_k|}[K(s) + 2kM(s)]$ the constant $C_n(s)$ in estimate (5.5) is given by

$$
C_n(s) := \frac{1}{|\gamma_n|}[K(s) + (n - 1)M(s)]. 
$$ (5.13)

This concludes the proof. \hfill \square

**Lemma 5.2.** Let $n$ be an odd number, $T, Z \in \mathcal{B}^{0,1}_n(V_n)$, let $s \in \rho_T(T)$, $s \not\in \sigma_L(T) \cup \sigma_L(Z)$ and suppose that

$$
\|T - Z\| < \frac{1}{C_n(s)}(|s| + \vartheta)^{-(n-1)}\|\mathcal{F}_n^{-1}(s, T)\|^{-1},
$$
where $C_n(s)$ is defined in (5.13). Then $s \in \rho_{\mathcal{F}}(Z)$ and
\[
\mathcal{F}_n^{-1}(s, Z) - \mathcal{F}_n^{-1}(s, T) = \mathcal{F}_n^{-1}(s, T) \sum_{m \geq 1} |(\mathcal{F}_n(s, T) - \mathcal{F}_n(s, Z))\mathcal{F}_n^{-1}(s, T)|^m. \tag{5.14}
\]

**Proof.** Let us recall (5.2), (5.4) and set
\[
A := \mathcal{F}_n(s, T), \quad B := \mathcal{F}_n(s, Z), \quad A^{-1} = \mathcal{F}_n^{-1}(s, T)(s, T). \tag{5.15}
\]
By Lemma 4.1, formula (4.1), for $B^{-1} := \mathcal{F}_n^{-1}(s, Z)$ we get
\[
\mathcal{F}_n^{-1}(s, Z) = \mathcal{F}_n^{-1}(s, T) \sum_{m \geq 0} |(\mathcal{F}_n(s, T) - \mathcal{F}_n(s, Z))\mathcal{F}_n^{-1}(s, T)|^m. \tag{5.16}
\]
Using the hypothesis, we have that the series converges since
\[
\|(\mathcal{F}_n(s, T) - \mathcal{F}_n(s, Z))\mathcal{F}_n^{-1}(s, T)\| \leq C_n(s)\|s + \delta\|^{-1}\|Z - T\|\mathcal{F}_n^{-1}(s, T)\| < 1. \tag*{□}
\]

**Theorem 5.3.** Let $n$ be an odd number, $T, Z \in \mathcal{BC}_{n,1}(V_n)$, $s \in \rho_{\mathcal{F}}(T)$, $s \notin \sigma_{L}(\overline{T}) \cup \sigma_{L}(\overline{Z})$. Let $\varepsilon > 0$ and consider the $\varepsilon$-neighborhood $B(\sigma_{\mathcal{F}}(T) \cup \sigma_{L}(\overline{T}), \varepsilon)$ of $\sigma_{\mathcal{F}}(T) \cup \sigma_{L}(\overline{T})$. Then there exists $\delta > 0$ such that, for $\|T - Z\| < \delta$, we have
\[
\sigma_{\mathcal{F}}(Z) \subseteq B(\sigma_{\mathcal{F}}(T) \cup \sigma_{L}(\overline{T}), \varepsilon)
\]
and
\[
\|\mathcal{F}_n^{-1}(s, Z) - \mathcal{F}_n^{-1}(s, T)\| < \varepsilon, \quad \text{for} \quad s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_{L}(\overline{T}), \varepsilon).
\]

**Proof.** Let $\overline{T}, \overline{Z} \in \mathcal{BC}_{n,1}(V_n)$ and let $\varepsilon > 0$. Thanks to Lemma 4.1 there exists a $\eta > 0$ such that if
\[
\|T - Z\| < \eta,
\]
then $\sigma_{L}(\overline{Z}) \subseteq B(\sigma_{L}(\overline{T}), \varepsilon)$, where $B(\sigma_{L}(\overline{T}), \varepsilon)$ is the $\varepsilon$-neighborhood of $\sigma_{L}(\overline{T})$. So we can always choose $\eta$ such that $\sigma_{L}(\overline{Z}) \subseteq B(\sigma_{\mathcal{F}}(T) \cup \sigma_{L}(\overline{T}), \varepsilon)$. Consider the function $C_n(s)$ defined in (5.13). The constant $C_{n,\varepsilon}$ defined as
\[
C_{n,\varepsilon} = \sup_{s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_{L}(\overline{T}), \varepsilon)} C_n(s)
\]
is finite because $s \notin B(\sigma_{\mathcal{F}}(T) \cup \sigma_{L}(\overline{T}), \varepsilon)$ and
\[
\lim_{s \to \infty} \|(sI - \overline{Z})^{-1}\| = \lim_{s \to \infty} \|(sI - \overline{T})^{-1}\| = 0.
\]
Observe that the since $s \in \rho_{\mathcal{F}}(T)$ map $s \mapsto \|\mathcal{F}_n^{-1}(s, T)\|$ is continuous and
\[
\lim_{s \to \infty} \|\mathcal{F}_n^{-1}(s, T)\| = 0,
\]
and so for $s$ in the complement set of $B(\sigma_{\mathcal{F}}(T) \cup \sigma_{L}(\overline{T}), \varepsilon)$ we have that there exists a positive constant $M_{\varepsilon}$ such that
\[
\|\mathcal{F}_n^{-1}(s, T)\| \leq M_{\varepsilon}.
\]
From Lemma 5.2 if $\delta_1 > 0$ is such that
\[
\|Z - T\| < \frac{1}{C_{n,\varepsilon} M_{\varepsilon}} := \delta_3,
\]
then $s \in \rho_F(Z)$ and
\[
\|\mathcal{F}^{-1}_n(s, Z) - \mathcal{F}^{-1}_n(s, T)\| \\
\leq \frac{\|\mathcal{F}^{-1}_n(s, T)\|^2 \|\mathcal{F}_n(s, T) - \mathcal{F}_n(s, Z)\|}{1 - \|\mathcal{F}^{-1}_n(s, T)\| \|\mathcal{F}_n(s, T) - \mathcal{F}(s, Z)\|} \\
\leq \frac{M^2 C_{n,\varepsilon} \|Z - T\|}{1 - M_{\varepsilon} C_{n,\varepsilon} \|Z - T\|} < \varepsilon
\]
if we take
\[
\|Z - T\| < \delta_4 := \frac{\varepsilon}{C_{n,\varepsilon} (M^2 + \varepsilon M_{\varepsilon})}.
\]
To get the statement it suffices to set $\delta = \min\{\eta, \delta_3, \delta_4\}$. \hfill \Box

**Theorem 5.4.** Let $n$ be an odd number, $T, Z \in \mathcal{BC}_{n,1}^0(V_n)$, $f \in \mathcal{SM}_{\sigma_F(T)}$ and let $\varepsilon > 0$. Then there exists $\delta > 0$ such that, for $\|Z - T\| < \delta$, we have $f \in \mathcal{SM}_{\sigma_F(Z)}$ and
\[
\|\tilde{f}(Z) - \tilde{f}(T)\| < \varepsilon.
\]

**Proof.** We recall that
\[
\tilde{f}(T) = \frac{1}{2\pi} \int_{\partial(U \cap C_I)} \mathcal{F}^{-1}(s, T) \ ds_I \ f(s)
\]
and $U \subset \mathbb{R}^{n+1}$ is a domain as in Definition 2.10, $ds_I = ds/I$ for $I \in \mathbb{S}$. Let $U$ be an $\varepsilon$-neighborhood of $\sigma_F(T) \cup \sigma_L(T)$ contained in the domain in which $f$ is s-monogenic. By Lemma 5.3 there is a $\delta_1 > 0$ such that $\sigma_F(Z) \subset U$ for $\|Z - T\| < \delta_1$. Consequently, $f \in \mathcal{SM}_{\sigma_F(Z)}$ for $\|Z - T\| < \delta_1$. By Lemma 5.3, $\mathcal{F}^{-1}_n(s, T)$ is uniformly near to $\mathcal{F}^{-1}_n(s, Z)$ with respect to $s \in \partial(U \cap C_I)$ for $I \in \mathbb{S}$ if $\|Z - T\|$ is small enough, so for some positive $\delta \leq \delta_1$ we get
\[
\|\tilde{f}(T) - \tilde{f}(Z)\| = \frac{1}{2\pi} \left\| \int_{\partial(U \cap C_I)} [\mathcal{F}^{-1}_n(s, T) - \mathcal{F}^{-1}_n(s, Z)] \ ds_I \ f(s) \right\| < \varepsilon. \hfill \Box
\]

**References**


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