On Axiomatic Systems for Arbitrary Systems of Sentences

Part I
Sentences of the First Degree
(On Axiomatic Systems of the Smallest Number of Sentences and the Concept of the Ideal Element)

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1 Introduction

Whenever a system of sentences is recognized to be valid, it is often not necessary to convey each and every sentence to memory; it is sufficient to choose some of them from which the rest can follow. Such sentences, as it is generally known, are called axioms. The choice of these axioms is to a certain degree arbitrary. One can ask, however, if the property of a system of sentences to have several axiom systems is interconnected with other remarkable properties, and if there are systematic approaches to find, as the case may be, that axiomatic system which contains the least possible number of sentences. In the following some thoughts shall be communicated, which might be useful as a pre-stage for the treatment of these or related problems.

In fact the actual problem of interest is so entangled, that initially it seems appropriate to be content with an immense simplification: We only consider sentences of a certain type, sentences that we can write symbolically: $(a_1, \ldots, a_n) \to b$ and that can be expressed linguistically by formulations such as: If $(a_1, \ldots, a_n)$ altogether holds, so does b. In addition, a second simplification will be introduced in the present first part, by only considering sentences of type $a \to b$; however, we will liberate ourselves from this limitation in a following part. Further we assume rules according to which from certain sentences other ones follow: So, e.g., the validity of the sentences $a \to b$, $b \to c$ should result in the holding of the sentence $a \to c$.

However, what is actually meant by such a sentence, what the symbol $\to$ means in the combination of characters $a \to b$ or the word 'if' in the corresponding linguistic formulation, does not have to be indicated here. At this point, it cannot be our task to lay out where we take the inference rules from, in what sense such sentences appear in ordinary life and which relation our question might have with the configuration of a scientific discipline. Also, the reason cannot be given here, why our considerations do not even begin to reach that level of generality which would be necessary if we wanted to reach by them a full understanding of the connection between mathematical or physical sentences and

Translated by J. Legris, with kind permission, from: Hertz, P., "Über Axiomensysteme für beliebige Satzsysteme, Teil 1: Sätze ersten Grades", Mathematische Annalen Vol. 87(3–4), pp. 246–269.

J.-Y. Béziau (ed.), *Universal Logic: An Anthology*, 11–29

Studies in Universal Logic, DOI 10.1007/978-3-0346-0145-0_2, © Springer Basel AG 2012

not only being prepared for such an understanding. All those questions will be addressed on another occasion and at a different place[1]. It is completely sufficient if in this paper we stick to those inference rules according to which from certain sentences other ones follow. From the formal point of view assumed here, our sentences are just character strings, of which we examine certain sets; namely sets with the property that if they contain certain sentences, others necessarily occur in them, which are formed from those according to a certain rule.

As far as possible, the appeal to the direct geometrical intuition has been avoided in this paper. But, because of this, the easy understanding of the sentences to be dealt with might be jeopardized. That is why in this introduction the content of what is to be proved in the first part shall be summarized and explained by a geometrical illustration; although this illustration is only applicable in this way to laws of type $a \to b$ which are the ones we limit ourselves to in the first part[2].

The elements a, b, $c \ldots$ are presented as dots, and the sentences $a \to b$ as a solid arrow drawn from a to b; furthermore, it needs to be taken care that if such an arrow leads from a to b and another one from b to c, there also must lead an arrow from a to c. If now according to this rule all other sentences arise from certain sentences, then those sentences are called axioms and the corresponding arrows shall be drawn out, whereas the other arrows are drawn as dotted lines only.

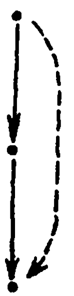

Fig. 1

Now Fig. 1 on the one hand and Figs. 2a and 2b on the other hand provide examples for the case in which there is only one axiomatic system and for the case in which there are several axiomatic systems. Figures 3a and 3b reveal that if the choice of the axiomatic system is not unique, a varying number of axioms may be employed to present the system of sentences. Particularly, they present the case in which every element is connected with every other element in both directions. If that is the case, one obtains the least number of axioms by connecting the elements with each other cyclically by axioms. Furthermore it emerges that the choice of the axiomatic system is unique if, and only if, there are no two elements that are mutually connected with each other by sentences. The proof for that given in this paper can be easily translated into the realm of geometry.

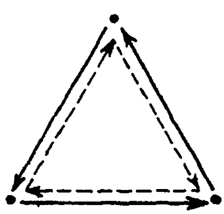

Fig. 2a

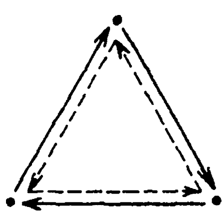

Fig. 2b

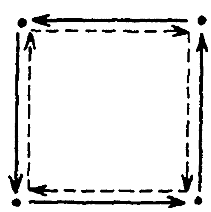

Fig. 3a

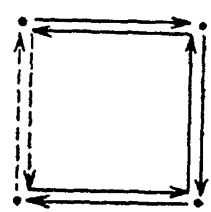

Fig. 3b

[1]It might be added though, that our sentences $a \to b$ are nothing other than formal "implications" in the sense of Russell [Whitehead–Russell **1** (1900), p. 15], and that the scheme of inference used as a base in the first part is the Theorem listed by Russell as No. 10, 3, p. 150, or put differently: Our sentences are judgements of subsumptions, our inferences are syllogisms of modus Barbara.

[2]Similar geometrical illustrations in the papers by Zaremba, Enseignement mathématique (1916), p. 5; G. Pólya, Schweizer Pädagog. Zeitschr. 1919, Hft. 2.

Additionally it will be possible to trace back the most general case to that one where there is only *one* independent axiomatic system, namely by at first substituting certain groups of interconnected elements with *one* element each, and then turning to the axiomatic system for the reduced system and subsequently reintroducing the omitted elements. Therefore, we only have to consider the case of uniqueness.

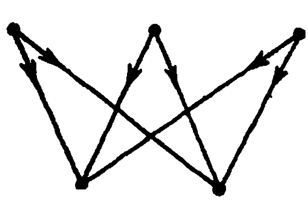

Fig. 4a

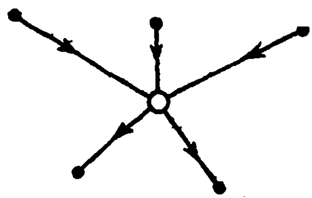

Fig. 4b

Now, in the case of uniqueness, however, one can reduce the number of axioms even further by introducing *ideal* elements. If, e.g., each of three elements is connected to each of two elements with one sentence, as it is visualized in Fig. 4a, then six sentences are needed; by introducing an ideal element—as indicated with a small circle in Fig. 4b—one can lower that number to five. Such an ideal element thus has the only purpose to avail us of a convenient presentation of the sentences between real elements; the sentences between ideal elements or between real and ideal elements have no meaning per se, but only those are followed from them obtaining between real elements.

Ideal elements are used constantly in all of the sciences, especially within physics and mathematics. Counted among them is, e.g., the concept of force, by which we can better describe the connection between the real elements: position of the acting bodies and movement of those bodies that are affected. From a philosophical point of view the ideal element has been made an object of thorough investigation by Vaihinger in his philosophy of the "As-If"[3].

One can ask now: How does one get to the least number of axioms if ideal elements are admitted? This problem will not be entirely solved here; it will only be shown for a very special case, how to get to that axiomatic system with the least number of sentences among those axiomatic systems in which no ideal elements are related to each other.

2 Problem Statement

1. (Definition.) We consider finitely many elements. By a sentence we understand the embodiment [*Inbegriff*] of a complex that may only consist of a single element, called *antecedens*, and an element, called *succedens*. We represent sentences by formulas like $(a_1, \ldots, a_n) \to b$ respectively $a \to b$. If the antecedens consists of only one element, then the sentence is called a *sentence of the first degree*.

In this part only sentences of the first degree are considered and therefore "sentence" is always to be understood as one of the first degree. Furthermore, we only want to consider here sentences of the first degree whose antecedens and succedens differ.

[3] 2nd edition, Berlin 1913.

2. (Definition.) A system of sentences

$$\begin{array}{r}
\text{I. } a \to b \\
\text{II. } b \to c \\
\hline
\text{III. } a \to c
\end{array}$$

is called an *inference*, I. the *minor sentence*, II. the *major sentence*, III. the *conclusion*. Minor sentence and major sentence together are called *premises*. Of the conclusion we also say that it follows from the premises.

3. (Definition.) Regarding a system $\mathfrak{T}$ of sentences, we denote by chain inference a simply ordered finite series of inferences with the property that each sentence in the inferences of this series which does not belong to $\mathfrak{T}$ coincides with a conclusion of an earlier sentence.

4. (Definition.) A given sentence is called properly *provable* from a given system of sentences $\mathfrak{T}$ if there is a chain inference belonging to $\mathfrak{T}$ in which the sentence occurs as conclusion. That chain inference itself is called proof of the sentence. A sentence is called provable from $\mathfrak{T}$, if it belongs to $\mathfrak{T}$ or if it is actually provable from $\mathfrak{T}$.

5. (Definition.) A system of sentences is called *closed* if each provable sentence regarding the system occurs within this system.

6. (Definition.) For every system of sentences $\mathfrak{S}$, a system of sentences $\mathfrak{A}$ is called *an axiomatic system* if each sentence of $\mathfrak{S}$ is provable from $\mathfrak{A}$.

7. (Definition.) An axiomatic system $\mathfrak{A}$ that belongs to a system of sentences $\mathfrak{S}$ is called *independent* if no sentence within $\mathfrak{A}$ is provable using other sentences within $\mathfrak{A}$.

8. (Proposition.) If every sentence within a system of sentences $\mathfrak{C}$ is provable from a system of sentences $\mathfrak{B}$, and if a sentence $\mathfrak{a}$ is provable from $\mathfrak{C}$, then $\mathfrak{a}$ is provable from $\mathfrak{B}$.

9. (Proposition.) For every closed system $\mathfrak{S}$ at least one independent axiomatic system exists.

PROOF. $\mathfrak{S}$ is an axiomatic system itself. If it is not independent, there has to be at least one sentence which follows from the others; we leave out that one; if the residual system is not independent, we leave out another sentence, and so on, until the method comes to an end. The remaining system is an axiomatic system as outlined by Proposition 8 and it is independent.

Now it will be our task to get an overview of the number of possibilities for independent axiomatic systems. We especially want to examine, when there is only one axiomatic system and how to systematically arrive at axiomatic systems that contain as few sentences as possible.

3 Possibility of a Unique Choice of the Axiomatic System

10. (Definition.) A system of elements and their connecting sentences is called a *net* if there are two sentences $x \to y$, $y \to x$ for each pair of elements x, y.

11. (Proposition.) All sentences within a net form a closed system of sentences.

12. (Proposition.) If a number of elements is ordered in a series, such that every following succedens is in a sentence in which the former is antecedens, and the first succedens is in a sentence in which the last is antecedens, then those sentences are axioms of the

net and inversely there exists an axiomatic system for every net whose sentences can be ordered in such a way.

13. (Proposition.) For all sentences of a net which consists of n elements, there exists an independent axiomatic system consisting of n sentences, and, if $n \geq 3$, there exist independent systems of axioms which consist of more than n sentences, but no independent axiomatic system which consists of less than n sentences.

PROOF. It follows from Proposition 12 that there are independent axiomatic systems which consist of n sentences, therefore an axiomatic system for a net of n elements is the following:

$$a_1 \to a_2,\ a_2 \to a_3, \ldots, a_n \to a_1.$$

An independent axiomatic system of more than n elements ($n \geq 3$) is the following:

$$a_1 \to a_2,\ a_2 \to a_3, \ldots, a_{n-1} \to a_n,$$
$$a_2 \to a_1,\ a_3 \to a_2, \ldots, a_n \to a_{n-1}.$$

One realizes that there is no independent axiomatic system with less than n elements by noticing that every element needs to be antecedens and succedens of an axiom. Therefore, one has at least $2n$ axioms that need to be found within the system and of those, at most 2 can be identical.

14. (Proposition.) If two nets share an element, they are parts of one and the same net.

15. (Definition.) Regarding a closed system of sentences $\mathfrak{S}$, a net which is not part of another one that can be found within $\mathfrak{S}$, is called *maximal*.

16. (Proposition.) Maximal nets are disjoint.

17. (Definition.) Two elements, x and y, of a system of sentences, are called *disconnected*, if it is possible to divide all elements into two classes such that x and y belong to different classes and if there is no sentence which connects two elements of different classes.

18. (Definition.) Sentences that belong to a system of sentences, are called *disconnected*, if it is possible to divide all sentences of the system into two disjoint classes, such that the given sentences belong to different classes.

19. (Definition.) Elements are said to be *connected*, if they are not disconnected.

20. (Definition.) Sentences are said to be *connected*, if they are not disconnected.

21. (Proposition.) If two elements are connected with a third element, they are connected with each other.

PROOF. Let a be connected with c and b be connected with c. If a and b were disconnected, then we could divide all elements into two classes A and B, such that a would belong to A and b would belong to B, and there would be no sentence that would contain both one element belonging to A and one belonging to B. Now then, if c belonged to A, c would be disconnected with b, that conflicts with our condition. In the same manner, it would also lead to a contradiction if c belonged to B.

22. (Proposition.) If two sentences are connected with a third sentence, they are connected with each other.

PROOF. Let the sentence $\mathfrak{a}$ be connected with sentence $\mathfrak{c}$, and $\mathfrak{b}$ be connected with sentence $\mathfrak{c}$. If $\mathfrak{a}$ and $\mathfrak{b}$ were disconnected, all sentences of the system could be divided into two disjoint classes $\mathfrak{A}$ and $\mathfrak{B}$. Class $\mathfrak{A}$ would contain $\mathfrak{a}$, and class $\mathfrak{B}$ would contain $\mathfrak{b}$. Now then, if $\mathfrak{c}$ belonged to $\mathfrak{A}$, $\mathfrak{c}$ would be disconnected with $\mathfrak{b}$, that conflicts with our condition. The same contradiction follows if we assume that $\mathfrak{c}$ belongs to $\mathfrak{B}$.

23. (Proposition.) The necessary and sufficient condition for two sentences to be connected, is that the four elements determinated by them are connected.

PROOF. 1. Let the sentences $\mathfrak{e} \equiv (a \to b)$; $\mathfrak{f} \equiv (c \to d)$ be connected. It is immediately clear that a, b and c, d are connected. If a, c were disconnected, there would be two disjoint classes A and C, to which a and c respectively would belong, such that there would be no sentence which connects an element of class A with an element of class C. All sentences could be divided in two disjoint classes $\mathfrak{E}$ and $\mathfrak{F}$ where the elements of these sentences would either belong only to A or to C. Hence, e belongs to $\mathfrak{E}$, and f to $\mathfrak{F}$, e and f were disconnected which would be contrary to the condition.

2. Let a and c be connected. If e and f were disconnected, there would be two disjoint classes of sentences $\mathfrak{E}$ and $\mathfrak{F}$ to which e and f would belong. The elements that belong to $\mathfrak{E}$ form a class A, while elements that belong to $\mathfrak{F}$ form class C. So every sentence either belongs to $\mathfrak{E}$, meaning it only contains elements of A, or $\mathfrak{F}$, meaning it only contains elements of C, then according to our assumption, there is no sentence that connects an element of A and C. Then the elements a and c which belong to A and C respectively, would be disconnected, which would conflict with our condition. That is why the assumption, that $\mathfrak{e}$ and $\mathfrak{f}$ are disconnected, is wrong. One can reason in a similar way if assuming a connected with d, or b with c, or b with d.

Remark. If two elements, say a and c are identical, then of course $\mathfrak{e}$ and $\mathfrak{f}$ are connected as well. That is because if $\mathfrak{e}$ and $\mathfrak{f}$ were disconnected, there would have to be two disjoint systems of sentences $\mathfrak{E}$ and $\mathfrak{F}$ which $\mathfrak{e}$ and $\mathfrak{f}$ belonged to. However, $\mathfrak{E}$ and $\mathfrak{F}$ would share elements because $\mathfrak{e} \equiv a \to b$; $\mathfrak{f} \equiv c \to d$ and $a \equiv c$.

24. (Definition.) Regarding a system of sentences $\mathfrak{S}$ we call a system of connected sentences a *chain*, if that system does not contain a sentence that belongs to a net of $\mathfrak{S}$.

Remark. However, a chain may contain elements that belong to a net.

25. (Proposition.) Two chains that share an element form a new chain together.

PROOF. Let $\mathfrak{K}'$ and $\mathfrak{K}''$ share an element x. Let $\mathfrak{k}'$ and $\mathfrak{k}''$ be sentences that belong to $\mathfrak{K}'$ and $\mathfrak{K}''$ which contain the element x. Then $\mathfrak{k}'$ and $\mathfrak{k}''$ are connected by the remark according to Proposition 23. Also by Proposition 22, each sentence of $\mathfrak{K}'$ is connected with each sentence of $\mathfrak{K}''$ or all sentences of ($\mathfrak{K}'$, $\mathfrak{K}''$) amongst each other. Now if within ($\mathfrak{K}'$, $\mathfrak{K}''$), there was a sentence that belonged to a net, this sentence would either have to be found in $\mathfrak{K}'$ or $\mathfrak{K}''$. Therefore these two would not both be chains.

26. (Definition.) Regarding a system of sentences $\mathfrak{S}$ we call a chain that is not contained in any other chain, a *maximal* chain.

27. (Proposition.) A maximal chain and a chain that is not contained within it have no element in common.

PROOF. Follows from Proposition 25 and Definition 26.

28. (Proposition.) In a closed system $\mathfrak{S}$ the conclusion of two sentences of a maximal chain belongs to the maximal chain as well.

PROOF. If $a \to b$, $b \to c$ belong to the maximal chain $\mathfrak{M}$, then $a \to c$ must belong to $\mathfrak{M}$ as well. If that were not the case, $a \to c$ could not be a chain by Proposition 27, $a \to c$, therefore, would belong to a net by Definition 24. That is why $c \to a$ would be a sentence of $\mathfrak{S}$, and because $b \to c$ is a sentence of $\mathfrak{S}$, $b \to a$ would be as well. Then $a \to b$ would belong to a net, and because this sentence is contained in $\mathfrak{M}$, so $\mathfrak{M}$ would be no chain according to Definition 24 because that would conflict with the condition.

29. (Proposition.) A maximal chain in a closed system is a closed system.

PROOF. Follows from Definition 5 and Proposition 28.

30. (Proposition.) Two maximal chains have no sentence in common.

PROOF. Follows from Proposition 27.

31. (Proposition.) A sentence in a closed system does not belong to a maximal net and a maximal chain at the same time.

PROOF. Follows from Definitions 24 and 26.

32. (Proposition.) Every sentence in a closed system belongs either to a maximal net or a maximal chain.

PROOF. If a sentence $\mathfrak{a}$ does not belong to any net, it represents a chain by Definition 24. Let $\mathfrak{M}$ be the entirety of all sentences that are connected with $\mathfrak{a}$, and do not belong to any net. Then those sentences are connected amongst each other. Therefore, $\mathfrak{M}$ is a chain, and, as is easily recognizable, a maximal chain as well. But, if $\mathfrak{a}$ belongs to a net, it belongs to a maximal net as well.

33. (Proposition.) The entirety of all sentences in a closed system together form maximal nets and maximal chains which have no sentence in common.

34. (Proposition.) If a sentence $\mathfrak{a} \equiv a \to b$ is provable from a system of sentences $\mathfrak{T}$, there is a series of sentences $a \to \alpha_1,\ \alpha_1 \to \alpha_2, \ldots, \alpha_{\rho-1} \to \alpha_\rho,\ \alpha_\rho \to b$ that all belong to $\mathfrak{T}$.

PROOF. If $\mathfrak{a} \equiv a \to b$ belongs to $\mathfrak{T}$, this sentence is already proved. In the other case, according to Definition 4, $a \to b$ has to be found as a conclusion in an inference that belongs to a chain inference whose premises $a \to x$, $x \to b$ would be provable from $\mathfrak{T}$. These two sentences again belong to $\mathfrak{T}$, or are conclusions of earlier sentences. In the last case, $a \to x'$, $x' \to x$, $x \to x''$, $x'' \to b$ would be provable from $\mathfrak{T}$. One can see that this claim is true by continuing this procedure, which has to be finite.

35. (Theorem.) There is only one independent axiomatic system for a closed system of sentences $\mathfrak{S}$ without nets.

Proof. According to Proposition 9 there is at least *one* independent axiomatic system. Assuming that for $\mathfrak{S}$ there were two independent axiomatic systems $\mathfrak{A}$ and $\mathfrak{A}'$, then each axiomatic system would have to contain an axiom which is not contained in the other. Let $\mathfrak{a} \equiv a \to b$ be an axiom of $\mathfrak{A}$ which is not contained in $\mathfrak{A}'$. Then this sentence would have to be provable from $\mathfrak{A}'$, and by Proposition 34 there must exist a series of at least two sentences that belong to $\mathfrak{A}'$:

$$a \to \alpha_1,\ \alpha_1 \to \alpha_2, \ldots, \alpha_{\rho-1} \to \alpha_\rho,\ \alpha_\rho \to b.$$

Each of those sentences, however, has to be provable from $\mathfrak{A}$ again. That is why we again obtain by Proposition 34, a series of at least two sentences:

$$a \to \beta_1,\ \beta_1 \to \beta_2, \ldots, \beta_{\sigma-1} \to \beta_\sigma,\ \beta_\sigma \to b,$$

which all belong to $\mathfrak{A}$. Amongst these sentences, however, $\mathfrak{a}$ must be found, because if not, the axiomatic system $\mathfrak{A}$ would not be independent. From this it follows that the series of above sentences has to contain at least three sentences. Hence, three cases are possible:

1. It is $\mathfrak{a} \equiv a \to \beta_1$. Then $\beta_1 \equiv b$ and $\mathfrak{A}$ would contain the net $b \to \beta_2, \ldots, \beta_\sigma \to b$, therefore, $\mathfrak{S}$ would contain a net contrary to the condition.

2. It is $\mathfrak{a} \equiv a \to b_\sigma$. Then $b_\sigma \equiv a$ and $\mathfrak{A}$ would contain the net $a \to \beta_1, \ldots, \beta_{\sigma-1} \to a$, therefore, $\mathfrak{S}$ would contain a net contrary to the condition.

3. The sentence $\mathfrak{a}$ is identical with one middle sentence $\beta_{\mu-1} \to \beta_\mu$, then $\beta_{\mu-1} \equiv a$, $b_\mu \equiv b$ and $\mathfrak{A}$ would contain the nets $a \to \beta_1, \ldots, \beta_{\mu-2} \to a$ and $b \to b_{\mu+1}, \ldots, \beta_\sigma \to b$, therefore $\mathfrak{S}$ would contain a net contrary to the condition.

36. (Definition.) To a given closed system of sentences $\mathfrak{S}$, we call $\mathfrak{S}'$ a *reduced system* of sentences with the following properties:

1. For every element a in $\mathfrak{S}$ there is an element a' in $\mathfrak{S}'$ that does not belong to a net.

2. To all elements $c_1, c_2, \ldots$ of a maximal net of $\mathfrak{S}$ together corresponds a single element c' in $\mathfrak{S}'$.

3. Every sentence $a \to b$ in $\mathfrak{S}$, where a and b both do not belong to a net, correspond to a sentence $a' \to b'$ in $\mathfrak{S}'$ where a' and b' are the corresponding elements to a and b.

4. All sentences together, $a \to \overline{c}_i$ respectively $\overline{\overline{c}}_j \to b$, where a and b do not belong to a maximal net, but where all $\overline{c}_i$ and $\overline{\overline{c}}_j$ belong to one and the same maximal net, correspond to a sentence $a' \to \overline{c}'$ respectively $\overline{\overline{c}}' \to b$ in $\mathfrak{S}'$.

All sentences $\overline{c}_k \to \overline{\overline{c}}_l$ together, where all $\overline{c}_k$ belong to one and the same maximal net, and where all $\overline{\overline{c}}_l$ belong to one and the same but another maximal net, correspond to one sentence $\overline{c}' \to \overline{\overline{c}}'$ in $\mathfrak{S}'$.

5. $\mathfrak{S}'$ contains no elements other than those corresponding to $\mathfrak{S}$, according to 1 and 2.

6. $\mathfrak{S}'$ contains no sentences other than those corresponding to $\mathfrak{S}$, according to 3 and 4.

37. (Assumption.) There exists a reduced system of sentences for every closed one.

Remark. Of course, Theorem 35 can be derived from simpler assumptions. One can point out, that if it is known how many elements the system contains and how these elements are connected, then it can be said that it exists. The meaning of this claim, however, will only be clear after discussing the concept of existence in detail. One could assume, for instance, that the elements a of $\mathfrak{S}$ and the elements $c_1, \ldots, c_n$ together are assigned to the symbols a' and c'. But these symbols exist in our minds. Or one could say that the totality of the things $c_1, \ldots, c_n$ is another thing again. Finally, one could also regard the reduced system as a means to portray the rules of a non-reduced system.

In Definition 36 we introduced the reduced system in hypothetical form and stated that we call a reduced system one that has certain relations to the given system. We also could have done that differently and might have said: instead of each element a we put an element a', instead of the elements $c_1,\ c_2, \ldots, c_n$ of a net we put an element c', and so on (the requirements of 5 and 6 would then be dispensible). We will call the resulting system a reduced system. This sentence, then, would not have the form of a definition but of a statement.

We had explained in the introduction that we only want to consider sentences of the form $a \to b$. This seems to be in contradiction with the expression we get now. But the reduced system on its own underlies the same rules as the initial one and is only used to be able to infer certain sentences about the initial one more easily.

38. (Proposition.) A reduced system $\mathfrak{S}'$ that belongs to a closed system $\mathfrak{S}$ contains no net.

Proof. If $\mathfrak{S}'$ contained a net, there would have to be two elements a', b' such that $a' \to b'$, $b' \to a'$ would exist in $\mathfrak{S}'$. According to Definition 36, there would be two sentences $a_1 \to b_1$, $b_2 \to a_2$ in $\mathfrak{S}$ that would correspond to these sentences. But because b_1 and b_2 would correspond to the same element b' they then would also have to belong to the same maximal net (according to Definition 36), so $a_2 \to a_1$ would occur in $\mathfrak{S}$. Because it is closed, $\mathfrak{S}$ would then contain $b_1 \to a_1$, i.e. a_1 and b_1 would belong to a maximal net. Then different elements in $\mathfrak{S}'$ could not correspond to them, which would be contrary to the condition.

39. (Proposition.) The reduced system $\mathfrak{S}'$ which belongs to a closed system is a maximal chain regarding to $\mathfrak{S}$ or can be broken down into several maximal chains that do not have any element or sentence in common.

PROOF. Follows from Definition 26 and Proposition 38.

40. (Proposition.) A reduced system $\mathfrak{S}'$ which belongs to a closed system $\mathfrak{S}$ is itself closed.

PROOF. Let $\mathfrak{e}' \equiv a' \to b'$ and $\mathfrak{f}' \equiv b' \to c'$ be two sentences that are contained in $\mathfrak{S}'$, then $a \not\equiv c$, according to Proposition 38. Then there have to be two corresponding sentences by Definition 36 that correspond to $\mathfrak{e}'$ and $\mathfrak{f}'$ in $\mathfrak{S}$: $a_1 \to b_1$, $b_2 \to c_2$. Now either $b_1 \equiv b_2$, then $a_1 \to c_2$ would be a sentence in $\mathfrak{S}$. Or $b_1 \not\equiv b_2$, then b_1 and b_2 would belong to a maximal net by Definition 36. Therefore $b_1 \to b_2$ would be contained in $\mathfrak{S}$, so $a_1 \to c_2$ would be a sentence in $\mathfrak{S}$ again. But because $a' \not\equiv c'$, a_1 and c_2 do not belong to the same maximal net and, again by Definition 36, $a' \to c'$ is a sentence in $\mathfrak{S}'$.

41. (Proposition.) There is one and only one independent axiomatic system for every reduced system.

PROOF. Follows from Theorem 35 and Proposition 38.

42. (Proposition.) The following procedure may be used to obtain an axiomatic system $\mathfrak{A}$ from a closed system $\mathfrak{S}$: one constructs an axiomatic system $\mathfrak{A}'$ from $\mathfrak{S}'$, furthermore, for every sentence in $\mathfrak{A}'$, one looks for a corresponding sentence in $\mathfrak{S}'$ (Definition 36) and adds an axiomatic system for each maximal net that is contained in $\mathfrak{S}$.

PROOF. Let $\mathfrak{e} \equiv a \to b$ a sentence of $\mathfrak{S}$, it needs to be shown that it is provable from $\mathfrak{A}$. If $\mathfrak{e}$ is a sentence of a maximal net, then the proof is already accomplished. So let us assume that $\mathfrak{e}$ does not belong to a maximal net. Then, according to Definition 36, there has to be a corresponding sentence $\mathfrak{e}' \equiv a' \to b'$ in $\mathfrak{S}'$. This one has to be provable by axioms in $\mathfrak{A}'$. Thus, according to Proposition 34, a series of sentences must exist that belong to $\mathfrak{A}'$:

$$a' \to \alpha'_1,\ \alpha'_1 \to \alpha'_2, \ldots, \alpha'_{\rho-1} \to \alpha'_\rho,\ \alpha'_\rho \to b'.$$

Let the corresponding sentences in $\mathfrak{A}$ be the following:

$$\overline{a} \to \overline{\alpha}_1,\ \overline{\overline{\alpha}}_1 \to \overline{\alpha}_2, \ldots, \overline{\grave{\alpha}}_{\rho-1} \to \overline{\alpha}_\rho,\ \overline{\dot{\alpha}}_\rho \to \overline{b}.$$

Now either two consecutive elements $\overline{\alpha}_i$, $\overline{\ddot{\alpha}}_i$ are identical to each other, or they belong to the same maximal net, because they correspond to the same α'_i, such that $\overline{\alpha}_i \to \overline{\ddot{\alpha}}_i$ is provable from $\mathfrak{A}$. Additionally a and $\overline{a}$, as well as b and $\overline{b}$, are identical or belong to the same maximal net.

Consequently, there is a system of sentences

$$a \to \alpha_1,\ \alpha_1 \to \alpha_2, \ldots, \alpha_\rho \to b$$

that belong to $\mathfrak{A}$, so we have provided the proof.

43. (Proposition.) The following procedure may be used to obtain an independent axiomatic system $\mathfrak{A}$ from a closed system $\mathfrak{S}$: one constructs the independent axiomatic system (Proposition 41) of $\mathfrak{S}'$, furthermore one looks for a corresponding sentence in $\mathfrak{S}$ for every sentence in $\mathfrak{A}'$, and adds an independent axiomatic system for every maximal net that is contained in $\mathfrak{S}$.

PROOF. According to Proposition 42, one obtains an axiomatic system by using this procedure. Suppose that this system is not independent, and that $\mathfrak{e} \equiv a \to b$ is a sentence of $\mathfrak{A}$ which is provable from the other axioms. Then $\mathfrak{e}$ can either belong to a maximal net of $\mathfrak{S}$ or not.

a) If $\mathfrak{e}$ belonged to a maximal net of $\mathfrak{S}$, a series of two or more sentences would exist, according to Proposition 34:

$$a \to \alpha_1,\ \alpha_1 \to \alpha_2, \ldots, \alpha_\rho \to b,$$

which all would belong to $\mathfrak{A}$ and each would be different from $\mathfrak{e}$. Of these, according to the condition, not all could belong to the same maximal net as $\mathfrak{e}$. Therefore, not all could belong to the same maximal net amongst each other, which is why in the series there has to be a sentence which belongs to no maximal net. Let this sentence be $\alpha_i \to \alpha_{i+1}$.

On the other hand, $b \to a$, $a \to \alpha_1$ are sentences of $\mathfrak{S}$, because $a \to b$ would belong to a net, so $\alpha_{i+1} \to \alpha_i$ are sentences of $\mathfrak{S}$ as well, and therefore, $\alpha_i \to \alpha_{i+1}$ would belong to a net nevertheless.

b) If $\mathfrak{e}$ did not belong to a maximal net, again according to Proposition 34, there would be a series $\mathfrak{R}$ of two or more sentences $a \to \alpha_1, \ldots, \alpha_\rho \to b$, which would belong to $\mathfrak{A}$, which each would be different from $e \equiv a \to b$, and which could not all belong to a maximal net. Let $\alpha_k \to \alpha_{k+1}$ be such a sentence which does not belong to a maximal net and would be different from $a \to b$. However, such sentences of $\mathfrak{R}$ which belong to no maximal net, correspond to sentences in $\mathfrak{S}'$ that belong to $\mathfrak{A}'$, and which form a series $\mathfrak{R}'$ $a' \to \beta_1',\ \beta_1' \to \beta_2', \ldots, \beta_\rho' \to b'$ which has the property that the antecedens of each sentence (except the first) is identical with the succedens of the preceding sentence. Furthermore the series $\mathfrak{R}'$ contains the sentence which is corresponding to $\alpha_k \to \alpha_{k+1}$ and which is different from $a' \to b'$ (because every sentence in $\mathfrak{A}'$ only corresponds to one sentence in $\mathfrak{A}$), thus it contains at least two sentences. If one of the sentences was $\mathfrak{R}' \equiv a' \to b'$, the series would contain a net, in contradiction with Proposition 38, but if $a' \to b'$ were not to be found in $\mathfrak{R}'$, then the system $\mathfrak{A}'$ would not be independent, contrary to the condition.

44. (Proposition.) For a given closed system of sentences there are no other independent axiomatic systems than those which can be found using the method described in Proposition 43.

PROOF. Let $\mathfrak{A}$ be an independent axiomatic system. It needs to be shown:

1. Those sentences of $\mathfrak{A}$ which belong to a maximal net, form an independent axiomatic system for this.

2. If $\mathfrak{A}^*$ is the embodiment of those sentences of $\mathfrak{A}$ which do not belong to a maximal net and $\mathfrak{A}^{*'}$ is the corresponding system to $\mathfrak{A}^*$ in $\mathfrak{S}'$, then $\mathfrak{A}^{*'}$ is the independent axiomatic system of $\mathfrak{S}'$.

3. $\mathfrak{A}$ contains only one of several sentences which correspond to the same sentence in $\mathfrak{S}'$.

ad 1. It is sufficient to show that the system $\overline{\mathfrak{A}}$ of those sentences in $\mathfrak{A}$, which do belong to a maximal net $\mathfrak{M}$, is an axiomatic system for $\mathfrak{M}$.

If that were not be the case, and if $a \to b$ were a sentence belonging to $\mathfrak{M}$ that was not provable from $\overline{\mathfrak{A}}$, then according to Proposition 34, a series of sentences $a \to \alpha_1, \ldots, \alpha_\rho \to b$ would exist that would belong to $\mathfrak{A}$ and of which at least one $\alpha_i \to \alpha_{i+1}$ would not belong to $\overline{\mathfrak{A}}$ and therefore also would not belong to $\mathfrak{M}$.

Then $\alpha_{i+1} \to b$, $b \to a$ would have to belong to $\mathfrak{S}$, which means that $\alpha_{i+1} \to \alpha_i$ would belong to $\mathfrak{S}$ as well, i.e. α_i and α_{i+1} would belong to a maximal net, and therefore, also to $\mathfrak{M}$ which is contrary to our assumption.

ad 2. It needs to be shown: a) that each sentence of $\mathfrak{S}'$ is provable from $\mathfrak{A}^{*'}$, b) that the sentences are independent of $\mathfrak{A}^{*'}$.

a) Suppose that the sentence $\mathfrak{e}' \equiv a' \to b'$ is not provable by $\mathfrak{A}^{*'}$. But now the corresponding sentence $\mathfrak{e} \equiv a \to b$ is provable from $\mathfrak{A}$. Thus, there are sentences $a \to \alpha_1, \ldots, \alpha_\rho \to b$ which belong to $\mathfrak{A}$. By finding the corresponding sentences in $\mathfrak{S}'$ of those sentences among them that do not belong to a maximal net, one obtains a series of sentences that belong to $\mathfrak{A}^{*'}$ contrary to our assumption:

$$a' \to \beta_1', \ldots, \beta_\sigma' \to b'.$$

b) If there was a sentence $a' \to b'$ in $\mathfrak{A}^{*'}$ which was provable from other sentences in $\mathfrak{A}^{*'}$, one would obtain a series of at least two sentences $a' \to \beta_1', \ldots, \beta_\sigma' \to b'$ which would belong to $\mathfrak{A}^{*'}$. The corresponding series of at least two sentences in $\mathfrak{A}$ would be

$$\overline{a} \to \overline{\alpha}_1,\ \overline{\overline{\alpha}}_2 \to \overline{\alpha}_2, \ldots, \overline{\overline{\alpha}}_\sigma \to \overline{b}.$$

But if two neighbouring elements like $\overline{\alpha}_1$, $\overline{\overline{\alpha}}_2$ are different, they belong to a net, according to Definition 1, there exists a series of sentences that belong to $\mathfrak{A}$ which connects a and b. If $a \to b$ were found within this series, then a, b would belong to a net, but this is impossible because a sentence $a' \to b'$ corresponds to $a \to b$ in $\mathfrak{S}$. Therefore, $a \to b$ which belongs to $\mathfrak{A}$, would be provable from other sentences in $\mathfrak{A}$, and $\mathfrak{A}$ would not be independent.

ad 3. $\mathfrak{A}$ cannot contain two sentences that correspond to the same sentence in $\mathfrak{S}'$. Let $a_1 \to b_1$ and $a_2 \to b_2$ be two such sentences, then either a_1 and a_2 have to be identical and b_1 and b_2 would belong to the same net, or vice versa, or a_1 and a_2 on the one hand, and b_1 and b_2 on the other hand, would belong to the same net.

Let us examine the last case for instance. According to Definition 1, a series of sentences would exist, which would belong to $\mathfrak{A}$ and the same maximal net, which would therefore be different from $a_1 \to b_1$ and from which $a_1 \to a_2$ would be provable. We will call this series $a_1 \to \alpha_1, \ldots, \alpha_\rho \to a_2$. Likewise, there would be a series $b_2 \to \beta_1, \ldots, \beta_\sigma \to b_1$ which does not contain $a_1 \to b_1$. Thus, one could prove $a_1 \to b_1$ by the series:

$$a_1 \to \alpha_1, \ldots, \alpha_\rho \to a_2,\ a_2 \to b_2,\ b_2 \to \beta_1, \ldots, \beta_\sigma \to b_1,$$

which cannot contain $a_1 \to b_1$, i.e. $\mathfrak{A}$ would not be independent.

45. (Definition.) By a *sequence* we understand a system of elements and a closed system of sentences that exists between these elements, with the property that each pair of elements is connected by one and only one sentence.

46. (Proposition.) There is no net within a sequence.

PROOF. Follows from Definitions 10 and 45.

47. (Proposition and Definition.) There is one and only one element in a sequence which is succedens of no other element—we call this element the *highest element*—and there is one and only one element in a sequence which is antecedens of no other element—we call this element the *lowest element*.

PROOF. We consider an arbitrary element, if there exists an antecedens of it, so we locate that one, then the one which is the antecedens of that last one, and so on. Because the system of sentences is closed, one can never meet an element already encountered, thus the series has to terminate, therefore there has to be a highest element. In the same way one can recognize the existence of the lowest element. From the Definition 45 it is also clear that there can only be one highest and only one lowest element.

48. (Proposition and Definition.) The elements of a sequence can be arranged in one and only one way, such that each element, with the exception of the lowest, is antecedens to the following, and each, with the exception of the highest, is succedens to the preceding and that there is no element for two consecutive elements which can be antecedens to one and succedens to the other. We call this order the *natural order*.

PROOF. As first element we pick the highest, as the next one we pick the highest of the rest, and so on. If now x for example is one element and y is the following, so $x \to y$ has to be an element of the sequence, because if $y \to x$ was an element of the sequence, we would have committed a mistake when choosing x, because y was still available. Furthermore one can see, that if $x \to z$ is an element of the sequence where $z \not\equiv y$, $z \to y$ does not belong to the sequence. One can also easily show that this order is the only one that has the required property.

49. (Proposition.) One obtains an independent axiomatic system for the sequence, which is also the only one, by ordering the elements in natural order and using as axioms all those sentences which connect two consecutive elements.

PROOF. That these sentences form an axiomatic system is immediately apparent. That they are independent follows by Proposition 34 from the fact, that for two consecutive elements, there is none which is succedens of the higher one and antecedens of the lower one. That there is no other independent axiomatic system follows from Theorem 35 and Proposition 46.

50. (Definition.) By a *maximal sequence* in a closed system of sentences we understand a sequence that is contained in no other sequence.

51. (Proposition.) Every sentence in a closed system belongs to at least one maximal sequence.

52. (Proposition.) We can obtain the independent axiomatic system of a closed system of sentences without net, by forming axiomatic systems for all maximal sequences (thereby sentences might possibly occur as axioms of several maximal sequences).

PROOF. Let $\mathfrak{A}$ be the system of sentences which consists of all axiomatic systems of all maximal sequences.

1. Let $a \to b$ be a sentence of $\mathfrak{S}$. According to Proposition 51, it belongs to a maximal sequence, and is therefore provable from $\mathfrak{A}$.

2. Let $a \to b$ be a sentence of $\mathfrak{A}$. It needs to be shown that it cannot be proven from other sentences of $\mathfrak{A}$. Let us assume just that, however. Because $a \to b$ belongs to $\mathfrak{A}$, there must be a maximal sequence Z, in which b is the following element of a. But if $a \to b$ was provable from other sentences in $\mathfrak{A}$, a series of sentences would exist that would be different from $a \to b$ and would belong to $\mathfrak{A}$:

$$a \to \alpha_1, \ldots, \alpha_\rho \to b.$$

These sentences cannot belong to Z, because there should not be an element x of Z, for which $a \to x$, $x \to b$ holds. On the other hand we again get a sequence if we add these sentences to Z. Because for element x that belongs to Z, either $x \to a$, as well as $x \to \alpha_i$ holds, or $b \to x$, as well as $\alpha_i \to x$. Then Z would be no maximal sequence, which would be in conflict with our condition.

4 Ideal Elements

53. (Definition.) If $\mathfrak{S}$ is the totality of elements and of a closed system of sentences that exist between these elements, then we call the *extended system* $\hat{\mathfrak{S}}$ a totality of elements and of a closed system of sentences that exist between these elements if the following holds:

1. Each element a in $\mathfrak{S}$ corresponds to an element $\hat{a}$ in $\hat{\mathfrak{S}}$. We call such elements *real* elements of $\mathfrak{S}$, and we call the non-real elements of $\hat{\mathfrak{S}}$ *ideal* elements.

2. Each sentence $a \to b$ of $\mathfrak{S}$ corresponds to a sentence $\hat{a} \to \hat{b}$ of $\hat{\mathfrak{S}}$ between the corresponding real elements in $\hat{\mathfrak{S}}$.

3. Each sentence of $\hat{\mathfrak{S}}$ between real elements corresponds to a sentence in $\mathfrak{S}$ between corresponding elements.

Remark 1. Sentences in $\hat{\mathfrak{S}}$ between ideal elements or between a real and an ideal element correspond to no sentences in $\mathfrak{S}$.

Remark 2. Because of property 3, by examining the extended system, we gain an overview of the sentences of the original system at the same time.

Remark 3. In the future, for the sake of convenience, we shall consider the extended system as having been formed from elements of the original system on the one hand, which will become real elements of the extended system, and from added ideal elements on the other hand.

Remark 4. The extended system can have an independent axiomatic system which consists of fewer sentences than the original system. If, for instance, the original system consists of the elements a', a'', b', b'', b''', and of the six sentences between them:

$$\begin{aligned} a' &\to b', \\ a' &\to b'', \\ a' &\to b''', \\ a'' &\to b', \\ a'' &\to b'', \\ a'' &\to b''', \end{aligned}$$

then there is an extended system of these five sentences:

$$\begin{aligned} a' &\to \mathfrak{a}, \\ a'' &\to \mathfrak{a}, \\ \mathfrak{a} &\to b', \\ \mathfrak{a} &\to b'', \\ \mathfrak{a} &\to b''', \end{aligned}$$

where $\mathfrak{a}$ is an ideal element.

Remark 5. By introducing ideal elements we circumvent the usage of the word ‘or’ in the antecedens, as well as the usage of the word ‘and’ in the succedens.

54. (Definition.) In a closed system of sentences $\mathfrak{S}$ without net, we call a *pair of groups* a pair of two disjoint groups A and B of elements with the property that there is a sentence between each element of A as antecedens and each element of B as succedens

which belongs to the independent axiomatic system (Theorem 35). A is called *group of antecedens*, B is called *group of succedens*.

Remark. For A and B to be a pair of groups, it is therefore not sufficient that there is a sentence between every element of A as antecedens and B as succedens.

55. (Definition.) A *maximal* pair of groups is a pair of groups A, B, if there is no element, except the elements of A, where there is an axiom between such an element, as antecedens, and each element of B. In the same way there must not be an element, except the elements of B, where there is an axiom between such an element, as succedens, and each element of A.

56. (Proposition.) Each axiom can be found in at least one maximal pair of groups.

57. (Definition.) A *small maximal* pair of groups is such a pair which contains only one element in either the antecedens group or the succedens group. A *medium maximal* pair of groups is such a pair, which contains two elements in each of the antecedens group and the succedens group. A *large maximal* pair of groups is such a pair, which is neither a small, nor a medium one.

58. (Proposition.) There are no sentences between two elements of an antecedens group and two elements of the succedens group of a closed system of sentences $\mathfrak{S}$ without net.

PROOF. Let e.g. $a_i \to a_k$ be a sentence that exists between elements of A, then there will be a series which belongs to the axiomatic system:

$$a_i \to \alpha_1, \ldots, \alpha_\rho \to a_k.$$

On the other hand there are axioms for $\mathfrak{S}$ $a_i \to b_l$, $a_k \to b_l$, where b_l is an element of B. We can also substitute $a_i \to b_l$ by a series of axioms

$$a_i \to \alpha_1, \ldots, \alpha_\rho \to a_k,\ a_k \to b_l.$$

Because these axioms should be independent from each other, $a_i \to b_l$ would have to occur in this series, i.e. it would contain a net contrary to the condition.

59. (Definition.) A pair of groups A, B is called *closed* if there are no elements, except those in A, where there is an axiom between such an element, as antecedens, and an element of B as succedens. In the same way there must not be an element, except the elements of B, where there is an axiom between such an element, as succedens, and an element of A as antecedens.

60. (Proposition.) Each closed pair of groups is a maximal pair of groups.

61. (Proposition.) The antecedens groups and succedens groups of two closed pair of groups are disjoint.

62. (Definition.) A closed system of sentences is called *simple* if it does not contain a net and each of its maximal pairs of groups is closed.

63. (Proposition.) In an arbitrary simple system of sentences each axiom belongs to one and only one maximal pair of groups.

64. (Proposition.) In a simple system of sentences, if there is a sentence $x \to b_i$, where b_i belongs to the succedens group of a maximal pair of groups, but x does not belong to the corresponding antecedens group of A, then there is a sentence $x \to a_l$ in $\mathfrak{S}$, where a_l belongs to A. Likewise: if there is a sentence $a_k \to y$ in $\mathfrak{S}$, where a_k belongs to A, but y does not belong to B, then there is a sentence $b_m \to y$ in $\mathfrak{S}$, where b_m belongs to B.

PROOF. According to Proposition 34, in the first case, there is a series of at least two sentences that consists of axioms

$$x \to \alpha_1, \ldots, \alpha_\rho \to b_i.$$

Because A, B are a closed pair of groups now, α_ρ has to belong to A. From that follows the first part of the proposition, and the second part as well.

65. (Definition.) If $\mathfrak{S}$ is a simple system of sentences and $\hat{\mathfrak{S}}$ an extended system to it, then an independent axiomatic system $\hat{\mathfrak{A}}$ of $\mathfrak{S}$, in which no two ideal elements are connected via axioms, is called *an axiomatic system of the first degree to* $\mathfrak{S}$.

66. (Proposition.) If $\mathfrak{S}$ is a simple system and $\hat{\mathfrak{A}}$ is an axiomatic system of the first degree of $\mathfrak{S}$, and if a, b are elements of an antecedens group and of a succedens group of a maximal pair of groups of $\mathfrak{S}$ (there is an axiom $a \to b$ of $\mathfrak{S}$), then in $\hat{\mathfrak{A}}$, there is either an axiom $\hat{a} \to \hat{b}$ or a pair of axioms $\hat{a} \to \mathfrak{a}$, $\mathfrak{a} \to b$, where $\mathfrak{a}$ is an ideal element.

PROOF. Let $\hat{\mathfrak{S}}$ be the extended system that $\hat{\mathfrak{A}}$ belongs to. According to Definition 53, there is a sentence $\hat{a} \to \hat{b}$ in $\hat{\mathfrak{S}}$. If this sentence does not belong to $\hat{\mathfrak{A}}$, there would be a series $\mathfrak{R}$ in $\hat{\mathfrak{A}}$,

$$\hat{a} \to \hat{\alpha}_1,\ \hat{\alpha}_1 \to \hat{\alpha}_2, \ldots, \hat{\alpha}_\rho \to b,$$

of at least two axioms where $\hat{\alpha}_1 \not\equiv b$. If now $\hat{\alpha}_1$ is real, then by Definition 53.3 the sentence $\hat{a} \to \hat{\alpha}_1$ would correspond to $a \to \alpha_1$ in $\mathfrak{S}$, which would mean that by Proposition 64 there would have to be a sentence $b_m \to \alpha_1$ in $\mathfrak{S}$, where b_m would belong to B. $\mathfrak{S}$ therefore would contain both sentences $b_m \to \alpha_1$, $\alpha_1 \to b$. If now $b_m \equiv b$, then $\mathfrak{S}$ would contain a net, and if $b_m \not\equiv b$, there would be a conflict with Proposition 58. Both are impossible, α_1 must be ideal.

Now if $\mathfrak{R}$ contained more than two sentences, $\hat{\alpha}_2$ would have to be real, because $\hat{\mathfrak{A}}$ has to be of the first degree. Thus, there would have to be a sentence in $\mathfrak{S}$, $a \to \alpha_2$, and therefore, by Proposition 64, also a sentence $b_n \to a_2$, where b_n would belong to B. This would again be in conflict with $\alpha_2 \to b$. That is why the series $\mathfrak{R}$ does not contain another element.

67. (Definition.) If A, B are two disjoint complexes of elements we call a *connecting system* a system of these elements and other elements $\mathfrak{v}$ (which we call connecting elements), and of sentences between the elements $\mathfrak{v}$, the elements a of A, and the elements b of B with the following properties:

1. An element $\mathfrak{v}$ has only to be antecedens to one element of B and succedens to one element of A.

2. For each pair a, b there is a sentence $a_i \to b_k$ or a pair of sentences $a_i \to \mathfrak{v}_{ik}$, $\mathfrak{v}_{ik} \to b_k$.

68. (Definition.) A connecting system without connecting elements we call a *disparate* connecting system.

69. (Definition.) If a connecting system has only one connecting element and if this element is antecedens to all elements of B and succedens to all elements of A and if the connecting system does not contain any more sentences, then we call it a *centralized* connecting system.

70. (Proposition.) If $\mathfrak{S}$ is a simple system of sentences, $\hat{\mathfrak{A}}$ a corresponding axiomatic system of the first degree, furthermore A, B a maximal pair of groups of $\mathfrak{S}$, then regarding A, B, there is always a connecting system $\mathfrak{v}$ that belongs to $\hat{\mathfrak{A}}$. The connecting elements are ideal elements of $\hat{\mathfrak{S}}$.

PROOF. Follows from Proposition 66.

71. (Proposition.) In a simple system of sentences the connecting systems of two different maximal pairs of groups do not share an ideal element.

PROOF. From Proposition 61.

72. (Assumption.) If A, B are complexes of elements in a system $\mathfrak{T}$ of elements and sentences, then we can assume the existence of a different system $\grave{\mathfrak{T}}$, such that each element and each sentence of $\mathfrak{T}$ corresponds to an element and a sentence in $\grave{\mathfrak{T}}$, however, $\grave{\mathfrak{T}}$ does additionally contain a connecting system to the complexes A, B.

About the character of this Assumption, the same can be said as about Assumption 37. For the following it will be more convenient if we express ourselves in such a way that we say: We can add a connecting system to the pair of complexes A, B.

73. (Proposition.) If $\mathfrak{S}$ is a simple system of sentences, A, B is a maximal pair of groups, $\hat{\mathfrak{A}}$ is an axiomatic system of the first degree of $\mathfrak{S}$, $\hat{\mathfrak{B}}$ is a connecting system to $\hat{\mathfrak{A}}$ for the maximal pair of groups A, B (Proposition 70), $\hat{\mathfrak{B}}^*$ some other added connecting system for A, B (see Assumption 72), then each sentence between real elements of $\hat{\mathfrak{S}}$, which is provable from $\hat{\mathfrak{A}}$ and $\hat{\mathfrak{B}}^*$, corresponds to a sentence of $\mathfrak{S}$.

(It is assumed that the added elements of $\hat{\mathfrak{B}}^*$ are different from those ideal elements that were originally contained in $\hat{\mathfrak{S}}$.)

PROOF. Let $\hat{c} \to \hat{d}$ be such a sentence which is provable by the series of sentences that belongs to $\hat{\mathfrak{A}}$ and $\hat{\mathfrak{B}}^*$:

$$\hat{c} \to \alpha_1, \ldots, \alpha_\rho \to \hat{d}.$$

If now $\alpha_\lambda \to \alpha_{\lambda+1}$ did not belong to $\hat{\mathfrak{A}}$ here, then this sentence would have to belong to $\hat{\mathfrak{B}}^*$, thus, it would have to have the form $a \to b$ or $a \to \mathfrak{a}$, $\mathfrak{a} \to b$.

(Now and in the following a shall always be an element of A and b an element of B, $\mathfrak{a}$ shall be a non-real element of $\hat{\mathfrak{S}}$, meaning an ideal or connecting element.)

In the first case $\alpha_\lambda \to \alpha_{\lambda+1}$ is provable by $\hat{\mathfrak{A}}$. In the second case $\alpha_{\lambda+1}$ is a connecting element, therefore different from $\hat{d}$, and there still is another sentence $\alpha_{\lambda+1} \to \alpha_{\lambda+2}$ that would have to have the form $\mathfrak{a} \to b$ necessarily. The pair of sentences $\alpha_\lambda \to \alpha_{\lambda+1}$, $\alpha_{\lambda+1} \to \alpha_{\lambda+2}$ can be substituted by axioms $\hat{\mathfrak{A}}$. The same is true in the third case and for all sentences of $\mathfrak{R}$. Now follows the claim from Definition 53.3.

74. (Proposition.) (Denomination as in Proposition 73.) If $\hat{\mathfrak{A}}^*$ is a system of sentences that is generated from $\hat{\mathfrak{A}}$, by removing the sentences of $\hat{\mathfrak{B}}$ and adding the sentences of $\hat{\mathfrak{B}}^*$, and if the sentence $\hat{c} \to \hat{d}$, which exists between real elements, is provable from $\hat{\mathfrak{A}}^*$ and $\hat{\mathfrak{A}}$, then the corresponding sentence $c \to d$ belongs to $\mathfrak{S}$.

PROOF. Follows from Proposition 73.

75. (Proposition.) (Denomination as in Propositions 73 and 74.) If $\hat{c} \to \hat{d}$ is provable from $\hat{\mathfrak{A}}^*$, then the sentence $c \to d$ holds in $\mathfrak{S}$.

PROOF. Follows from Proposition 74.

76. (Proposition.) (Denomination as in Propositions 73 and 74.) The embodiment $\hat{\mathfrak{S}}^*$ of all elements of $\hat{\mathfrak{A}}^*$ and all sentences provable from $\hat{\mathfrak{A}}^*$ is an extended system of $\mathfrak{S}$.

PROOF. It follows from the definition that $\hat{\mathfrak{S}}^*$ is a closed system. Furthermore it follows from Proposition 75 that each sentence between real elements $\hat{c} \to \hat{d}$ in $\hat{\mathfrak{S}}^*$ corresponds to a sentence $c \to d$ in $\mathfrak{S}$. So we still need to show that each sentence $c \to d$ corresponds to a sentence $\hat{c} \to \hat{d}$ in $\hat{\mathfrak{S}}^*$ or that each sentence $\hat{c} \to \hat{d}$ is provable from $\hat{\mathfrak{A}}^*$.

Because $\hat{c} \to \hat{d}$ is provable from $\hat{\mathfrak{A}}$, there is a series that belongs to $\hat{\mathfrak{A}}$

$$\hat{c} \to \alpha_1, \ldots, \alpha_\rho \to \hat{d}.$$

Now if a contained sentence $\alpha_\lambda \to \alpha_{\lambda+1}$ did not belong to $\hat{\mathfrak{A}}^*$, then it would have to belong to $\hat{\mathfrak{B}}$, it would therefore have to have the form $\hat{a} \to \hat{b}$ or $\hat{a} \to \mathfrak{a}$ or $\mathfrak{a} \to \hat{b}$.

In the first case we substitute the sentence with sentences that belong to $\hat{\mathfrak{B}}^*$, and thus to $\hat{\mathfrak{A}}^*$. In the second case $\alpha_{\lambda+1}$ cannot be the last element because it is ideal. Therefore, it has to follow a sentence $\alpha_{\lambda+1} \to \alpha_{\lambda+2}$. In this connection $\alpha_{\lambda+2}$ has to be real, because $\hat{\mathfrak{A}}$ is of the first degree. So the sentence $\alpha_{\lambda+1} \to \alpha_{\lambda+2}$ has the form $\mathfrak{a} \to \hat{y}$. If y belonged to B, one could substitute the pair of sentences $\alpha_\lambda \to \alpha_{\lambda+1},\ \alpha_{\lambda+1} \to \alpha_{\lambda+2}$ with a sentence of the form $\hat{a} \to \hat{b}$ or with a sentences of $\hat{\mathfrak{A}}^*$. But if y did not belong to B, according to Proposition 64, there would have to be a sentence $b_m \to y$ in $\mathfrak{S}$, thus, there would have to be a series

$$\hat{b}_m \to \beta_1,\ \beta_1 \to \beta_2, \ldots, \beta_\rho \to \hat{y}$$

which belongs to $\hat{\mathfrak{A}}$. In this case we substitute $\alpha_\lambda \to \alpha_{\lambda+1},\ \alpha_{\lambda+1} \to \alpha_{\lambda+2}$ with the series

$$\hat{a} \to \overline{\mathfrak{a}},\ \overline{\mathfrak{a}} \to \hat{b}_m,\ \hat{b}_m \to \beta_1,\ \beta_1 \to \beta_2, \ldots, \beta_\rho \to \hat{y},$$

or with the series $\hat{a} \to \hat{b}_m,\ \hat{b}_m \to \beta_1,\ \beta_1 \to \beta_2, \ldots, \beta_\rho \to \hat{y}$, where the first two sentences and the first sentence respectively belong to $\hat{\mathfrak{B}}^*$, thus also to $\hat{\mathfrak{A}}^*$. The other sentences, however, belong to $\hat{\mathfrak{A}}$. The procedure works the same way, if the sentence $\alpha_\lambda \to \alpha_{\lambda+1}$ has the form $\mathfrak{a} \to \beta$.

It is therefore shown: for each sentence or each pair of sentences in our series that does not belong to $\hat{\mathfrak{A}}^*$, we can introduce solely a sentence that belongs to $\hat{\mathfrak{A}}^*$ or a pair of sentences $\hat{a} \to \mathfrak{a},\ \mathfrak{a} \to \hat{b}$ that belongs to $\hat{\mathfrak{A}}^*$, in connection with further sentences in $\hat{\mathfrak{A}}$. We want to repeat this procedure. It can only terminate if the series contains only sentences of $\hat{\mathfrak{A}}^*$. Now we want to assume that this procedure never terminated. From what was formerly said one realizes that this is only the case if we continuously introduce new sentences of the form $\hat{a} \to \hat{b}$, meaning pairs of sentences of the form $\hat{a} \to \overline{\mathfrak{a}},\ \overline{\mathfrak{a}} \to \hat{b}$, into our series $\mathfrak{R}$. It has to happen then that our series contains two identical sentences or pairs of sentences, for instance $\hat{a}_i \to \mathfrak{a},\ \mathfrak{a} \to \hat{b}_k$. Then by Proposition 74 the sentences $a_i \to b_k$, $b_k \to a_i$ would exist in $\mathfrak{S}$, thus $\mathfrak{S}$ would contain a net.

From this we can realize that our assumption was wrong. The series has to terminate, i.e. $\hat{c} \to \hat{d}$ has to be provable from $\hat{\mathfrak{A}}^*$.

Second proof.[4] For each sentence $c \to d$ there is a series of sentences $c \to \alpha_1, \ldots, \alpha_\rho \to d$ that belongs to $\mathfrak{A}$. Each series belongs to one and only one maximal net, according to Proposition 63. If $\alpha_\iota \to \alpha_{\iota+1}$ belongs to a maximal pair of groups that is different from $(A,\ B)$, then by Proposition 66, $\hat{\alpha}_\iota \to \hat{\alpha}_{\iota+1}$ belongs either to $\hat{\mathfrak{A}}$ and by Proposition 61 not to $\hat{\mathfrak{B}}$, so it does belong to $\hat{\mathfrak{A}}^*$, or it can be substituted by two sentences of $\hat{\mathfrak{A}}$ that, according to Proposition 61 or 71, do not belong to $\hat{\mathfrak{B}}$, such that they also belong to $\hat{\mathfrak{A}}^*$. If $\alpha_\kappa \to \alpha_{\kappa+1}$ belongs to $(A,\ B)$, then $\hat{\alpha}_\kappa \to \hat{\alpha}_{\kappa+1}$ belongs to $\hat{\mathfrak{B}}^*$ and such also to $\hat{\mathfrak{A}}^*$, or it is substitutable by two such sentences.

77. (Definition.) (Cf. Definition 57.) We call a pair of element complexes A, B a *small complex* pair, if each complex contains only one element, a *medium complex* pair, if A and B both contain two elements, and a *large complex* pair, if it is neither small, nor medium.

78. (Proposition.) For each complex pair with a number of sentences m and n there is always a connecting system with $m+n$ sentences.

PROOF. In fact it is a centralized connecting system (see Definition 69).

[4] Additional remark after proof-reading.

79. (Proposition.) A non-centralized connecting system $\mathfrak{B}$ for a large complex pair with m, n as number of elements always contains more than $m+n$ sentences.

PROOF. 1. $\mathfrak{B}$ is disparate (see Definition 68).

2. $\mathfrak{B}$ is not disparate and not centralized.

ad 1. Then the number of sentences is $m \cdot n$. However, if $m > n$, it is $m+n < 2m \leq m \cdot n$ and $m+n = 2m < n \cdot m$, if $m = n$.

ad 2. It is sufficient to show that the number of sentences can be reduced. Let $\mathfrak{v}'$, $\mathfrak{v}''$ be two connecting elements and let $a' \to \mathfrak{v}'$, $\mathfrak{v}' \to b'$ and $a'' \to \mathfrak{v}''$, $\mathfrak{v}'' \to b''$ be two corresponding pairs of sentences of $\mathfrak{B}$, where $a' \not\equiv a''$ and $b' \not\equiv b''$. Now we let $\mathfrak{v}'$ and $\mathfrak{v}''$ coincide and we can either drop a sentence that existed formerly between a' and b'', or, in case we already had the sentences $a' \to \mathfrak{v}'''$, $\mathfrak{v}''' \to \mathfrak{v}''$, we can have $\mathfrak{v}'''$ coincide with $\mathfrak{v}'$ and $\mathfrak{v}''$, which also leads to a reduction. One can reason similarly, if $a' \not\equiv a''$, $b' \equiv b''$ or $a' \equiv a''$, $b' \not\equiv b''$ or $a' \equiv a''$, $b' \equiv b''$. If there exists only one connecting element, and either only one antecedens element or only one succedens element corresponding to it, then the connecting element can be left out. If there is only one connecting element, and more than one antecedens element and more than one succedens element corresponding to it, and if x is an element of A or B respectively, that is not connected to this element, then adding the sentence $x \to \mathfrak{v}$ or $\mathfrak{v} \to x$ respectively, allows us a reduction.

80. (Proposition.) The number of sentences in a connecting system for a medium complex pair is 4, and 4 only then, if the connecting system is disparate or centralized.

81. (Proposition.) The number of sentences for a complex pair with $(m, 1)$ or $(1, m)$ number of elements is at least m, and m only then, if the connecting system is disparate.

82. (Definition.) For a closed system of sentences $\mathfrak{S}$ without a net, an independent axiomatic system of the first degree is called an independent minimal axiomatic system of the first degree, if there is no other axiomatic system for $\mathfrak{S}$ of the first degree which contains fewer sentences.

83. (Definition.) For a simple system of sentences $\mathfrak{S}$ without a net we define *canonical system* to be a system of sentences that consists of connecting systems of the maximal pairs of groups of $\mathfrak{S}$, which in fact contains for each large maximal pair of groups of $\mathfrak{S}$ a centralized connecting system, for each small maximal pair of groups of $\mathfrak{S}$ a disparate connecting system and for each medium maximal pair of groups of $\mathfrak{S}$ either a centralized or a disparate connecting system.

84. (Proposition.) For each closed system of sentences $\mathfrak{S}$ there is a minimal system of the first degree.

PROOF. There is an axiomatic system of the first degree, in fact the axiomatic system $\mathfrak{A}$ of $\mathfrak{S}$ itself. Therefore there also must be an independent axiomatic system of the first degree with the lowest number of sentences.

85. (Proposition.) Each minimal system of the first degree of a simple system of sentences $\mathfrak{S}$ is a canonical system.

PROOF. Let $\hat{\mathfrak{A}}$ be a minimal system of the first degree to $\mathfrak{S}$. By Proposition 70, each maximal pair of groups of $\mathfrak{S}$ is connected by a connecting system. Now if a connecting system of a maximal pair of groups would not satisfy the requirements of Proposition 83, then it could be substituted by a different one. In that way, by Propositions 79 to 81, the number of sentences would decrease, but according to Proposition 76, the system could not cease to be an axiomatic system. Thus, $\hat{\mathfrak{A}}$ could not be a minimal system. Therefore $\hat{\mathfrak{A}}$ contains a connecting system for each maximal pair of groups which fulfills the requirements of Definition 83. According to Proposition 56, however, all axioms of $\mathfrak{S}$ are already derivable from the sentences of these connecting systems. The axiomatic system $\hat{\mathfrak{A}}$ of the first degree can therefore not contain any more sentences, because it is supposed to be a minimal system.

86. (Proposition.) Every canonical system of sentences is a minimal axiomatic system of the first degree.

PROOF. We assume an arbitrary minimal axiomatic system of the first degree. According to Proposition 85, the canonical system at hand can differ from it only by the connecting system of the medium maximal pairs of groups. If one substitutes these connecting systems in the minimal system of the first degree in this sense, the number of sentences does not change and by Proposition 76 we get an axiomatic system of the first degree which has the lowest number of sentences. It is therefore independent and a minimal axiomatic system of the first degree.

Göttingen, September 15 1921

Acknowledgements Javier Legris thanks Peter Arndt and Arnold Koslow for their support.

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http://www.springer.com/978-3-0346-0144-3

Universal Logic: An Anthology
From Paul Hertz to Dov Gabbay
Béziau, J.-Y. (Ed.)
2012, XVIII, 410 p. 155 illus., Softcover
ISBN: 978-3-0346-0144-3
A product of Birkhäuser Basel