Preface

The unit group $\mathscr{U}(FG)$ of the group ring FG is an important object of study. It is interesting to explore the conditions under which this group satisfies various identities. Early results of this type appear in Sehgal's classic book [94]. My purpose here is to pick up where that book leaves off and present the fascinating results that have appeared, for the most part, since the mid nineties.

A group G (or a subset S of G) is said to satisfy a group identity if there is a nontrivial reduced word $w(x_1,\dots,x_n)$ in the free group $\langle x_1,x_2,\dots\rangle$ such that $w(g_1,\dots,g_n)=1$ for all $g_i \in G$ (or in S, as the case may be). A series of papers by Giambruno et al. [31, 38, 40], Liu [75], Liu and Passman [76] and Passman [84] presented the conditions under which $\mathscr{U}(FG)$ satisfies a group identity.

But we can also consider the involution $*$ on FG given by

$$\left(\sum_{g\in G} \alpha_g g\right)^* = \sum_{g\in G} \alpha_g g^{-1}.$$

Denote by $\mathscr{U}^+(FG)$ the set of symmetric units; that is, $\mathscr{U}^+(FG) = \{\alpha \in \mathscr{U}(FG) : \alpha^* = \alpha\}$. There have been quite a few papers since the late nineties devoted to discovering the extent to which the symmetric units determine the group structure of $\mathscr{U}(FG)$. In particular, Giambruno et al. [39] and Sehgal and Valenti [96] determined when $\mathscr{U}^+(FG)$ satisfies a group identity. Furthermore, a series of papers by Lee [67], Lee et al. [69] and Lee and Spinelli [73, 74] examined the conditions under which $\mathscr{U}^+(FG)$ satisfies specific group identities. The quaternion group Q_8 tends to provide a counterexample to every question asked, since it is easily verified that the symmetric elements of FQ_8 are central. It is, however, an engrossing problem to classify the counterexamples.

In an effort to maintain consistency, I always assume that our group ring is over a field, even though some of the results have been extended to suitable domains or other rings. Also, the involution on FG will always be the classical one described above, although I break this rule in Chapter 7 in order to mention a few recent results allowing more general involutions.

I assume that the reader has had an introductory graduate course on groups and rings, and so use standard results freely. Some prior exposure to the concept of a group ring (in particular, to the famous books of Passman [82] and Sehgal [94]) would be an asset, but it is not essential. To be sure, I am not attempting anything so ambitious as these books, where essentially, all of the major results concerning the ring structure of FG and the group structure of $\mathcal{U}(FG)$ were discussed. Given the explosion of work on group rings in recent years, such an attempt now would be all but impossible. (It would most definitely be impossible for this author!) Instead, this book is largely a continuation of the chapters *Lie Properties in KG* and *Units in Group Rings II* in [94]. I freely borrow theorems from [82] and [94], but the purloined results are all stated in full for the reader's convenience. Also, it will be clear to the reader that results labeled "Theorem" in this book are the main theorems on group identities and Lie identities of group rings. A number of major classical results are, therefore, presented as propositions.

This book contains seven chapters and an appendix. Here is a brief summary of their contents.

In Chapter 1, we discuss the conditions under which $\mathcal{U}(FG)$ satisfies a group identity. The first step is to establish Hartley's conjecture; namely, if $\mathcal{U}(FG)$ satisfies a group identity, and G is torsion, then FG satisfies a polynomial identity. After that, necessary and sufficient conditions are found, for both torsion and nontorsion groups G. When G is nontorsion, a suitable restriction must be imposed upon G modulo its torsion part for the sufficiency. A well-known conjecture due to Kaplansky states that if G is torsion-free, then the only units of FG are the trivial units, namely λg, with $0 \neq \lambda \in F$ and $g \in G$. It is certainly true if G is a u.p. (unique product) group, and we impose this as our restriction upon G modulo its torsion part.

Chapter 2 covers the same territory for the symmetric units. We find that if F is infinite and G is torsion, then FG satisfies a polynomial identity whenever $\mathcal{U}^+(FG)$ satisfies a group identity. Necessary and sufficient conditions are then presented for $\mathcal{U}^+(FG)$ to satisfy a group identity, for both torsion and nontorsion groups G.

Chapter 3 contains proofs of theorems concerning Lie identities satisfied by the set of symmetric elements, $(FG)^+$. (The corresponding results for all of FG are classical.) These results are essential to our later discussion, as there is an intimate connection between the Lie properties of FG (resp., $(FG)^+$), and corresponding group identities of $\mathcal{U}(FG)$ (resp., $\mathcal{U}^+(FG)$).

In Chapters 4, 5 and 6, we consider particular group identities. Chapter 4 contains classical results establishing when $\mathcal{U}(FG)$ is nilpotent and recent results for $\mathcal{U}^+(FG)$. Similarly, Chapters 5 and 6 contain results pertaining to the bounded Engel and solvability properties respectively, for both $\mathcal{U}(FG)$ and $\mathcal{U}^+(FG)$.

In Chapter 7, some related results are mentioned. In particular, we discuss some identities related to those mentioned earlier, and some recent results concerning involutions other than the classical one on FG.

The appendix contains some results concerning the central closure of a prime ring. These are needed in order to prove Proposition 2.2.2.

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