Chapter 2
Fundamentals of Fractional-order Systems

2.1 Fractional-order Operators: Definitions and Properties

2.1.1 Introduction

Essentially, the mathematical problem for defining fractional-order derivatives and integrals consists of the following [2,7]: to establish, for each function $f(z), z = x + \mathrm{j}y$ of a general enough class, and for each number α (rational, irrational or complex), a correspondence with a function $g(z) = \mathscr{D}_c^{\alpha} f(z)$ fulfilling the following conditions:

- If $f(z)$ is an analytic function of the variable z, the derivative $g(z) = \mathscr{D}_c^{\alpha} f(z)$ is an analytic function of z and α.
- The operation $\mathscr{D}_c^{\alpha}$ and the usual derivative of order $n \in \mathbb{Z}^+$, $\alpha = n$ give the same result.
- The operation $\mathscr{D}_c^{\alpha}$ and the usual n-fold integral with $n \in \mathbb{Z}^-$, $\alpha = -n$ give the same result.
- $\mathscr{D}_c^{\alpha} f(z)$ and its first $(n-1)$th-order derivatives must vanish to zero at $z = c$.
- The operator of order $\alpha = 0$ is the identity operator.
- The operator must be linear: $\mathscr{D}_c^{\alpha}\left[af(z) + bh(z)\right] = a\mathscr{D}_c^{\alpha} f(z) + b\mathscr{D}_c^{\alpha} h(z)$.
- For the fractional-order integrals of arbitrary order, $\Re(\alpha) > 0, \Re(\beta) > 0$, it holds the additive law of exponents (semigroup property): $\mathscr{D}_c^{\alpha}\mathscr{D}_c^{\beta} f(z) = \mathscr{D}_c^{\alpha+\beta} f(z)$.

In the following sections the reader can find some of the more usual definitions of the fractional-order operators, fulfilling the above conditions, following mainly [3,7].

2.1.2 Fractional-order Integrals

In agreement with Riemann–Liouville's conception, the notion of fractional-order integral of order $\Re(\alpha) > 0$ is a natural consequence of Cauchy's formula for repeated integrals, which reduces the computation of the primitive corresponding to the n-fold integral of a function $f(t)$ to a simple convolution. This formula can be expressed as

$$\mathscr{I}_c^n f(t) \triangleq \mathscr{D}_c^{-n} f(t) = \frac{1}{(n-1)!} \int_c^t (t-\tau)^{n-1} f(\tau) \mathrm{d}\tau, \quad t > c, \ n \in \mathbb{Z}^+. \quad (2.1)$$

In (2.1) we can see that $\mathscr{I}_c^n f(t)$ and its derivatives of orders $1, 2, 3, \cdots, n$–1 become zero for $t = c$.

In a natural way, we can extend the validity of (2.1) to $n \in \mathbb{R}^+$. Taking into account that $(n-1)! = \Gamma(n)$, and introducing the positive real number α, the Riemann–Liouville fractional-order integral is defined as

$$\mathscr{I}_c^\alpha f(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_c^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \quad t > c, \ \alpha \in \mathbb{R}^+. \quad (2.2)$$

It can be proved that this operator fulfils the aforementioned conditions.

When we deal with dynamic systems it is usual that $f(t)$ be a causal function of t, and so in what follows the definition for the fractional-order integral to be used is

$$\mathscr{I}^\alpha f(t) \triangleq \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) \mathrm{d}\tau, \quad t > 0, \ \alpha \in \mathbb{R}^+. \quad (2.3)$$

In (2.3) we can see that the fractional-order integral can be expressed as a causal convolution of the form

$$\mathscr{I}^\alpha f(t) = \Phi_\alpha(t) * f(t), \quad \alpha \in \mathbb{R}^+, \quad (2.4)$$

with

$$\Phi_\alpha(t) = \frac{t_+^{\alpha-1}}{\Gamma(\alpha)}, \quad \alpha \in \mathbb{R}^+ \quad (2.5)$$

the causal kernel of the convolution and

$$t_+^{\alpha-1} = 0, \ \text{for} \ \ t < 0; \quad t_+^{\alpha-1} = t^{\alpha-1}, \ \text{for} \ \ t \geqslant 0. \quad (2.6)$$

2.1.3 Fractional-order Derivatives

The definition (2.3) cannot be used for the fractional-order derivative by direct substitution of α by $-\alpha$, because we have to proceed carefully in order to guarantee the convergence of the integrals involved in the definition, and to preserve the properties of the ordinary derivative of integer-order.

Denoting the derivative operator of order $n \in \mathbb{N}$ by $\mathscr{D}^n$, and the identity operator by $\mathbb{I}$, we can verify that

$$\mathscr{D}^n \mathscr{I}^n = \mathbb{I}, \quad \mathscr{I}^n \mathscr{D}^n \neq \mathbb{I}, \quad n \in \mathbb{N}. \tag{2.7}$$

In other words, the operator $\mathscr{D}^n$ is only a left-inverse of the operator $\mathscr{I}^n$. In fact, from (2.1) we can deduce that

$$\mathscr{I}^n \mathscr{D}^n f(t) = f(t) - \sum_{k=0}^{n-1} f^{(k)}(0^+)\frac{t^k}{k!}, \quad t > 0, \tag{2.8}$$

where $f^{(k)}(\cdot)$ is the kth-order derivative of the function $f(\cdot)$. Consequently, it must be verified whether $\mathscr{D}^\alpha$ is a left-inverse of $\mathscr{I}^\alpha$ or not. For this purpose, introducing the positive integer m so that $m-1 < \alpha < m$, the Riemann–Liouville definition for the fractional-order derivative of order $\alpha \in \mathbb{R}^+$ has the following form:

$${}_{\mathrm{R}}\mathscr{D}^\alpha f(t) \triangleq \mathscr{D}^m \mathscr{I}^{m-\alpha} f(t) = \frac{\mathrm{d}^m}{\mathrm{d}t^m}\left[\frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau\right], \tag{2.9}$$

where $m-1 < \alpha < m,\ m \in \mathbb{N}$.

An alternative definition for the fractional-order derivative was introduced by Caputo as

$${}_{\mathrm{C}}\mathscr{D}^\alpha f(t) \triangleq \mathscr{I}^{m-\alpha} \mathscr{D}^m f(t) = \frac{1}{\Gamma(m-\alpha)}\int_0^t \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}}\mathrm{d}\tau, \tag{2.10}$$

where $m-1 < \alpha < m,\ m \in \mathbb{N}$.

This definition is more restrictive than the Riemann–Liouville one because it requires the absolute integrability of the mth-order derivative of the function $f(t)$. It is clear that, in general,

$${}_{\mathrm{R}}\mathscr{D}^\alpha f(t) \triangleq \mathscr{D}^m \mathscr{I}^{m-\alpha} f(t) \neq \mathscr{I}^{m-\alpha} \mathscr{D}^m f(t) \triangleq_{\mathrm{C}} \mathscr{D}^\alpha f(t), \tag{2.11}$$

except in the case of being zero at $t = 0^+$ for the function $f(t)$ and its first $(m-1)$th-order derivatives. In fact, between the two definitions there are the following relations:

$${}_{\mathrm{R}}\mathscr{D}^\alpha f(t) = {}_{\mathrm{C}}\mathscr{D}^\alpha f(t) + \sum_{k=0}^{m-1} \frac{t^{k-\alpha}}{\Gamma(k-\alpha+1)} f^{(k)}\left(0^+\right), \tag{2.12}$$

$${}_{\mathrm{R}}\mathscr{D}^\alpha \left(f(t) - \sum_{k=0}^{m-1} f^{(k)}\left(0^+\right)\frac{t^k}{k!}\right) = {}_{\mathrm{C}}\mathscr{D}^\alpha f(t). \tag{2.13}$$

Due to its importance in applications, we will consider here the Grünwald–Letnikov's definition, based on the generalization of the backward difference. This definition has the form

$$\mathscr{D}^\alpha f(t)\big|_{t=kh} = \lim_{h\to 0}\frac{1}{h^\alpha}\sum_{j=0}^{k}(-1)^j\binom{\alpha}{j} f(kh-jh). \tag{2.14}$$

An alternative definition of the Grünwald–Letnikov's derivative in integral form is [3]

$$ {}_{\mathrm{L}}\mathscr{D}^{\alpha} f(t) = \sum_{k=0}^{m} \frac{f^{(k)}(0^+)t^{k-\alpha}}{\Gamma(m+1-\alpha)} + \frac{1}{\Gamma(m+1-\alpha)} \int_0^t (t-\tau)^{m-\alpha} f^{(m+1)}(\tau)\mathrm{d}\tau, \tag{2.15} $$

where $m > \alpha - 1$.

2.1.4 Laplace and Fourier Transforms

Laplace and Fourier integral transforms are fundamental tools in systems and control engineering. For this reason, we will give here the equation of these transforms for the defined fractional-order operators. These equations are

$$ \mathscr{L}\left[\mathscr{I}^{\alpha} f(t)\right] = s^{-\alpha} F(s), \tag{2.16} $$

$$ \mathscr{L}\left[{}_{\mathrm{R}}\mathscr{D}^{\alpha} f(t)\right] = s^{\alpha} F(s) - \sum_{k=0}^{m-1} s^{k} \left[{}_{\mathrm{R}}\mathscr{D}^{\alpha-k-1} f(t)\right]_{t=0}, \tag{2.17} $$

$$ \mathscr{L}\left[{}_{\mathrm{C}}\mathscr{D}^{\alpha} f(t)\right] = s^{\alpha} F(s) - \sum_{k=0}^{m-1} s^{\alpha-k-1} f^{(k)}(0), \tag{2.18} $$

$$ \mathscr{L}\left[{}_{\mathrm{L}}\mathscr{D}^{\alpha} f(t)\right] = s^{\alpha} F(s), \tag{2.19} $$

$$ \mathscr{F}\left[\mathscr{I}^{\alpha} f(t)\right] = \mathscr{F}\left[\frac{t_{+}^{\alpha-1}}{\Gamma(\alpha)}\right] \mathscr{F}\left\{f(t)\right\} = (\mathrm{j}\omega)^{-\alpha} F(\omega), \tag{2.20} $$

$$ \mathscr{F}\left[\mathscr{D}^{\alpha} f(t)\right] = \mathscr{F}\left\{\mathscr{D}^{m} \mathscr{I}^{m-\alpha} f(t)\right\} = (\mathrm{j}\omega)^{\alpha} F(\omega), \tag{2.21} $$

where $(m - 1 \leqslant \alpha < m)$. More on Laplace transform can be found in the Appendix.

2.2 Fractional-order Differential Equations

Once the basic definitions of the fractional calculus have been established, and as a prelude for the study of fractional-order linear time invariant systems, we will briefly review in this section the fundamentals of fractional-order ordinary differential equations. We will start with the two-term equations (relaxation and oscillation equations) and continue with the general equations for the solutions of n-term equations. A detailed study of fractional-order differential equations can be found in [3, 8].

2.2.1 Relaxation and Oscillation Equations

It is known that the classical problems of relaxation and oscillation are described by linear ordinary differential equations of orders 1 and 2, respectively (for control community, normal relaxation is equivalent to first-order dynamics). We can generalize the equations

$$\mathscr{D}u(t) + u(t) = q(t) \tag{2.22}$$

and

$$\mathscr{D}^2u(t) + u(t) = q(t) \tag{2.23}$$

by simply substituting the integer-order derivatives by the fractional order α. If we want to preserve the usual initial conditions, we will use Caputo's fractional-order derivatives for obtaining

$${}_{\mathrm{C}}\mathscr{D}^{\alpha}u(t)+u(t)={}_{\mathrm{R}}\mathscr{D}^{\alpha}\left[u(t)-\sum_{k=0}^{m-1}\frac{t^k}{k!}u^{(k)}\left(0^+\right)\right]+u(t)=q(t),\ t>0, \tag{2.24}$$

where m defined by $m-1<\alpha<m$ is a positive integer which determines the number of initial conditions $u^{(k)}(0^+) = b_k, k = 0, 1, 2, \cdots, m-1$. It is obvious that in the case of $\alpha = m$, (2.24) becomes an ordinary differential equation, whose solution can be expressed as

$$u(t) = \sum_{k=0}^{m-1} b_k u_k(t) + \int_0^t q(t-\tau)u_\delta(\tau)\mathrm{d}\tau, \tag{2.25}$$

$$u_k(t) = \mathscr{I}^k u_0(t), \quad u_k^{(h)}(0^+) = \delta_{kh}, \quad u_\delta(t) = -\mathscr{D}u_0(t), \tag{2.26}$$

for $h, k = 0, 1, \cdots, m-1$, the m functions $u_k(t)$ are the fundamental solutions of the homogeneous differential equation, and the function $u_\delta(t)$ is the impulse response (the particular solution for $q(t) = \delta(t)$ under zero initial conditions).

It can be proved [3, 8] that the solution of (2.24) can be expressed in the same form as

$$u(t) = \sum_{k=0}^{m-1} b_k \mathscr{I}^k \mathscr{E}_\alpha(-t^\alpha) - \int_0^t q(t-\tau)\mathscr{E}'_\alpha(-\tau^\alpha)\mathrm{d}\tau, \tag{2.27}$$

where $\mathscr{E}_\alpha(-t^\alpha)$ is the *Mittag–Leffler function* defined by [3, 9, 10]

$$\mathscr{E}_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + 1)}. \tag{2.28}$$

Comparing (2.25) and (2.27) we can see that:

- The Mittag–Leffler function in (2.27) has the role of the exponential function in (2.25).

- When α is non-integer, $m-1 < \alpha < m$, $m-1$ is the integer part of α, $\left(m-1 \triangleq [\alpha]\right)$ and m the number of initial conditions for uniqueness of the solution, $u(t)$.
- The m functions $\mathscr{I}^k \mathscr{E}_\alpha(-t^\alpha)$, with $k = 0, 1, \cdots, m-1$ are the particular solutions of the homogeneous equation which satisfy the initial conditions, *i.e.*, the fundamental solutions of the homogenous equation.
- The function $\mathscr{E}'_\alpha(-t^\alpha)$, the first-order derivative of the function $\mathscr{E}_\alpha(-t^\alpha)$, is the impulse response.

It is clear that the form of the solutions is given by the properties of the Mittag–Leffler function. Figures 2.1 and 2.2 give the curves of the function for different values of α. As we can see, the behavior corresponds to an anomalous relaxation) (non-standard first-order decay) for $\alpha < 1$, is exponential for $\alpha = 1$, becomes a damped oscillation for $1 < \alpha < 2$, and oscillates for $\alpha = 2$.

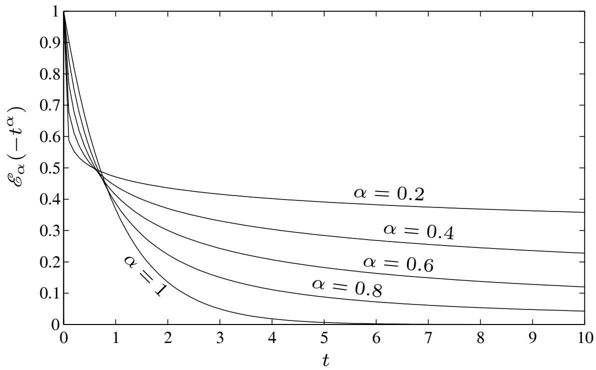

Figure 2.1 Mittag–Leffler functions $\mathscr{E}_\alpha(-t^\alpha)$ for $\alpha = 0.2, 0.4, 0.6, 0.8, 1$

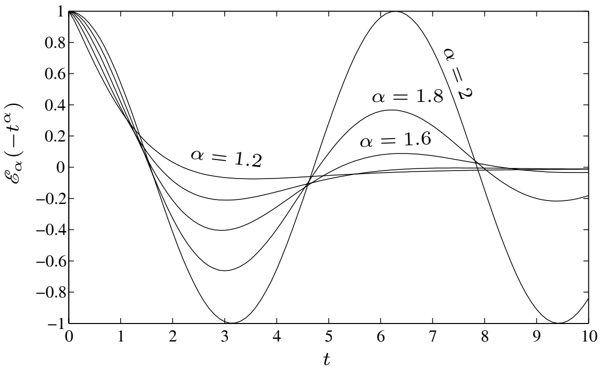

Figure 2.2 Mittag–Leffler functions $\mathscr{E}_\alpha(-t^\alpha)$ for $\alpha = 1.2, 1.4, 1.6, 1.8, 2$

For the general two-term equation with zero initial conditions

$$a\mathscr{D}^{\alpha}u(t) + bu(t) = q(t), \tag{2.29}$$

we can obtain the solution by applying the Laplace transform method. So, in the Laplace domain, the solution can be expressed as

$$U(s) = Q(s)\frac{1/a}{s^{\alpha} + b/a}, \tag{2.30}$$

and in the time domain as

$$u(t) = q(t) * \frac{1}{a}t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}\left(-\frac{b}{a}t^{\alpha}\right), \tag{2.31}$$

where $t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}\left(-bt^{\alpha}/a\right)/a$ is the impulse response, and $\mathscr{E}_{\alpha,\alpha}\left(-bt^{\alpha}/a\right)$ is the so-called *Mittag–Leffler function in two parameters* defined as [3]

$$\mathscr{E}_{\alpha,\beta}(z) = \sum_{k=0}^{\infty}\frac{z^k}{\Gamma\left(\alpha k + \beta\right)}, \quad \Re(\alpha) > 0,\ \Re(\beta) > 0. \tag{2.32}$$

2.2.2 Numerical Solutions

In order to obtain a numerical solution for the fractional-order differential equations, we can make use of the Grünwald–Letnikov's definition, and approximate

$$\mathscr{D}^{\alpha}f(t) \approx \Delta_h^{\alpha}f(t), \tag{2.33}$$

$$\Delta_h^{\alpha}f(t)\Big|_{t=kh} = h^{-\alpha}\sum_{j=0}^{k}(-1)^j\binom{\alpha}{j}f(kh - jh). \tag{2.34}$$

So, for the two-term equation (2.29) with $a = 1$ and zero initial conditions, this approximation leads to

$$h^{-\alpha}\sum_{j=0}^{k}w_j^{(\alpha)}y_{k-j} + by_k = q_k, \tag{2.35}$$

where $t_k = kh,\ y_k = y\left(t_k\right),\ y_0 = 0,\ q_k = q(t_k),\ k = 0, 1, 2, \cdots$, and

$$w_j^{(\alpha)} = (-1)^j\binom{\alpha}{j}, \tag{2.36}$$

for $j = 0, 1, 2, \cdots$, and the algorithm to obtain the numerical solution will be

$$y_i = 0, \quad i = 1, 2, \cdots, n - 1, \tag{2.37}$$

$$y_k = -bh^{\alpha}y_{k-1} - \sum_{j=1}^{k}w_j^{(\alpha)}y_{k-j} + h^{\alpha}q_k, \quad k = n, n + 1, \cdots. \tag{2.38}$$

As we can see in former equations, as t grows, we need more and more terms to add for computing the solution, in other words, we need unlimited memory. To solve this problem, the *short memory principle* was proposed [11]. This principle is based on the observation that for large t the coefficients of the Grünwald–Letnikov's definition corresponding to values of the function near $t = 0$ (or any other point considered as initial) have little influence in the solution. This fact allows us to approximate the numerical solution by using the information of the "recent past," in other words, the interval $[t - L, t]$, L being the length of memory, a moving low limit to compute the derivatives. So, we will use

$$\mathscr{D}^{\alpha} f(t) \approx {}_{t-L}\mathscr{D}^{\alpha} f(t), \quad t > L, \tag{2.39}$$

and the number of terms to add is limited by the value of L/h. The error of the approximation when $| f(t) | \leqslant M, \ (0 < t \leqslant t_1)$ is bounded by

$$\varepsilon(t) = |\mathscr{D}^{\alpha} f(t) - {}_{t-L}\mathscr{D}^{\alpha} f(t)| \leqslant \frac{ML^{-\alpha}}{|\Gamma(1-\alpha)|}, \quad L \leqslant t \leqslant t_1, \tag{2.40}$$

which can be used to determine the necessary memory length, L, to obtain a certain error bound, as

$$\varepsilon(t) < \epsilon, \quad L \leqslant t \leqslant t_1 \Rightarrow L \geqslant \left(\frac{M}{\epsilon\,|\Gamma(1-\alpha)|} \right)^{1/\alpha}. \tag{2.41}$$

For the computation of the coefficients to obtain the numerical solution, several methods can be used. For the case of a fixed value of derivative order α, we can use the following recursive formula:

$$w_0^{(\alpha)} = 1; \quad w_k^{(\alpha)} = \left(1 - \frac{\alpha+1}{k}\right) w_{k-1}^{(\alpha)}, \quad k = 1, 2, \cdots. \tag{2.42}$$

For a non-fixed α (for example, if we need to identify a system and α is a parameter to be determined) it is more convenient to use the fast Fourier transform (FFT). In such a case, it should be noted that the coefficients can be considered as the coefficients of the series expansion for the function $(1-z)^{\alpha}$

$$(1-z)^{\alpha} = \sum_{k=0}^{\infty} (-1)^k \binom{\alpha}{k} z^k = \sum_{k=0}^{\infty} w_k^{(\alpha)} z^k, \tag{2.43}$$

and with $z = \mathrm{e}^{-\mathrm{j}\omega}$

$$\left(1 - \mathrm{e}^{-\mathrm{j}\omega}\right)^{\alpha} = \sum_{k=0}^{\infty} w_k^{(\alpha)} \mathrm{e}^{-\mathrm{j}k\omega}, \tag{2.44}$$

we can express the coefficients in terms of the inverse Fourier transform

$$w_k^{(\alpha)} = \frac{1}{2\pi} \int_0^{2\pi} \left(1 - \mathrm{e}^{-\mathrm{j}\omega}\right)^{\alpha} \mathrm{e}^{\mathrm{j}k\omega} \mathrm{d}\omega \tag{2.45}$$

that can be computed by using FFT algorithms.

2.3 Fractional-order Systems

After establishing the fundamental definitions of the fractional calculus in the previous sections and determining the kind of solutions of the differential equations of fractional order, this section will deal with the analysis of the systems described by such equations. This analysis, as is usual for the systems of integer-order in the classical control theory framework, will start with the input-output models or representations of these systems in different domains (*e.g.*, time, Laplace, and Z) to study their performance, both transient and steady state, discussing the conditions and criteria for stability, and determining the steady-state error coefficients.

2.3.1 Models and Representations

The equations for a continuous-time dynamic system of fractional-order can be written as follows:

$$H\left(\mathscr{D}^{\alpha_0\alpha_1\alpha_2\cdots\alpha_m}\right)(y_1, y_2, \cdots, y_l) = G\left(\mathscr{D}^{\beta_0\beta_1\beta_2\cdots\beta_n}\right)(u_1, u_2, \cdots, u_k), \quad (2.46)$$

where y_i, u_i are functions of time and $H(\cdot)$,$G(\cdot)$ are the combination laws of the fractional-order derivative operator. For the linear time-invariant single-variable case, the following equation would be obtained:

$$H\left(\mathscr{D}^{\alpha_0\alpha_1\alpha_2\cdots\alpha_n}\right) y(t) = G\left(\mathscr{D}^{\beta_0\beta_1\beta_2\cdots\beta_m}\right) u(t), \quad (2.47)$$

with

$$H\left(\mathscr{D}^{\alpha_0\alpha_1\alpha_2\cdots\alpha_n}\right) = \sum_{k=0}^{n} a_k \mathscr{D}^{\alpha_k}; \quad G\left(\mathscr{D}^{\beta_0\beta_1\beta_2\cdots\beta_m}\right) = \sum_{k=0}^{m} b_k \mathscr{D}^{\beta_k}, \quad (2.48)$$

where $a_k, b_k \in \mathbb{R}$. Or, explicitly,

$$\begin{aligned} &a_n \mathscr{D}^{\alpha_n} y(t) + a_{n-1} \mathscr{D}^{\alpha_{n-1}} y(t) + \cdots + a_0 \mathscr{D}^{\alpha_0} y(t) \\ &\quad = b_m \mathscr{D}^{\beta_m} u(t) + b_{m-1} \mathscr{D}^{\beta_{m-1}} u(t) + \cdots + b_0 \mathscr{D}^{\beta_0} u(t). \end{aligned} \quad (2.49)$$

If in the previous equation all the orders of derivation are integer multiples of a base order, α, that is, $\alpha_k, \beta_k = k\alpha,\ \alpha \in \mathbb{R}^+$, the system will be of *commensurate-order*, and (2.49) becomes

$$\sum_{k=0}^{n} a_k \mathscr{D}^{k\alpha} y(t) = \sum_{k=0}^{m} b_k \mathscr{D}^{k\alpha} u(t). \quad (2.50)$$

If in (2.50) $\alpha = 1/q,\ q \in \mathbb{Z}^+$, the system will be of *rational-order*.

This way, linear time-invariant systems can be classified as follows:

$$\text{LTI Systems}\begin{cases}\text{Non-integer}\begin{cases}\text{Commensurate}\begin{cases}\text{Rational}\\ \text{Irrational}\end{cases}\\ \text{Non-commensurate}\end{cases}\\ \text{Integer}\end{cases}$$

In the case of discrete-time systems (or discrete equivalents of continuous-time systems) we can use (2.33) and (2.34) to obtain models of the form

$$\begin{aligned}&a_n\Delta_h^{\alpha_n}y(t)+a_{n-1}\Delta_h^{\alpha_{n-1}}y(t)+\cdots+a_0\Delta_h^{\alpha_0}y(t)\\&\quad=b_m\Delta_h^{\beta_m}u(t)+b_{m-1}\Delta_h^{\beta_{m-1}}u(t)+\cdots+b_0\Delta_h^{\beta_0}u(t).\end{aligned}\tag{2.51}$$

Applying the Laplace transform to (2.49) with zero initial conditions, or the Z transform to (2.51), the input-output representations of fractional-order systems can be obtained. In the case of continuous models, a fractional-order system will be given by a transfer function of the form

$$G(s)=\frac{Y(s)}{U(s)}=\frac{b_ms^{\beta_m}+b_{m-1}s^{\beta_{m-1}}+\cdots+b_0s^{\beta_0}}{a_ns^{\alpha_n}+a_{n-1}s^{\alpha_{n-1}}+\cdots+a_0s^{\alpha_0}}\,.\tag{2.52}$$

In the case of discrete-time systems, the discrete-time transfer function will be of the form

$$G(z)=\frac{b_m\left(\omega\left(z^{-1}\right)\right)^{\beta_m}+b_{m-1}\left(\omega\left(z^{-1}\right)\right)^{\beta_{m-1}}+\cdots+b_0\left(\omega\left(z^{-1}\right)\right)^{\beta_0}}{a_n\left(\omega\left(z^{-1}\right)\right)^{\alpha_n}+a_{n-1}\left(\omega\left(z^{-1}\right)\right)^{\alpha_{n-1}}+\cdots+a_0\left(\omega\left(z^{-1}\right)\right)^{\alpha_0}},\tag{2.53}$$

where $\left(\omega\left(z^{-1}\right)\right)$ is the Z transform of the operator Δ_h^1, or, in other words, the discrete equivalent of Laplace's operator, s.

As can be seen in the previous equations, a fractional-order system has an irrational-order transfer function in Laplace's domain or a discrete transfer function of unlimited order in the Z domain, since only in the case of $\alpha_k\in\mathbb{Z}$ will there be a limited number of coefficients $(-1)^l\binom{\alpha_k}{l}$ different from zero. Because of this, it can be said that a fractional-order system has an unlimited memory or is infinite-dimensional, and obviously the systems of integer-order are just particular cases.

In the case of a commensurate-order system, the continuous-time transfer function is given by

$$G(s)=\frac{\sum\limits_{k=0}^{m}b_k(s^\alpha)^k}{\sum\limits_{k=0}^{n}a_k(s^\alpha)^k},\tag{2.54}$$

which can be considered as a pseudo-rational function, $H(\lambda)$, of the variable

$\lambda = s^\alpha$,

$$H(\lambda) = \frac{\sum\limits_{k=0}^{m} b_k \lambda^k}{\sum\limits_{k=0}^{n} a_k \lambda^k}. \tag{2.55}$$

2.3.2 Stability

2.3.2.1 Previous Considerations

In a general way, the study of the stability of fractional-order systems can be carried out by studying the solutions of the differential equations that characterize them. An alternative way is the study of the transfer function of the system (2.52). To carry out this study it is necessary to remember that a function of the type

$$a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + ... + a_0 s^{\alpha_0}, \tag{2.56}$$

with $\alpha_i \in \mathbb{R}^+$, is a multi-valued function of the complex variable s whose domain can be seen as a Riemann surface [12, 13] of a number of sheets which is finite only in the case of $\forall i,\ \alpha_i \in \mathbb{Q}^+$, being the principal sheet defined by $-\pi < \arg(s) < \pi$. In the case of $\alpha_i \in \mathbb{Q}^+$, that is, $\alpha = 1/q$, q being a positive integer, the q sheets of the Riemann surface are determined by

$$s = |s|\, \mathrm{e}^{\mathrm{j}\phi}, \quad (2k+1)\pi < \phi < (2k+3)\pi, \quad k = -1, 0, \cdots, q-2. \tag{2.57}$$

Correspondingly, the case of $k = -1$ is the *principal sheet*. For the mapping $w = s^\alpha$, these sheets become the regions of the plane w defined by

$$w = |w|\, \mathrm{e}^{\mathrm{j}\theta}, \quad \alpha(2k+1)\pi < \theta < \alpha(2k+3)\pi. \tag{2.58}$$

This mapping is illustrated in Figures 2.3 and 2.4 for the case of $w = s^{1/3}$. Figure 2.3 represents the Riemann surface that corresponds to the transformation introduced above, and Figure 2.4 represents the regions of the complex plane w that correspond to each sheet of the Riemann surface. These three sheets correspond to

$$k = \begin{cases} -1, & -\pi < \arg(s) < \pi, \text{ (the principal sheet)} \\ 0, & \pi < \arg(s) < 3\pi, \text{ (sheet 2)} \\ 1\ (= 3-2), & 3\pi < \arg(s) < 5\pi, \text{ (sheet 3)} \end{cases}$$

Thus, an equation of the type

$$a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0} = 0, \tag{2.59}$$

which in general is not a polynomial, will have an infinite number of roots, among which only a finite number of roots will be on the principal sheet of

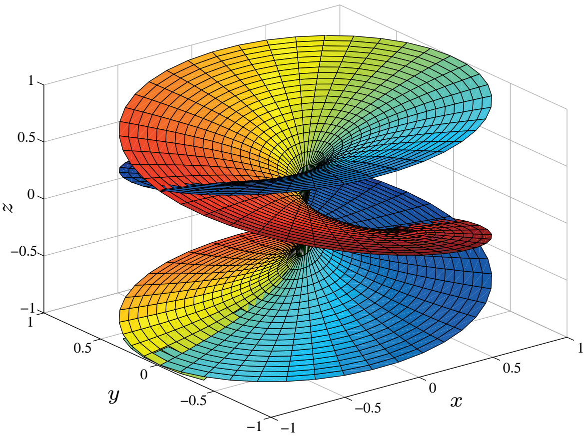

Figure 2.3 Riemann surface for $w = s^{1/3}$

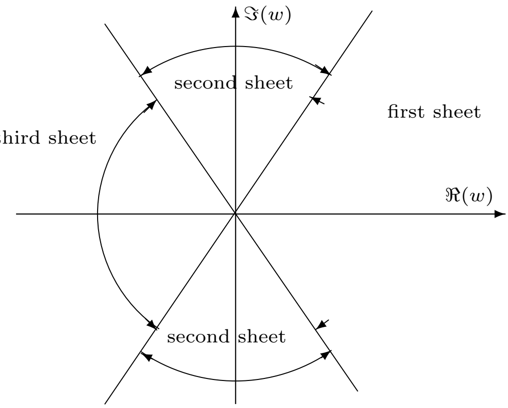

Figure 2.4 w-plane regions corresponding to the Riemann surface for $w = s^{1/3}$

the Riemann surface. It can be said that the roots which are in the secondary sheets of the Riemann surface are related to solutions that are always monotonically decreasing functions (they go to zero without oscillations when $t \to \infty$) and only the roots that are in the principal sheet of the Riemann surface are responsible for a different dynamics: damped oscillation, oscillation of constant amplitude, oscillation of increasing amplitude with monotonic growth.

For example, for equation

$$s^{\alpha} + b = 0, \tag{2.60}$$

the solutions are given by

$$s = (-b)^{1/\alpha} = |b|^{1/\alpha} \angle \frac{\arg(b) + 2l\pi}{\alpha}, \quad l = 0, \pm 1, \pm 2, \cdots ,$$

and only the roots satisfying the condition

$$\left| \frac{\arg(b) + 2l\pi}{\alpha} \right| < \pi$$

will be on the principal sheet.

This definition of the principal sheet, which assumes a cut along $\mathbb{R}^-$, corresponds to the Cauchy principal value of the integral corresponding to the inverse transformation of Laplace, that is, to that obtained by direct application of the residue theorem. The roots which are in this sheet are called *structural roots* [14] or *relevant roots* [8].

For example, for the function

$$f(s) = \frac{1}{s^\alpha + b}, \quad \alpha = \frac{1}{\pi}, \quad b \in \mathbb{R}^+,$$

the roots of the denominator are given by the equation

$$s_l = \left.(-b)^\pi\right|_l = |b|^\pi \angle\pi\,(\pi + 2l\pi)\,, \quad l = 0, \pm 1, \pm 2, \cdots .$$

For the roots to be in the principal sheet, they must fulfil the following condition:

$$|\arg(s_l)| < \pi \Longrightarrow |\pi\,(\pi + 2l\pi)| < \pi \Longrightarrow |\pi\,(1 + 2l)| < 1.$$

As can be seen, there is no value of l to fulfil this condition, so there are no structural roots of this function. This fact can be observed in Figure 2.5 for $b = 1$: the function is analytical for every s, $|\arg(s)| < \pi$ with a maximum at $s = 0 + \mathrm{j}0$, and the point $s^\alpha = -1$ is not a pole, but a branch point.

Studying the function for $\alpha = 4/\pi$, the condition becomes

$$\frac{|(\pi + 2l\pi)|}{4} < 1,$$

and it is fulfilled for $l = 0$ and $l = -1$, being the corresponding arguments $\angle s_0 = \pi^2/4$, and $\angle s_{-1} = -\pi^2/4$, respectively. In Figure 2.6 it can be observed that the function has poles at s_0 and s_{-1}.

2.3.2.2 Stability Conditions

After the previous considerations, the stability conditions of the fractional-order systems can be established.

In general, it can be said that a fractional-order system, with an irrational-order transfer function $G(s) = P(s)/Q(s)$, is bounded-input bounded-output stable (BIBO stable) if and only if the following condition is fulfilled (see [14] for more details):

$$\exists M, \quad |G(s)| \leqslant M, \quad \forall s \ \ \Re(s) \geqslant 0. \tag{2.61}$$

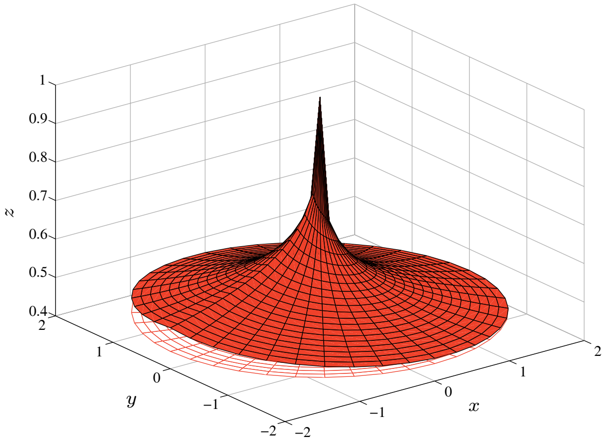

Figure 2.5 Magnitude of the function $f(s), \alpha = 1/\pi, b = 1$

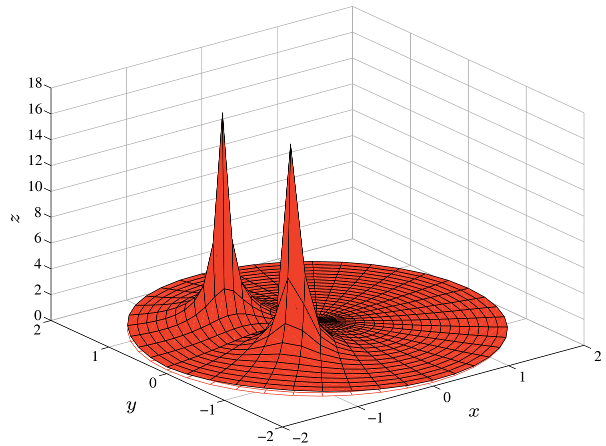

Figure 2.6 Magnitude of the function $f(s), \alpha = 4/\pi, b = 1$

The previous condition is satisfied if all the roots of $Q(s) = 0$ in the principal Riemann sheet, not being roots of $P(s) = 0$, have negative real parts.

For the case of commensurate-order systems, whose characteristic equation is a polynomial of the complex variable $\lambda = s^\alpha$, the stability condition is expressed as

$$|\arg(\lambda_i)| > \alpha\frac{\pi}{2}, \tag{2.62}$$

where λ_i are the roots of the characteristic polynomial in λ. For the particular case of $\alpha = 1$ the well known stability condition for linear time-invariant systems of integer-order is recovered:

$$|\arg(\lambda_i)| > \frac{\pi}{2}, \quad \forall\lambda_i / Q(\lambda_i) = 0. \tag{2.63}$$

An equivalent result was previously obtained by Mittag–Leffler in [9].

2.3.2.3 Stability Criteria

Nowadays there are no polynomial techniques, either Routh or Jury type, to analyze the stability of fractional-order systems. Only the geometrical techniques of complex analysis based on the Cauchy's argument principle can be applied, since they are techniques that inform about the number of singularities of the function within a rectifiable curve by observing the evolution of the function's argument through this curve.

In this way, applying the argument principle to the curve generally known as the Nyquist path (a curve that encloses the right half-plane of the Riemann principal sheet), the stability of the system can be determined by determining the number of revolutions of the resultant curve around the origin. To determine the closed-loop stability, it will be enough to check whether the evaluation curve encloses the critical point $(-1, \mathrm{j}0)$ or not.

For the case of rational-order systems, a similar procedure can be applied. Given a system defined by the transfer function

$$G(s) = \frac{1}{a_n s^{n\alpha} + a_{n-1} s^{(n-1)\alpha} + \cdots + a_1 s^{\alpha} + a_0}, \tag{2.64}$$

where $\alpha = 1/q,\ q, n \in \mathbb{Z}^+,\ a_k \in \mathbb{R}$, we can introduce the mapping $\lambda = s^\alpha$ to obtain the function $G(\lambda)$, and applying the condition (2.62) the stability of the system can be studied by evaluating the function $G(\lambda)$ along the curve Γ defined in the λ-plane in Figure 2.7:

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \tag{2.65}$$

with

$$\begin{aligned}
\Gamma_1 &: \lambda \diagup \arg(\lambda) = -\alpha \frac{\pi}{2}, \quad |\lambda| \in [0, \infty)\,, \\
\Gamma_2 &: \lambda = \lim_{R \to \infty} R^{\mathrm{j}\phi}, \quad \phi \in \left(-\alpha \frac{\pi}{2}, \alpha \frac{\pi}{2}\right), \\
\Gamma_3 &: \lambda \diagup \arg(\lambda) = \alpha \frac{\pi}{2}, \quad |\lambda| \in (\infty, 0)\,.
\end{aligned}$$

If the transfer function has the form

$$H(s) = \frac{1}{a_n s^{p_n/q_n} + a_{n-1} s^{p_{n-1}/q_{n-1}} + \cdots + a_1 s^{p_1/q_1} + a_0}, \tag{2.66}$$

where $p_i,\ q_i \in \mathbb{Z}^+$, and $p_n/q_n > p_{n-1}/q_{n-1} > \cdots > p_1/q_1$, the same procedure can be applied by using the function

$$H(\lambda), \quad \lambda = \frac{1}{q}, \quad q = \operatorname{lcm}\left(q_n, q_{n-1}, \cdots, q_1\right), \tag{2.67}$$

where $\operatorname{lcm}(\cdot)$ stands for the least common multiplier.

To illustrate the application of this procedure, two examples are given.

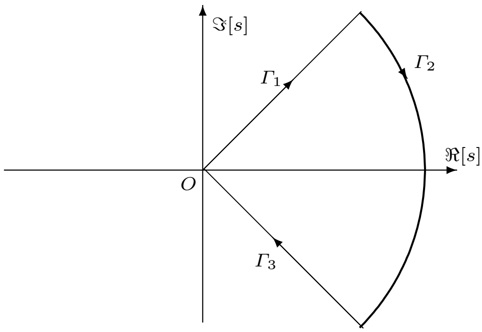

Figure 2.7 The Γ evaluation contour

Example 1 Given a system with transfer function

$$G(s) = \frac{1}{s^{2/3} - s^{1/2} + 1/2},$$

in a unity negative feedback structure with gain K, its evaluation along the Nyquist path defined in the Riemann principal sheet gives the result shown in Figure 2.8. It can be observed that the evaluation curve does not enclose the point $(-1, \mathrm{j}0)$. So, we can conclude that the closed-loop system is stable for any $K > 0$.

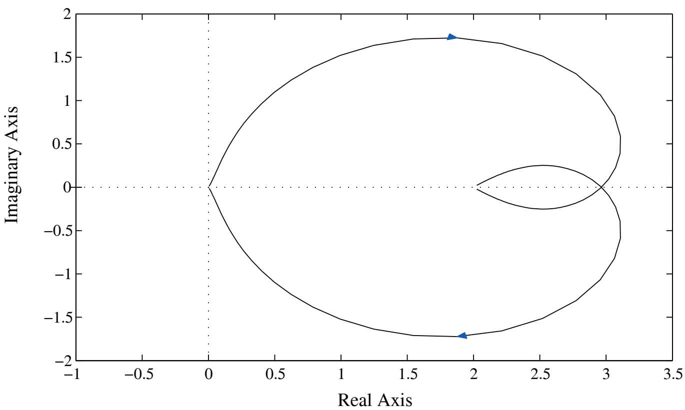

Figure 2.8 Evaluation result for Example 1

An equivalent result can be obtained by evaluating the function

$$G(\lambda) = \frac{1}{\lambda^4 - \lambda^3 + 1/2}, \quad \lambda = s^{1/6},$$

along the curve Γ of the complex plane λ defined by

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \tag{2.68}$$

with

$$\begin{aligned}
&\Gamma_1: \quad \lambda \big/ \arg(\lambda) - \frac{\pi}{12}, \quad |\lambda| \in [0, \infty),\\
&\Gamma_2: \quad \lambda = \lim_{R\to\infty} R^{j\phi}, \quad \phi \in \left(-\frac{\pi}{12}, \frac{\pi}{12}\right),\\
&\Gamma_3: \quad \lambda \big/ \arg(\lambda) = \frac{\pi}{12}, \quad |\lambda| \in (\infty, 0).
\end{aligned}$$

Effectively, the roots of the characteristic equation can be obtained from the roots of the polynomial

$$Q(\lambda) = \lambda^4 - \lambda^3 + \frac{3}{2},$$

which are $\lambda_{1,2} = 1.0891 \pm j0.6923 = 1.2905\angle \pm 0.5662$, $\lambda_{3,4} = -0.5891 \pm j0.7441 = 0.9491\angle \pm 2.2404$, being the roots of the denominator of $G(s)$:

$$\begin{aligned}
s_{1,2} &= (\lambda_{1,2})^6 = 4.6183\angle \pm 3.3975, \quad |\arg(s_{1,2})| > \pi,\\
s_{3,4} &= (\lambda_{3,4})^6 = 0.7308\angle \pm 13.4423, \quad |\arg(s_{3,4})| > \pi.
\end{aligned}$$

As can be observed, there are no structural roots (roots on the Riemann principal sheet defined by $|\arg(s)| < \pi$), which indicates the closed-loop system stability. □

Example 2 If now we deal with the stability of the closed-loop system whose transfer function is

$$G(s) = \frac{1}{s - 2s^{1/2} + 1.25},$$

evaluating the function

$$G(\lambda) = \frac{1}{\lambda^2 - 2\lambda + 1.25}, \qquad \lambda = s^{1/2},$$

along the curve defined by

$$\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3, \tag{2.69}$$

with

$$\begin{aligned}
&\Gamma_1: \quad \lambda \big/ \arg(\lambda) = -\frac{\pi}{4}, \quad |\lambda| \in [0, \infty),\\
&\Gamma_2: \quad \lambda = \lim_{R\to\infty} R^{j\phi}, \quad \phi \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) 2\\
&\Gamma_3: \quad \lambda \big/ \arg(\lambda) = \frac{\pi}{4}, \quad |\lambda| \in (\infty, 0),
\end{aligned}$$

the result shown in Figure 2.9 is obtained.

As can be observed, the resultant curve encloses twice the critical point $(-1, j0)$ in the negative direction for $K > 0.75$. Taking into account that $G(\lambda)$ has two unstable poles, $\lambda_{1,2} = 1 \pm j0.5, \arg(\lambda_{1,2}) < \pi/4$, it can be concluded that the system is stable for $K > 0.75$. For $K < 0.75$ the closed-loop system

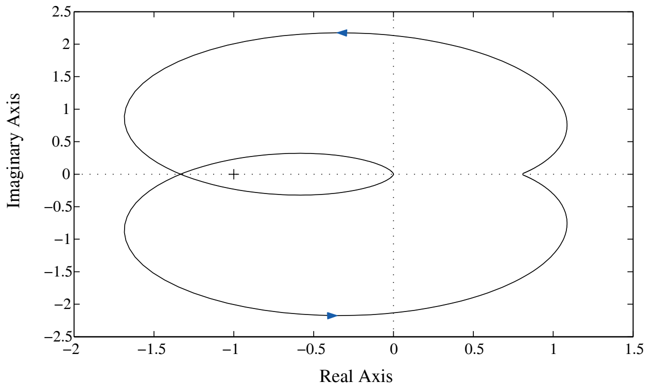

Figure 2.9 Evaluation result for Example 2

is unstable, with two structural roots in the right half-plane of the Riemann principal sheet.

Effectively, the roots of the polynomial

$$Q(\lambda) = \lambda^2 - 2\lambda + 1.25 + K$$

are, for $K = 1$,

$$\lambda_{1,2} = 1.0 \pm \mathrm{j}1.1180,$$

and the structural poles of $G(s)$ are

$$s_{1,2} = (\lambda_{1,2})^2 = 2.25\angle \pm 0.9273, \quad |\arg(s_{1,2})| < \frac{\pi}{2} = 1.5708. \qquad \square$$

It is also important to note that the root locus technique can be applied to a commensurate-order system as easily as it can be applied to an integer-order one. Only the interpretation changes, that is, the relation of the complex plane points $\lambda = s^\alpha$ with the dynamic characteristics of the system.

2.3.3 Analysis of Time and Frequency Domain Responses

2.3.3.1 Transient Response

The general equation for the response of a fractional-order system in the time domain can be determined by using the analytical methods described previously.

The response will depend on the roots of the characteristic equation, having six different cases:

- There are no roots in the Riemann principal sheet. In this case the response will be a monotonically decreasing function.
- There are roots in the Riemann principal sheet, located in $\Re(s)<0$, $\Im(s)=0$. In this case the response will be a monotonically decreasing function.
- There are roots in the Riemann principal sheet, located in $\Re(s)<0$, $\Im(s)\neq 0$. In this case the response will be a function with damped oscillations.
- There are roots in the Riemann principal sheet, located in $\Re(s)=0$, $\Im(s)\neq 0$. In this case the response will be a function with oscillations of constant amplitude.
- There are roots in the Riemann principal sheet, located in $\Re(s)>0$, $\Im(s)\neq 0$. In this case the response will be a function with oscillations of increasing amplitude.
- There are roots in the Riemann principal sheet, located in $\Re(s)>0$, $\Im(s)=0$. In this case the response will be a monotonically increasing function.

In the particular case of commensurate-order systems, the impulse response can be written as follows:

$$\mathscr{L}^{-1}\left\{H(\lambda), \lambda = s^{\alpha}\right\} = \mathscr{L}^{-1}\left\{\frac{\sum\limits_{k=0}^{m} a_k\lambda^k}{\sum\limits_{k=0}^{n} b_k\lambda^k}\right\} = \mathscr{L}^{-1}\left\{\sum_{k=0}^{n}\frac{r_k}{\lambda-\lambda_k}\right\}.$$

Taking into account the general equation

$$\mathscr{L}^{-1}\left\{\frac{s^{\alpha-\beta}}{s^{\alpha}-\lambda_k}\right\} = t^{\beta-1}\mathscr{E}_{\alpha,\beta}\left(\lambda_k t^{\alpha}\right), \tag{2.70}$$

the impulse response, $g(t)$, can be obtained by setting $\alpha=\beta$ in the previous equation as follows:

$$g(t) = \sum_{k=0}^{n} r_k t^{\alpha-1}\mathscr{E}_{\alpha,\alpha}\left(\lambda_k t^{\alpha}\right). \tag{2.71}$$

The step response, given by the equation

$$y(t) = \mathscr{L}^{-1}\left\{\sum_{k=0}^{n}\frac{r_k s^{-1}}{(s^{\alpha}-\lambda_k)}\right\}, \tag{2.72}$$

can be obtained setting $\alpha=\beta-1$ in (2.71), in the following form:

$$y(t) = \sum_{k=0}^{n} r_k t^{\alpha}\mathscr{E}_{\alpha,\alpha+1}\left(\lambda_k t^{\alpha}\right). \tag{2.73}$$

The form of these responses will be:

- monotonically decreasing if $|\arg(\lambda_k)| \geqslant \alpha\pi$;
- oscillatory with decreasing amplitude if $\alpha\pi/2 < |\arg(\lambda_k)| < \alpha\pi$;

- oscillatory with constant amplitude if $|\arg(\lambda_k)| = \alpha\pi/2$;
- oscillatory with increasing amplitude if $|\arg(\lambda_k)| < \alpha\pi/2,\ \ |\arg(\lambda_k)| \neq 0$;
- monotonically increasing if $|\arg(\lambda_k)| = 0$.

These responses are depicted in Figure 2.10.

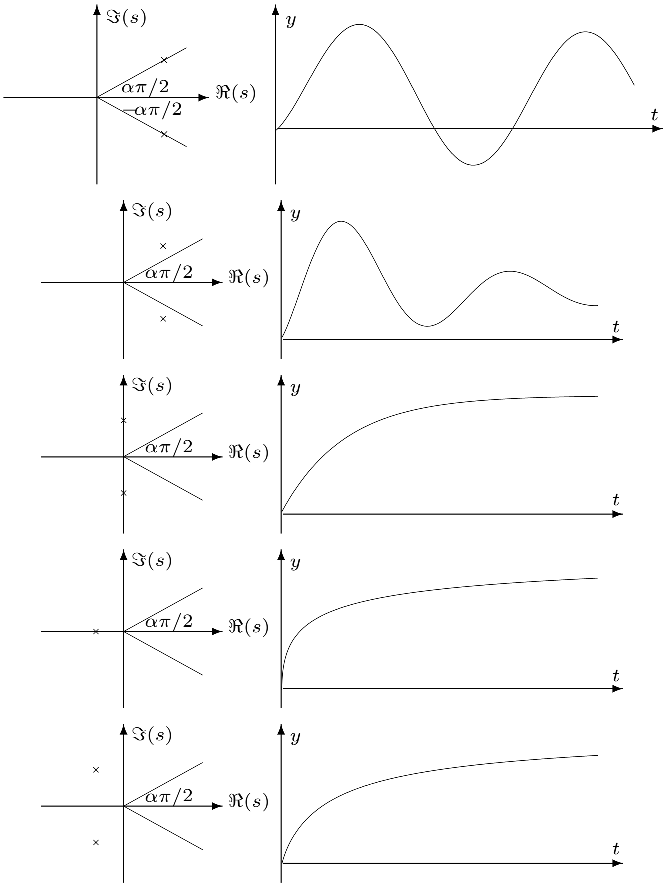

Figure 2.10 Root locations and the corresponding time responses

2.3.3.2 Frequency Domain Response

In general, the frequency response has to be obtained by the direct evaluation of the irrational-order transfer function of the fractional-order system along the imaginary axis for $s = \mathrm{j}\omega,\ \omega \in (0, \infty)$. However, for the commensurate-order systems we can obtain Bode-like plots, in other words, the frequency response can be obtained by the addition of the individual contributions of the terms of order α resulting from the factorization of the function

$$G(s) = \frac{P(s^\alpha)}{Q(s^\alpha)} = \frac{\prod_{k=0}^{m} (s^\alpha + z_k)}{\prod_{k=0}^{n} (s^\alpha + \lambda_k)}, \quad z_k, P(z_k) = 0, \quad \lambda_k, Q(\lambda_k) = 0, \quad z_k \neq \lambda_k.$$

For each of these terms, referred to as $(s^\alpha + \gamma)^{\pm 1}$, the magnitude curve will have a slope which starts at zero and tends to $\pm\alpha 20\,\text{dB/dec}$ for higher frequencies, and the phase plot will go from 0 to $\pm\alpha\pi/2$. Besides, there will be resonances for $\alpha > 1$. To illustrate this, Figure 2.11 shows the frequency response of the system whose transfer function is

$$G(s) = \frac{1}{s^{3/2} + 1}, \tag{2.74}$$

in which can be observed a slope which goes from 0 to $-30\,\text{dB/dec}$, a phase which starts at 0 and tends to $-3\pi/4$, and a resonant frequency of $\omega \approx 0.8\,\text{rad/sec}$.

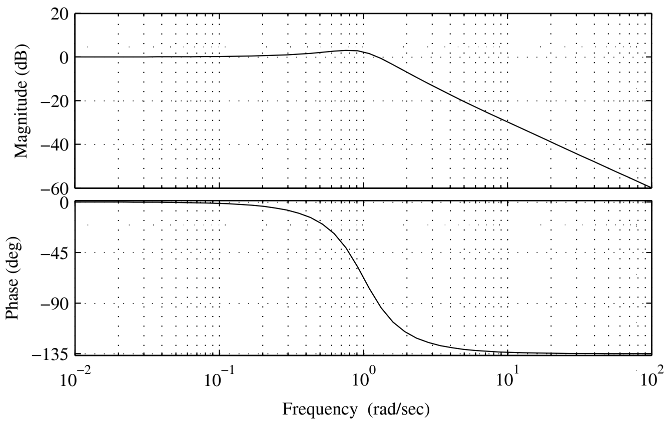

Figure 2.11 Bode plot of the system (2.74)

2.3.3.3 Steady-state Response

To finish this brief analysis of the response, the behavior of the fractional-order systems in steady-state will be considered now, starting from the typical system with unity negative feedback and using the usual definitions of steady-state error coefficients. These definitions are:

- Position error coefficient:

$$K_p = \lim_{s \to 0} G(s). \tag{2.75}$$

- Velocity error coefficient:

$$K_v = \lim_{s \to 0} sG(s). \tag{2.76}$$

- Acceleration error coefficient:

$$K_a = \lim_{s \to 0} s^2 G(s). \tag{2.77}$$

For a fractional-order system whose transfer function can be expressed by

$$G(s) = \frac{K\left(a_m s^{\beta_m} + a_{m-1} s^{\beta_{m-1}} + \cdots + 1\right)}{s^\gamma \left(b_n s^{\alpha_n} + b_{n-1} s^{\alpha_{n-1}} + \cdots + 1\right)}, \tag{2.78}$$

the following relations are obtained:

$$K_{\rm p} = \lim_{s \to 0} \frac{K}{s^\gamma} = \lim_{s \to 0} K s^{-\gamma}, \quad e_{\rm p} = \frac{1}{1 + K_{\rm p}}, \tag{2.79}$$

$$K_{\rm v} = \lim_{s \to 0} K \frac{s}{s^\gamma} = \lim_{s \to 0} K s^{1-\gamma}, \quad e_{\rm v} = \frac{1}{K_{\rm v}}, \tag{2.80}$$

$$K_{\rm a} = \lim_{s \to 0} K \frac{s^2}{s^\gamma} = \lim_{s \to 0} K s^{2-\gamma}, \quad e_{\rm a} = \frac{1}{K_{\rm a}}. \tag{2.81}$$

In Table 2.1, the steady-state errors and steady-state error coefficients are summarized for different values of γ. As can be observed, the fractional-order systems always have steady-state error coefficients 0 or ∞, and this shows that their behavior, also in steady-state, has to do with the behavior of the integer-order systems of higher or lower order than the fractional order. These systems will have finite coefficients only for inputs whose temporal dependence is of the form

$$r(t) = At^\gamma, \tag{2.82}$$

or, in Laplace domain,

$$\mathscr{L}\{r(t)\} = R(s) = A\frac{\Gamma(\gamma+1)}{s^{\gamma+1}}. \tag{2.83}$$

Table 2.1 Steady-state error coefficients

Steady state					Steady state				
γ	$K_{\rm p}, e_{\rm p}$	$K_{\rm v}, e_{\rm v}$	$K_{\rm a},\ e_{\rm a}$	Type	γ	$K_{\rm p},\ e_{\rm p}$	$K_{\rm v},\ e_{\rm v}$	$K_{\rm a},\ e_{\rm a}$	Type
0	$K,\ 1/(1+K)$	$0,\ \infty$	$0,\ \infty$	0	(0,1)	∞, 0	$0,\ \infty$	$0,\ \infty$	0/1
1	∞, 0	$K,\ 1/K$	$0,\ \infty$	1	(1,2)	∞, 0	∞, 0	$0,\ \infty$	1/2
2	∞, 0	∞, 0	$K,\ 1/K$	2	(2,3)	∞, 0	∞, 0	∞, 0	2/3

2.3.4 Bode's Ideal Loop Transfer Function as Reference System

2.3.4.1 Introduction

In the previous sections it has been shown that the system

$$G(s) = \frac{A}{s^\alpha + A}, \quad 0 < \alpha < 2, \tag{2.84}$$

can exhibit behaviors that range from relaxation to oscillation, including the behaviors corresponding to first- and second-order systems as particular cases. For this reason, it is interesting to take this system as the reference system. It was first proposed in [15] and is the starting point for the CRONE control [16, 17]. This function can be considered as the result of the closed-loop connection of a fractional-order integrator with gain A and order α (see Figure 2.12), that is, a system whose open-loop transfer function is given by

$$F(s) = \frac{A}{s^\alpha}, \quad 0 < \alpha < 2. \tag{2.85}$$

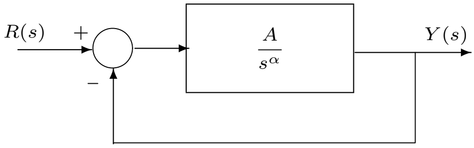

Figure 2.12 Reference system

Bode called this transfer function, $F(s)$, *the ideal open-loop transfer function* [3, 18].

2.3.4.2 General Characteristics

This ideal transfer function has the following characteristics:

1. Open-loop:

- The magnitude curve has a constant slope of -20α dB/dec.
- The gain crossover frequency depends on A.
- The phase plot is a horizontal line of value $-\alpha\pi/2$.
- The Nyquist plot is a straight line which starts from the origin with argument $-\alpha\pi/2$.

2. Closed-loop with unity negative feedback:

- The gain margin is infinite.
- The phase margin is constant with value $\varphi_\mathrm{m} = \pi(1-\alpha/2)$, only depending on α.
- The step response is of the form

$$y(t) = At^\alpha \mathscr{E}_{\alpha,\alpha+1}(-At^\alpha). \tag{2.86}$$

2.3.4.3 Step Response and Characteristic Parameters

In Figure 2.13 the step responses of the system $F(s)$ for $A = 1$ and different values of α are represented.

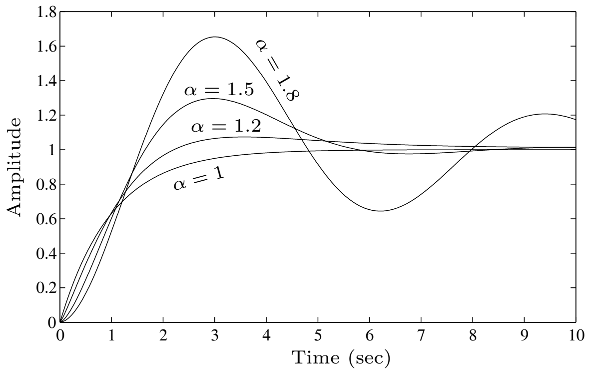

Figure 2.13 Step response (2.86) with $A = 1$

These curves correspond to the damping ratios and the natural frequency that can be obtained from the structural roots of the denominator of $F(s)$. These roots are

$$s_{1,2} = A^{1/\alpha} \mathrm{e}^{\mathrm{j}\pi/\alpha} = A^{1/\alpha}\left(\cos\frac{\pi}{\alpha} + \mathrm{j}\sin\frac{\pi}{\alpha}\right). \tag{2.87}$$

By using the well known relations

$$\omega_n = |s_{1,2}|, \quad -\delta\omega_\mathrm{n} = \Re(s_{1,2}), \quad \omega_\mathrm{p} = \omega_\mathrm{n}\sqrt{1-\delta^2}, \tag{2.88}$$

for the natural frequency ω_n, the damping ratio δ, and the damped natural frequency of the system ω_p as functions of the position of the poles, these characteristic parameters can be determined by

$$\delta = -\cos\frac{\pi}{\alpha}, \ \omega_\mathrm{n} = A^{1/\alpha}, \ \omega_\mathrm{p} = A^{1/\alpha}\sqrt{1-\left(-\cos\frac{\pi}{\alpha}\right)^2} = A^{1/\alpha}\sin\frac{\pi}{\alpha}. \tag{2.89}$$

Other alternative definitions can be found in [15, 16].

2.3.4.4 Frequency Response and Characteristic Parameters

To complete the characterization of the system as done for the integer-order ones, the frequency and the resonant peak can be determined. For this purpose, we will set $s = \mathrm{j}\omega$ to obtain

$$F(\mathrm{j}\omega) = \frac{A}{(\mathrm{j}\omega)^\alpha + A} = \frac{A}{(\omega^\alpha \cos \alpha\pi/2 + A) + \mathrm{j}\omega^\alpha \sin \alpha\pi/2}. \tag{2.90}$$

The magnitude of this function is given by

$$|F(\mathrm{j}\omega)| = \frac{A}{\sqrt{\omega^{2\alpha} + 2A\omega^\alpha \cos \alpha\pi/2 + A^2}}, \tag{2.91}$$

having the maximum at

$$\omega^\alpha = -A \cos \alpha \frac{\pi}{2} \Longrightarrow \omega_\mathrm{r} = \left(-A \cos \alpha \frac{\pi}{2}\right)^{1/\alpha}, \quad \alpha > 1. \tag{2.92}$$

By substituting the equation obtained for the resonant frequency ω_r in the equation of the magnitude, the equation for the resonant peak is

$$M_\mathrm{r} = \frac{1}{\sin \alpha\pi/2}. \tag{2.93}$$

As can be seen, the resonant peak, like the damping ratio, only depends on α. Figure 2.14 shows the magnitude of the frequency responses for $A = 1, \alpha = 1, 1.2, 1.5, 1.8$.

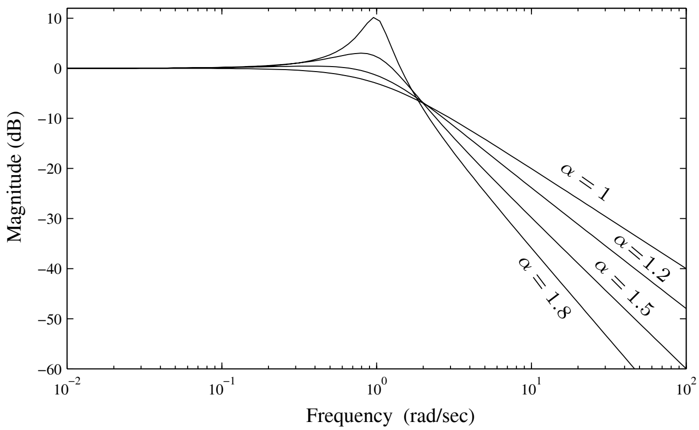

Figure 2.14 Magnitude of the frequency responses of the system (2.84)

2.4 Summary

The aim of this chapter has been to provide the reader with the essentials of input-output models (external representations) for fractional-order linear time invariant systems, as well as the dynamical properties (stability, time transient and steady-state responses, and frequency response) usually considered in classical control theory. With this aim, we have introduced two preliminary sections, the first devoted to the fundamental definitions of fractional-order operators in both time and Laplace domain, and the second to the analytical and numerical solutions of the fractional-order ordinary differential equations. As an introduction to the fractional-order control, a brief study of the so-called *Bode's ideal function* has been included. The necessary tools for using modern control theory (state-space or internal representations) will be given in the following chapter. Numerical implementations of the content of this chapter can be found in Chapter 13.

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